Hyperbolicity of general algebraic hypersurfaces of high degree

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Bernie at the Skoda Conference in Paris, 2005

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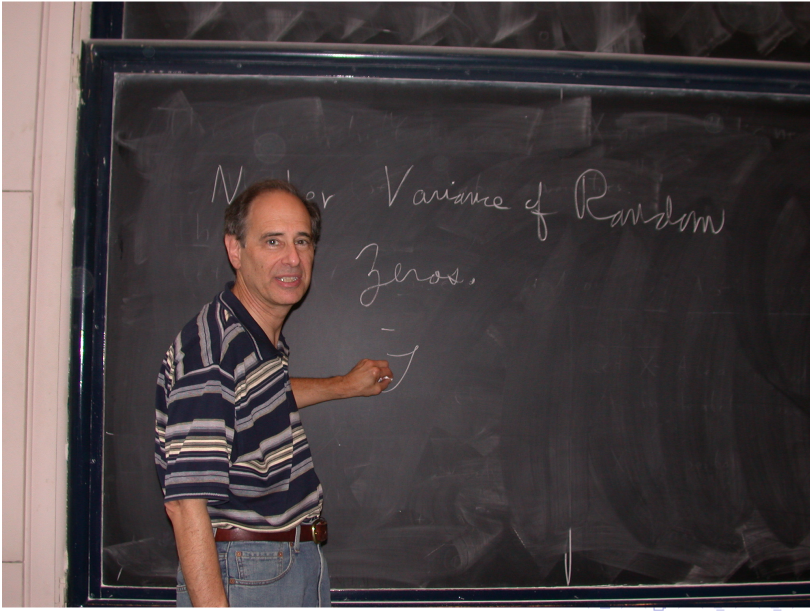

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Let X be a complex space, $\dim_{\mathbb{C}} X = n$.
The **Kobayashi pseudodistance** d_X^K is defined to be
$d_X^K(p, q) =$ infimum of sums $\sum_{i=0}^{N-1} d_{\mathrm{Poinc}}(p_i, p_{i+1})$
for all chains of points $p = p_0, p_1, \ldots, p_N = q \in X$ such that $p_i = g_i(t_i), p_{i+1} = g_i(u_i)$ lie in the image of an analytic disc $g_i : \mathbb{D} \to X$, and $d_{\mathrm{Poinc}}(p_i, p_{i+1}) = d_{\mathrm{Poinc}}(t_i, u_i)$ on that disc.

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Theorem (D, Lempert, Shiffman, Duke Math. J. 1994)

If X is projective algebraic, one obtains the same pseudodistance by taking algebraic curves C_i containing p_i, p_{i+1} and computing $d_{\rm Poinc}(p_i,p_{i+1})$ as the Poincaré distance on the universal cover $\widehat{C}_i$ of the normalization of C_i (with $d_{\rm Poinc}(p_i,p_{i+1}) = 0$ if $g(C_i) \le 1$).

Kobayashi conjecture

Definition

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Green-Griffiths-Lang conjecture

Useful characterization of Kobayashi hyperbolicity (Brody, 1978)

A compact complex space X is Kobayashi hyperbolic iff it is Brody hyperbolic, namely if $\nexists$ entire curves $f : \mathbb{C} \to X$, $f \neq$ const.

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A projective variety X is (Kobayashi/Brody) hyperbolic iff all its subvarieties Y (possibly singular, including X) are of general type.

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The latter property has been characterized by work of Zaidenberg, Clemens, Ein, Voisin, Pacienza. In this way, the optimal bound is expected to be $d_1 = 4$, $d_n = 2n + 1$ for $2 \leq n \leq 4$ and $d_n = 2n$ for $n \geq 5$.

Geometric approaches of the Kobayashi conjecture

The main idea is to use the geometry of jet bundles, and more specifically Semple bundles introduced in this context by D. in 1995.

Theorem (Brotbek, April 2016)

Let Z be a projective $n+1$-dimensional projective manifold and $A \to Z$ a very ample line bundle. Let $\sigma \in H^0(Z, dA)$ be a generic section. Then for $d \gg 1$ the hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic.

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The initial proof of Brotbek didn't provide effective bounds. Through various improvements, Deng Ya got in his PhD thesis (May 2016) the explicit bound $d_n = (n+1)n^{2n+3} = O(n^{2n+4})$.

Theorem (D-, 2018, with a substantially simplified proof)

In the above setting, a general hypersurface $X_\sigma = \sigma^{-1}(0)$ is hyperbolic as soon as

$$d \geq d_n = \lfloor (en)^{2n+2}/3 \rfloor.$$

Use of algebraic differential operators

Let $\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X. It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!} \nabla^s f(0)$$

where ∇ is the trivial connection on T_X.

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One considers polynomials on the k-jet bundle $J^k X$ defined locally in coordinate charts by

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V.$$

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where the a_α's are holomorphic on X. One can view them as algebraic differential operators on k-jets of curves $f : (\mathbb{C}, 0)_k \to X$ or entire curves $f : \mathbb{C} \to X$

$$P(f; f', \ldots, f^{(k)})(t) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\, f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}.$$

Invariant differential operators

Some operators $f \mapsto P(f)$, such as Wronskians $f' \wedge \ldots \wedge f^{(k)}$, or more generally polynomial operators of the form $Q(f', f' \wedge f'', \ldots, f' \wedge \ldots \wedge f^{(k)})$ satisfy the additional property that for any reparametrization $\varphi : \mathbb{C} \to \mathbb{C}$, one has

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

where m is the weighted degree

$$m = |\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k|.$$

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Fundamental vanishing theorem

[Green-Griffiths 1979], [D. 1995], [Siu-Yeung 1996]
$\forall X$ projective manifold, $\forall A$ ample line bundle on X,
$\forall P \in H^0(X, E_{k,m}T^*_X \otimes \mathcal{O}(-A))$: global diff. operator on X,
$\forall f : \mathbb{C} \to X$ entire curve, one has $P(f) \equiv 0$.

Proof of the fundamental vanishing theorem

Proof in a simple case. If X has an entire curve $g : \mathbb{C} \to X$, then the Brody reparametrization technique produces a Brody curve $f : \mathbb{C} \to X$, such that $\sup_{t\in\mathbb{C}} \|f'(t)\|_\omega < +\infty$ for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

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General case. On can use the logarithmic derivative lemma (Siu-Yeung, 1996), or the Alhfors lemma in case P is invariant (D. 1995), combined with an induction on $m = \deg P$ when P is non invariant.

Semple bundles and direct image formula

Semple bundles

One can construct of tower of $\mathbb{P}^{n-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(n-1)$,
X_k being equipped with a certain rank n subbundle $V_k \subset T_{X_k}$,
and inductively $X_k = P(V_{k-1})$ over X_{k-1},
such that X_k is a desingularization of $J^k X /\!/ \mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of biholomorphisms $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$.

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Direct image formula

The tautological line bundles $\mathcal{O}_{X_k}(m)$ on $X_k = P(V_{k-1})$ have a direct image $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) \simeq E_{k,m} T^*_X$

Reinterpretation of the vanishing theorem

Any global differential operator $P \in H^0(X, E_{k,m}T^*_X \otimes \mathcal{O}(-A))$ corresponds to a section $\sigma_P \in H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi^*_{k,0}\mathcal{O}(-A))$.

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Saying that a curve $f : \mathbb{C} \to X$ satisfies the differential equation $P(f; f', \ldots, f^{(k)}) = 0$ means that the k-jet $f_{[k]} : \mathbb{C} \to X_k$ has its image $f_{[k]}(\mathbb{C})$ contained in the hypersurface $\sigma_P^{-1}(0) \subset X_k$.

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Consequence

All entire curves lie in the so called Green-Griffiths locus

$$\mathrm{GG}(X) = \bigcap_k \pi_{k,0}(\mathrm{GG}(X_k), \quad \mathrm{GG}(X_k) = \bigcap_P \sigma_P^{-1}(0) \subset X_k,$$

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Hope: existence theorem for jet differentials (D-, 2010)

Let X be of general type, i.e. K_X big. Then for $m \gg k \gg 1$

$$h^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^*\mathcal{O}(-A)) \geq c_k m^{\dim X_k}, \quad c_k > 0.$$

Sketch of proof of the existence theorem

One starts with a hermitian metric h_X on T_X such that its determinant yields a singular hermitian metric $\det(h_X^*)$ on K_X with strictly positive curvature current. One can then equip J^kX with a natural Finsler metric that induces a singular hermitian metric on $\mathcal{O}_{X_k}(1)$. One finally applies holomorphic Morse inequalities to

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{\varepsilon}{kn}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big), \quad \varepsilon \in \mathbb{Q}_+^*,$$

$$\eta = \Theta_{K_X,\det h_X^*} - \varepsilon\Theta_{A,h_A}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, one gets upper and lower bounds [$q = 0$ most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq c_{k,n} m^{n+k(n-1)} \bigg(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\bigg)$$

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Wronskian operators are easier to deal with ...

Let $L \to X$ be a line bundle, and let

$$s_0, \ldots, s_k \in H^0(X, L)$$

be arbitrary sections. One defines Wronskian operators acting on $f : \mathbb{C} \to X$, $t \mapsto f(t)$ by $D = \frac{d}{dt}$ and

$$W(s_0, \ldots, s_k)(f) = \begin{vmatrix} s_0(f) & s_1(f) & \ldots & s_k(f) \\ D(s_0(f)) & D(s_1(f)) & \ldots & D(s_k(f)) \\ \vdots & & & \vdots \\ D^k(s_0(f)) & D^k(s_1(f)) & \ldots & D^k(s_k(f)) \end{vmatrix}$$

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This actually does not depend on the trivialization of L and defines

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Problem. One has to take $L > 0$, hence $L^{k+1} > 0$: seems useless!

Wronskian operators can sometimes be simplified !

Take e.g. $X = \mathbb{P}^N$, $A = \mathcal{O}(1)$ very ample, $k \le N$, $d \ge k$ and

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$$P := \prod_{0 \le i \le k+1} z_i^{-(d-k)} W(s_0, \ldots s_k) \in H^0(X, E_{k,k'} T_X^* \otimes A^{k(k+2)-d}).$$

A result of Shiffman-Zaidenberg

By the vanishing theorem, if $d \geq k(k+2)+1 = N^2$, then $W(s_0, \ldots, s_k)(f) = 0$, which means that f satisfies an extra Fermat type equation $\sum_{j=0}^{N-1} c_j z_j^d = 0$, and one can then use induction on N to show that the f_j's are pairwise proportional (Toda '71, Fujimoto '74, Green '75).

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Using this result, inspired J. Noguchi's construction of explicit n-dim hyperbolic hypersurfaces (1996), Shiffman-Zaidenberg proved

Theorem (Shiffman-Zaidenberg 2001)

For $N = 2n$ and $d \geq N^2 = 4n^2$, the intersection of the Fermat hypersurface $H = \{\sum z_j^d = 0\} \subset \mathbb{P}^{2n}$ with a sufficiently general projective linear subspace $\Lambda \simeq \mathbb{P}^{n+1} \subset \mathbb{P}^{2n}$ yields a hyperbolic hypersurface $X := H \cap \Lambda \subset \mathbb{P}^{n+1}$.

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However, one gets here only one Wronskian, so that the genericity of hyperbolicity in such low degrees $d = O(n^2)$ is hard to establish.

Getting more jet differentials from Wronskians

A "better choice" than the Fermat hypersurface is to take $X = \sigma^{-1}(0) \subset \mathbb{P}^{n+1}$ with $\sigma \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(d))$ given by

$$\sigma = \sum_{0\le i\le N} a_i(z) m_i(z)^\delta,\ a_i \text{ "random"},\ \deg a_i = \rho \ge k,\ m_i(z) = \prod_{J\ni i} \tau_J(z),$$

where the J's run over all subsets $J \subset \{0, 1, \ldots, N\}$ with $\mathrm{card}\, J = n$, $\tau_J \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(1))$ is a sufficiently general linear section and $\delta \gg 1$.

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Then, for $k \ge N$ and all $J \subset \{0, 1, ..., N\}$, $\operatorname{card} J = n$, the Wronskians

$$W_{q,\widehat{\tau},k,J} = W(q_1\widehat{\tau}_1^{d-k}, ..., q_r\widehat{\tau}_r^{d-k}, (a_i m_i^\delta)_{i\in\complement J}), \quad r = k - N + n$$

with $\deg q_j = k$ are divisible by $(\widehat{\tau}_j^{d-2k})_{1\le j\le n}$ and $(m_i^{\delta-k})_{i\in\complement J}$ $\Rightarrow$

$$P_{q,\widehat{\tau},k,J} := \prod_{i\in\complement J} m_i^{-(\delta-k)} \prod_j \widehat{\tau}_j^{d-2k} W_{k,r} \in H^0(X, E_{k,k'} T_X^* \otimes A^{c_n})$$

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where $c_n = k(k+1) \deg m_j = O((en)^{n+5/2})$. As $a_i m_i^\delta = -\sum_{j\ne i} a_j m_j^\delta$ on X, we infer the divisibility of $P_{q,\hat\tau,k,J}$ by the extra factor $\tau_J^{\delta-k}$.

Strategy of proof of the Kobayashi conjecture (Brotbek 2016, simplified by D. in 2018)

Let $\pi : \mathcal{X} \to S$ be family of smooth projective varieties, and let $\mathcal{X}_k \to S$ be the relative Semple tower of $(\mathcal{X}, T_{\mathcal{X}/S})$.
If $X_t = \pi^{-1}(t)$, $t \in S$, is the general fiber, then the fiber of $\mathcal{X}_k \to S$ is the k-stage of the Semple tower $X_{t,k} \to X_t$

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Basic observation

Assume that there exists $t_0 \in S$ such that we get on $X_{t_0,k}$ a nef "twisted tautological sheaf" $\mathcal{G}_{|X_{t_0,k}}$ where

$$\mathcal{G} := \mathcal{O}_{\mathcal{X}_k}(m) \otimes \mathcal{I}_{k,m} \otimes \pi_{k,0} A^{-1}$$

(in the sense that a log resolution of $\mathcal{G}$ is nef), and $\mathcal{I}_{k,m}$ is a suitable "functorial" multiplier ideal with support in the set $\mathcal{X}_k^{\rm sing}$ of singular jets. Then X_t is Kobayashi hyperbolic for general $t \in S$.

Simplified proof of the Kobayashi conjecture

Proof. By hypothesis, One can take a resolution $\mu_{k,m} : \widehat{\mathcal{X}}_k \to \mathcal{X}_k$ of the ideal $\mathcal{I}_{k,m}$ as an invertible sheaf $\mu_{k,m}^*\mathcal{I}_{k,m}$ on $\widehat{\mathcal{X}}_{k,m}$, so that $\mu_{k,m}^*\mathcal{G}_{|\widehat{X}_{t_0,k}}$ is a nef line bundle.

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Then one can add a small $\mathbb{Q}$-divisor $\mathcal{P}_\varepsilon$ that is a combination of the lower stages $\mathcal{O}_{\mathcal{X}_\ell}(m')$, $\ell < k$, and of the exceptional divisor of $\mu_{k,m}$ so that $(\mu_{k,m}^*\mathcal{G} \otimes \mathcal{P}_\varepsilon)_{|\widehat{X}_{t_0,k}}$ is an ample line bundle.

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Since ampleness is a Zariski open property, one concludes that $(\mu_{k,m}^* \mathcal{G} \otimes G_\varepsilon)_{|\widehat{X}_{t,k}}$ is ample for general $t \in S$. The fundamental vanishing theorem then implies that X_t is Kobayashi hyperbolic. $\square$

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The next idea is to produce a very particular hypersurface X_{t_0} on which there are a lot of non trivial Wronskian operators that generate the required sheaf

$$\mathcal{G} = \mathcal{O}_{\mathcal{X}_k}(m) \otimes \mathcal{I}_{k,m} \otimes \pi_{k,0}A^{-1}.$$

Then $\mathcal{G}_{|X_{k,t_0}}$ is nef and we are done.

Conclusion: analyzing base loci of Wronskians

We need $\delta > k + c_n$ to reach a negative exponent $A^{<0}$

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For $k \geq n^3 + n^2 + 1$, the k-jets of the coefficients a_j are general enough, the simplified Wronskians $\widetilde{P}_{q,\widehat{\tau},k,J}$ generate the universal Wronskian ideal $\mathcal{I}_{k,k'}$ outside of the hyperplane sections $\tau_J^{-1}(0)$.

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The proof is achieved by induction on $\dim X$. By taking $X' = \tau_J^{-1}(0)$ one can define further simplified Wronskian sections that generate the universal line bundle $\mathcal{G}$ everywhere on $\widehat{X}_{t_0,k}$. $\square$

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In order to improve the bounds (and eventually to prove the GGL conjecture), one would need to achieve a betting understanding of the geometry of Semple jet bundles and of base loci of jet differentials.

Concept of algebraic jet hyberbolicity

Fix X projective. An irreducible algebraic subset $Z \subset X_k$ of the Semple k-jet bundle is said admissible if it does not project into the intermediate "vertical divisors" $P(T_{X_{\ell-1}/X_{\ell-2}}) \subset X_\ell$, $2 \leq \ell \leq k$.

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One then defines an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by putting $W = \overline{T_{Z'} \cap V_k}$, where the intersection is taken on a suitable Zariski open set $Z' \subset Z_{\rm reg}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

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Definition

Let X be of general type. We say that X is "algebraically jet hyperbolic", [resp. "strongly of general type"], if for every admissible alg. subvariety $Z \subsetneq X_k$ [resp. such that $\pi_{k,0}(Z) = X$)], the induced directed structure $(Z, W) \subset (X_k, V_k)$ is of general type modulo $X_k \to X$, i.e. $\mathcal{K}_W \otimes \mathcal{O}_{X_k}(m)_{|Z}$ is big for some $m \in \mathbb{Q}_+$.

Relation between the GGL conjecture and hyperbolicity

Theorem (D., 2015)

Let X be a projective variety.

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Another important result was obtained in 2018 by Eric Riedl and David Yang, thanks to a general Grassmannian construction.

Theorem (Riedl, Yang 2018 - rough statement)

Assume that a general hypersurface $X \subset \mathbb{P}^n$ satisfies the GGL conjecture for $d \geq d_{\mathrm{GGL}}(n)$. Then a general hypersurface $X \subset \mathbb{P}^{n+1}$ satisfies the Kobayashi conjecture for $d \geq d_{\mathrm{GGL}}(2n)$.

Recent result of J. Merker

Using the Riedl-Yang approach, J. Merker recently proved

Theorem (J. Merker, January 2019)

Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d.

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(At this date – February 2019 – this is the best known estimate).

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Part (i) is obtained by producing jet differentials via holomorphic morse inequalities, and applying Siu's technique of slanted holomorphic vector fields, along with careful estimates.

Part (ii) is now a consequence of the above theorem of Riedl-Yang.

(At this date – February 2019 – this is the best known estimate).

In 2010, G. Bérczi has formulated a combinatorial conjecture for Thom polynomials that would imply polynomial bounds.

Best wishes Bernie!