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GRENOBLE 1

A sharp lower bound for the log canonical threshold

Jean-Pierre Demailly / Phạm Hoàng Hiệp

Institut Fourier, Université de Grenoble I, France

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dediated to the memory of Mikael Passare
in honor of Urban Cegrell, on the occasion of his retirement

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log canonical threshold of psh functions

Singularities of psh (plurisubharmonic) functions can be measured by Lelong numbers. Another useful invariant is the log canonical threshold.

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$$c_p(\varphi) = \sup \left\{ c \geq 0 \ : \ e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } p \right\},$$

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Here we will take $p = 0$ and denote $c(\varphi) = c_0(\varphi)$.

log canonical threshold of coherent ideals

The log canonical threshold is a subtle invariant. A special interesting case is

$$\varphi(z) = \frac{1}{2}\log(|g_1|^2 + \ldots + |g_N|^2)$$

associated to some ideal $\mathcal{J} = (g_1, \ldots, g_N)$ of polynomials or holomorphic functions on some complex manifold X.

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$$c(\varphi) = \min_{E_j,\, \mu(E_j) \ni 0} \frac{1 + b_j}{a_j} \in \mathbb{Q}_+^*.$$

Proof of the formula for the log canonical threshold

In fact, we have to find the supremum of $c > 0$ such that

$$I = \int_{V \ni 0} \frac{d\lambda(z)}{\left(|g_1|^2 + \ldots + |g_N|^2\right)^c} < +\infty.$$

Let us perform the change of variable $z = \mu(w)$. Then

$$d\lambda(z) = |\mathrm{Jac}(\mu)(w)|^2 \sim \Big|\prod w_j^{b_j}\Big|^2 d\lambda(w)$$

with respect to coordinates on the blow-up $\widetilde{V}$ of V, and

$$I \sim \int_{\widetilde{V}} \frac{\left|\prod w_j^{b_j}\right|^2 d\lambda(w)}{\left|\prod w_j^{a_j}\right|^{2c}}$$

so convergence occurs if $ca_j - b_j < 1$ for all j.

Notations and basic facts

- A domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if $\exists \psi \in \mathcal{PSH}(\Omega)$, $\psi \leq 0$, such that $\{z : \psi(z) < c\} \Subset \Omega$ for all $c < 0$.

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- $\mathcal{E}_0(\Omega) = \left\{ \varphi \in \mathcal{PSH} \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < +\infty \right\}$

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- $\mathcal{F}(\Omega) = \Big\{ \varphi \in \mathcal{PSH}(\Omega) : \exists\, \mathcal{E}_0(\Omega) \ni \varphi_p \searrow \varphi$, and
$$\sup_{p \geq 1} \int_\Omega (dd^c \varphi_p)^n < +\infty \Big\},$$
- $\widetilde{\mathcal{E}}(X) = \{\varphi \in \mathcal{PSH}(X)$ locally in $\mathcal{F}(\Omega)$ mod $C^\infty(\Omega)\}$

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Theorem (U. Cegrell)

$\widetilde{\mathcal{E}}(X)$ is the largest subclass of psh functions defined on a complex manifold X for which the complex Monge-Ampère operator is locally well-defined.

Intermediate Lelong numbers

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$$e_j(\varphi) = \nu\big((dd^c\varphi)^j, 0\big).$$

In other words

$$e_j(\varphi) = \int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c \log \|z\|)^{n-j}\,.$$

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One has $e_0(\varphi) = 1$ and $e_1(\varphi) = \nu(\varphi, 0)$ (usual Lelong number). When

$$\varphi(z) = \frac{1}{2}\log(|g_1|^2 + \ldots + |g_N|^2),$$

one has $e_j(\varphi) \in \mathbb{N}$.

The main result

Main Theorem (Demailly & Phạm)

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Otherwise, we have

$$c(\varphi) \geq \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)}.$$

The lower bound improves a classical result of H. Skoda (1972), according to which

$$\frac{1}{e_1(\varphi)} \leq c(\varphi) \leq \frac{n}{e_1(\varphi)}.$$

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Remark: The above theorem is optimal, with equality for

$$\varphi(z) = \log(|z_1|^{a_1} + \ldots + |z_n|^{a_n}), \quad 0 < a_1 \leq a_2 \leq \ldots \leq a_n.$$

Then $e_j(\varphi) = a_1 \ldots a_j, \quad c(\varphi) = \dfrac{1}{a_1} + \ldots + \dfrac{1}{a_n}.$

Geometric applications

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Another important application is to birational rigidity.

Theorem (Pukhlikov 1998, Corti 2000, de Fernex 2011)

Let X be a smooth hypersurface of degree d in $\mathbb{CP}^{n+1}$. Then if $d = n + 1$, $\mathrm{Bir}(X) \simeq \mathrm{Aut}(X)$

It was first shown by Manin-Iskovskih in the early 70's that a 3-dim quartic in $\mathbb{CP}^4$ ($n = 3$, $d = 4$) is not rational.

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Question

For $3 \leq d \leq n + 1$, when is it true that $\mathrm{Bir}(X) \simeq \mathrm{Aut}(X)$ (birational rigidity) ?

Lemma 1

Let $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ and $0 \in \Omega$. Then we have that

$$e_j(\varphi)^2 \leq e_{j-1}(\varphi)e_{j+1}(\varphi),$$

for all $j = 1, \ldots, n-1$.

In other words $j \mapsto \log e_j(\varphi)$ is convex, thus we have $e_j(\varphi) \geq e_1(\varphi)^j$ and the ratios $e_{j+1}(\varphi)/e_j(\varphi)$ are increasing.

Corollary

If $e_1(\varphi) = \nu(\varphi, 0) = 0$, then $e_j(\varphi) = 0$ for $j = 1, 2, \ldots, n-1$.

A hard conjecture by V. Guedj and A. Rashkovskii ($\sim$ 1998) states that $\varphi \in \widetilde{\mathcal{E}}(\Omega)$, $e_1(\varphi) = 0$ also implies $e_n(\varphi) = 0$.

Proof of Lemma 1

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Without loss generality we can assume that Ω is the unit ball and $\varphi \in \mathcal{E}_0(\Omega)$. For h, $\psi \in \mathcal{E}_0(\Omega)$ an integration by parts and the Cauchy-Schwarz inequality yield

$$\left[\int_\Omega -h(dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2$$

$$= \left[\int_\Omega d\varphi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h\right]^2$$

$$\leq \int_\Omega d\psi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h$$

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Proof of Lemma 1, continued

Now, as $p \to +\infty$, take

$$h(z) = h_p(z) = \max\left(-1, \frac{1}{p}\log\|z\|\right) \nearrow \begin{cases} 0 & \text{if } z \in \Omega\setminus\{0\} \\ -1 & \text{if } z = 0. \end{cases}$$

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By the monotone convergence theorem we get in the limit that

$$\left[\int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2 \le \int_{\{0\}} (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j+1}$$
$$\int_{\{0\}} (dd^c\varphi)^{j+1} \wedge (dd^c\psi)^{n-j-1}.$$

For $\psi(z) = \ln\|z\|$, this is the desired estimate. $\square$

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$$\sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)} \leq \sum_{j=0}^{n-1} \frac{e_j(\psi)}{e_{j+1}(\psi)} .$$

The argument if based on the monotonicity of Lelong numbers with respect to the relation $\varphi \leq \psi$, and on the monotonicity of the right hand side in the relevant range of values.

Proof of Lemma 2

Set

$$D = \{t=(t_1, ..., t_n)\in[0, +\infty)^n : t_1^2\leq t_2, t_j^2\leq t_{j-1}\, t_{j+1}, \forall j = 2, ..., n-1\}.$$

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Then D is a convex set in $\mathbb{R}^n$, as can be checked by a straightforward application of the Cauchy-Schwarz inequality. Next, consider the function $f : \mathrm{int}\, D \to [0, +\infty)$ defined by

$$f(t_1, \ldots, t_n) = \frac{1}{t_1} + \frac{t_1}{t_2} \ldots + \frac{t_{n-1}}{t_n}.$$

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We have

$$\frac{\partial f}{\partial t_j}(t) = -\frac{t_{j-1}}{t_j^2} + \frac{1}{t_{j+1}} \leq 0, \qquad \forall t \in D.$$

Proof of Lemma 2, continued

For $a, b \in \text{int } D$ such that $a_j \geq b_j$, $j = 1, \ldots, n$, the function

$$[0, 1] \ni \lambda \to f(b + \lambda(a - b))$$

is decreasing.

Proof of Lemma 2, continued

For $a, b \in \operatorname{int} D$ such that $a_j \geq b_j$, $j = 1, \ldots, n$, the function

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$$f(a) \leq f(b) \qquad \text{for all } a, b \in \operatorname{int} D,\ a_j \geq b_j,\ j = 1, \ldots, n\,.$$

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On the other hand, the hypothesis $\varphi \leq \psi$ implies that $e_j(\varphi) \geq e_j(\psi)$, $j = 1, \ldots, n$, by the comparison principle. Therefore we have that

$$f(e_1(\varphi), \ldots, e_n(\varphi)) \leq f(e_1(\psi), \ldots, e_n(\psi))\,.$$

□

Proof of the Main Theorem

It will be convenient here to introduce Kiselman's refined Lelong number.

Definition

Let $\varphi \in \mathcal{PSH}(\Omega)$. Then the function defined by

$$\nu_\varphi(x) = \lim_{t \to -\infty} \frac{\max\left\{\varphi(z) : |z_1| = e^{x_1 t}, \ldots, |z_n| = e^{x_n t}\right\}}{t}$$

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The refined Lelong number of φ at 0 is increasing in each variable x_j, and concave on $\mathbb{R}^n{}_+$.

Proof of the Main Theorem

The proof is divided into the following steps:

- Proof of the theorem in the toric case, i.e. $\varphi(z_1, \ldots, z_n) = \varphi(|z_1|, \ldots, |z_n|)$ depends only on $|z_j|$ and therefore we can without loss of generality assume that $\Omega = \Delta^n$ is the unit polydisk.

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- Reduction to the case of plurisubharmonic functions with analytic singularity, i.e. $\varphi = \log(|f_1|^2 + \ldots + |f_N|^2)$, where $f_1, \ldots, f_N$ are germs of holomorphic functions at 0.

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- Reduction to the case of plurisubharmonic functions with analytic singularity, i.e. $\varphi = \log(|f_1|^2 + \ldots + |f_N|^2)$, where $f_1, \ldots, f_N$ are germs of holomorphic functions at 0.

- Reduction to the case of monomial ideals, i.e. for $\varphi = \log(|f_1|^2 + \ldots . + |f_N|^2)$, where $f_1, \ldots, f_N$ are germs of monomial elements at 0.

Proof of the theorem in the toric case

Set

$$\Sigma = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{j=1}^{n} x_j = 1 \right\}.$$

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By Theorem 5.8 in [Kis94] we have the following formula

$$c(\varphi) = \frac{1}{\nu_\varphi(x^0)}.$$

Proof of the theorem in the toric case, continued

Set $\zeta(x) = \nu_\varphi(x^0) \min\left(\frac{x_1}{x_1^0}, \ldots, \frac{x_n}{x_n^0}\right), \qquad \forall x \in \Sigma.$

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Then ζ is the smallest nonnegative concave increasing function on Σ such that $\zeta(x^0) = \nu_\varphi(x^0)$, hence $\zeta \leq \nu_\varphi$.

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$$\begin{aligned}\varphi(z_1, \ldots, z_n) \le -\nu_\varphi(-\ln|z_1|, \ldots, -\ln|z_n|)\\ \le -\zeta(-\ln|z_1|, \ldots, -\ln|z_n|)\\ \le \nu_\varphi(x^0)\max\left(\frac{\ln|z_1|}{x_1^0}, \ldots, \frac{\ln|z_n|}{x_n^0}\right) := \psi(z_1, \ldots, z_n).\end{aligned}$$

Proof of the theorem in the toric case, continued

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$$\begin{aligned}\varphi(z_1, \ldots, z_n) &\leq -\nu_\varphi(-\ln|z_1|, \ldots, -\ln|z_n|)\\ &\leq -\zeta(-\ln|z_1|, \ldots, -\ln|z_n|)\\ &\leq \nu_\varphi(x^0) \max\left(\frac{\ln|z_1|}{x_1^0}, \ldots, \frac{\ln|z_n|}{x_n^0}\right) := \psi(z_1, \ldots, z_n).\end{aligned}$$

By Lemma 2 we get that

$$f(e_1(\varphi), ..., e_n(\varphi)) \leq f(e_1(\psi), ..., e_n(\psi)) = c(\psi) = \frac{1}{\nu_\varphi(x^0)} = c(\varphi)\,.$$

Reduction to the case of plurisubharmonic functions with analytic singularity

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and let $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$ where $\{g_{m,k}\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H}_{m\varphi}(\Omega)$.

Reduction to the case of plurisubharmonic functions with analytic singularity

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$$\int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty ,$$

and let $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$ where $\{g_{m,k}\}_{k\geq 1}$ be an orthonormal basis for $\mathcal{H}_{m\varphi}(\Omega)$. Using $\bar\partial$-equation with L^2-estimates (D-Kollár), there are constants $C_1, C_2 > 0$ independent of m such that

$$\varphi(z) - \frac{C_1}{m} \leq \psi_m(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$.

Reduction to the case of plurisubharmonic functions with analytic singularity, continued

and

$$\nu(\varphi) - \frac{n}{m} \le \nu(\psi_m) \le \nu(\varphi), \qquad \frac{1}{c(\varphi)} - \frac{1}{m} \le \frac{1}{c(\psi_m)} \le \frac{1}{c(\varphi)}.$$

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By Lemma 2, we have that

$$f(e_1(\varphi), \ldots, e_n(\varphi)) \le f(e_1(\psi_m), \ldots, e_n(\psi_m)), \qquad \forall m \ge 1.$$

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The above inequalities show that in order to prove the lower bound of $c(\varphi)$ in the Main Theorem, we only need prove it for $c(\psi_m)$ and then let $m \to \infty$.

Reduction to the case of monomial ideals

For $j = 0, \ldots, n$ set

$$\mathcal{J} = (f_1, \ldots, f_N), \; c(\mathcal{J}) = c(\varphi), \; \text{and } e_j(\mathcal{J}) = e_j(\varphi)\,.$$

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Now, by fixing a multiplicative order on the monomials

$$z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

it is well known that one can construct a flat family $(\mathcal{J}_s)_{s\in\mathbb{C}}$ of ideals of $\mathcal{O}_{\mathbb{C}^n,0}$ depending on a complex parameter $s \in \mathbb{C}$, such that $\mathcal{J}_0$ is a monomial ideal, $\mathcal{J}_1 = \mathcal{J}$ and

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In fact $\mathcal{J}_0$ is just the initial ideal associated to $\mathcal{J}$ with respect to the monomial order.

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compute the intermediate multiplicities

$$e_p(\mathcal{J}_s) = \lim_{t\to+\infty} \frac{p!}{t^p} \dim\left(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J}_{s|\mathbb{C}^p})^t\right) = e_p(\mathcal{J}),$$

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in particular, $e_p(\mathcal{J}_0) = e_p(\mathcal{J})$ for all p. The semicontinuity property of the log canonical threshold implies that $c(\mathcal{J}_0) \leq c(\mathcal{J}_s) = c(\mathcal{J})$ for all s, so the lower bound is valid for $c(\mathcal{J})$ if it is valid for $c(\mathcal{J}_0)$.

About the continuity of Monge-Ampère operators

Conjecture

Let $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ and $\Omega \ni 0$. Then the analytic approximations ψ_m satisfy $e_j(\psi_m) \to e_j(\varphi)$ as $m \to +\infty$, in other words, we have "strong continuity" of Monge-Ampère operators and higher Lelong numbers with respect to Bergman kernel approximation.

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In the 2-dimensional case, $e_2(\varphi)$ can be computed as follows (at least when $\varphi \in \widetilde{\mathcal{E}}(\omega)$ has analytic singularities). Let $\mu : \widetilde{\Omega} \to \Omega$ be the blow-up of Ω at 0. Take local coordinates (w_1, w_2) on $\widetilde{\Omega}$ so that the exceptional divisor E can be written $w_1 = 0$.

About the continuity of Monge-Ampère operators (II)

With $\gamma = \nu(\varphi, 0)$, we get that

$$\widetilde{\varphi}(w) = \varphi \circ \mu(w) - \gamma \log |w_1|$$

is psh with generic Lelong numbers equal to 0 along E, and therefore there are at most countably many points $x_\ell \in E$ at which $\gamma_\ell = \nu(\widetilde{\varphi}, x_\ell) > 0$. Set $\Theta = dd^c\varphi$, $\widetilde{\Theta} = dd^c\widetilde{\varphi} = \mu^*\Theta - \gamma[E]$. Since $E^2 = -1$ in cohomology, we have $\{\widetilde{\Theta}\}^2 = \{\mu^*\Theta\}^2 - \gamma^2$ in $H^2(E, \mathbb{R})$, hence

$$(*) \qquad \int_{\{0\}} (dd^c\varphi)^2 = \gamma^2 + \int_E (dd^c\widetilde{\varphi})^2.$$

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If $\widetilde{\varphi}$ only has ordinary logarithmic poles at the x_ℓ's, then $\int_E (dd^c\widetilde{\varphi})^2 = \sum \gamma_\ell^2$, but in general the situation is more complicated. Let us blow-up any of the points x_ℓ and repeat the process k times.

About the continuity of Monge-Ampère operators (III)

We set $\ell = \ell_1$ in what follows, as this was the first step, and at step $k = 0$ we omit any indices as 0 is the only point we have to blow-up to start with. We then get inductively $(k+1)$-iterated blow-ups depending on multi-indices $\ell = (\ell_1, \ldots, \ell_k) = (\ell', \ell_k)$ with $\ell' = (\ell_1, \ldots, \ell_{k-1})$,

$$\mu_\ell : \widetilde{\Omega}_\ell \to \widetilde{\Omega}_{\ell'}, \quad k \geq 1, \;\; \mu_\emptyset = \mu : \widetilde{\Omega}_\emptyset = \widetilde{\Omega} \to \Omega, \;\; \gamma_\emptyset = \gamma$$

and exceptional divisors $E_\ell \subset \widetilde{\Omega}_\ell$ lying over points $x_\ell \in E_{\ell'} \subset \widetilde{\Omega}_{\ell'}$, where

$$\gamma_\ell = \nu(\widetilde{\varphi}_{\ell'}, x_\ell) > 0,$$
$$\widetilde{\varphi}_\ell(w) = \widetilde{\varphi}_{\ell'} \circ \mu_\ell(w) - \gamma_\ell \log|w_1^{(\ell)}|,$$
$$(w_1^{(\ell)} = 0 \text{ an equation of } E_\ell \text{ in the relevant chart}).$$

About the continuity of Monge-Ampère operators (IV)

Formula $(*)$ implies

$$(**) \qquad e_2(\varphi) \geq \sum_{k=0}^{+\infty} \sum_{\ell \in \mathbb{N}^k} \gamma_\ell^2$$

with equality when φ has an analytic singularity at 0. We conjecture that $(**)$ is always an equality whenever $\varphi \in \widetilde{\mathcal{E}}(\Omega)$.

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Notice that the currents $\Theta_\ell = dd^c \widetilde{\varphi}_\ell$ satisfy inductively $\Theta_\ell = \mu_\ell^* \Theta_{\ell'} - \gamma_\ell [E_\ell]$, hence the cohomology class of Θ_ℓ restricted to E_ℓ is equal to γ_ℓ times the fundamental generator of E_ℓ. As a consequence we have

$$\sum\nolimits_{\ell_{k+1} \in \mathbb{N}} \gamma_{\ell, \ell_{k+1}} \leq \gamma_\ell,$$

in particular $\gamma_\ell = 0$ for all $\ell \in \mathbb{N}^k$ if $\gamma = \nu(\varphi, 0) = 0$.

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