UNIVERSITÉ Grenoble Alpes

INSTITUT DE FRANCE
Académie des sciences

Ricci curvature and geometry of compact Kähler varieties

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

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General plan of the Kähler main lectures

Ricci curvature and geometry of compact Kähler varieties

- Lecture 1: Positivity concepts in Kähler geometry
  – definitions and characterizations of the concept of ample, nef, big and pseudoeffective line bundles and (1,1)-classes
  – Numerical characterization of the Kähler cone
  – Approximate analytic Zariski decomposition and abundance
- Lecture 2: Uniruledness, rational connectedness and $-K_X$ nef
  – Orthogonality estimates and duality of positive cones
  – Criterion for uniruledness and rational connectedness
  – Examples of compact Kähler mflds X with $-K_X \geq 0$ or nef.
- Lecture 3: Holonomy and main structure theorems
  – concept of holonomy of euclidean & hermitian vector bundles
  – De Rham splitting theorem and Berger's classification of holonomy groups
  – Generalized holonomy principle and structure theorems
  – investigation of the case when $-K_X$ is nef (Cao, Höring)

Lecture 1

Positivity concepts in Kähler geometry

A brief survey of the main positivity concepts in algebraic and analytic geometry.

Complex manifolds / (p, q)-forms

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- A complex n-dimensional manifold is given by coordinate charts equipped with local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$.

Complex manifolds / (p, q)-forms

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- A complex n-dimensional manifold is given by coordinate charts equipped with
  local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$.
- A differential form u of type (p, q) can be written as a sum

$$u(z) = \sum_{|J|=p,|K|=q} u_{JK}(z)\, dz_J \wedge d\overline{z}_K$$

where $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$,

$$dz_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}, \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_q}.$$

Complex manifolds / Currents

- A current is a differential form with distribution coefficients

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- The current T is said to be positive if the distribution $\sum \lambda_j \overline{\lambda}_k T_{JK}$ is a positive real measure for all $(\lambda_J) \in \mathbb{C}^N$ (so that $T_{KJ} = \overline{T}_{JK}$, hence $\overline{T} = T$).
- The coefficients T_{JK} are then complex measures – and the diagonal ones T_{JJ} are positive real measures.

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- The coefficients T_{JK} are then complex measures – and the diagonal ones T_{JJ} are positive real measures.
- T is said to be closed if $dT = 0$ in the sense of distributions.

- The current of integration over a codimension p analytic cycle $A = \sum c_j A_j$ is defined by duality as $[A] = \sum c_j [A_j]$ with

$$\langle [A_j], u \rangle = \int_{A_j} u_{|A_j}$$

for every $(n-p, n-p)$ test form u on X.

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- Hessian forms of plurisubharmonic functions :

$$\varphi \ \text{ plurisubharmonic} \Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right) \geq 0$$

then

$$T = i\partial\overline{\partial}\varphi \qquad \text{is a closed positive } (1,1)\text{-current.}$$

Complex manifolds / Kähler metrics

- A Kähler metric is a smooth positive definite $(1,1)$-form

$$\omega(z) = i \sum_{1\le j,k\le n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k \qquad \text{such that } d\omega = 0.$$

- The manifold X is said to be Kähler (or of Kähler type) if it possesses at least one Kähler metric ω [Kähler 1933]

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- The manifold X is said to be Kähler (or of Kähler type) if it possesses at least one Kähler metric ω [Kähler 1933]
- Every complex analytic and locally closed submanifold $X \subset \mathbb{P}^N_{\mathbb{C}}$ in projective space is Kähler when equipped with the restriction of the Fubini-Study metric

$$\omega_{FS} = \frac{i}{2\pi} \log(|z_0|^2 + |z_1|^2 + \ldots + |z_N|^2).$$

- Especially projective algebraic varieties are Kähler.

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- $\Omega_X^p = \mathcal{O}(\Lambda^p T_X^*) =$ sheaf of holomorphic p-forms on X.

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- $\Omega^p_X=\mathcal{O}(\Lambda^p T^*_X)=$ sheaf of holomorphic p-forms on X.
- Cohomology classes [forms / currents yield same groups]

$$\alpha\ d\text{-closed } k\text{-form/current to } \mathbb{C} \longmapsto \{\alpha\}\in H^k(X,\mathbb{C})$$

$$\alpha\ \overline{\partial}\text{-closed } (p,q)\text{-form/current to } F \longmapsto \{\alpha\}\in H^{p,q}(X,F)$$

Dolbeault isomorphism (Dolbeault - Grothendieck 1953)

$$H^{0,q}(X,F)\simeq H^q(X,\mathcal{O}(F)),$$

$$H^{p,q}(X,F)\simeq H^q(X,\Omega^p_X\otimes\mathcal{O}(F))$$

Bott-Chern and Aeppli cohomology

The Bott-Chern cohomology groups are defined as

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) := \{(p,q)-\text{forms } u \text{ such that } \partial u = \overline{\partial} u = 0\} \;/\; \{(p,q)-\text{forms } u = \partial\overline{\partial} v\}$$

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These groups are dual each other via Serre duality:

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \times H^{n-p,n-q}_{A}(X,\mathbb{C}) \to \mathbb{C}, \quad (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

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One always has morphisms

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C}) \to H^{p,q}_{\mathrm{A}}(C,\mathbb{C}).$$

They are not always isomorphisms, but are if X is Kähler.

Hodge decomposition theorem

- **Theorem.** *If (X,ω) is compact Kähler, then*
$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}).$$
- *Each group $H^{p,q}(X,\mathbb{C})$ is isomorphic to the space of (p,q) harmonic forms α with respect to ω, i.e. $\Delta_\omega\alpha = 0$.*

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Hodge Conjecture (a millenium problem!).

If X is a projective algebraic manifold,
Hodge (p,p)-classes $= H^{p,p}(X,\mathbb{C}) \cap H^{2p}(X,\mathbb{Q})$
are generated by classes of algebraic cycles of codimension p with $\mathbb{Q}$-coefficients.

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Theorem (Claire Voisin, 2001)

There exists a 4-dimensional complex torus X possessing a non trivial Hodge class of type $(2,2)$, such that every coherent analytic sheaf $\mathcal{F}$ on X satisfies $c_2(\mathcal{F}) = 0$.

Kodaira embedding theorem

Theorem (Kodaira 1953)

Let X be a compact complex n-dimensional manifold. Then the following properties are equivalent.

- X can be embedded in some projective space $\mathbb{P}^N_{\mathbb{C}}$ as a closed analytic submanifold (and such a submanifold is automatically algebraic by Chow's thorem).
- X carries a hermitian holomorphic line bundle (L, h) with positive definite smooth curvature form $i\Theta_{L,h} > 0$.
  For $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|^2_h = |\xi|^2 e^{-\varphi(x)}$,
$$i\Theta_{L,h} = i\partial\overline{\partial}\varphi = -i\partial\overline{\partial}\log h,$$
$$c_1(L) = \left\{\frac{i}{2\pi}\Theta_{L,h}\right\}.$$
- X possesses a Hodge metric, i.e., a Kähler metric ω such that $\{\omega\} \in H^2(X, \mathbb{Z})$.

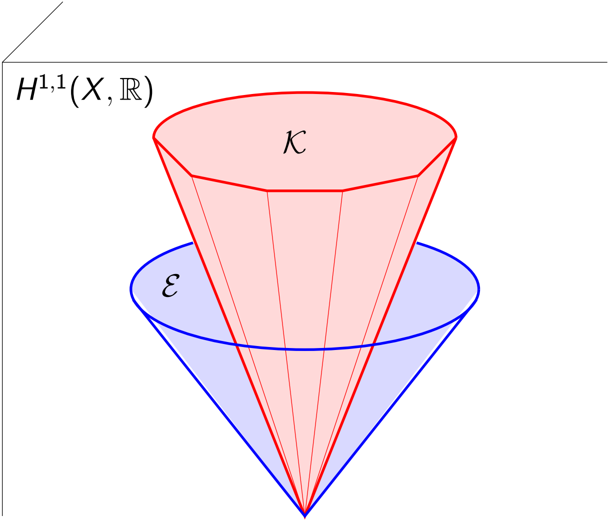
Positive cones

Definition

Let X be a compact Kähler manifold.

- The Kähler cone is the set $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The pseudo-effective cone is the set $\mathcal{E} \subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone. (by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.
- One can have: $\overline{\mathcal{K}} \subsetneq \mathcal{E}$: if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Kähler (red) cone and pseudoeffective (blue) cone



Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$\begin{aligned}
\mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big),\\
\mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R},\\
\mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X),\\
\mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).
\end{aligned}$$

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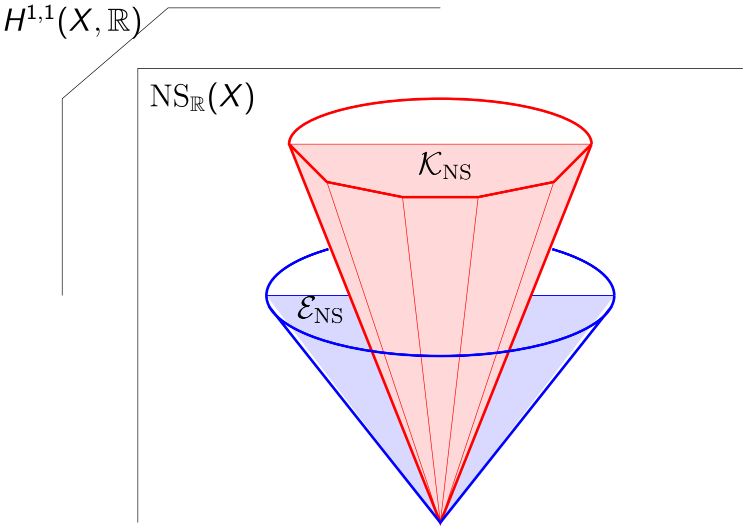


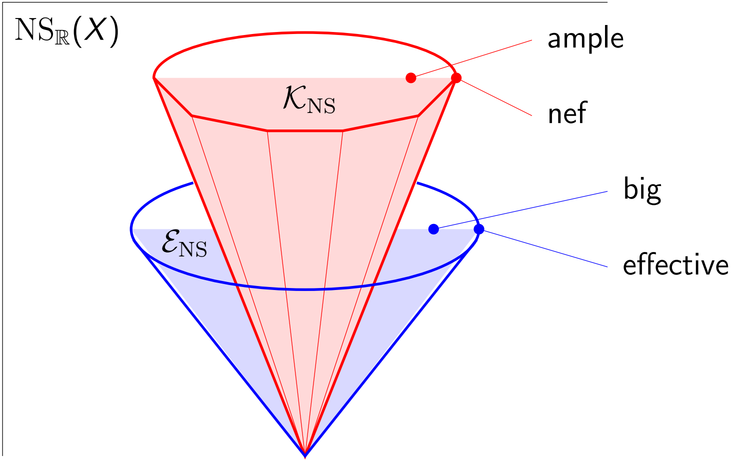
ample / nef / effective / big divisors

Theorem (Kodaira + successors, D 90)

Assume X projective.

- $\mathcal{K}_{\rm NS}$ is the open cone generated by ample (or very ample) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).
- The closed cone $\overline{\mathcal{K}}_{\rm NS}$ consists of the closure of the cone of nef divisors D (or nef line bundles L), namely effective integral divisors D such that $D \cdot C \geq 0$ for every curve C.
- $\mathcal{E}_{\rm NS}$ is the closure of the cone of effective divisors, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.
- The interior $\mathcal{E}^\circ_{\rm NS}$ is the cone of big divisors, namely divisors D such that $h^0(X, \mathcal{O}(kD)) \geq c\, k^{\dim X}$ for k large.

Proof: L^2 estimates for $\overline{\partial}$ / Bochner-Kodaira technique


$NS_{\mathbb{R}}(X)$
ample
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Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*

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- **Proposition.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha = \{T\}$, $T =$ *a Kähler current.*

  We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

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Theorem (D– 92)

Any Kähler current T can be written

$$T = \lim T_m$$

where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e.
$\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_m = [E_m] + \beta_m,$$

where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Idea of proof of analytic Zariski decomposition (1)

Locally one can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ on X. The approximating potentials φ_m of φ are defined as

$$\varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\}.$$

The Ohsawa-Takegoshi L^2 extension theorem (applied to extension from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$. On the other hand $\varphi = \lim_{m\to+\infty}\varphi_m$ by a Bergman kernel trick and by the mean value inequality.

Idea of proof of analytic Zariski decomposition (2)

The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. The modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, with

$$\mu_m^\star \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective $\mathbb{Q}$-divisor E_m with normal crossings on $\widetilde{X}_m$. Now, we set $T_m = i\partial\overline{\partial}\varphi_m$ and $\beta_m = \mu_m^* T_m - [E_m]$. Then $\beta_m = i\partial\overline{\partial}\psi_m$ where

$$\psi_m = \frac{1}{2m} \log \sum_\ell |g_{\ell,m} \circ \mu_m / h|^2 \quad \text{locally on } \widetilde{X}_m$$

and h is a generator of $\mathcal{O}(-mE_m)$, and we see that β_m is a smooth semi-positive form on $\widetilde{X}_m$. The construction can be made global by using a gluing technique, e.g. via partitions of unity, and β_m can be made Kähler by a perturbation argument.

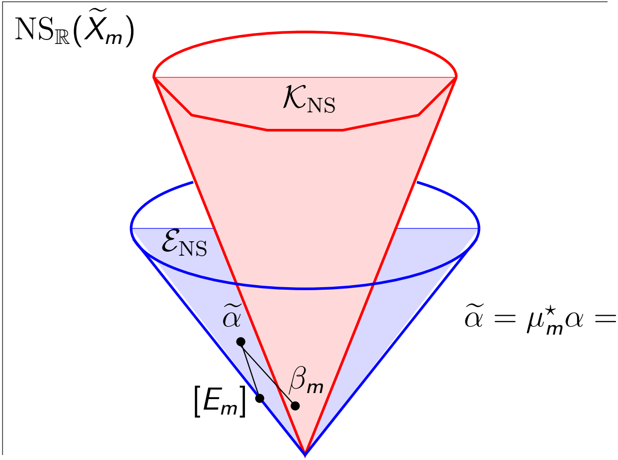
Algebraic analogue

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \ \text{ effective}, \quad D_m \ \text{ free.}$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.

Analytic Zariski decomposition



$$\widetilde{\alpha} = \mu_m^\star \alpha = [E_m] + \beta_m$$

Characterization of the Fujiki class $\mathcal{C}$

Theorem (Demailly-Păun 2004)

A compact complex manifold X is bimeromorphic to a Kähler manifold $\widetilde{X}$ (or equivalently, dominated by a Kähler manifold $\widetilde{X}$) if and only if it carries a Kähler current T.

Proof. If $\mu : \widetilde{X} \to X$ is a modification and $\widetilde{\omega}$ is a Kähler metric on $\widetilde{X}$, then $T = \mu_\star\widetilde{\omega}$ is a Kähler current on X.

Conversely, if T is a Kähler current, we take $\widetilde{X} = \widetilde{X}_m$ and $\widetilde{\omega} = \beta_m$ for m large enough.

Definition

The class of compact complex manifolds X bimeromorphic to some Kähler manifold $\widetilde{X}$ is called
the Fujiki class $\mathcal{C}$.
Hodge decomposition still holds true in $\mathcal{C}$.

Numerical characterization of the Kähler cone

Theorem (Demailly-Păun 2004)

Let X be a compact Kähler manifold. Let

$$\mathcal{P} = \left\{\alpha \in H^{1,1}(X,\mathbb{R})\,;\ \int_Y \alpha^p > 0,\ \forall Y \subset X,\ \dim Y = p\right\}.$$

"cone of numerically positive classes".

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Corollary (Projective case)

If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.

Note: this is a "transcendental version" of the Nakai-Moishezon criterion. The proof relies in an essential way on Monge-Ampère equations (Calabi-Yau theorem).

Example (non projective) for which $\mathcal{K} \subsetneq \mathcal{P}$.

Take X = generic complex torus $X = \mathbb{C}^n/\Lambda$.

Then X does not possess any analytic subset except finite subsets and X itself.

Hence $\mathcal{P} = \{\alpha \in H^{1,1}(X,\mathbb{R})\,;\ \int_X \alpha^n > 0\}$.

Since $H^{1,1}(X,\mathbb{R})$ is in one-to-one correspondence with constant hermitian forms, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, i.e.
possessing an even number of negative eigenvalues.

$\mathcal{K}$ is the component with all eigenvalues > 0.

Proof of the theorem : use Monge-Ampère

Fix $\alpha \in \overline{\mathcal{K}}$ so that $\int_X \alpha^n > 0$.

If ω is Kähler, then $\{\alpha + \varepsilon\omega\}$ is a Kähler class $\forall\varepsilon > 0$.

Use the Calabi-Yau theorem (Yau 1978) to solve the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = f_\varepsilon$$

where $f_\varepsilon > 0$ is a suitably chosen volume form.

Necessary and sufficient condition :

$$\int_X f_\varepsilon = (\alpha + \varepsilon\omega)^n \quad \text{in } H^{n,n}(X, \mathbb{R}).$$

In other terms, the infinitesimal volume form of the Kähler metric $\alpha_\varepsilon = \alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon$ can be distributed randomly on X.

Proof of the theorem : concentration of mass

In particular, the mass of the right hand side f_ε can be distributed in an ε-neighborhood U_ε of any given subvariety $Y \subset X$.

If $\mathrm{codim}\, Y = p$, on can show that

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^p \to \Theta \quad \text{weakly}$$

where Θ positive (p,p)-current and $\Theta \geq \delta[Y]$ for some $\delta > 0$.

Now, "diagonal trick": apply the above result to

$$\widetilde{X} = X \times X, \qquad \widetilde{Y} = \text{diagonal} \subset \widetilde{X}, \qquad \widetilde{\alpha} = \mathrm{pr}_1^*\,\alpha + \mathrm{pr}_2^*\,\alpha.$$

As $\widetilde{\alpha}$ is nef on $\widetilde{X}$ and $\int_{\widetilde{X}}(\widetilde{\alpha})^{2n} > 0$, it follows by the above that the class $\{\widetilde{\alpha}\}^n$ contains a Kähler current Θ such that $\Theta \geq \delta[\widetilde{Y}]$ for some $\delta > 0$. Therefore

$$T := (\mathrm{pr}_1)_*(\Theta \wedge \mathrm{pr}_2^*\,\omega)$$

is numerically equivalent to a multiple of α and dominates $\delta\omega$, and we see that T is a Kähler current.

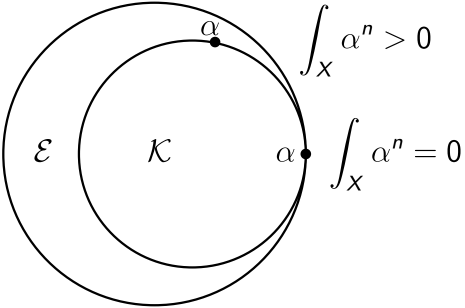
Generalized Grauert-Riemenschneider result

This implies the following result.

Theorem (Demailly-Păun, Annals of Math. 2004)

Let X be a compact Kähler manifold and consider a class $\{\alpha\} \in \overline{\mathcal{K}}$ such that $\int_X \alpha^n > 0$.
Then $\{\alpha\}$ contains a Kähler current T, i.e. $\{\alpha\} \in \mathcal{E}^\circ$.

Illustration:



Final step of proof

Clearly the open cone $\mathcal{K}$ is contained in $\mathcal{P}$, hence in order to show that $\mathcal{K}$ is one of the connected components of $\mathcal{P}$, we need only show that $\mathcal{K}$ is closed in $\mathcal{P}$, i.e. that $\overline{\mathcal{K}} \cap \mathcal{P} \subset \mathcal{K}$. Pick a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X \alpha^n > 0$. Hence $\{\alpha\}$ contains a Kähler current T.

Now, an induction on dimension using the assumption $\int_Y \alpha^p > 0$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on Y.

We conclude that $\{\alpha\}$ is a Kähler class by results of Paun (PhD 1997), therefore $\{\alpha\} \in \mathcal{K}$.

Variants of the main theorem

Corollary 1 (DP 2004). *Let X be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \ \text{ is Kähler} \Leftrightarrow \exists\omega \text{ Kähler s.t. } \int_Y \alpha^k \wedge \omega^{p-k} > 0$$

for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.

Proof. Argue with $(1-t)\alpha + t\omega$, $t \in [0,1]$.

Corollary 2 (DP 2004). *Let X be a compact Kähler manifold.*

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \ \text{ is nef } (\alpha \in \overline{\mathcal{K}}) \Leftrightarrow \forall\omega \text{ Kähler } \int_Y \alpha \wedge \omega^{p-1} \geq 0$$

for all $Y \subset X$ irreducible and all $k = 1, 2, \ldots, p = \dim Y$.

Consequence

The dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}(X,\mathbb{R})$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1} \in H^{n-1,n-1}(X,\mathbb{R})$.

Theorem on deformation stability of Kähler cones

Theorem (Demailly-Păun 2004)

Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are $\nabla^{1,1}$-invariant over $S \smallsetminus S'$ under parallel transport with respect to the $(1,1)$-projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇ in the decomposition of

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Positive cones in $H^{n-1,n-1}(X)$ and Serre duality

Definition

Let X be a compact Kähler manifold.

- Cone of $(n-1, n-1)$ positive currents
  $\mathcal{N} = \overline{\mathrm{cone}}\big\{\{T\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ T \text{ closed} \ge 0\big\}$.
- Cone of effective curves
  $$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap \mathrm{NS}_{\mathbb{R}}^{n-1,n-1}(X),$$
  $$= \overline{\mathrm{cone}}\big\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ C \text{ effective curve}\big\}.$$
- Cone of movable curves : with $\mu : \widetilde{X} \to X$, let
  $\mathcal{M}_{\mathrm{NS}} = \overline{\mathrm{cone}}\big\{\{C\} \in H^{n-1,n-1}(X,\mathbb{R})\,;\ [C] = \mu_\star(H_1 \cdots H_{n-1})\big\}$
  where H_j = ample hyperplane section of $\widetilde{X}$.
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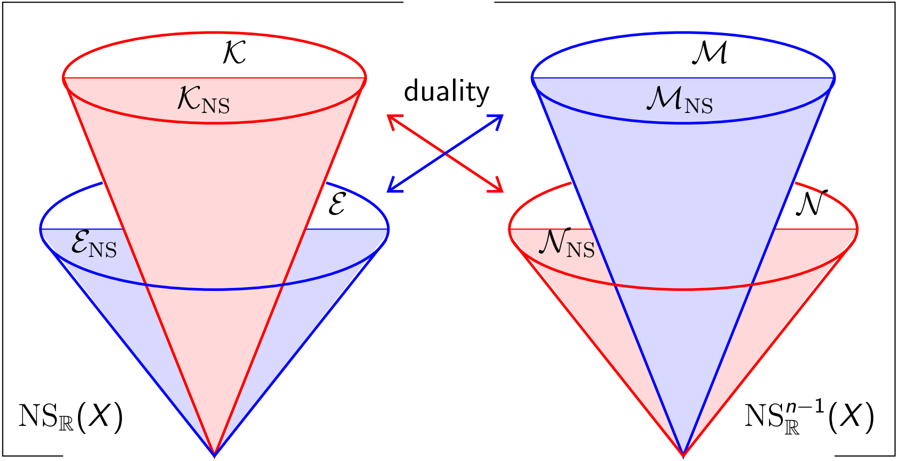
  where $\widetilde{\omega}_j$ = Kähler metric on $\widetilde{X}$.

Main duality theorem



$H^{1,1}(X,\mathbb{R})$ ← Serre duality → $H^{n-1,n-1}(X,\mathbb{R})$

Precise duality statement

Recall that the Serre duality pairing is

$$(\alpha^{(p,q)}, \beta^{(n-p,n-q)}) \longmapsto \int_X \alpha \wedge \beta.$$

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If X is compact Kähler, then $\mathcal{K}$ and $\mathcal{N}$ are dual cones.
(well known since a long time : $\mathcal{K}_{\mathrm{NS}}$ and $\mathcal{N}_{\mathrm{NS}}$ are dual).

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Conjecture (Boucksom-Demailly-Paun-Peternell 2004)

If X is Kähler, then $\mathcal{E}$ and $\mathcal{M}$ should be dual cones.

Concept of volume (very important !)

Definition (Boucksom 2002)

The volume (movable self-intersection) of a big class $\alpha \in \mathcal{E}^\circ$ is

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde X} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^\star T = [E] + \beta$ with respect to some modification $\mu : \widetilde X \to X$.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Theorem (Boucksom 2002)

If L is a big line bundle and $\mu_m^*(mL) = [E_m] + [D_m]$ (where E_m = fixed part, D_m = moving part), then

$$\mathrm{Vol}(c_1(L)) = \lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m\to+\infty} D_m^n.$$

Approximate Zariski decomposition

In other words, the volume measures the amount of sections and the growth of the degree of the images of the rational maps

$$\Phi_{|mL|} : X \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$$

By Fujita 1994 and Demailly-Ein-Lazarsfeld 2000, one has

Theorem

Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that

$$\mathrm{Vol}(L) - \varepsilon \le \mathrm{Vol}(\beta) \le \mathrm{Vol}(L).$$

Movable intersection theory

Theorem (Boucksom, PhD 2002)

Let X be a compact Kähler manifold and

$$H^{k,k}_{\geq 0}(X) = \big\{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\; T \text{ closed} \geq 0\big\}.$$

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- $\forall k = 1, 2, \ldots, n$, $\exists$ canonical "movable intersection product"

$$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\geq 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever α is a big class.

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- The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.

$$\langle \alpha_1 \cdots (\alpha_j' + \alpha_j'') \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha_j' \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha_j'' \cdots \alpha_k \rangle.$$

  It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

Divisorial Zariski decomposition

Using the above intersection product, one can easily derive the following divisorial Zariski decomposition result.

Theorem (Boucksom, PhD 2002)

- For $k = 1$, one gets a "divisorial Zariski decomposition"

$$\alpha = \{N(\alpha)\} + \langle\alpha\rangle$$

where :

- $N(\alpha)$ is a uniquely defined effective divisor which is called the "negative divisorial part" of α. The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive ;
- $\langle\alpha\rangle$ is "nef in codimension 1".

Construction of the movable intersection product

First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous more and more accurate log-resolutions $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \beta_{j,m}.$$

We define

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m\to+\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the limit is unique in cohomology ; this is based on "monotonicity properties" of the Zariski decomposition.

Generalized abundance conjecture

Definition

For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, the numerical dimension $\nu(\alpha)$ is

- $\nu(\alpha) = -\infty$ if α is not pseudo-effective,
- $\nu(\alpha) = \max\{p \in \mathbb{N}\,;\ \langle\alpha^p\rangle \neq 0\} \quad \in \{0, 1, \ldots, n\}$ if α is pseudo-effective.

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Conjecture ("generalized abundance conjecture")

For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :

$$\kappa(X) = \nu(c_1(K_X)).$$

Remark. The generalized abundance conjecture holds true when $\nu(c_1(K_X)) = -\infty$, 0, n (cases $-\infty$ and n being easy).

Lecture 2

Uniruledness, rational connectedness and $-K_X$ nef

We start by proving an orthogonality estimate, which in its turn identifies the dual of the cone of (pseudo)-effective divisors.

From there, we derive necessary and sufficient conditions for uniruledness and rational connectedness.

We conclude this lecture by presenting examples of compact Kähler manifolds X such that $-K_X$ is semipositive or nef.

Orthogonality estimate

Theorem

Let X be a projective manifold. Let $\alpha = \{T\} \in \mathcal{E}^{\circ}_{\mathrm{NS}}$ be a big class represented by a Kähler current T, and consider an approximate Zariski decomposition

$$\mu_m^\star T_m = [E_m] + [D_m].$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \leq 20\,(C\omega)^n \big(\,\mathrm{Vol}(\alpha) - D_m^n\big)$$

where $\omega = c_1(H)$ is a Kähler form and $C \geq 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

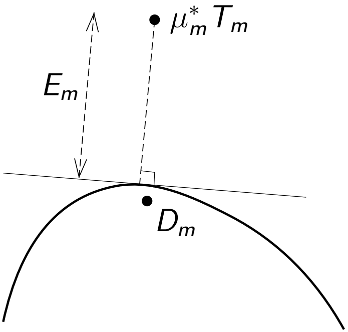
By going to the limit, one gets

Corollary

$\alpha \cdot \langle \alpha^{n-1} \rangle - \langle \alpha^n \rangle = 0.$

Proof of the orthogonality estimate

The argument for the "almost" orthogonality of the two parts in $\mu_m^* T_m = E_m + D_m$ is similar to the one used for projections from Hilbert space onto a closed convex set, where the segment to closest point is orthogonal to tangent plane.



Proof of duality between $\mathcal{E}_{\rm NS}$ and $\mathcal{M}_{\rm NS}$

Theorem (Boucksom-Demailly-Păun-Peternell 2004)

For X projective, a class α is in $\mathcal{E}_{\rm NS}$ (pseudo-effective) if and only if it is dual to the cone $\mathcal{M}_{\rm NS}$ of moving curves.

Proof of the theorem. We want to show that $\mathcal{E}_{\rm NS} = \mathcal{M}_{\rm NS}^\vee$. By obvious positivity of the integral pairing, one has in any case

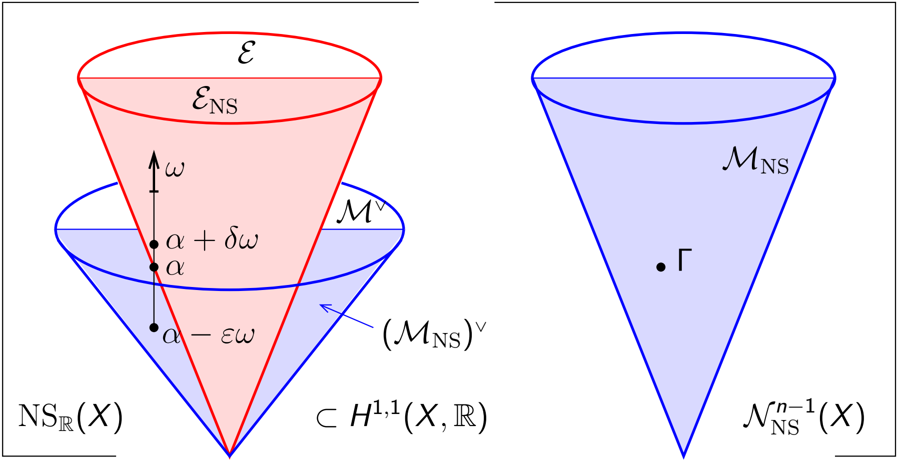
$$\mathcal{E}_{\rm NS} \subset (\mathcal{M}_{\rm NS})^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\rm NS}$ on the boundary of $\mathcal{E}_{\rm NS}$ which is in the interior of $\mathcal{N}_{\rm NS}^\vee$. Hence

$$(*) \qquad \alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve Γ, while $\langle \alpha^n \rangle = \mathrm{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$ (the family is assumed to be algebraic, and "covered" means that a Zariski open set of the variety at least is covered).

Theorem (Boucksom-Demailly-Paun-Peternell 2004)

A projective manifold X is not uniruled if and only if K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.

Proof (of the non trivial implication). If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Criterion for rational connectedness

Definition

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(c) $\forall$ invertible subsheaf $\mathcal{F} \subset \mathcal{O}((T_X^*)^{\otimes p})$, $p \geq 1$, $\mathcal{F}$ is not psef.

(d) For some (resp. for any) ample line bundle A on X, there exists a constant $C_A > 0$ such that

$$H^0(X, (T_X^*)^{\otimes m} \otimes A^{\otimes k}) = 0 \quad \forall m,\, k \in \mathbb{N}^* \text{ with } m \geq C_A k.$$

Proof of the RC criterion

Proof (essentially from Peternell 2006)

(a) $\Rightarrow$ (d) is easy (RC implies there are many rational curves on which T_X, so $T_X^* < 0$), (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are trivial.

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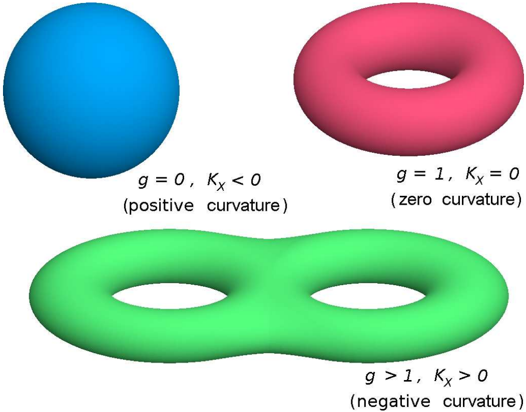
By BDPP (2004), Y not uniruled $\Rightarrow$ K_Y psef. Then $\pi^* K_Y \hookrightarrow \Omega_X^p$ where $p = \dim Y$, which is a contradiction unless $p = 0$, and therefore X is RC.

Structure of projective/compact Kähler varieties

As is well known since the beginning of the XX^{th} century at the geometry of projective or compact Kähler manifolds X depends on the sign of the curvature of the canonical line bundle K_X.



$K_X = \Lambda^n T_X^*$, $\deg(K_X) = 2g - 2$

Goal: Kähler manifolds with $-K_X \geq 0$ or $-K_X$ nef

Recall: By the Calabi-Yau theorem,

$-K_X \geq 0 \Leftrightarrow \exists\omega$ Kähler with Ricci$(\omega) \geq 0$,

$-K_X$ nef $\Leftrightarrow \forall\varepsilon > 0,\ \exists\omega_\varepsilon = \omega + i\partial\overline{\partial}\varphi_\varepsilon$ such that Ricci$(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$.

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- the rather large class of rationally connected manifolds Z with $-K_Z \geq 0$
- all products $T \times \prod S_j \times \prod Y_k \times \prod Z_\ell$.

Let us first give examples of varieties in each category.

Example of Ricci flat manifolds

– Examples of holomorphic symplectic manifolds:
Hilbert schemes $X = S^{[n]}$ of length n subschemes of a K3 surface and similar "Kummer varieties" $X = A^{[n+1]}/A$ associated with a complex 2-dimensional torus. Some "sporadic" examples have been constructed by O'Grady.

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– Examples of Calabi-Yau manifolds:

Smooth hypersurface of degree $n+2$ in $\mathbb{P}^{n+1}$, suitable complete intersections in (weighted) projective space.

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Hilbert schemes $X = S^{[n]}$ of length n subschemes of a K3 surface and similar "Kummer varieties" $X = A^{[n+1]}/A$ associated with a complex 2-dimensional torus. Some "sporadic" examples have been constructed by O'Grady.

– Examples of Calabi-Yau manifolds:
Smooth hypersurface of degree $n+2$ in $\mathbb{P}^{n+1}$, suitable complete intersections in (weighted) projective space.

Following work by Bogomolov and Fujiki, Beauville has shown:

Beauville-Bogomolov decomposition theorem (1983)

Every compact Kähler manifold X with $c_1(X) = 0$ admits a finite étale cover $\widetilde{X}$ such that

$$\widetilde{X} \simeq T \times \prod S_j \times \prod Y_k \text{ (isometrically)}$$

where T is a torus, S_j holomorphic symplectic and Y_k Calabi-Yau.

Examples of RC manifolds with $-K_X \geq 0$ or nef

Let X be the rational surface obtained by blowing up $\mathbb{P}^2$ in 9 distinct points $\{p_i\}$ on a smooth (cubic) elliptic curve $C \subset \mathbb{P}^2$, $\mu : X \to \mathbb{P}^2$ and $\hat{C}$ the strict transform of C.

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$$K_X = \mu^* K_{\mathbb{P}^2} \otimes \mathcal{O}(\sum E_i) \Rightarrow -K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}(-\sum E_i),$$

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One has

$$-K_X \cdot \Gamma = \hat{C} \cdot \Gamma \geq 0 \qquad \text{if } \Gamma \neq \hat{C},$$

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Brunella has shown that $-K_X$ is C^∞ semi-positive if $c_1(G)$ satisfies a diophantine condition found by T. Ueda, but otherwise it need not be semi-positive (although nef).

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Lecture 3

Holonomy principle and main structure theorems

We describe here a structure theorem for compact Kähler manifolds with $-K_X \geq 0$. It depends in an essential way on the concept of holonomy and its implications on the geometry of the manifold.

Then, following work of Junyan Cao and Andreas Höring, we discuss some results describing the structure of compact Kähler manifolds with $-K_X$ nef.

Structure theorem for manifolds with $-K_X \geq 0$

Theorem [Campana, D., Peternell, 2012]

Let X be a compact Kähler manifold with $-K_X \geq 0$. Then:

(a) $\exists$ holomorphic and isometric splitting in irreducible factors

$$\widetilde{X} = \text{universal cover of } X \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where Y_j = Calabi-Yau (holonomy $\mathrm{SU}(n_j)$), S_k = holomorphic symplectic (holonomy $\mathrm{Sp}(n'_k/2)$), and Z_ℓ = RC with $-K_{Z_\ell} \geq 0$ (holonomy $\mathrm{U}(n''_\ell)$) or compact symmetric space).

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(b) There exists a finite étale Galois cover $\widehat{X} \to X$ such that the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is an (isometrically) locally trivial holomorphic fiber bundle whose fibers are products $\prod Y_j \times \prod S_k \times \prod Z_\ell$, as described in (a).

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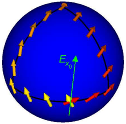
(c) $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q} \rtimes \Gamma$, Γ finite ("almost abelian" group).

Concept of holonomy / restricted holonomy

Let (E, h) be a Euclidean vector bundle over X, and ∇_h a compatible connection. For every path $\gamma : [0,1] \to X$ joining $p, q \in X$, one considers the (metric preserving) parallel transport operator $\tau_{p,q} : E_p \to E_q$, $v(0) \mapsto v(1)$ where $\frac{\nabla_h}{dt}(\frac{dv}{dt}) = 0$.



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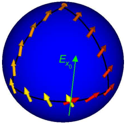

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The holonomy group $\mathrm{Hol}(E, h)_p$ (resp. restricted holonomy group $\mathrm{Hol}^\circ(E, h)_p$) of a Euclidean vector bundle $E \to X$ is the subgroup of $\mathrm{SO}(E_p)$ generated by parallel transport operators $\tau_{p,p}$ over loops based at p (resp. contractible loops).

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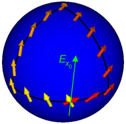

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It is independent of p, up to conjugation. In the hermitian case, $\mathrm{Hol}^\circ(E, h)_p$ is contained in the unitary group $\mathrm{U}(E_p)$.

The De Rham splitting theorem

In the important case when $E = T_X$, we have

De Rham splitting theorem

If (X, h) is complete and the holonomy representation of $H = \mathrm{Hol}^\circ(T_X, h)_p$ splits into irreductible representations $T_{X,p} = S_1 \oplus \ldots \oplus S_k$, then the universal cover $\widetilde{X}$ splits metrically as

$$\widetilde{X} = X_1 \times \ldots \times X_k$$

where the holonomy of X_j yields the irreducible representation on $S_j \subset T_{X_j,p}$.
This means in particular that the pull-back metric $\widetilde{h}$ splits as a direct sum of metrics $h_1 \oplus \ldots \oplus h_k$ on the factors X_j.

Berger's classification of holonomy groups

The Berger classification of holonomy groups of non locally symmetric Riemannian manifolds stands as follows:

Hol(M,g)	**dim(M)**	**Type of manifold**	**Comments**
SO(n)	n	Orientable manifold	Generic Riemannian
U(n)	$2n$	Kähler manifold	Generic Kähler
SU(n)	$2n$	Calabi–Yau manifold	Ricci-flat, Kähler
Sp(n)	$4n$	Hyperkähler manifold	Ricci-flat, Kähler
Sp(n) · Sp(1)	$4n$	Quaternion-Kähler manifold	Einstein
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The complex cases are shown by the red lines.

Generalized holonomy principle

Generalized holonomy principle

Let $(E, h) \to X$ be a hermitian holomorphic vector bundle of rank r over X compact$/\mathbb{C}$. Assume that

$$\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \quad B \in \mathrm{Herm}(E, E), \quad B \geq 0 \ \text{ on } X.$$

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Let H the restricted holonomy group of (E, h). Then

(a) If there exists a psef invertible sheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$, then $\mathcal{L}$ is flat and invariant under parallel transport by the connection of $(E^*)^{\otimes m}$ induced by the Chern connection ∇ of (E, h); moreover, H acts trivially on $\mathcal{L}$.

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Proof. $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ which has trace of curvature ≤ 0 while $\Theta_{\mathcal{L}} \geq 0$, use Bochner formula. □

Proof of the generalized holonomy principle

Assume that we have an invertible sheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ that is pseudoeffective. For a local non zero section f of $\mathcal{L}$, one considers

$$\psi = \frac{|f|^2_{h^{*m}}}{|f|^2_{h_{\mathcal{L}}}}.$$

Writing $|f|^2_{h_{\mathcal{L}}} = e^{-\varphi}$, a standard Bochner type inequality yields

$$\Delta_\omega \psi \geq |f|^2_{h^{*m}}\ e^{\varphi}(\Delta_\omega \varphi + m\lambda_1) + |\nabla^{1,0}_h f + f\partial\varphi|^2_{\omega,h^{*m}}\ e^{\varphi}$$

where $\lambda_1(z) \geq 0$ is the lowest eigenvalue of the hermitian endomorphism $B = \mathrm{Tr}_\omega \Theta_{E,h}$ at point $z \in X$.

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By a conformal change, one can arrange ω to be a Gauduchon metric $(\partial\overline{\partial}(\omega^{n-1}) = 0)$, and then observe by Stokes' theorem that

$$\int_X \Delta_\omega \psi\, \omega^n = \int_X i\partial\overline{\partial}\psi \wedge \omega^{n-1} = 0.$$

Then in particular $\nabla_h^{1,0} f + f\partial\varphi = 0$ and the theorem follows.

Proof of the structure theorem for $-K_X \geq 0$

Cheeger-Gromoll theorem (J. Diff. Geometry 1971)

Let (X, g) be a complete Riemannian manifold of nonnegative Ricci curvature. Then the universal cover $\widetilde{X}$ splits as

$$\widetilde{X} = \mathbb{R}^q \times Z$$

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Proof of the structure theorem for $-K_X \geq 0$, using the generalized holonomy principle. Let (X, ω) be compact Kähler with $-K_X \geq 0$. By the De Rham and Cheeger-Gromoll theorems, write $\widetilde{X}$ as a product of manifolds with irreducible holonomy

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where $\mathrm{Hol}^\circ(Y_j) = \mathrm{SU}(n_j))$ (Calabi-Yau), $\mathrm{Hol}^\circ(S_k) = \mathrm{Sp}(n'_k/2))$ (holomorphic symplectic), and Z_ℓ either compact hermitian symmetric, or $\mathrm{Hol}^\circ(Z_\ell) = \mathrm{U}(n''_\ell) \Rightarrow Z_\ell$ rationally connected (H.P.)

Surjectivity of the Albanese morphism

Recall that if X is a compact Kähler manifold, the Albanese map

$$\alpha_X : X \to \mathrm{Alb}(X) := \mathbb{C}^q/\Lambda$$

is the holomorphic map given by

$$z \mapsto \alpha_X(z) = \Big(\int_{z_0}^{z} u_j\Big)_{1\le j\le q} \quad \text{mod period subgroup } \Lambda \subset \mathbb{C}^q,$$

where $(u_1, \ldots, u_q)$ is a basis of $H^0(X, \Omega^1_X)$.

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Theorem [Qi Zhang, 1996, 2005]

If X is projective and $-K_X$ is nef, then α_X is surjective.

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Proof. Based on characteristic p techniques.

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Proof. Based on variation arguments for twisted Kähler-Einstein metrics.

Approach via generically nef vector bundles (J.Cao)

Definition

Let X be a compact Kähler manifold, $\mathcal{E} \to X$ a torsion free sheaf.

(a) $\mathcal{E}$ is stable with respect to a Kähler class ω if

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(c) Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$$

be a filtration of $\mathcal{E}$ by torsion free coherent subsheaves such that the quotients $\mathcal{E}_j/\mathcal{E}_{j-1}$ are ω-stable subsheaves of $\mathcal{E}/\mathcal{E}_{j-1}$ of maximal rank. We call such a sequence a refined Harder-Narasimhan (HN) filtration w.r.t. ω.

Characterization of generically nef vector bundles

It is a standard fact that refined HN-filtrations always exist. Bando and Siu have proved that the graded pieces $G_j := \mathcal{E}_j/\mathcal{E}_{j-1}$ then possess a Hermite-Einstein metric h_j such that

$$\mathrm{Tr}_\omega\, \Theta_{G_j,h_j} = \mu_\omega(G_j)\cdot \mathrm{Id}_{G_j},$$

and that h_j is smooth outside of the codim 2 locus where $G_j := \mathcal{E}_j/\mathcal{E}_{j-1}$ is not locally free. Moreover one always has

$$\mu_\omega(\mathcal{E}_j/\mathcal{E}_{j-1}) \geq \mu_\omega(\mathcal{E}_{j+1}/\mathcal{E}_j), \quad \forall j.$$

Proposition

Let (X,ω) be a compact Kähler manifold and $\mathcal{E}$ a torsion free sheaf on X. Then $\mathcal{E}$ is ω-generically nef if and only if

$$\mu_\omega(\mathcal{E}_j)/\mathcal{E}_{j-1}) \geq 0$$

for some refined HN-filtration.

Proof. This is done by easy arguments on filtrations. □

A result of J. Cao about manifolds with $-K_X$ nef

Theorem

(Junyan Cao, 2013) Let X be a compact Kähler manifold with $-K_X$ nef. Then the tangent bundle T_X is ω-generically nef for all Kähler classes ω.

Proof. use the fact that $\forall\varepsilon > 0$, $\exists$ Kähler metric with $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\,\omega_\varepsilon$ (Yau, DPS 1995).

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From this, one can deduce

Theorem

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the bundles $T_X^{\otimes m}$ are ω-generically nef for all Kähler classes ω and all positive integers m. In particular, the bundles $S^k T_X$ and $\bigwedge^p T_X$ are ω-generically nef.

A lemma on sections of contravariant tensors

Lemma

Let (X, ω) be a compact Kähler manifold with $-K_X$ nef and

$$\eta \in H^0(X, (\Omega_X^1)^{\otimes m} \otimes \mathcal{L})$$

where $\mathcal{L}$ is a numerically trivial line bundle on X.

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Proof. By Cao's theorem there exists a refined HN-filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = T_X^{\otimes m}$$

with ω-stable quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ such that $\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$ for all i.
Then we use the fact that any section in a (semi-)negative slope reflexive sheaf $\mathcal{E}_{i+1}/\mathcal{E}_i \otimes \mathcal{L}$ is parallel w.r.t. its Bando-Siu metric (resp. vanishes). □

Smoothness of the Albanese morphism (after Cao)

Theorem (Junyan Cao 2013)

Non-zero holomorphic p-forms on a compact Kähler manifold X with $-K_X$ nef vanish only on the singular locus of the refined HN filtration of T^*X.

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This already implies the following result.

Corollary

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map $\alpha_X : X \to \mathrm{Alb}(X)$ is a submersion on the complement of the HN filtration singular locus in X [$\Rightarrow \alpha_X$ surjects onto $\mathrm{Alb}(X)$].

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Cao's thm $\Rightarrow$ rank of $(du_1, \ldots, du_q)$ is $= q$ generically. □

Isotriviality of the Albanese map

Theorem [Junyan Cao, arXiv:1612.05921]

Let X be a projective manifold with nef anti-canonical bundle. Then the Albanese map $\alpha_X : X \to Y = \mathrm{Alb}(X)$ is locally trivial, i.e., for any small open set $U \subset Y$, $\alpha_X^{-1}(U)$ is biholomorphic to the product $U \times F$, where F is the generic fiber of α_X.

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Proof. Let A be a (large) ample line bundle on X and $E = (\alpha_X)_*A$ its direct image. Then $E = (\alpha_X)_*(mK_{X/Y} + L)$ with $L = A - mK_{X/Y} = A - mK_X$ nef. By results of Berndtsson-Păun on direct images, one can show that det E is pseudoeffective.

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The simply connected case

The above results reduce the study of projective manifolds with $-K_X$ nef to the case when $\pi_1(X) = 0$.

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Theorem [Junyan Cao, Andreas Höring, arXiv:1706.08814]

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Corollary [Junyan Cao, Andreas Höring]

Let X be a projective manifold such that $-K_X$ is nef. Then after replacing X with a finite étale cover, the Albanese map α_X is locally trivial and its fibers are of the form $\prod S_j \times \prod Y_k \times \prod Z_\ell$ with S_j holomorphic symplectic, Y_k Calabi-Yau and Z_ℓ rationally connected.

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Possible general definition of singular Calabi-Yau manifolds

A compact Kähler manifold X is a singular Calabi-Yau manifold if X has a non singular model X' satisfying $\pi_1(X') = 0$ and $K_{X'} = E$ for an effective divisor E of numerical dimension 0 (an exceptional divisor), and $H^0(X', \Omega^p_{X'}) = 0$ for $0 < p < \dim X$.

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A compact Kähler manifold $X = X^{2p}$ is a singular hyperkähler manifold if X has a non singular model X' satisfying $\pi_1(X') = 0$ and possessing a section $\sigma \in H^0(X', \Omega^2_{X'})$ such that the zero divisor $E = \mathrm{div}(\sigma^p)$ has numerical dimension 0,
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Conjecture (known by BDPP for X projective!)

Let X be compact Kähler, and let $X \to Y$ be the MRC fibration (after replacing X by a suitable blow-up to make $X \to Y$ a genuine morphism). Then K_Y is psef.

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Proof ? Take the part of slope > 0 in the HN filtration of T_X, w.r.t. to classes in the dual of the psef cone, show that this corresponds to the MRC fibration, and apply duality.

Further problems (III)

- According to F. Campana, one should be able to factorize "special subvarieties" of Y (i.e. essentially the RC, singular Calabi-Yau and hyperkähler subvarieties) to get a morphism $Y \to Z$, along with a ramification divisor $\Delta \subset Z$ of that morphism, in such a way that the pair (Z, Δ) is of general type, i.e. $K_Z + \Delta$ is big.

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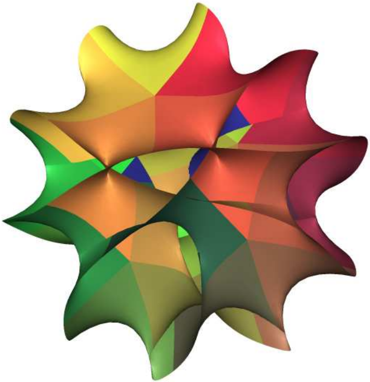
  One can hopefully expect similar decomposition theorems for varieties in this class. They might possibly include all rationally connected varieties.

Thank you for your attention!



A representation of the real points of a quintic Calabi-Yau manifold