

On the cohomology of pseudoeffective line bundles

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 - solution of MMP (BCHM 2006), D-Hacon-Păun (2010)

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Any subspace $V_m \subset H^0(X, L^{\otimes m})$ define a meromorphic map

$$\begin{aligned} \Phi_{mL} : X \setminus Z_m &\longrightarrow \mathbb{P}(V_m) \quad (\text{hyperplanes of } V_m) \\ x &\longmapsto H_x = \{\sigma \in V_m; \sigma(x) = 0\} \end{aligned}$$

where $Z_m = \text{base locus } B(mL) = \bigcap \sigma^{-1}(0)$.

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Given sections $\sigma_1, \dots, \sigma_n \in H^0(X, L^{\otimes m})$, one gets a **singular hermitian metric** on L defined by

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and the curvature is $\Theta_{L,h} = \frac{1}{m} dd^c \log \varphi \geq 0$
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One has

$$(\Theta_{L,h})|_{X \setminus B} = \frac{1}{m} \Phi_{mL}^* \omega_{\text{FS}} \quad \text{where} \quad \Phi_{mL} : X \setminus B \rightarrow \mathbb{P}(V_m) \simeq \mathbb{P}^{N_m}.$$

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Definition

- L is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\text{loc}}$, (possibly singular) such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X , in the sense of currents.

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- L is positive if $\exists h = e^{-\varphi}$ smooth such that $\Theta_{L,h} = -dd^c \log h > 0$ on X .

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The well-known Kodaira embedding theorem states that L is positive if and only if L is ample, namely:

$Z_m = B(mL) = \emptyset$ and

$$\Phi_{|mL|} : X \rightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

is an embedding for $m \geq m_0$ large enough.

Positive cones

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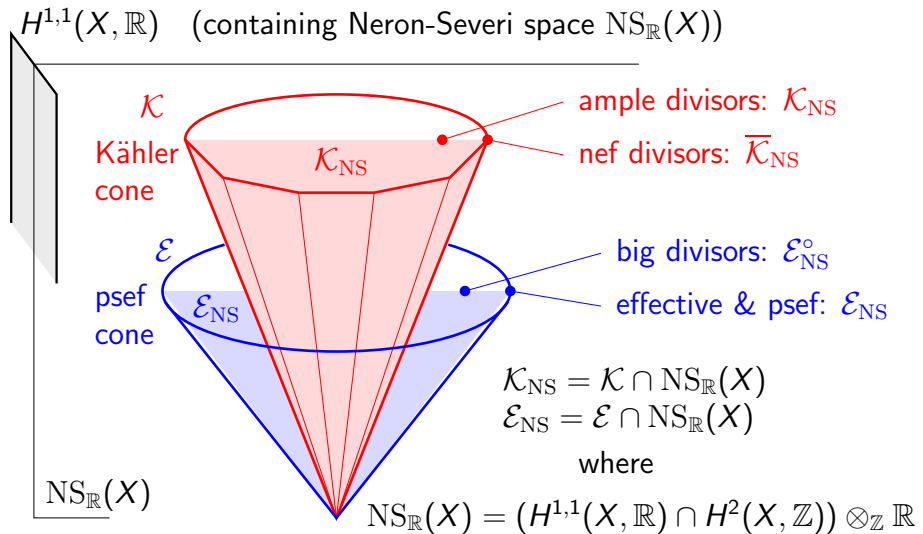
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- $\overline{\mathcal{K}}$ is the cone of “nef classes”. One has $\overline{\mathcal{K}} \subset \mathcal{E}$.
- It may happen that $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
if X is the surface obtained by blowing-up \mathbb{P}^2 in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Ample / nef / effective / big divisors

Positive cones can be visualized as follows :



Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a **Kähler current** T is a closed positive $(1,1)$ -current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

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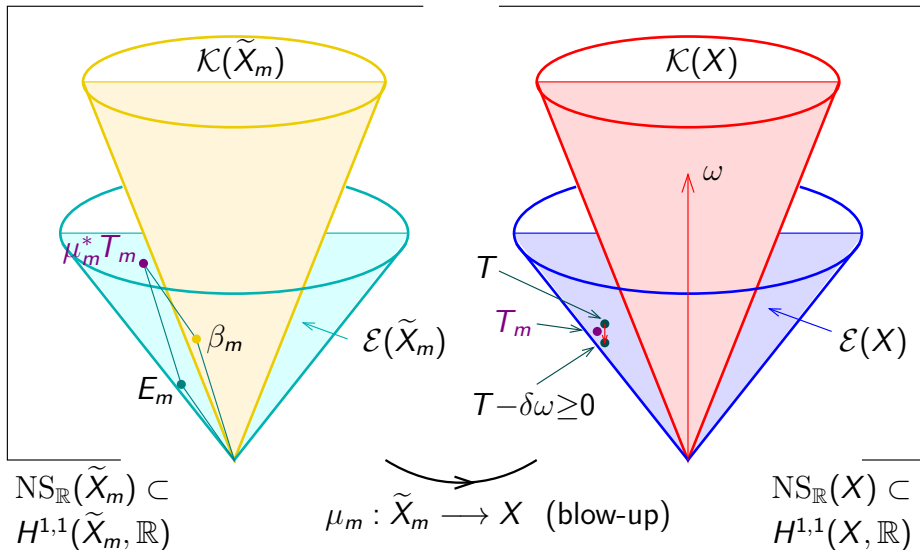
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Theorem on approximate Zariski decomposition (D, '92)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has **analytic singularities & logarithmic poles**,
i.e. \exists **modification** $\mu_m : \tilde{X}_m \rightarrow X$ such that $\mu_m^* T_m = [E_m] + \beta_m$
where E_m is an effective \mathbb{Q} -divisor on \tilde{X}_m with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on \tilde{X}_m .

Schematic picture of Zariski decomposition



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- Write locally

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- Approximate T (again locally) as

$$T_m = i\partial\bar{\partial}\varphi_m, \quad \varphi_m(z) = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \left\{ f \in \mathcal{O}(\Omega); \int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty \right\}.$$

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- Further, $\varphi = \lim_{m \rightarrow +\infty} \varphi_m$ by the mean value inequality.

“Movable” intersection of currents

Let $\mathcal{P}(X) =$ closed positive $(1, 1)$ -currents on X

$$H_{\geq 0}^{k,k}(X) = \{ \{T\} \in H^{k,k}(X, \mathbb{R}) ; T \text{ closed } \geq 0 \}.$$

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Theorem (Boucksom PhD 2002, Junyan Cao PhD 2012)

$\forall k = 1, 2, \dots, n$, \exists canonical “movable intersection product”

$$\mathcal{P} \times \dots \times \mathcal{P} \rightarrow H_{\geq 0}^{k,k}(X), \quad (T_1, \dots, T_k) \mapsto \langle T_1 \cdot T_2 \cdots T_k \rangle$$

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Method. $T_j = \lim_{\varepsilon \rightarrow 0} T_j + \varepsilon \omega$, can assume T_j Kähler.

Approximate each T_j by Kähler currents $T_{j,m}$ with logarithmic poles, take a **simultaneous log-resolution** $\mu_m : \tilde{X}_m \rightarrow X$ such that

$$\mu_m^* T_j = [E_{j,m}] + \beta_{j,m}.$$

and define

$$\langle T_1 \cdot T_2 \cdots T_k \rangle = \lim_{m \rightarrow +\infty} \uparrow \{ (\mu_m)_* (\beta_{1,m} \wedge \beta_{2,m} \wedge \dots \wedge \beta_{k,m}) \}.$$

Volume and numerical dimension of currents

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Special case. The **volume** of a class $\alpha \in H^{1,1}(X, \mathbb{R})$ is

$$\text{Vol}(\alpha) = \sup_{T \in \alpha} \langle T^n \rangle \quad \text{if } \alpha \in \mathcal{E}^\circ \text{ (big class),}$$

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Numerical dimension of a hermitian line bundle (L, h)

$$\text{nd}(L, h) = \text{nd}(\Theta_{L,h}).$$

Generalized abundance conjecture

Numerical dimension of a class $\alpha \in H^{1,1}(X, \mathbb{R})$

If α is **not pseudoeffective**, set $\text{nd}(\alpha) = -\infty$, otherwise

$$\text{nd}(\alpha) = \max \{ p \in \mathbb{N}; \exists T_\varepsilon \in \{\alpha + \varepsilon \omega\}, \lim_{\varepsilon \rightarrow 0} \langle T_\varepsilon^p \rangle \wedge \omega^{n-p} \geq C > 0 \}.$$

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L is said to be **abundant** if $\kappa(L) = \text{nd}(L)$.

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 $0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow \mathcal{O}_C \rightarrow 0$ on $C =$ elliptic curve, let $X = \mathbb{P}(E)$
(ruled surface over C) and $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Then $\text{nd}(L) = 1$ but
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Generalized abundance conjecture

For X compact Kähler, K_X is **abundant**, i.e. $\kappa(X) = \text{nd}(K_X)$.

Hard Lefschetz theorem with pseudoeffective coefficients

Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n , and for $h = e^{-\varphi}$, let $\mathcal{I}(h) = \mathcal{I}(\varphi)$ be the multiplier ideal sheaf:

$$\mathcal{I}(\varphi)_x := \left\{ f \in \mathcal{O}_{X,x}; \exists V \ni x, \int_V |f|^2 e^{-\varphi} dV_\omega < +\infty \right\}.$$

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$$\Theta_{L,h} \geq \varepsilon \omega \implies H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0 \text{ for } q \geq 1.$$

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Hard Lefschetz theorem (D-Peternell-Schneider 2001)

Assume merely $\Theta_{L,h} \geq 0$. Then, the Lefschetz map :
 $u \mapsto \omega^q \wedge u$ induces a **surjective morphism** :

$$\Phi_{\omega,h}^q : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

Idea of proof of Hard Lefschetz theorem

Main tool. “Equisingular approximation theorem”:

$$\varphi = \lim \downarrow \varphi_\nu \Rightarrow h = \lim h_\nu$$

with:

- $\varphi_\nu \in C^\infty(X \setminus Z_\nu)$, where Z_ν is an increasing sequence of analytic sets,
- $\mathcal{I}(h_\nu) = \mathcal{I}(h)$, $\forall \nu$,
- $\Theta_{L, h_\nu} \geq -\varepsilon_\nu \omega$.

(Again, the proof uses in several ways the Ohsawa-Takegoshi theorem).

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Then, use the fact that $X \setminus Z_\nu$ is Kähler complete, so one can apply (non compact) [harmonic form theory](#) on $X \setminus Z_\nu$, and pass to the limit to get rid of the errors ε_ν .

Generalized Nadel vanishing theorem

Theorem (Junyan Cao, PhD 2012)

Let X be compact Kähler, and let (L, h) be pseudoeffective on X . Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \text{ for } q \geq n - \text{nd}(L, h) + 1,$$

where

$$\mathcal{I}_+(h) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}(h^{1+\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}((1 + \varepsilon)\varphi)$$

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Remark 2. In the projective case, one can use a hyperplane section argument, provided one first shows that $\text{nd}(L, h)$ coincides with H. Tsuji’s algebraic definition ($\dim Y = p$) :

$$\text{nd}(L, h) = \max \{ p \in \mathbb{N} ; \exists Y^p \subset X, h^0(Y, (L^{\otimes m} \otimes \mathcal{I}(h^m))|_Y) \geq cm^p \}.$$

Proof of generalized Nadel vanishing (projective case)

Hyperplane section argument (projective case). Take A = very ample divisor, $\omega = \Theta_{A,h_A} > 0$, and $Y = A_1 \cap \dots \cap A_{n-p}$, $A_j \in |A|$. Then

$$\langle \Theta_{L,h}^p \rangle \cdot Y = \int_X \langle \Theta_{L,h}^p \rangle \cdot Y = \int_X \langle \Theta_{L,h}^p \rangle \wedge \omega^{n-p} > 0.$$

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Lemma (J. Cao)

When (L, h) is big, i.e. $\langle \Theta_{L,h}^n \rangle > 0$, there exists a metric \tilde{h} such that $\mathcal{I}(\tilde{h}) = \mathcal{I}_+(h)$ with $\Theta_{L,\tilde{h}} \geq \varepsilon \omega$ [Riemann-Roch].

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Conclude by **induction on dim X** and the exact cohomology sequence for the restriction to a **hyperplane section**.

Proof of generalized Nadel vanishing (Kähler case)

Kähler case. Assume $c_1(L)$ nef for simplicity. Then $c_1(L) + \varepsilon\omega$ Kähler. By Yau's theorem, solve **Monge-Ampère equation**:

$$\exists h_\varepsilon \text{ on } L, \quad (\Theta_{L, h_\varepsilon} + \varepsilon\omega)^n = C_\varepsilon \omega^n.$$

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so $\lambda_{q+1} \dots \lambda_n \leq C$ on a large open set $U \subset X$ and

$$\lambda_q^q \geq \lambda_1 \dots \lambda_q \geq c\varepsilon^{n-p} \Rightarrow \lambda_q \geq c\varepsilon^{(n-p)/q} \text{ on } U,$$

$$\sum_{j=1}^q (\lambda_j - \varepsilon) \geq \lambda_q - q\varepsilon \geq c\varepsilon^{(n-p)/q} - q\varepsilon > 0 \text{ for } q > n - p.$$

Final step: use Bochner-Kodaira formula

$$\lambda_j = \text{eigenvalues of } (\Theta_{L, h_\varepsilon} + \varepsilon \omega) \Rightarrow (\text{eigenvalues of } \Theta_{L, h_\varepsilon}) = \lambda_j - \varepsilon.$$

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Then one has to show that one can take the limit by assuming integrability with $e^{-(1+\delta)\varphi}$, thus introducing $\mathcal{I}_+(h)$.

Application to Kähler geometry

Definition (Campana)

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It is expected that simple compact Kähler manifolds are either **generic complex tori**, **generic hyperkähler manifolds** and their **finite quotients**, up to modification.

On simple Kähler 3-folds

Theorem (Campana - D - Verbitsky, 2013)

Let X be a compact Kähler 3-fold without any positive dimensional analytic subset $A \subsetneq X$. Then

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- Hilbert polynomial $P(m) = \chi(X, K_X^{\otimes m})$ is bounded, hence $\chi(X, \mathcal{O}_X) = 0$.
- Albanese map $\alpha : X \rightarrow \text{Alb}(X)$ is a biholomorphism.

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