

# General extension theorem for cohomology classes on non reduced analytic subspaces

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble Alpes & Académie des Sciences de Paris

The 12th Korean Conference on Several Complex Variables KSCV12  
in honor of Kang-Tae Kim on the occasion of his 60th birthday  
The Kolon Hotel in Gyeong-Ju, South Korea, July 3–7, 2017

# References

This is a joint work with [Junyan Cao & Shin-ichi Matsumura](#)  
[J.-P. Demailly](#), *Extension of holomorphic functions defined on non reduced analytic subvarieties*, arXiv:1510.05230v1, Advanced Lectures in Mathematics Volume 35.1, the legacy of Bernhard Riemann after one hundred and fifty years, 2015.

[J.Y. Cao](#), [J.-P. Demailly](#), [S-i. Matsumura](#), *A general extension theorem for cohomology classes on non reduced analytic subspaces*, arXiv:1703.00292v2, Science China Mathematics, **60** (2017) 949–962

[T. Ohsawa](#), *On a curvature condition that implies a cohomology injectivity theorem of Kollár-Skoda type*, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 565–577.

[T. Ohsawa](#), [K. Takegoshi](#), *On the extension of  $L^2$  holomorphic functions*, Math. Zeitschrift **195** (1987), 197–204.

# The extension problem

Let  $(X, \omega)$  be a **possibly noncompact**  $n$ -dimensional Kähler manifold,  $\mathcal{J} \subset \mathcal{O}_X$  a coherent ideal sheaf,  $Y = V(\mathcal{J})$  its zero variety and

$$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}.$$

Here  $Y$  may be non reduced, i.e.  $\mathcal{O}_Y$  may have nilpotent elements.

# The extension problem

Let  $(X, \omega)$  be a **possibly noncompact**  $n$ -dimensional Kähler manifold,  $\mathcal{J} \subset \mathcal{O}_X$  a coherent ideal sheaf,  $Y = V(\mathcal{J})$  its zero variety and

$$\mathcal{O}_Y = \mathcal{O}_X / \mathcal{J}.$$

Here  $Y$  may be non reduced, i.e.  $\mathcal{O}_Y$  may have **nilpotent elements**.

Also, let  $(L, h_L)$  be a hermitian holomorphic line bundle on  $X$ , and

$$\Theta_{L, h_L} = i \partial \bar{\partial} \log h_L^{-1}$$

its curvature current (we allow singular metrics,  $h_L = e^{-\varphi}$ ,  $\varphi \in L^1_{\text{loc}}$ ,  $\Theta_{L, h_L}$  being computed in the sense of currents).

## Question

Under which conditions on  $X$ ,  $Y = V(\mathcal{J})$ ,  $(L, h_L)$  is

$$H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y) = H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X / \mathcal{J})$$

a surjective restriction morphism?

# First (too restrictive) answer

Consider the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J} \rightarrow 0$$

twisted by  $\mathcal{O}_X(K_X \otimes L)$ ,

# First (too restrictive) answer

Consider the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J} \rightarrow 0$$

twisted by  $\mathcal{O}_X(K_X \otimes L)$ , and the corresponding long exact sequence of cohomology groups

$$\begin{aligned} \cdots \rightarrow H^q(X, K_X \otimes L) &\rightarrow H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J}) \\ &\rightarrow H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \cdots \end{aligned}$$

# First (too restrictive) answer

Consider the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J} \rightarrow 0$$

twisted by  $\mathcal{O}_X(K_X \otimes L)$ , and the corresponding long exact sequence of cohomology groups

$$\begin{aligned} \cdots \rightarrow H^q(X, K_X \otimes L) \rightarrow H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J}) \\ \rightarrow H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \cdots \end{aligned}$$

It is therefore enough to have

$$H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) = 0.$$

In order to kill  $H^{q+1}$  it is enough to make a **strict positivity** (ampleness) assumption, by the Kodaira-Nakano / Nadel vanishing theorems.

# First (too restrictive) answer

Consider the exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J} \rightarrow 0$$

twisted by  $\mathcal{O}_X(K_X \otimes L)$ , and the corresponding long exact sequence of cohomology groups

$$\begin{aligned} \cdots \rightarrow H^q(X, K_X \otimes L) \rightarrow H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{O}_X/\mathcal{J}) \\ \rightarrow H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) \cdots \end{aligned}$$

It is therefore enough to have

$$H^{q+1}(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{J}) = 0.$$

In order to kill  $H^{q+1}$  it is enough to make a **strict positivity** (ampleness) assumption, by the Kodaira-Nakano / Nadel vanishing theorems. **But we do not want to make such strong assumptions!**



# Assumptions (1)

We assume  $X$  to be **holomorphically convex**. By the Cartan-Remmert theorem, this is the case iff  $X$  admits a **proper holomorphic map**  $p : X \rightarrow S$  only a Stein complex space  $S$ .

# Assumptions (1)

We assume  $X$  to be **holomorphically convex**. By the Cartan-Remmert theorem, this is the case iff  $X$  admits a **proper holomorphic map**  $p : X \rightarrow S$  only a Stein complex space  $S$ .

Observation : cohomology is then always Hausdorff

Let  $X$  be a holomorphically convex complex space and  $\mathcal{F}$  a coherent analytic sheaf over  $X$ . Then all cohomology groups  $H^q(X, \mathcal{F})$  are **Hausdorff** with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

# Assumptions (1)

We assume  $X$  to be **holomorphically convex**. By the Cartan-Remmert theorem, this is the case iff  $X$  admits a **proper holomorphic map**  $p : X \rightarrow S$  only a Stein complex space  $S$ .

**Observation :** cohomology is then always Hausdorff

Let  $X$  be a holomorphically convex complex space and  $\mathcal{F}$  a coherent analytic sheaf over  $X$ . Then all cohomology groups  $H^q(X, \mathcal{F})$  are **Hausdorff** with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

**Proof.**  $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$  is a Fréchet space.

# Assumptions (1)

We assume  $X$  to be **holomorphically convex**. By the Cartan-Remmert theorem, this is the case iff  $X$  admits a **proper holomorphic map**  $p : X \rightarrow S$  only a Stein complex space  $S$ .

**Observation :** cohomology is then always Hausdorff

Let  $X$  be a holomorphically convex complex space and  $\mathcal{F}$  a coherent analytic sheaf over  $X$ . Then all cohomology groups  $H^q(X, \mathcal{F})$  are **Hausdorff** with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

**Proof.**  $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$  is a Fréchet space.

## Corollary

To solve an equation  $\bar{\partial}u = v$  on a holomorphically convex manifold  $X$ , it is enough to solve it approximately:

$$\bar{\partial}u_\varepsilon = v + w_\varepsilon, \quad w_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

# Assumptions (2)

We assume that the subvariety  $Y \subset X$  is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where  $\psi$  is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood  $V$  of an arbitrary point  $x_0 \in X$  we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

# Assumptions (2)

We assume that the subvariety  $Y \subset X$  is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where  $\psi$  is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood  $V$  of an arbitrary point  $x_0 \in X$  we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

and  $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$  is the **multiplier ideal sheaf**

$$\mathcal{I}(e^{-\psi})_{x_0} = \left\{ f \in \mathcal{O}_{X,x_0}; \exists U \ni x_0, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\}$$

# Assumptions (2)

We assume that the subvariety  $Y \subset X$  is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where  $\psi$  is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood  $V$  of an arbitrary point  $x_0 \in X$  we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

and  $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$  is the **multiplier ideal sheaf**

$$\mathcal{I}(e^{-\psi})_{x_0} = \left\{ f \in \mathcal{O}_{X,x_0}; \exists U \ni x_0, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\}$$

## Claim (Nadel)

$\mathcal{I}(e^{-\psi})$  is a coherent ideal sheaf.

# Assumptions (2)

We assume that the subvariety  $Y \subset X$  is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where  $\psi$  is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood  $V$  of an arbitrary point  $x_0 \in X$  we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

and  $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$  is the **multiplier ideal sheaf**

$$\mathcal{I}(e^{-\psi})_{x_0} = \left\{ f \in \mathcal{O}_{X,x_0}; \exists U \ni x_0, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\}$$

## Claim (Nadel)

$\mathcal{I}(e^{-\psi})$  is a coherent ideal sheaf.

Moreover  $\mathcal{I}(e^{-\psi})$  is always an integrally closed ideal.



# Assumptions (2)

We assume that the subvariety  $Y \subset X$  is defined by

$$Y = V(\mathcal{I}(e^{-\psi})), \quad \mathcal{O}_Y := \mathcal{O}_X / \mathcal{I}(e^{-\psi})$$

where  $\psi$  is a quasi-psh function with *analytic singularities*, i.e. locally on a neighborhood  $V$  of an arbitrary point  $x_0 \in X$  we have

$$\psi(z) = c \log \sum |g_j(z)|^2 + v(z), \quad g_j \in \mathcal{O}_X(V), \quad c > 0, \quad v \in C^\infty(V),$$

and  $\mathcal{I}(e^{-\psi}) \subset \mathcal{O}_X$  is the **multiplier ideal sheaf**

$$\mathcal{I}(e^{-\psi})_{x_0} = \left\{ f \in \mathcal{O}_{X,x_0}; \exists U \ni x_0, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\}$$

## Claim (Nadel)

$\mathcal{I}(e^{-\psi})$  is a coherent ideal sheaf.

Moreover  $\mathcal{I}(e^{-\psi})$  is always an integrally closed ideal.

**Typical choice:**  $\psi(z) = c \log |s(z)|_{h_E}^2, \quad c > 0, \quad s \in H^0(X, E).$

# Log resolution / reduction to the divisorial case

The simplest case is when  $Y = \sum m_j Y_j$  is an effective simple normal crossing divisor and  $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{O}_X(-Y)$ . We can then take

$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|_{h_j}^2, \quad c_j > 0, \quad [c_j] = m_j,$$

for some smooth hermitian metric  $h_j$  on  $\mathcal{O}_X(Y_j)$ . Then

$$\mathcal{I}(e^{-\psi}) = \mathcal{O}_X(-\sum m_j Y_j), \quad i\partial\bar{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j), h_j})$$

# Log resolution / reduction to the divisorial case

The simplest case is when  $Y = \sum m_j Y_j$  is an effective simple normal crossing divisor and  $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{O}_X(-Y)$ . We can then take

$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|_{h_j}^2, \quad c_j > 0, \quad [c_j] = m_j,$$

for some smooth hermitian metric  $h_j$  on  $\mathcal{O}_X(Y_j)$ . Then

$$\mathcal{I}(e^{-\psi}) = \mathcal{O}_X(-\sum m_j Y_j), \quad i\partial\bar{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j), h_j})$$

The case of a higher codimensional multiplier ideal scheme  $\mathcal{I}(e^{-\psi})$  can easily be reduced to the divisorial case by using a suitable log resolution (a composition of blow ups, thanks to Hironaka's desingularization theorem).

# Main results

Theorem (JY. Cao, D- , S-i. Matsumura, January 2017)

Take  $(X, \omega)$  to be **Kähler and holomorphically convex**,  
and let  $(L, h_L)$  be a hermitian line bundle such that

$$(**) \quad \Theta_{L, h_L} + (1 + \alpha\delta)i\partial\bar{\partial}\psi \geq 0 \quad \text{in the sense of currents}$$

for some  $\delta(x) > 0$  continuous and  $\alpha = 0, 1$ . Then:

# Main results

Theorem (JY. Cao, D- , S-i. Matsumura, January 2017)

Take  $(X, \omega)$  to be **Kähler and holomorphically convex**,  
and let  $(L, h_L)$  be a hermitian line bundle such that

$$(**) \quad \Theta_{L, h_L} + (1 + \alpha\delta)i\partial\bar{\partial}\psi \geq 0 \quad \text{in the sense of currents}$$

for some  $\delta(x) > 0$  continuous and  $\alpha = 0, 1$ . Then:

the morphism induced by the natural inclusion  $\mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L)$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L))$$

is injective for every  $q \geq 0$ ,

# Main results

Theorem (JY. Cao, D- , S-i. Matsumura, January 2017)

Take  $(X, \omega)$  to be **Kähler and holomorphically convex**,  
and let  $(L, h_L)$  be a hermitian line bundle such that

$$(**) \quad \Theta_{L, h_L} + (1 + \alpha\delta)i\partial\bar{\partial}\psi \geq 0 \quad \text{in the sense of currents}$$

for some  $\delta(x) > 0$  continuous and  $\alpha = 0, 1$ . Then:

the morphism induced by the natural inclusion  $\mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L)$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L))$$

is injective for every  $q \geq 0$ , in other words, the sheaf morphism  
 $\mathcal{I}(h) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$  yields a surjection

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

# Main results

Theorem (JY. Cao, D- , S-i. Matsumura, January 2017)

Take  $(X, \omega)$  to be **Kähler and holomorphically convex**,  
and let  $(L, h_L)$  be a hermitian line bundle such that

$$(**) \quad \Theta_{L, h_L} + (1 + \alpha\delta)i\partial\bar{\partial}\psi \geq 0 \quad \text{in the sense of currents}$$

for some  $\delta(x) > 0$  continuous and  $\alpha = 0, 1$ . Then:

the morphism induced by the natural inclusion  $\mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L)$

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L))$$

is injective for every  $q \geq 0$ , in other words, the sheaf morphism  
 $\mathcal{I}(h) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$  yields a surjection

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) \rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

Corollary (take  $h_L$  smooth  $\Rightarrow \mathcal{I}(h_L) = \mathcal{O}_X$ , and  $Y = V(\mathcal{I}(e^{-\psi}))$ )

If  $h_L$  is smooth,  $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}(e^{-\psi})$  and  $h_L, \psi$  satisfy (\*\*), then  
 $H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y)$  is surjective.

# Comments / algebraic consequences

The exact sequence  $0 \rightarrow \mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}) \rightarrow 0$  implies that both injectivity and surjectivity hold when

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) = 0,$$

and for this it is enough to have a **strict curvature assumption**

(\*\*\*)  $\Theta_{L, h_L} + i\partial\bar{\partial}\psi \geq \delta\omega > 0$  in the sense of currents.



# Comments / algebraic consequences

The exact sequence  $0 \rightarrow \mathcal{I}(h_L e^{-\psi}) \rightarrow \mathcal{I}(h_L) \rightarrow \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}) \rightarrow 0$  implies that both injectivity and surjectivity hold when

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-\psi})) = 0,$$

and for this it is enough to have a **strict curvature assumption**

$$(***) \quad \Theta_{L, h_L} + i\partial\bar{\partial}\psi \geq \delta\omega > 0 \quad \text{in the sense of currents.}$$

## Corollary (purely algebraic)

Assume that  $X$  is projective (or that one has a projective morphism  $X \rightarrow S$  over an affine algebraic base  $S$ ). Let  $Y = \sum m_j Y_j$  be an effective divisor and  $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{O}_X(-Y)$ . If (as  $\mathbb{Q}$ -divisors)

$$(**) \quad L - (1 + \delta) \sum c_j Y_j = G_\delta + U_\delta, \quad [c_j] = m_j$$

with  $\delta = 0$  or  $\delta_0 \in \mathbb{Q}_+^*$ ,  $G_\delta$  semiample and  $U_\delta \in \text{Pic}^0(X)$ , then

$$H^q(X, K_X \otimes L) \rightarrow H^q(Y, (K_X \otimes L)|_Y)$$

is surjective.

# Possible use for abundance by induction on $\dim X$ ?

For a line bundle  $L$ , one defines the Kodaira-Iitaka dimension  $\kappa(L) = \limsup_{m \rightarrow +\infty} \log \dim H^0(X, L^{\otimes m}) / \log m$  and the numerical dimension  $\text{nd}(L) = \text{maximum power of non zero positive intersection of a positive current } T \in c_1(L)$ , if  $L$  is pseudo-effective, and  $\text{nd}(L) = -\infty$  otherwise. They always satisfy

$$-\infty \leq \kappa(L) \leq \text{nd}(L) \leq n = \dim X.$$

# Possible use for abundance by induction on $\dim X$ ?

For a line bundle  $L$ , one defines the Kodaira-Iitaka dimension  $\kappa(L) = \limsup_{m \rightarrow +\infty} \log \dim H^0(X, L^{\otimes m}) / \log m$  and the numerical dimension  $\text{nd}(L) = \text{maximum power of non zero positive intersection of a positive current } T \in c_1(L)$ , if  $L$  is pseudo-effective, and  $\text{nd}(L) = -\infty$  otherwise. They always satisfy

$$-\infty \leq \kappa(L) \leq \text{nd}(L) \leq n = \dim X.$$

One says that  $L$  is **abundant** if  $\kappa(L) = \text{nd}(L)$ . The **abundance conjecture** states that  $K_X$  is always abundant if  $X$  is nonsingular and projective (or even compact Kähler). More generally, it is expected that  $K_X + \Delta$  is abundant for every effective klt  $\mathbb{Q}$ -divisor  $\Delta$ .

# Possible use for abundance by induction on $\dim X$ ?

For a line bundle  $L$ , one defines the Kodaira-Iitaka dimension  $\kappa(L) = \limsup_{m \rightarrow +\infty} \log \dim H^0(X, L^{\otimes m}) / \log m$  and the numerical dimension  $\text{nd}(L) = \text{maximum power of non zero positive intersection of a positive current } T \in c_1(L)$ , if  $L$  is pseudo-effective, and  $\text{nd}(L) = -\infty$  otherwise. They always satisfy

$$-\infty \leq \kappa(L) \leq \text{nd}(L) \leq n = \dim X.$$

One says that  $L$  is **abundant** if  $\kappa(L) = \text{nd}(L)$ . The **abundance conjecture** states that  $K_X$  is always abundant if  $X$  is nonsingular and projective (or even compact Kähler). More generally, it is expected that  $K_X + \Delta$  is abundant for every effective klt  $\mathbb{Q}$ -divisor  $\Delta$ .

When  $X$  is not uniruled, i.e.  $K_X$  is pseudoeffective, one can ask whether the following **generalized abundance property** holds true: let  $L$  be a line bundle such that  $L - \varepsilon K_X$  is pseudoeffective,  $0 < \varepsilon \ll 1$ ; does there exist  $G \in \text{Pic}^0(X)$  such that  $L + G$  is abundant ?

# Twisted Bochner-Kodaira-Nakano inequality (Ohsawa-Takegoshi)

Let  $(X, \omega)$  be a Kähler manifold and let  $\eta, \lambda > 0$  be smooth functions on  $X$ .

For every compactly supported section  $u \in C_c^\infty(X, \Lambda^{p,q} T_X^* \otimes L)$  with values in a hermitian line bundle  $(L, h_L)$ , one has

$$\begin{aligned} & \|(\eta + \lambda)^{\frac{1}{2}} \bar{\partial}^* u\|^2 + \|\eta^{\frac{1}{2}} \bar{\partial} u\|^2 + \|\lambda^{\frac{1}{2}} \partial u\|^2 + 2\|\lambda^{-\frac{1}{2}} \partial \eta \wedge u\|^2 \\ & \geq \int_X \langle B_{L, h_L, \omega, \eta, \lambda}^{p,q} u, u \rangle dV_{X, \omega} \end{aligned}$$

where  $dV_{X, \omega} = \frac{1}{n!} \omega^n$  is the Kähler volume element and  $B_{L, h_L, \omega, \eta, \lambda}^{p,q}$  is the Hermitian operator on  $\Lambda^{p,q} T_X^* \otimes L$  such that

$$B_{L, h_L, \omega, \eta, \lambda}^{p,q} = [\eta i\Theta_L - i\partial\bar{\partial}\eta - i\lambda^{-1}\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega].$$

# Approximate solutions to $\bar{\partial}$ -equations

## Main $L^2$ estimate

Let  $(X, \omega)$  be a Kähler manifold possessing a complete Kähler metric let  $(E, h_E)$  be a Hermitian vector bundle over  $X$ . Assume that  $B = B_{E, h, \omega, \eta, \lambda}^{n, q}$  satisfies  $B + \varepsilon \text{Id} > 0$  for some  $\varepsilon > 0$  (so that  $B$  can be just semi-positive or even slightly negative).

# Approximate solutions to $\bar{\partial}$ -equations

## Main $L^2$ estimate

Let  $(X, \omega)$  be a Kähler manifold possessing a complete Kähler metric let  $(E, h_E)$  be a Hermitian vector bundle over  $X$ . Assume that  $B = B_{E, h, \omega, \eta, \lambda}^{n, q}$  satisfies  $B + \varepsilon \text{Id} > 0$  for some  $\varepsilon > 0$  (so that  $B$  can be just semi-positive or even slightly negative).

Take a section  $v \in L^2(X, \Lambda^{n, q} T_X^* \otimes E)$  such that  $\bar{\partial}v = 0$  and

$$M(\varepsilon) := \int_X \langle (B + \varepsilon \text{Id})^{-1} v, v \rangle dV_{X, \omega} < +\infty.$$

# Approximate solutions to $\bar{\partial}$ -equations

## Main $L^2$ estimate

Let  $(X, \omega)$  be a Kähler manifold possessing a complete Kähler metric let  $(E, h_E)$  be a Hermitian vector bundle over  $X$ . Assume that  $B = B_{E, h, \omega, \eta, \lambda}^{n, q}$  satisfies  $B + \varepsilon \text{Id} > 0$  for some  $\varepsilon > 0$  (so that  $B$  can be just semi-positive or even slightly negative).

Take a section  $v \in L^2(X, \Lambda^{n, q} T_X^* \otimes E)$  such that  $\bar{\partial}v = 0$  and

$$M(\varepsilon) := \int_X \langle (B + \varepsilon \text{Id})^{-1} v, v \rangle dV_{X, \omega} < +\infty.$$

Then there exists an approximate solution  $u_\varepsilon \in L^2(X, \Lambda^{n, q-1} T_X^* \otimes E)$  and a **correction term**  $w_\varepsilon \in L^2(X, \Lambda^{n, q} T_X^* \otimes E)$  such that

$$\bar{\partial}u_\varepsilon = v + w_\varepsilon \quad \text{and}$$

$$\int_X (\eta + \lambda)^{-1} |u_\varepsilon|^2 dV_{X, \omega} + \frac{1}{\varepsilon} \int_X |w_\varepsilon|^2 dV_{X, \omega} \leq M(\varepsilon).$$



# Proof: setting up the relevant $\bar{\partial}$ equation (1)

Every cohomology class in

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$$

is represented by a holomorphic Čech  $q$ -cocycle with respect to a Stein covering  $\mathcal{U} = (U_i)$ , say  $(c_{i_0 \dots i_q})$ ,

$$c_{i_0 \dots i_q} \in H^0(U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

# Proof: setting up the relevant $\bar{\partial}$ equation (1)

Every cohomology class in

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$$

is represented by a holomorphic Čech  $q$ -cocycle with respect to a Stein covering  $\mathcal{U} = (U_i)$ , say  $(c_{i_0 \dots i_q})$ ,

$$c_{i_0 \dots i_q} \in H^0(U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})).$$

By the standard sheaf theoretic isomorphism with Dolbeault cohomology, this class is represented by a smooth  $(n, q)$ -form

$$f = \sum_{i_0, \dots, i_q} c_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

by means of a partition of unity  $(\rho_i)$  subordinate to  $(U_i)$ . This form is to be interpreted as a form on the (non reduced) analytic subvariety  $Y$  associated with the colon ideal sheaf

$\mathcal{J} = \mathcal{I}(h e^{-\psi}) : \mathcal{I}(h)$  and the structure sheaf  $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$ .

# Proof: setting up the relevant $\bar{\partial}$ equation (2)

We get an extension of  $f$  as a smooth (no longer  $\bar{\partial}$ -closed)  $(n, q)$ -form on  $X$  by taking

$$\tilde{f} = \sum_{i_0, \dots, i_q} \tilde{c}_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

where  $\tilde{c}_{i_0 \dots i_q} =$  extension of  $c_{i_0 \dots i_q}$  from  $U_{i_0} \cap \dots \cap U_{i_q} \cap Y$  to  $U_{i_0} \cap \dots \cap U_{i_q}$

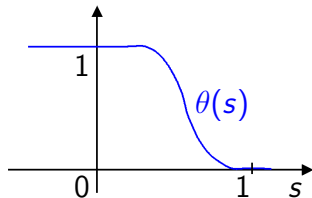
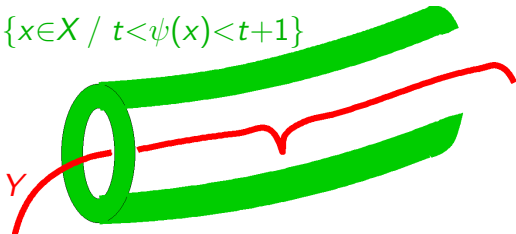
# Proof: setting up the relevant $\bar{\partial}$ equation (2)

We get an extension of  $f$  as a smooth (no longer  $\bar{\partial}$ -closed)  $(n, q)$ -form on  $X$  by taking

$$\tilde{f} = \sum_{i_0, \dots, i_q} \tilde{c}_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

where  $\tilde{c}_{i_0 \dots i_q} =$  extension of  $c_{i_0 \dots i_q}$  from  $U_{i_0} \cap \dots \cap U_{i_q} \cap Y$  to  $U_{i_0} \cap \dots \cap U_{i_q}$

$\{x \in X / t < \psi(x) < t+1\}$



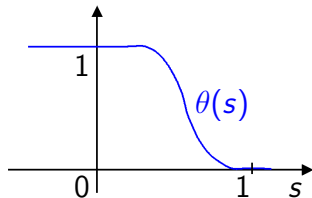
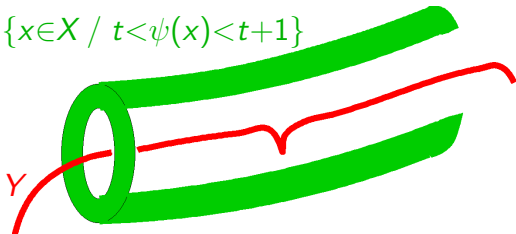
# Proof: setting up the relevant $\bar{\partial}$ equation (2)

We get an extension of  $f$  as a smooth (no longer  $\bar{\partial}$ -closed)  $(n, q)$ -form on  $X$  by taking

$$\tilde{f} = \sum_{i_0, \dots, i_q} \tilde{c}_{i_0 \dots i_q} \rho_{i_0} \bar{\partial} \rho_{i_1} \wedge \dots \wedge \bar{\partial} \rho_{i_q}$$

where  $\tilde{c}_{i_0 \dots i_q} =$  extension of  $c_{i_0 \dots i_q}$  from  $U_{i_0} \cap \dots \cap U_{i_q} \cap Y$  to  $U_{i_0} \cap \dots \cap U_{i_q}$

$\{x \in X / t < \psi(x) < t+1\}$



Now, truncate  $\tilde{f}$  as  $\theta(\psi - t) \cdot \tilde{f}$  on the green hollow tubular neighborhood, and solve an approximate  $\bar{\partial}$ -equation

$$(*) \quad \bar{\partial} u_{t,\varepsilon} = \bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) + w_{t,\varepsilon}$$

# Proof: setting up the relevant $\bar{\partial}$ equation (3)

Here we have

$$\bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) = \theta'(\psi - t)\bar{\partial}\psi \wedge \tilde{f} + \theta(\psi - t) \cdot \bar{\partial}\tilde{f}$$

where the first term vanishes near  $Y$  and the second one is  $L^2$  with respect to  $h_L e^{-\psi}$  (as the image of  $\bar{\partial}\tilde{f}$  in  $\mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$  is  $\bar{\partial}f = 0$ ).

# Proof: setting up the relevant $\bar{\partial}$ equation (3)

Here we have

$$\bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) = \theta'(\psi - t) \bar{\partial}\psi \wedge \tilde{f} + \theta(\psi - t) \cdot \bar{\partial}\tilde{f}$$

where the first term vanishes near  $Y$  and the second one is  $L^2$  with respect to  $h_L e^{-\psi}$  (as the image of  $\bar{\partial}\tilde{f}$  in  $\mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$  is  $\bar{\partial}f = 0$ ).

With ad hoc “twisting functions”  $\eta = \eta_t := 1 - \delta\chi_t(\psi)$ ,  $\lambda := \pi(1 + \delta^2\psi^2)$  and a suitable adjustment  $\varepsilon = e^{(1+\beta)t}$ ,  $\beta \ll 1$ , we can find approximate  $L^2$  solutions of the  $\bar{\partial}$ -equation such that

$$\bar{\partial}u_{t,\varepsilon} = \bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) + w_{t,\varepsilon}, \quad \int_X |u_{t,\varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X,\omega} < +\infty$$

and

$$\lim_{t \rightarrow -\infty} \int_X |w_{t,\varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X,\omega} = 0.$$

# Proof: setting up the relevant $\bar{\partial}$ equation (3)

Here we have

$$\bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) = \theta'(\psi - t) \bar{\partial}\psi \wedge \tilde{f} + \theta(\psi - t) \cdot \bar{\partial}\tilde{f}$$

where the first term vanishes near  $Y$  and the second one is  $L^2$  with respect to  $h_L e^{-\psi}$  (as the image of  $\bar{\partial}\tilde{f}$  in  $\mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi})$  is  $\bar{\partial}f = 0$ ).

With ad hoc “twisting functions”  $\eta = \eta_t := 1 - \delta\chi_t(\psi)$ ,  $\lambda := \pi(1 + \delta^2\psi^2)$  and a suitable adjustment  $\varepsilon = e^{(1+\beta)t}$ ,  $\beta \ll 1$ , we can find approximate  $L^2$  solutions of the  $\bar{\partial}$ -equation such that

$$\bar{\partial}u_{t,\varepsilon} = \bar{\partial}(\theta(\psi - t) \cdot \tilde{f}) + w_{t,\varepsilon}, \quad \int_X |u_{t,\varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X,\omega} < +\infty$$

and

$$\lim_{t \rightarrow -\infty} \int_X |w_{t,\varepsilon}|_{\omega, h_L}^2 e^{-\psi} dV_{X,\omega} = 0.$$

The estimate on  $u_{t,\varepsilon}$  with respect to the weight  $h_L e^{-\psi}$  shows that  $\theta(\psi - t) \cdot \tilde{f} - u_{t,\varepsilon}$  is an approximate extension of  $f$ . □



# Can one get estimates for the extension ?

The answer is **yes if  $\psi$  is log canonical**, namely  $\mathcal{I}(e^{-(1-\varepsilon)\psi}) = \mathcal{O}_X$  for all  $\varepsilon > 0$ .

# Can one get estimates for the extension ?

The answer is **yes if  $\psi$  is log canonical**, namely  $\mathcal{I}(e^{-(1-\varepsilon)\psi}) = \mathcal{O}_X$  for all  $\varepsilon > 0$ .

## Ohsawa's residue measure

If  $\psi$  is log canonical, one can also associate in a natural way a measure  $dV_{Y^\circ, \omega}[\psi]$  on the set  $Y^\circ$  of regular points of  $Y$  as follows. If  $g \in \mathcal{C}_c(Y^\circ)$  is a compactly supported continuous function on  $Y^\circ$  and  $\tilde{g}$  a compactly supported extension of  $g$  to  $X$ , one sets

$$\int_{Y^\circ} g dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} e^{-\psi} dV_{X, \omega}$$

# Can one get estimates for the extension ?

The answer is **yes if  $\psi$  is log canonical**, namely  $\mathcal{I}(e^{-(1-\varepsilon)\psi}) = \mathcal{O}_X$  for all  $\varepsilon > 0$ .

## Ohsawa's residue measure

If  $\psi$  is log canonical, one can also associate in a natural way a measure  $dV_{Y^\circ, \omega}[\psi]$  on the set  $Y^\circ$  of regular points of  $Y$  as follows. If  $g \in \mathcal{C}_c(Y^\circ)$  is a compactly supported continuous function on  $Y^\circ$  and  $\tilde{g}$  a compactly supported extension of  $g$  to  $X$ , one sets

$$\int_{Y^\circ} g dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} e^{-\psi} dV_{X, \omega}$$

## Theorem

If  $\psi$  is lc and the curvature hypothesis is satisfied, for any  $f$  in  $H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L)/\mathcal{I}(h_L e^{-\psi}))$  s.t.  $\int_{Y^\circ} |f|_{\omega, h_L}^2 dV_{Y^\circ, \omega}[\psi] < +\infty$ , there exists  $\tilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L))$  which extends  $f$ , such that

$$\int_X (1 + \delta^2 \psi^2)^{-1} e^{-\psi} |\tilde{f}|_{\omega, h_L}^2 dV_{X, \omega} \leq \frac{34}{\delta} \int_{Y^\circ} |f|_{\omega, h_L}^2 dV_{Y^\circ, \omega}[\psi].$$

# Can one get estimates for the extension ? (sequel)

If  $\psi$  is not log canonical, consider the “last jumps”  $m_{p-1} < m_p \leq 1$  such that  $\mathcal{I}(h_L e^{-m_{p-1}\psi}) \supsetneq \mathcal{I}(h_L e^{-m_p\psi}) = \mathcal{I}(h_L e^{-\psi})$  and assume

$$f \in H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}) / \mathcal{I}(h_L e^{-m_p\psi})),$$

i.e.,  $f$  vanishes just a little bit less than prescribed by the sheaf  $\mathcal{I}(h_L e^{-\psi})$ . Then there is still a corresponding residue measure:

# Can one get estimates for the extension ? (sequel)

If  $\psi$  is not log canonical, consider the “last jumps”  $m_{p-1} < m_p \leq 1$  such that  $\mathcal{I}(h_L e^{-m_{p-1}\psi}) \not\supseteq \mathcal{I}(h_L e^{-m_p\psi}) = \mathcal{I}(h_L e^{-\psi})$  and assume

$$f \in H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}) / \mathcal{I}(h_L e^{-m_p\psi})),$$

i.e.,  $f$  vanishes just a little bit less than prescribed by the sheaf  $\mathcal{I}(h_L e^{-\psi})$ . Then there is still a corresponding residue measure:

## Higher multiplicity residue measure

If  $f$  is as above, and  $\tilde{f}$  is a local extension, one can associate a higher multiplicity residue measure  $|f|^2 dV_{Y^\circ, \omega}[\psi]$  (formal notation) as follows. If  $g \in \mathcal{C}_c(Y^\circ)$  and  $\tilde{g}$  a compactly supported extension of  $g$  to  $X$ , one sets

$$\int_{Y^\circ} g |f|^2 dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} |\tilde{f}|^2 e^{-m_p \psi} dV_{X, \omega}$$

# Can one get estimates for the extension ? (sequel)

If  $\psi$  is not log canonical, consider the “last jumps”  $m_{p-1} < m_p \leq 1$  such that  $\mathcal{I}(h_L e^{-m_{p-1}\psi}) \supsetneq \mathcal{I}(h_L e^{-m_p\psi}) = \mathcal{I}(h_L e^{-\psi})$  and assume

$$f \in H^0(Y, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}) / \mathcal{I}(h_L e^{-m_p\psi})),$$

i.e.,  $f$  vanishes just a little bit less than prescribed by the sheaf  $\mathcal{I}(h_L e^{-\psi})$ . Then there is still a corresponding residue measure:

## Higher multiplicity residue measure

If  $f$  is as above, and  $\tilde{f}$  is a local extension, one can associate a higher multiplicity residue measure  $|f|^2 dV_{Y^\circ, \omega}[\psi]$  (formal notation) as follows. If  $g \in \mathcal{C}_c(Y^\circ)$  and  $\tilde{g}$  a compactly supported extension of  $g$  to  $X$ , one sets

$$\int_{Y^\circ} g |f|^2 dV_{Y^\circ, \omega}[\psi] = \lim_{t \rightarrow -\infty} \int_{\{x \in X, t < \psi(x) < t+1\}} \tilde{g} |\tilde{f}|^2 e^{-m_p \psi} dV_{X, \omega}$$

Then a global extension  $\tilde{f} \in H^0(X, K_X \otimes L \otimes \mathcal{I}(h_L e^{-m_{p-1}\psi}))$  exists, that satisfies the expected  $L^2$  estimate.

# Special case / limitations of the $L^2$ estimates

In the special case when  $\psi$  is given by  $\psi(z) = r \log |s(z)|_{h_E}^2$  for a section  $s \in H^0(X, E)$  generically transverse to the zero section of a rank  $r$  vector bundle  $E$  on  $X$ , the subvariety  $Y = s^{-1}(0)$  has codimension  $r$ , and one can check easily that

$$dV_{Y^\circ, \omega}[\psi] = \frac{dV_{Y^\circ, \omega}}{|\Lambda^r(ds)|_{\omega, h_E}^2}.$$

# Special case / limitations of the $L^2$ estimates

In the special case when  $\psi$  is given by  $\psi(z) = r \log |s(z)|_{h_E}^2$  for a section  $s \in H^0(X, E)$  generically transverse to the zero section of a rank  $r$  vector bundle  $E$  on  $X$ , the subvariety  $Y = s^{-1}(0)$  has codimension  $r$ , and one can check easily that

$$dV_{Y^\circ, \omega}[\psi] = \frac{dV_{Y^\circ, \omega}}{|\Lambda^r(ds)|_{\omega, h_E}^2}.$$

Thus one sees that the residue measure takes into account in a very precise manner the singularities of  $Y$ . It may happen that  $dV_{Y^\circ, \omega}[\psi]$  has infinite mass near the singularities of  $Y$ , as is the case when  $Y$  is a simple normal crossing divisor.



# Special case / limitations of the $L^2$ estimates

In the special case when  $\psi$  is given by  $\psi(z) = r \log |s(z)|_{h_E}^2$  for a section  $s \in H^0(X, E)$  generically transverse to the zero section of a rank  $r$  vector bundle  $E$  on  $X$ , the subvariety  $Y = s^{-1}(0)$  has codimension  $r$ , and one can check easily that

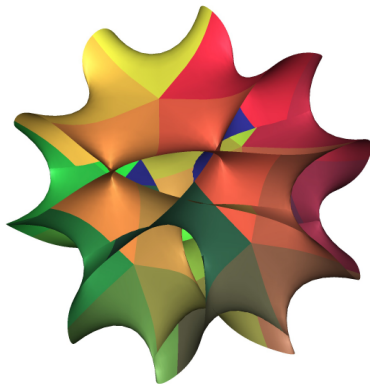
$$dV_{Y^\circ, \omega}[\psi] = \frac{dV_{Y^\circ, \omega}}{|\Lambda^r(ds)|_{\omega, h_E}^2}.$$

Thus one sees that the residue measure takes into account in a very precise manner the singularities of  $Y$ . It may happen that  $dV_{Y^\circ, \omega}[\psi]$  has infinite mass near the singularities of  $Y$ , as is the case when  $Y$  is a simple normal crossing divisor.

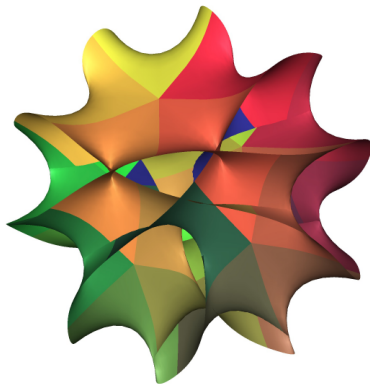
Therefore, sections  $s \in H^0(Y, (K_X \otimes L)|_Y)$  may not be  $L^2$  with respect to  $dV_{Y^\circ, \omega}[\psi]$ , and the  $L^2$  estimate of the approximate extension can blow up as  $\varepsilon \rightarrow 0$ . The surprising fact is this is however sufficient to prove the qualitative extension theorem, but without any effective  $L^2$  estimate in the limit.

The end

# Happy birthday Kang-Tae !



# Happy birthday Kang-Tae !



Thanks and congratulations to the organizers!  
Jisoo Byun, Hong Rae Cho, Sung-Yeon Kim,  
Kang-Hyurk Lee, Jong-Do Park