On the Green-Griffiths-Lang and Kobayashi conjectures for the hyperbolicity of algebraic varieties

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Kobayashi hyperbolicity and entire curves

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Theorem (Brody, 1978)

For a compact complex manifold X, $dim_{\mathbb{C}} X = n$, TFAE:
(i) X is Kobayashi hyperbolic
(ii) X is Brody hyperbolic, i.e. $\nexists$ entire curves $f : \mathbb{C} \to X$
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Our interest is the study of hyperbolicity for projective varieties.
In dim 1, X is hyperbolic iff genus $g \geq 2$.

Positivity concepts/General type varieties

Definition

Let $L \to X$ be a line bundle on a nonsingular complex projective variety X.

• L is said to be ample if for $m \gg 1$ the space of sections $S_m = H^0(X, L^{\otimes m})$ gives an embedding

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Example

A non singular hypersurface $X^n \subset \mathbb{P}^{n+1}$ of degree d has $K_X = \mathcal{O}(d - n - 2)$, X is of general type iff $d > n + 2$.

Main conjectures

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Arithmetic counterpart (Lang 1987): If X is projective and defined over a number field, the smallest locus $Y = \mathrm{GGL}(X)$ in GGL's conjecture is also the smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite $\forall \mathbb{K}$.

Results on the Kobayashi conjecture

Kobayashi conjecture (1970)

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Theorem (D., El Goul, 1998)

A very generic surface $X \subset \mathbb{P}^3$ of degree $d \geq 21$ is hyperbolic. Independently McQuillan got $d \geq 35$.

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In 2012, Yum-Tong Siu announced a proof of the case of arbitrary dimension n, with a very large d_n (and a rather involved proof).

Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

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The bound was improved by (D-, 2012) to

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Theorem (S. Diverio, S. Trapani, 2009)

Additionally, a generic hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 593$ is hyperbolic.

Category of directed manifolds

- **Goal.** More generally, we are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle of T_X (or singular linear subspace, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ linear).

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- **Definition.** *Category of directed manifolds :*
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  - "Absolute case" (X, T_X), i.e. $V = T_X$
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- **Fonctor "1-jet" :** $(X, V) \mapsto (\tilde{X}, \tilde{V})$ where :

$$\tilde{X} = P(V) = \text{bundle of projective spaces of lines in } V$$

$$\pi : \tilde{X} = P(V) \to X, \quad (x, [v]) \mapsto x, \quad v \in V_x$$

$$\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$$

Semple jet bundles

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

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- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \mathrm{rk}\,\tilde{V} = r = \mathrm{rk}\,V$$

$$0 \to \mathcal{O}_{\tilde{X}} \to \pi^\star V \otimes \mathcal{O}_{\tilde{X}}(1) \to T_{\tilde{X}/X} \to 0 \quad \text{(Euler)}$$

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$$0\to T_{X_k/X_{k-1}}\to V_k\xrightarrow{(\pi_k)_\star}\mathcal{O}_{X_k}(-1)\to 0\qquad\Rightarrow\text{rk }V_k=r$$

$$0\to\mathcal{O}_{X_k}\to\pi_k^\star V_{k-1}\otimes\mathcal{O}_{X_k}(1)\to T_{X_k/X_{k-1}}\to 0\quad\text{(Euler)}$$

Direct image formula

For $n = \dim X$ and $r = \operatorname{rk} V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rk} V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

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Theorem

X_k is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

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Direct image formula

$(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

Definition of algebraic differential operators

Let $(\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, $\quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k\xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0)$$

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One considers the Green-Griffiths bundle $E_{k,m}^{\mathrm{GG}}V^*$ of polynomials of weighted degree m written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}, \quad \xi_s \in V,$$

also viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2}\ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators [2]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X.

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The $\mathbb{G}_k$-action : $(f, \varphi) \mapsto f \circ \varphi$, yields in particular, $\varphi_\lambda(t) = \lambda t \Rightarrow (f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

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$E_{k,m}V^* \subset E^{\mathrm{GG}}_{k,m}V^*$ is the bundle of $\mathbb{G}_k$-"invariant" operators, i.e. such that

$$P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi, \quad \forall \varphi \in \mathbb{G}_k.$$

Canonical sheaf of a singular pair (X,V)

When V is nonsingular, we simply set $K_V = \det(V^*)$.
When V is singular, the canonical sheaf K_V is the rank 1 analytic sheaf defined as the integral closure of the image of the natural morphism

$$\Lambda^r T_X^* \to \Lambda^r V^* \to \mathcal{L}_V := \text{invert. sheaf } (\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$, one sets

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Sections of K_V^X can be seen as sections of $\mathcal{L}_V$ that are locally bounded with respect to a smooth hermitian metric ω on T_X.

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that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$, one sets

$$K_V^X = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V, \quad \overline{\mathcal{J}}_V = \text{integral closure of } \mathcal{J}_V.$$

Sections of K_V^X can be seen as sections of $\mathcal{L}_V$ that are locally bounded with respect to a smooth hermitian metric ω on T_X.

Definition

We say that (X,V) is of general type if there exist proper modifications $\mu = \hat\mu \circ \tilde\mu : \widehat{X} \to \widetilde{X} \to X$ such that $\hat\mu^* K_{\tilde V}^{\tilde X}$ is a big invertible sheaf on $\widehat{X}$, where $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\tilde\mu^{-1}(V)}$

Canonical sheaf of a singular pair (X,V) [2]

Alternatively, it is not hard to see that if $\mu : \tilde{X} \to X$ is a modification, there are for every $m > 0$ injective morphisms of rank one sheaves

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It would be important to know if one has invariance of "directed"

Generalized GGL conjecture

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If (X, V) is directed manifold of general type, i.e. K_V big, then $\exists Y \subsetneq X$ such that $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

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Strategy : fundamental vanishing theorem

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
$\forall P \in H^0(X, E^{\mathrm{GG}}_{k,m} V^* \otimes \mathcal{O}(-A))$: global diff. operator on X (A ample divisor), $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.

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Theorem (D-, 2010)

Let (X, V) be of general type, i.e. s.t. K_V is a big rank 1 sheaf. Then $\exists$ many global sections P, $m \gg k \gg 1 \Rightarrow \exists$ alg. hypersurface $Z \subsetneq X_k$ s.t. every entire $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset Z$.

Finsler metric on the k-jet bundles

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$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

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Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\rm GG}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{{\rm FS},k}(\xi) + \frac{i}{2\pi} \sum_{1 \le s \le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} \, dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{{\rm FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\rm GG} \to X$.

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The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

Probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

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The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \int_{u\in SV} \gamma(u)\, du.$$

As γ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r}\,\mathrm{Tr}(\gamma)$.

Main cohomological estimate

$\Rightarrow$ the leading term only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, that can be taken > 0 if $\det V^*$ is big.

Corollary (D-, 2010)

Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F,h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have upper and lower bounds [$q = 0$ most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$

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We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

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Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage $(\mathcal{X}_k, \mathcal{A}_k)$ of the "absolute Semple tower" associated with $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$ (so as to deal only with nonsingular ambient Semple bundles).

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This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that $\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \operatorname{rk} W < \operatorname{rk} V_k = \operatorname{rk} V$.

Partial solution of GGL conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "strongly of general type" if it is of general type and for every irreducible alg. subvariety $Z \subsetneq X_k$ that projects onto X, $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. $K_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$.

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Theorem (D-, 2014)

If (X, V) is strongly of general type, the Green-Griffiths-Lang conjecture holds true for (X, V), namely there $\exists Y \subsetneq X$ such that every non constant holomorphic curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfies $f(\mathbb{C}) \subset Y$.

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Proof: Induction on rank V, using existence of jet differentials.

Related stability property

Definition

Fix an ample divisor A on X. For every irreducible subvariety $Z \subset X_k$ that projects onto X_{k-1} for $k \geq 1$, $Z \not\subset D_k$, and $Z = X = X_0$ for $k = 0$, we define the slope of the corresponding directed variety (Z, W) to be $\mu_A(Z, W) =$

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Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

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Notice that (X, V) is of general type iff $\mu_A(X, V) < 0$.

We say that (X, V) is A-jet-stable (resp. A-jet-semi-stable) if $\mu_A(Z, W) < \mu_A(X, V)$ (resp. $\mu_A(Z, W) \leq \mu_A(X, V)$) for all $Z \subsetneq X_k$ as above.

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Observation. If (X, V) is of general type and A-jet-semi-stable, then (X, V) is strongly of general type.

Approach of the Kobayashi conjecture

Definition

Let (X, V) be a directed pair where X is projective algebraic. We say that (X, V) is "algebraically jet-hyperbolic" if for every irreducible alg. subvariety $Z \subsetneq X_k$ s.t. $X_k \not\subset D_k$, the induced directed structure $(Z, W) \subset (X_k, V_k)$ either has $W = 0$ or is of general type modulo $X_k \to X$.

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Theorem (D-, 2014)

If (X, V) is algebraically jet-hyperbolic, then (X, V) is Kobayashi (or Brody) hyperbolic, i.e. there are no entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.

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Now, the hope is that a (very) generic complete intersection $X = H_1 \cap \ldots \cap H_c \subset \mathbb{P}^{n+c}$ of codimension c and degrees $(d_1, \ldots, d_c)$ s.t. $\sum d_j \geq 2n + c$ yields (X, T_X) algebraically jet-hyperbolic.