Bergman bundles and applications to the geometry of compact complex manifolds

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Projective vs Kähler vs non Kähler varieties

Goal. Investigate positivity for general compact manifolds/$\mathbb{C}$.

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Surprising facts (?)

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The aim of this lecture is to investigate further this construction and explain potential applications to analytic geometry (invariance of plurigenera, transcendental holomorphic Morse inequalities...)

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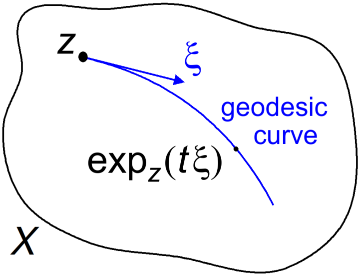
Assume that X is equipped with a real analytic hermitian metric γ, and let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.



Exponential map diffeomorphism and its inverse

Lemma

Denote by exph the "holomorphic" part of exp, so that for $z \in X$ and $\xi \in T_{X,z}$

$$\exp_z(\xi) = \sum_{\alpha,\beta\in\mathbb{N}^n} a_{\alpha\,\beta}(z)\xi^\alpha\overline{\xi}^\beta, \quad \mathrm{exph}_z(\xi) = \sum_{\alpha\in\mathbb{N}^n} a_{\alpha\,0}(z)\xi^\alpha.$$

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Then $d_\xi \exp_z(\xi)_{\xi=0} = d_\xi\, \mathrm{exph}_z(\xi)_{\xi=0} = \mathrm{Id}_{T_X}$, and so exph is a diffeomorphism from a neighborhood V of the 0 section of T_X to a neighborhood V' of the diagonal in $X \times X$.

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Notation

With the identification $\overline{X} \simeq_{\mathrm{diff}} X$, let $\mathrm{logh} : X \times \overline{X} \supset V' \to T_{\overline{X}}$ be the inverse diffeomorphism of exph and

$$U_\varepsilon = \{(z,w) \in V' \subset X \times \overline{X}\,;\ |\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

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Then, for $\varepsilon \ll 1$, U_ε is Stein and $\mathrm{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

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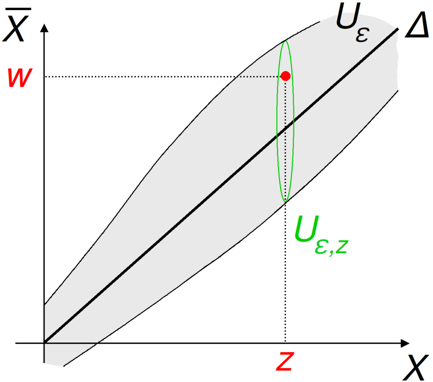

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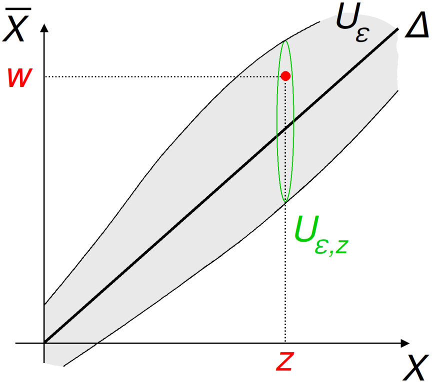

In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |\overline{z} - w| < \varepsilon\}$ is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\sum z_j w_j$$

and $(z,w) \mapsto \mathrm{Re}\sum z_j w_j$ is pluriharmonic.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

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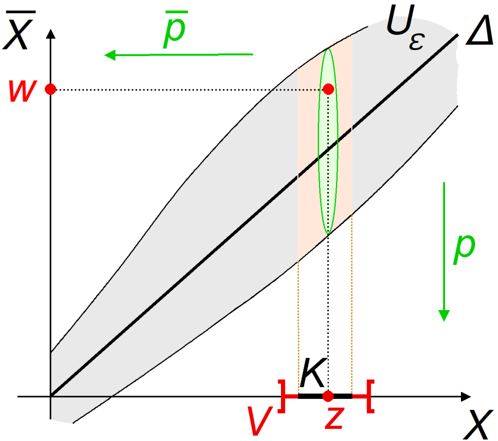

Bergman sheaves (continued)

Definition of the Bergman sheaf $\mathcal{B}_\varepsilon$

The Bergman sheaf $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is by definition the L^2 direct image

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i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$,

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$$f(z,w) = f_1(z,w)\, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f(z,w) \wedge \overline{f(z,w)} \wedge \gamma(z)^n < +\infty, \quad \forall K \Subset V.$$

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Clearly, $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module over X, but since it is a space of functions in w, it is of infinite rank.

Associated Bergman bundle and holom structure

Definition of the associated Bergman bundle B_ε

We consider the vector bundle $B_\varepsilon \to X$ whose fiber B_{ε,z_0} consists of all holomorphic functions f on $p^{-1}(z_0) \subset U_\varepsilon$ such that

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Then B_ε is a real analytic locally trivial Hilbert bundle whose fiber B_{ε,z_0} is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0,\varepsilon))$ of L^2 holomorphic n-forms on $p^{-1}(z_0) \simeq B(0,\varepsilon) \subset \mathbb{C}^n$.

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Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $\mathcal{B}_{\varepsilon',z_0} \to B_{\varepsilon,z_0}$ such that B_{ε,z_0} is the L^2 completion of $\mathcal{B}_{\varepsilon',z_0}/\mathfrak{m}_{z_0}\mathcal{B}_{\varepsilon',z_0}$.

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Question

Is there a "complex structure" on B_ε such that "$\mathcal{B}_\varepsilon = \mathcal{O}(B_\varepsilon)$" ?

Bergman Dolbeault complex

For this, consider the "Bergman Dolbeault" complex $\overline{\partial} : \mathcal{F}_\varepsilon^q \to \mathcal{F}_\varepsilon^{q+1}$ over X, with $\mathcal{F}_\varepsilon^q(V) =$ smooth (n, q)-forms

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

such that $f_J(z, w)$ is holomorphic in w, and for all $K \Subset V$ one has

$$f(z,w) \in L^2(p^{-1}(K)) \;\text{ and }\; \overline{\partial}_z f(z,w) \in L^2(p^{-1}(K)).$$

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An immediate consequence of this definition is:

Proposition

$\overline{\partial} = \overline{\partial}_z$ yields a complex of sheaves $(\mathcal{F}^\bullet_\varepsilon, \overline{\partial})$, and the kernel $\operatorname{Ker} \overline{\partial} : \mathcal{F}^0_\varepsilon \to \mathcal{F}^1_\varepsilon$ coincides with $\mathcal{B}_\varepsilon$.

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If we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of $L^2_{\rm loc}$ sections f of B_ε such that $\overline{\partial} f = 0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_\varepsilon) = \mathcal{B}_\varepsilon$ as a sheaf.

Bergman sheaves are “very ample”

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$.

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$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}_\varepsilon^\bullet \otimes \mathcal{O}(E)), \overline{\partial}\,\big) = 0, \quad \forall q \geq 1.$$

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Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In that sense, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).

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The proof is a direct consequence of Hörmander's L^2 estimates.

Caution !!

B_ε is NOT a locally trivial *holomorphic* bundle.

Embedding into a Hilbert Grassmannian

Corollary of the very ampleness of Bergman sheaves

Let X be an arbitrary compact complex manifold, $E \to X$ a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space $\mathbb{H} = H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

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$$\Psi : X \to \mathrm{Gr}(\mathbb{H}), \quad z \mapsto S_z,$$

mapping every point $z \in X$ to the infinite codimensional closed subspace S_z consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_{\varepsilon,z}$, i.e. $f_{|p^{-1}(z)} = 0$.

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The main problem with this "holomorphic embedding" is that the holomorphicity is to be understood in a weak sense, for instance the map Ψ is not even continuous with respect to the strong metric topology of $\mathrm{Gr}(\mathbb{H})$, given by
$d(S, S') =$ Hausdorff distance of the unit balls of S, S'.

Chern connection of Bergman bundles

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Then one sees that a orthonormal frame of B_ε is given by

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection on B_ε, and a natural hermitian metric as well, it follows from the usual formalism that B_ε can be equipped with a unique Chern connection.

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It is non holomorphic! The $(0,1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

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where $c_j = (0,\ldots,1,\ldots,0)\in\mathbb{N}^n$.

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and that one gets an associated quadratic Hermitian form on $T_X \otimes B_\varepsilon$ such that

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A simple calculation of ∇^2 in the orthonormal frame (e_α) leads to:

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However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, since then $\sum_\alpha (\varepsilon'/\varepsilon)^{2|\alpha|} |\xi_\alpha|^2 < +\infty$.

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A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by Wang Xu, expressing the curvature of weighted Bergman bundles $\mathcal{H}_t$ attached to a smooth family $\{D_t\}$ of strongly pseudoconvex domains.

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$$\begin{aligned}\mathrm{logh}_z(w) = w - \overline{z} &+ \sum z_j a_j(w - \overline{z}) + \sum \overline{z}_j a'_j(w - \overline{z}) \\ &+ \sum z_j z_k b_{jk}(w - \overline{z}) + \sum \overline{z}_j \overline{z}_k b'_{jk}(w - \overline{z}) \\ &+ \sum z_j \overline{z}_k c_{jk}(w - \overline{z}) + O(|z|^3),\end{aligned}$$

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which is used to compute the difference with the model case $\mathbb{C}^n$, for which $\mathrm{logh}_z(w) = w - \overline{z}$.

Potential application: invariance of plurigenera for polarized families of compact Kähler manifolds ?

Conjecture

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No algebraic proof is known in the latter case; one deeply uses the L^2 estimates of the Ohsawa-Takegoshi extension theorem.

Invariance of plurigenera: strategy of proof (1)

It is enough to consider the case of a family $\mathcal{X} \to \Delta$ over the disc, such that there exists a relatively ample line bundle $\mathcal{A}$ over $\mathcal{X}$.

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To produce $h = e^{-\varphi}$, one produces inductively (also by O-T !) sections of $\sigma_{p,j}$ of $\mathcal{L}_p := \mathcal{A} + pK_{\mathcal{X}}$ such that:

- $(\sigma_{p,j})$ generates $\mathcal{L}_p$ for $0 \leq p < m$
- $\sigma_{p,j}$ extends $(\sigma_{p-m,j}s)_{|X_0}$ to $\mathcal{X}$ for $p \geq m$
- $\displaystyle\int_{\mathcal{X}} \frac{\sum_j |\sigma_{p,j}|^2}{\sum_j |\sigma_{p-1,j}|^2} \leq C$ for $p \geq 1$.

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By Hölder, the L^2 estimates imply $\int_{\mathcal{X}} \big(\sum_j |\sigma_{p,j}|^2 \big)^{1/p} \le C$ for all p, and using the fact that $\lim \frac{1}{p}\Theta_{\mathcal{A}} = 0$, one can take

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In the polarized Kähler case $(\mathcal{X}, \omega)$, shrinking from U_ε to $U_{\rho\varepsilon}$ with $\rho < 1$, the B_ε curvature estimate gives

$$\varphi_p := \frac{1}{p}\log \sum_j \|\sigma_{p,j}\|^2_{U_{\rho\varepsilon}} \quad \Rightarrow \quad i\partial\overline{\partial}\varphi_p \ge -\frac{C}{\varepsilon^2\rho^2}(C' - \varphi_p)\omega.$$

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This implies that $\varphi = \limsup \varphi_p$ satisfies $\psi := -\log(C'' - \varphi)$ quasi-psh, but yields invariance of plurigenera only for $\varepsilon \to +\infty$.

Transcendental holomorphic Morse inequalities

Conjecture

Let X be a compact n-dimensional complex manifold and $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\overline{\partial}$ exact forms.

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Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha, \beta \in H^{1,1}(X,\mathbb{R})$ be nef classes. Then

$$\mathrm{Vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta.$$

Transcendental Morse: known facts & beyond

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Idea. In the general case, one can find a sequence of non holomorphic hermitian line bundles (L_m, h_m) such that

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Thank you for your attention