UNIVERSITÉ Grenoble Alpes

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Bergman bundles and applications to the geometry of compact complex manifolds

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

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Projective vs Kähler vs non Kähler varieties

Goal. Investigate positivity for general compact manifolds/$\mathbb{C}$.

Obviously, non projective varieties do not carry any ample line bundle.

In the Kähler case, a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$, $\omega > 0$, may sometimes be used as a substitute for a polarization.

What for non Kähler compact complex manifolds?

Surprising facts (?)

– Every compact complex manifold X carries a "very ample" complex Hilbert bundle, produced by means of a natural Bergman space construction.

– The curvature of this bundle is strongly positive in the sense of Nakano, and is given by a universal formula.

The aim of this lecture is to investigate further this construction and explain potential applications to analytic geometry (invariance of plurigenera, transcendental holomorphic Morse inequalities...)

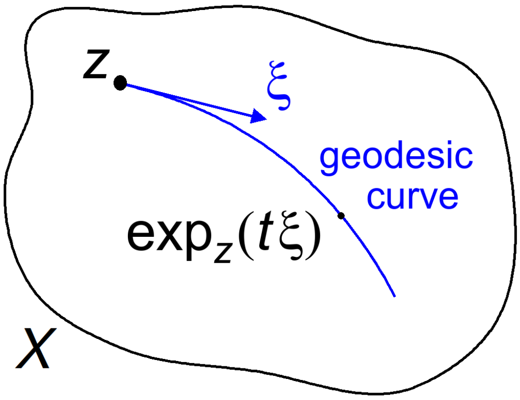
Tubular neighborhoods (thanks to Grauert)

Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.

Denote by $\overline{X}$ its complex conjugate $(X, -J)$, so that $\mathcal{O}_{\overline{X}} = \overline{\mathcal{O}_X}$.

The diagonal of $X \times \overline{X}$ is totally real, and by Grauert, we know that it possesses a fundamental system of Stein tubular neighborhoods.

Assume that X is equipped with a real analytic hermitian metric γ, and let $\exp : T_X \to X \times X$, $(z, \xi) \mapsto (z, \exp_z(\xi))$, $z \in X$, $\xi \in T_{X,z}$ be the associated geodesic exponential map.



Exponential map diffeomorphism and its inverse

Lemma

Denote by exph the "holomorphic" part of exp, so that for $z \in X$ and $\xi \in T_{X,z}$

$$\exp_z(\xi) = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha\beta}(z)\xi^\alpha\overline{\xi}^\beta, \quad \mathrm{exph}_z(\xi) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha 0}(z)\xi^\alpha.$$

Then $d_\xi \exp_z(\xi)_{\xi=0} = d_\xi \,\mathrm{exph}_z(\xi)_{\xi=0} = \mathrm{Id}_{T_X}$, and so exph is a diffeomorphism from a neighborhood V of the 0 section of T_X to a neighborhood V' of the diagonal in $X \times X$.

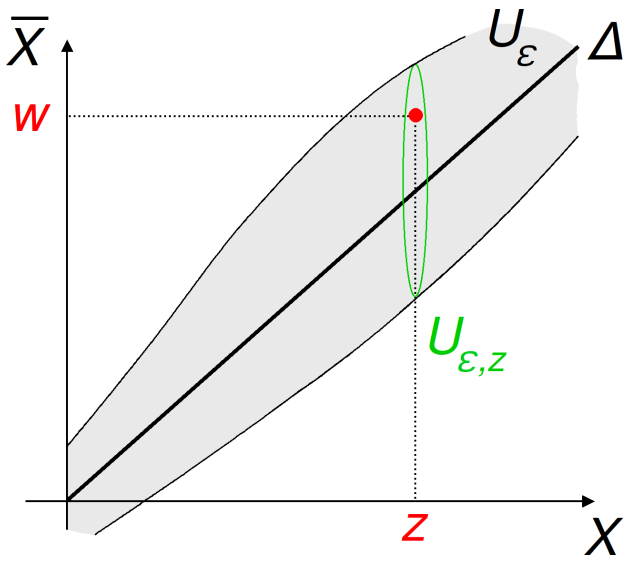
Notation

With the identification $\overline{X} \simeq_{\mathrm{diff}} X$, let $\mathrm{logh} : X \times \overline{X} \supset V' \to T_{\overline{X}}$ be the inverse diffeomorphism of exph and

$$U_\varepsilon = \{(z, w) \in V' \subset X \times \overline{X} \,;\, |\mathrm{logh}_z(w)|_\gamma < \varepsilon\}, \quad \varepsilon > 0.$$

Then, for $\varepsilon \ll 1$, U_ε is Stein and $\mathrm{pr}_1 : U_\varepsilon \to X$ is a real analytic locally trivial bundle with fibers biholomorphic to complex balls.

Such tubular neighborhoods are Stein



In the special case $X = \mathbb{C}^n$, $U_\varepsilon = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n\,;\ |\overline{z} - w| < \varepsilon\}$ is of course Stein since

$$|\overline{z} - w|^2 = |z|^2 + |w|^2 - 2\,\mathrm{Re}\sum z_j w_j$$

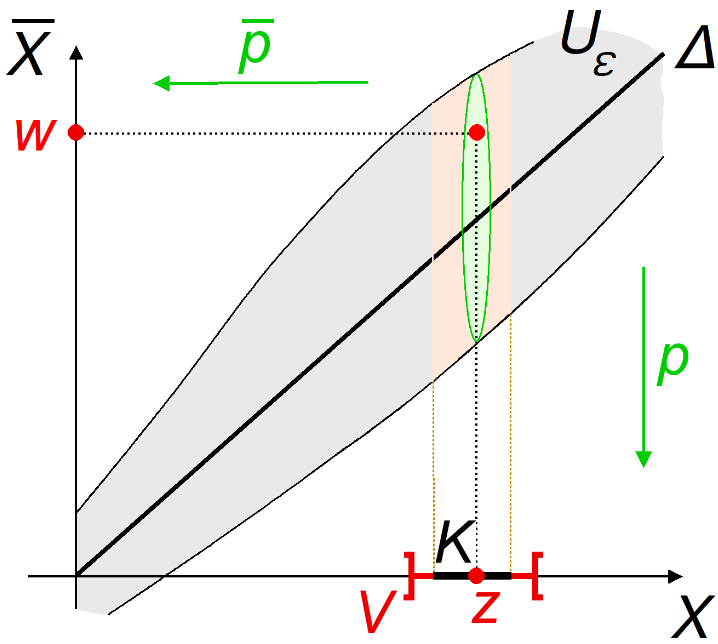
and $(z,w) \mapsto \mathrm{Re}\sum z_j w_j$ is pluriharmonic.

Bergman sheaves

Let $U_\varepsilon = U_{\gamma,\varepsilon} \subset X \times \overline{X}$ be the ball bundle as above, and

$$p = (\mathrm{pr}_1)_{|U_\varepsilon} : U_\varepsilon \to X, \qquad \overline{p} = (\mathrm{pr}_2)_{|U_\varepsilon} : U_\varepsilon \to \overline{X}$$

the natural projections.



Bergman sheaves (continued)

Definition of the Bergman sheaf $\mathcal{B}_\varepsilon$

The Bergman sheaf $\mathcal{B}_\varepsilon = \mathcal{B}_{\gamma,\varepsilon}$ is by definition the L^2 direct image

$$\mathcal{B}_\varepsilon = p_*^{L^2}(\overline{p}^*\mathcal{O}(K_{\overline{X}})),$$

i.e. the space of sections over an open subset $V \subset X$ defined by $\mathcal{B}_\varepsilon(V) =$ holomorphic sections f of $\overline{p}^*\mathcal{O}(K_{\overline{X}})$ on $p^{-1}(V)$,

$$f(z,w) = f_1(z,w)\, dw_1 \wedge \ldots \wedge dw_n, \quad z \in V,$$

that are in $L^2(p^{-1}(K))$ for all compact subsets $K \Subset V$:

$$\int_{p^{-1}(K)} i^{n^2} f(z,w) \wedge \overline{f(z,w)} \wedge \gamma(z)^n < +\infty, \quad \forall K \Subset V.$$

(This L^2 condition is the reason we speak of "L^2 direct image").

Clearly, $\mathcal{B}_\varepsilon$ is an $\mathcal{O}_X$-module over X, but since it is a space of functions in w, it is of infinite rank.

Associated Bergman bundle and holom structure

Definition of the associated Bergman bundle B_ε

We consider the vector bundle $B_\varepsilon \to X$ whose fiber B_{ε,z_0} consists of all holomorphic functions f on $p^{-1}(z_0) \subset U_\varepsilon$ such that

$$\|f(z_0)\|^2 = \int_{p^{-1}(z_0)} i^{n^2} f(z_0,w) \wedge \overline{f(z_0,w)} < +\infty.$$

Then B_ε is a real analytic locally trivial Hilbert bundle whose fiber B_{ε,z_0} is isomorphic to the Hardy-Bergman space $\mathcal{H}^2(B(0,\varepsilon))$ of L^2 holomorphic n-forms on $p^{-1}(z_0) \simeq B(0,\varepsilon) \subset \mathbb{C}^n$.
The Ohsawa-Takegoshi extension theorem implies that every $f \in B_{\varepsilon,z_0}$ can be extended as a germ $\tilde{f}$ in the sheaf $\mathcal{B}_{\varepsilon,z_0}$.
Moreover, for $\varepsilon' > \varepsilon$, there is a restriction map $\mathcal{B}_{\varepsilon',z_0} \to B_{\varepsilon,z_0}$ such that B_{ε,z_0} is the L^2 completion of $\mathcal{B}_{\varepsilon',z_0}/\mathfrak{m}_{z_0}\mathcal{B}_{\varepsilon',z_0}$.

Question

Is there a "complex structure" on B_ε such that "$\mathcal{B}_\varepsilon = \mathcal{O}(B_\varepsilon)$" ?

Bergman Dolbeault complex

For this, consider the "Bergman Dolbeault" complex $\overline{\partial} : \mathcal{F}^q_\varepsilon \to \mathcal{F}^{q+1}_\varepsilon$ over X, with $\mathcal{F}^q_\varepsilon(V) =$ smooth (n,q)-forms

$$f(z,w) = \sum_{|J|=q} f_J(z,w)\, dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{z}_J, \quad (z,w) \in U_\varepsilon \cap (V \times \overline{X}),$$

such that $f_J(z,w)$ is holomorphic in w, and for all $K \Subset V$ one has

$$f(z,w) \in L^2(p^{-1}(K)) \ \text{ and } \ \overline{\partial}_z f(z,w) \in L^2(p^{-1}(K)).$$

An immediate consequence of this definition is:

Proposition

$\overline{\partial} = \overline{\partial}_z$ yields a complex of sheaves $(\mathcal{F}^\bullet_\varepsilon, \overline{\partial})$, and the kernel $\mathrm{Ker}\,\overline{\partial} : \mathcal{F}^0_\varepsilon \to \mathcal{F}^1_\varepsilon$ coincides with $\mathcal{B}_\varepsilon$.

If we define $\mathcal{O}_{L^2}(B_\varepsilon)$ to be the sheaf of L^2_{loc} sections f of B_ε such that $\overline{\partial} f = 0$ in the sense of distributions, then we exactly have $\mathcal{O}_{L^2}(B_\varepsilon) = \mathcal{B}_\varepsilon$ as a sheaf.

Bergman sheaves are "very ample"

Theorem

Assume that $\varepsilon > 0$ is taken so small that $\psi(z,w) := |\log h_z(w)|^2$ is strictly plurisubharmonic up to the boundary on the compact set $\overline{U}_\varepsilon \subset X \times \overline{X}$. Then the complex of sheaves $(\mathcal{F}^\bullet_\varepsilon, \overline{\partial})$ is a resolution of $\mathcal{B}_\varepsilon$ by soft sheaves over X (actually, by $\mathcal{C}^\infty_X$-modules), and for every holomorphic vector bundle $E \to X$ we have

$$H^q(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E)) = H^q\big(\Gamma(X, \mathcal{F}^\bullet_\varepsilon \otimes \mathcal{O}(E)), \overline{\partial}\big) = 0, \quad \forall q \geq 1.$$

Moreover the fibers $B_{\varepsilon,z} \otimes E_z$ are always generated by global sections of $H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$.

In that sense, B_ε is a "very ample holomorphic vector bundle" (as a Hilbert bundle of infinite dimension).
The proof is a direct consequence of Hörmander's L^2 estimates.

Caution !!

B_ε is NOT a locally trivial *holomorphic* bundle.

Embedding into a Hilbert Grassmannian

Corollary of the very ampleness of Bergman sheaves

Let X be an arbitrary compact complex manifold, $E \to X$ a holomorphic vector bundle (e.g. the trivial bundle). Consider the Hilbert space $\mathbb{H} = H^0(X, \mathcal{B}_\varepsilon \otimes \mathcal{O}(E))$. Then one gets a "holomorphic embedding" into a Hilbert Grassmannian,

$$\Psi : X \to \mathrm{Gr}(\mathbb{H}), \quad z \mapsto S_z,$$

mapping every point $z \in X$ to the infinite codimensional closed subspace S_z consisting of sections $f \in \mathbb{H}$ such that $f(z) = 0$ in $B_{\varepsilon,z}$, i.e. $f_{|p^{-1}(z)} = 0$.

The main problem with this "holomorphic embedding" is that the holomorphicity is to be understood in a weak sense, for instance the map Ψ is not even continuous with respect to the strong metric topology of $\mathrm{Gr}(\mathbb{H})$, given by
$d(S, S') =$ Hausdorff distance of the unit balls of S, S'.

Chern connection of Bergman bundles

Since we have a natural $\nabla^{0,1} = \overline{\partial}$ connection on B_ε, and a natural hermitian metric as well, it follows from the usual formalism that B_ε can be equipped with a unique Chern connection.

Model case: $X = \mathbb{C}^n$, $\gamma =$ standard hermitian metric.
Then one sees that a orthonormal frame of B_ε is given by

$$e_\alpha(z,w) = \pi^{-n/2}\varepsilon^{-|\alpha|-n}\sqrt{\frac{(|\alpha|+n)!}{\alpha_1!\ldots\alpha_n!}}\,(w-\overline{z})^\alpha, \quad \alpha \in \mathbb{N}^n.$$

It is non holomorphic! The $(0,1)$-connection $\nabla^{0,1} = \overline{\partial}$ is given by

$$\nabla^{0,1}e_\alpha = \overline{\partial}_z e_\alpha(z,w) = \varepsilon^{-1}\sum_{1\le j\le n}\sqrt{\alpha_j(|\alpha|+n)}\;d\overline{z}_j \otimes e_{\alpha-c_j}$$

where $c_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^n$.

Curvature of Bergman bundles

Let $\Theta_{B_\varepsilon,h} = \nabla^2$ be the curvature tensor of B_ε with its natural Hilbertian metric h. Remember that

$$\Theta_{B_\varepsilon,h} = \nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0} \in C^\infty(X, \Lambda^{1,1}T^\star_X \otimes \mathrm{Hom}(B_\varepsilon, B_\varepsilon)),$$

and that one gets an associated quadratic Hermitian form on $T_X \otimes B_\varepsilon$ such that

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) = \langle \Theta_{B_\varepsilon,h}\sigma(v, Jv)\xi, \xi\rangle_h$$

for $v \in T_X$ and $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_\varepsilon$.

Definition

One says that the curvature tensor is Griffiths positive if

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) > 0, \quad \forall 0 \neq v \in T_X, \quad \forall 0 \neq \xi \in B_\varepsilon,$$

and Nakano positive if

$$\widetilde{\Theta}_\varepsilon(\tau) > 0, \quad \forall 0 \neq \tau \in T_X \otimes B_\varepsilon.$$

Calculation of the curvature tensor for $X = \mathbb{C}^n$

A simple calculation of ∇^2 in the orthonormal frame (e_α) leads to:

Formula

In the model case $X = \mathbb{C}^n$, the curvature tensor of the Bergman bundle (B_ε, h) is given by

$$\widetilde{\Theta}_\varepsilon(v\otimes\xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\left(\Big|\sum_j \sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\right).$$

Consequence

In $\mathbb{C}^n$, the curvature tensor $\Theta_\varepsilon(v\otimes\xi)$ is Nakano positive.

On should observe that $\widetilde{\Theta}_\varepsilon(v\otimes\xi)$ is an unbounded quadratic form on B_ε with respect to the standard metric $\|\xi\|^2 = \sum_\alpha |\xi_\alpha|^2$.

However there is convergence for all $\xi = \sum_\alpha \xi_\alpha e_\alpha \in B_{\varepsilon'}$, $\varepsilon' > \varepsilon$, since then $\sum_\alpha(\varepsilon'/\varepsilon)^{2|\alpha|}|\xi_\alpha|^2 < +\infty$.

Curvature of Bergman bundles (general case)

Bergman curvature formula on a general hermitian manifold

Let X be a compact complex manifold equipped with a C^ω hermitian metric γ, and $B_\varepsilon = B_{\gamma,\varepsilon}$ the associated Bergman bundle.
Then its curvature is given by an asymptotic expansion

$$\widetilde{\Theta}_\varepsilon(z, v\otimes\xi) = \sum_{p=0}^{+\infty} \varepsilon^{-2+p} Q_p(z, v\otimes\xi), \quad v\in T_X, \quad \xi\in B_\varepsilon$$

where $Q_0(z, v\otimes\xi) = Q_0(v\otimes\xi)$ is given by the model case $\mathbb{C}^n$:

$$Q_0(v\otimes\xi) = \varepsilon^{-2}\sum_{\alpha\in\mathbb{N}^n}\bigg(\Big|\sum_j \sqrt{\alpha_j}\,\xi_{\alpha-c_j}v_j\Big|^2 + \sum_j(|\alpha|+n)\,|\xi_\alpha|^2|v_j|^2\bigg).$$

The other terms $Q_p(z, v\otimes\xi)$ are real analytic; Q_1 and Q_2 depend respectively on the torsion and curvature tensor of γ.
In particular $Q_1 = 0$ is γ is Kähler.

A consequence of the above formula is that B_ε is strongly Nakano positive for $\varepsilon > 0$ small enough.

Idea of proof of the asymptotic expansion

The formula is in principle a special case of a more general result proved by Wang Xu, expressing the curvature of weighted Bergman bundles $\mathcal{H}_t$ attached to a smooth family $\{D_t\}$ of strongly pseudoconvex domains.
Wang's formula is however in integral form and not completely explicit.

Here, one simply uses the real analytic Taylor expansion of
$\mathrm{logh} : X\times\overline{X}\to T_X$ (inverse diffeomorphism of exph)

$$\begin{aligned}\mathrm{logh}_z(w) = w-\overline{z} &+ \sum z_j a_j(w-\overline{z}) + \sum \overline{z}_j a'_j(w-\overline{z})\\ &+ \sum z_j z_k b_{jk}(w-\overline{z}) + \sum \overline{z}_j\overline{z}_k b'_{jk}(w-\overline{z})\\ &+ \sum z_j\overline{z}_k c_{jk}(w-\overline{z}) + O(|z|^3),\end{aligned}$$

which is used to compute the difference with the model case $\mathbb{C}^n$, for which $\mathrm{logh}_z(w) = w-\overline{z}$.

Potential application: invariance of plurigenera for polarized families of compact Kähler manifolds ?

Conjecture

Let $\pi : \mathcal{X} \to S$ be a proper holomorphic map defining a family of smooth compact Kähler manifolds over an irreducible base S. Assume that the family admits a polarization, i.e. a closed smooth $(1,1)$-form ω such that $\omega_{|X_t}$ is positive definite on each fiber $X_t := \pi^{-1}(t)$. Then the plurigenera

$$p_m(X_t) = h^0(X_t, mK_{X_t}) \text{ are independent of } t \text{ for all } m \geq 0.$$

The conjecture is known to be true for a projective family $\mathcal{X} \to S$:

- Siu and Kawamata (1998) in the case of varieties of general type
- Siu (2000) and Păun (2004) in the arbitrary projective case

No algebraic proof is known in the latter case; one deeply uses the L^2 estimates of the Ohsawa-Takegoshi extension theorem.

Invariance of plurigenera: strategy of proof (1)

It is enough to consider the case of a family $\mathcal{X} \to \Delta$ over the disc, such that there exists a relatively ample line bundle $\mathcal{A}$ over $\mathcal{X}$.

Given $s \in H^0(X_0, mK_{X_0})$, the point is to show that it extends into $\widetilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}})$, and for this, one only needs to produce a hermitian metric $h = e^{-\varphi}$ on $K_{\mathcal{X}}$ such that:

- $\Theta_h = i\partial\overline{\partial}\varphi \geq 0$ in the sense of currents
- $|s|_h^2 = |s|^2 e^{-\varphi} \leq 1$, i.e. $\varphi \geq \log|s|$ on X_0.

The Ohsawa-Takegoshi theorem then implies the existence of $\widetilde{s}$.

To produce $h = e^{-\varphi}$, one produces inductively (also by O-T !) sections of $\sigma_{p,j}$ of $\mathcal{L}_p := \mathcal{A} + pK_{\mathcal{X}}$ such that:

- $(\sigma_{p,j})$ generates $\mathcal{L}_p$ for $0 \leq p < m$
- $\sigma_{p,j}$ extends $(\sigma_{p-m,j}s)_{|X_0}$ to $\mathcal{X}$ for $p \geq m$
- $\displaystyle\int_{\mathcal{X}} \frac{\sum_j |\sigma_{p,j}|^2}{\sum_j |\sigma_{p-1,j}|^2} \leq C$ for $p \geq 1$.

Invariance of plurigenera: strategy of proof (2)

By Hölder, the L^2 estimates imply $\int_{\mathcal{X}}\big(\sum_j|\sigma_{p,j}|^2\big)^{1/p}\le C$ for all p, and using the fact that $\lim\frac{1}{p}\Theta_{\mathcal{A}}=0$, one can take

$$\varphi=\limsup_{p\to+\infty}\varphi_p,\quad \varphi_p:=\tfrac{1}{p}\log\sum_j|\sigma_{p,j}|^2.$$

Idea. In the polarized Kähler case, use the Bergman bundle $B_\varepsilon\to\mathcal{X}$ instead of an ample line bundle $\mathcal{A}\to\mathcal{X}$. This amounts to applying the Ohsawa-Takegoshi L^2 extension on Stein tubular neighborhoods $U_\varepsilon\subset\mathcal{X}\times\overline{\mathcal{X}}$, with projections $\mathrm{pr}_1:U_\varepsilon\to\mathcal{X}$ and $\pi:\mathcal{X}\to\Delta$.

Proposition

In the polarized Kähler case $(\mathcal{X},\omega)$, shrinking from U_ε to $U_{\rho\varepsilon}$ with $\rho<1$, the B_ε curvature estimate gives

$$\varphi_p:=\frac{1}{p}\log\sum_j\|\sigma_{p,j}\|^2_{U_{\rho\varepsilon}}\quad\Rightarrow\quad i\partial\overline{\partial}\varphi_p\ge-\frac{C}{\varepsilon^2\rho^2}(C'-\varphi_p)\omega.$$

This implies that $\varphi=\limsup\varphi_p$ satisfies $\psi:=-\log(C''-\varphi)$ quasi-psh, but yields invariance of plurigenera only for $\varepsilon\to+\infty$.

Transcendental holomorphic Morse inequalities

Conjecture

Let X be a compact n-dimensional complex manifold and $\alpha\in H^{1,1}_{BC}(X,\mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\overline{\partial}$ exact forms. Set

$$\mathrm{Vol}(\alpha)=\sup_{T=\alpha+i\partial\overline{\partial}\varphi\ge0}\int_X T^n_{ac},\quad T\ge0\text{ current.}$$

Then

$$\mathrm{Vol}(\alpha)\ge\sup_{u\in\{\alpha\},\ u\in C^\infty}\int_{X(u,0)}u^n$$

where

$$X(u,0)=0\text{-index set of }u=\big\{x\in X\,;\ u(x)\text{ positive definite}\big\}.$$

Conjectural corollary (fundamental volume estimate)

Let X be compact Kähler, $\dim X=n$, and $\alpha,\beta\in H^{1,1}(X,\mathbb{R})$ be nef classes. Then

$$\mathrm{Vol}(\alpha-\beta)\ge\alpha^n-n\alpha^{n-1}\cdot\beta.$$

Transcendental Morse: known facts & beyond

The conjecture on Morse inequalities is known to be true when $\alpha = c_1(L)$ is the class of a line bundle ([D-1985]), and the corollary can be derived from this when α, β are integral classes (by [D-1993] and independently by [Trapani, 1993]).

Recently, the volume estimate for α, β transcendental has been established by D. Witt-Nyström when X is projective, and Xiao-Popovici even proved in general that $\mathrm{Vol}(\alpha - \beta) > 0$ if $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$.

Idea. In the general case, one can find a sequence of non holomorphic hermitian line bundles (L_m, h_m) such that

$$m\alpha = \Theta_{L_m,h_m} + \gamma_m^{2,0} + \overline{\gamma}_m^{0,2}, \quad \gamma_m \to 0.$$

As U_ε is Stein, $\overline{\gamma}_m^{0,2} = \overline{\partial} v_m$, $v_m \to 0$, and $\mathrm{pr}_1^* L_m$ becomes a holomorphic line bundle with curvature form $\Theta_{\mathrm{pr}_1^* L_m} \simeq m\,\mathrm{pr}_1^*\,\alpha$.

Then apply L^2 direct image $(\mathrm{pr}_1)_*^{L^2}$ and use Bergman estimates instead of dimension counts in Morse inequalities.

The end

Thank you for your attention