

# On the existence of logarithmic and orbifold jet differentials

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## 0. Introduction and main definitions

The present research is concerned with the existence of logarithmic and orbifold jet differentials on projective varieties. For the sake of generality, and in view of potential applications to the case of foliations, we work throughout this paper in the category of directed varieties, and generalize them by introducing the concept of directed orbifold.

**0.1. Definition.** Let  $X$  be a complex manifold or variety. A directed structure  $(X, V)$  on  $X$  is defined to be a subsheaf  $V \subset \mathcal{O}(T_X)$  such that  $\mathcal{O}(T_X)/V$  is torsion free. A morphism of directed varieties  $\Psi : (X, V) \rightarrow (Y, W)$  is a holomorphic map  $\Psi : X \rightarrow Y$  such that  $d\Psi(V) \subset \Psi^*W$ . We say that  $(X, V)$  is non singular if  $X$  is non singular and  $V$  is locally free, i.e., is a holomorphic subbundle of  $T_X$ .

We refer to the *absolute case* as being the situation when  $V = T_X$ , the *relative case* when  $V = T_{X/S}$  for some fibration  $X \rightarrow S$ , and the *foliated case* when  $V$  is integrable, i.e.  $[V, V] \subset V$ , that is,  $V$  is the tangent sheaf to a holomorphic foliation. We now combine these concepts with orbifold structures in the sense of Campana [Cam04].

**0.2. Definition.** A directed orbifold is a triple  $(X, V, D)$  where  $(X, V)$  is a directed variety and  $D = \sum (1 - \frac{1}{\rho_j}) \Delta_j$  an effective real divisor, where  $\Delta_j$  is an irreducible hypersurface and  $\rho_j \in ]1, \infty]$  an associated “ramification number”. We denote by  $[D] = \sum \Delta_j$  the corresponding reduced divisor, and by  $|D| = \bigcup \Delta_j$  its support.

- (a) We will say that  $(X, V, D)$  is non singular if  $(X, V)$  is non singular and  $D$  is a simple normal crossing divisor such that  $D$  is transverse to  $V$ . If  $r = \text{rank}(V)$ , we mean by this that there are at most  $r$  components  $\Delta_j$  meeting at any point  $x \in X$ , and that for any  $p$ -tuple  $(j_1, \dots, j_p)$  of indices,  $1 \leq p \leq r$ , we have  $\dim V_x \cap \bigcap_{j=1}^p T_{\Delta_{j_\ell}, x} = r - p$  at any point  $x \in \bigcap_{j=1}^p \Delta_{j_\ell}$ .
- (b) If  $(X, V, D)$  is non singular, the canonical divisor of  $(X, V, D)$  is defined to be

$$K_{V,D} = K_V + D$$

(in additive notation), where  $K_V = \det V^*$ .

- (c) The so called logarithmic case corresponds to all multiplicities  $\rho_j = \infty$  being taken infinite, so that  $D = \sum \Delta_j = [D]$ .

In case  $V = T_X$ , we recover the concept of orbifold introduced in [Cam04], except possibly for the fact that we allow here  $\rho_j \in \mathbb{R}$ ,  $\rho_j > 1$  (even though the case  $\rho_j \in \mathbb{N}^*$  is of greater interest). It would certainly be interesting to investigate the case when  $(X, V, D)$

is singular, by allowing singularities in  $V$  and tangencies between  $V$  and  $D$ , and to study whether the results discussed in this paper can be extended in some way, e.g. by introducing suitable multiplier ideal sheaves taking care of singularities, as was done in [Dem15] for the study of directed varieties  $(X, V)$ . For the sake of technical simplicity, we will refrain to do so here, and will therefore leave for future work the study of singular directed orbifolds.

**0.3. Definition.** *Let  $(X, V, D)$  be a singular directed orbifold. We say that  $f : \mathbb{C} \rightarrow X$  is an orbifold entire curve if  $f$  is a non constant holomorphic map such that :*

- (a)  *$f$  is tangent to  $V$  (i.e.  $f'(t) \in V_{f(t)}$  at every point, or equivalently  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$  is a morphism of directed varieties;*
- (b)  *$f(\mathbb{C})$  is not identically contained in  $|D|$ ;*
- (c) *at every point  $t_0 \in \mathbb{C}$  such that  $f(t_0) \in \Delta_j$ ,  $f$  meets  $\Delta_j$  with ramification number  $\geq \rho_j$ , i.e., if  $\Delta_j = \{z_j = 0\}$  near  $f(t_0)$ , then  $z_j \circ f(t)$  vanishes with multiplicity  $\geq \rho_j$  at  $t_0$ .*

*In the case of a logarithmic component  $\Delta_j$  ( $\rho_j = \infty$ ), condition (c) is to be replaced by the assumption*

- (c')  *$f(\mathbb{C})$  does not meet  $\Delta_j$ .*

One can now consider a category of directed orbifolds as follows.

**0.4. Definition.** *Consider directed orbifolds  $(X, V, D)$ ,  $(Y, W, \Lambda)$  with*

$$D = \sum \left(1 - \frac{1}{\rho_i}\right) \Delta_i, \quad D' = \sum \left(1 - \frac{1}{\rho'_j}\right) \Delta'_j.$$

*A morphism  $\Psi : (X, V, D) \rightarrow (Y, W, D')$  is a morphism  $\Psi : (X, V) \rightarrow (Y, W)$  of directed varieties satisfying the additional following properties (a,b,c).*

- (a) *for every component  $\Delta'_j$ ,  $\Psi^{-1}(\Delta'_j)$  consists of a union of components  $\Delta_i$ ,  $i \in I(j)$ , eventually after adding a number of extra components  $\Delta_i$  with  $\rho_i = 1$ ;*
- (b) *in case  $\rho'_j < \infty$ , for every  $i \in I(j)$  and  $z \in \Delta_i$ , the derivatives  $d^\alpha \Psi(z)$  of  $\Psi$  at  $z$ , computed in suitable local coordinates on  $X$  and  $Y$ , vanish for all multi-indices  $\alpha \in \mathbb{N}^n$  with  $0 < |\alpha| < \rho'_j / \rho_i$ ;*
- (c) *if  $\Delta'_j$  is a logarithmic component ( $\rho'_j = \infty$ ), then  $\Psi^{-1}(\Delta'_j) = \bigcup_{i \in I(j)} \Delta_i$  where the  $(\Delta_i)_{i \in I(j)}$  consist of logarithmic components ( $\rho_i = \infty$ ).*

It is easy to check that the composite of directed orbifold morphisms is actually a directed orbifold morphism, and that the composition of an orbifold entire curve  $f : \mathbb{C} \rightarrow (X, V, D)$  with a directed orbifold morphism  $\Psi : (X, V, D) \rightarrow (Y, W, D')$  produces an orbifold entire curve  $\Psi \circ f : \mathbb{C} \rightarrow (Y, W, D')$ . One of our main goals is to investigate the following orbifold generalization of the Green-Griffiths conjecture.

**0.5. Conjecture.** *Let  $(X, V, D)$  be a non singular directed orbifold of generated type, in the sense that the canonical divisor  $K_V + D$  is big. Then there exists an algebraic subvariety  $Y \subsetneq X$  containing all orbifold entire curves  $f : \mathbb{C} \rightarrow (X, V, D)$ .*

As in the absolute case ( $V = T_X$ ,  $D = 0$ ), the idea is to show, at least as a first step towards the conjecture, that orbifold entire curves must satisfy suitable algebraic differential equations. In section 1, we introduce graded algebras

$$(0.6) \quad \bigoplus_{m \in \mathbb{N}} E_{k,m} V^* \langle D \rangle$$

of sheaves of “orbifold jet differentials”. These sheaves correspond to algebraic differential operators  $P(f; f', f'', \dots, f^{(k)})$  acting on germs of  $k$ -jets of curves that are tangent to  $V$  and satisfy the ramification conditions prescribed by  $D$ . The strategy relies on the following standard vanishing theorem.

**0.7. Proposition.** *Let  $(X, V, D)$  be a projective non singular directed orbifold, and  $A$  an ample divisor on  $X$ . Then, for every orbifold entire curve  $f : \mathbb{C} \rightarrow (X, V, D)$  and every global jet differential operator  $P \in H^0(X, E_{k,m} V^* \langle D \rangle \otimes \mathcal{O}_X(-A))$ , we have  $P(f; f', f'', \dots, f^{(k)}) = 0$ .*

The next step consists precisely of finding sufficient conditions that ensure the existence of many global sections  $P \in H^0(X, E_{k,m} V^* \langle D \rangle \otimes \mathcal{O}_X(-A))$ . In this direction, among other more general results, we prove

**0.8. Theorem.** *Let  $D = \sum_j (1 - \frac{1}{\rho_j}) \Delta_j$  a simple normal crossing orbifold divisor on  $\mathbb{P}^n$  with  $\deg \Delta_j = d_j$ . Then there exist jet differentials of order  $n$  and large degree  $m$  on  $\mathbb{P}^n \langle D \rangle$ , with a small negative twist  $\mathcal{O}_{\mathbb{P}^n}(-m\varepsilon)$ , under any of the following two assumptions, where  $c_n = O((2n \log n)^n)$  :*

(a) *all components  $\Delta_j$  possess the same degree  $d$  and ramification number  $\rho > n$ , and the number of components satisfies*

$$N \geq c_n \max \left( \frac{1}{\rho}, \frac{2}{d} \right) \prod_{s=1}^n \left( 1 - \frac{s}{\rho} \right)^{-1},$$

(b)  *$D$  admits a component  $(1 - \frac{1}{\rho_1}) \Delta_1$  with  $\rho_1 \geq 2c_n$  and  $d_1 \geq 4c_n$ .*

The proof of Theorem 0.8 rests upon an application of holomorphic Morse inequalities. It is remarkable that a large part of the calculations use Chern forms and are non cohomological, although the final bounds are purely cohomological. At this point, we do not have a convincing or complete explanation of this “transcendental” phenomenon.

## 1. Logarithmic and orbifold jet differentials

### 1.A. Directed varieties and associated jet differentials

Let  $(X, V)$  be a non singular directed variety. We set  $n = \dim_{\mathbb{C}} X$ ,  $r = \text{rank}_{\mathbb{C}} V$ , and following the exposition of [Dem97], we denote by  $\pi_k : J^k V \rightarrow X$  the bundle of  $k$ -jets of holomorphic curves tangent to  $V$  at each point. The canonical bundle of  $V$  is defined to be

$$(1.1) \quad K_V = \det(V^*) = \Lambda^r V^*.$$

If  $f : (\mathbb{C}, 0) \rightarrow X$ ,  $t \mapsto f(t)$  is a germ of holomorphic curve tangent to  $V$ , we denote by  $f_{[k]}(0)$  its  $k$ -jet at  $t = 0$ . For  $x_0 \in X$  given, we take a coordinate system  $(z_1, \dots, z_n)$  centered at  $x_0$  such that  $V_{x_0} = \text{Span}(\frac{\partial}{\partial z_\mu})_{1 \leq \mu \leq r}$ . Then there exists a neighborhood  $U$  of  $x_0$  such that  $V|_U$  admits a holomorphic frame  $(e_\mu)_{1 \leq \mu \leq r}$  of the form

$$(1.2) \quad e_\mu(z) = \frac{\partial}{\partial z_\mu} + \sum_{r+1 \leq \lambda \leq n} a_{\lambda\mu}(z) \frac{\partial}{\partial z_\lambda}, \quad 1 \leq \mu \leq r,$$

with  $a_{\lambda\mu}(0) = 0$ . Germs of curves  $f : (\mathbb{C}, 0) \rightarrow X$  tangent to  $V|_U$  are obtained by integrating the system of ordinary differential equations

$$(1.3) \quad f'_\lambda(t) = \sum_{1 \leq \mu \leq r} a_{\lambda\mu}(f(t)) f'_\mu(t), \quad r+1 \leq \lambda \leq n,$$

when we write  $f = (f_1, \dots, f_n)$  in coordinates. Therefore any such germ of curve  $f$  is uniquely determined by its initial point  $z = f(0)$  and its projection  $\tilde{f} = (f_1, \dots, f_r)$  on the first  $r$  coordinates. By definition, every  $k$ -jet  $f_{[k]} \in J^k V_z = \pi_k^{-1}(z)$  is uniquely determined by its initial point  $f(0) = z \simeq (z_1, \dots, z_n)$  and the Taylor expansion of order  $k$

$$(1.4) \quad \tilde{f}(t) - \tilde{f}(0) = t\xi_1 + \frac{1}{2!}t^2\xi_2 + \dots + \frac{1}{k!}t^k\xi_k + O(t^{k+1}), \quad t \in \mathbb{D}(0, \varepsilon), \quad \xi_s \in \mathbb{C}^r, \quad 1 \leq s \leq k.$$

Alternatively, we can pick an arbitrary local holomorphic connection  $\nabla$  on  $V|_U$  and represent the  $k$ -jet  $f_{[k]}(0)$  by  $(\xi_1, \dots, \xi_k)$ , where  $\xi_s = \nabla^s f(0) \in V_z$  is defined inductively by  $\nabla^1 f = f'$  and  $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$ . This gives a local biholomorphic trivialization of  $J^k V|_U$  of the form

$$(1.5) \quad J^k V|_U \rightarrow V|_U^{\oplus k}, \quad f_{[k]}(0) \mapsto (\xi_1, \dots, \xi_k) = (\nabla f(0), \dots, \nabla f^k(0));$$

the particular choice of the “trivial connection”  $\nabla_0$  of  $V|_U$  that turns  $(e_\mu)_{1 \leq \mu \leq r}$  into a parallel frame precisely yields the components  $\xi_s \in V|_U \simeq \mathbb{C}^r$  appearing in (1.4). We could of course also use a  $C^\infty$  connection  $\nabla = \nabla_0 + \Gamma$  where  $\Gamma \in C^\infty(U, T_X^* \otimes \text{Hom}(V, V))$ , and in this case, the corresponding trivialization (1.5) is just a  $C^\infty$  diffeomorphism; the advantage, though, is that we can always produce such a global  $C^\infty$  connection  $\nabla$  by using a partition of unity on  $X$ , and then (1.5) becomes a global  $C^\infty$  diffeomorphism. Now, there is a global holomorphic  $\mathbb{C}^*$  action on  $J^k V$  given at the level of germs by  $f \mapsto \alpha \cdot f$  where  $\alpha \cdot f(t) := f(\alpha t)$ ,  $\alpha \in \mathbb{C}^*$ . With respect to our trivializations (1.5), this is the weighted  $\mathbb{C}^*$  action defined by

$$(1.6) \quad \alpha \cdot (\xi_1, \xi_2, \dots, \xi_k) = (\alpha\xi_1, \alpha^2\xi_2, \dots, \alpha^k\xi_k), \quad \xi_s \in V.$$

We see that  $J^k V \rightarrow X$  is an algebraic fiber bundle with typical fiber  $\mathbb{C}^{rk}$ , and that the projectivized  $k$ -jet bundle

$$(1.7) \quad X_k(V) := (J^k V \setminus \{0\})/\mathbb{C}^*, \quad \pi_k : X_k(V) \rightarrow X$$

is a  $\mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})$  weighted projective bundle over  $X$ , of total dimension

$$(1.8) \quad \dim X_k(V) = n + kr - 1.$$

**1.9. Definition.** We define  $\mathcal{O}_X(E_{k,m} V^*)$  to be the sheaf over  $X$  of holomorphic functions  $P(z; \xi_1, \dots, \xi_k)$  on  $J^k V$  that are weighted polynomials of degree  $m$  in  $(\xi_1, \dots, \xi_m)$ .

In coordinates and in multi-index notation, we can write

$$P(z; \xi_1, \dots, \xi_k) = \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{N}^r \\ |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k| = m}} a_{\alpha_1 \dots \alpha_k}(z) \xi_1^{\alpha_1} \dots \xi_k^{\alpha_k}$$

where the  $a_{\alpha_1 \dots \alpha_k}(z)$  are holomorphic functions in  $z = (z_1, \dots, z_n)$  and  $\xi_s^{\alpha_s}$  actually means

$$\xi_s^{\alpha_s} = \xi_{s,1}^{\alpha_{s,1}} \dots \xi_{s,r}^{\alpha_{s,r}} \quad \text{for } \xi_s = (\xi_{s,1}, \dots, \xi_{s,r}) \in \mathbb{C}^r, \quad \alpha_s = (\alpha_{s,1}, \dots, \alpha_{s,r}) \in \mathbb{N}^r,$$

and  $|\alpha_s| = \sum_{j=1}^r \alpha_{s,j}$ . Such sections can be interpreted as algebraic differential operators acting on holomorphic curves  $f : \mathbb{D}(0, R) \rightarrow X$  tangent to  $V$ , by putting  $P(f) := u$  where

$$(1.10) \quad u(t) = \sum_{\substack{\alpha_1, \dots, \alpha_k \in \mathbb{N}^r \\ |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k| = m}} a_{\alpha_1 \dots \alpha_k}(f(t)) f'(t)^{\alpha_1} \dots f^{(k)}(t)^{\alpha_k}.$$

Here  $f^{(s)}(t)^{\alpha_s}$  is actually to be expanded as

$$f^{(s)}(t)^{\alpha_s} = f_1^{(s)}(t)^{\alpha_{s,1}} \dots f_r^{(s)}(t)^{\alpha_{s,r}}$$

with respect to the components  $f_j^{(s)}$  defined in (1.4). We also set  $u = P(f; f', f'', \dots, f^{(k)})$  when we want to make more explicit the dependence of the expression in terms of the derivatives of  $f$ . We thus get a sheaf of graded algebras

$$(1.11) \quad \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*).$$

Locally in coordinates, the algebra is isomorphic to the weighted polynomial ring

$$(1.12) \quad \mathcal{O}_X[f_j^{(s)}]_{1 \leq j \leq r, 1 \leq s \leq k}, \quad \deg f_j^{(s)} = s$$

over  $\mathcal{O}_X$ . An immediate consequence of these definitions is :

**1.13. proposition.** *The projectivized bundle  $\pi_k : X_k(V) \rightarrow X$  can be identified with*

$$(a) \quad \text{Proj} \left( \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*) \right) \rightarrow X,$$

and, if  $\mathcal{O}_{X_k(V)}(m)$  denote the associated tautological sheaves, we have the direct image formula

$$(b) \quad (\pi_k)_* \mathcal{O}_{X_k(V)}(m) = \mathcal{O}_X(E_{k,m} V^*).$$

**1.14. Remark.** These objects were denoted  $X_k^{\text{GG}}$  and  $E_{k,m}^{\text{GG}} V^*$  in our previous paper [Dem97], as a reference to the work of Green-Griffiths [GG79], but we will avoid here the superscript GG to simplify the notation.

Thanks to the Faà di Bruno formula, a change of coordinates  $w = \psi(z)$  on  $X$  leads to a transformation rule

$$(\psi \circ f)^{(k)} = \psi' \circ f \cdot f^{(k)} + Q_\psi(f', \dots, f^{(k-1)})$$

where  $Q_\psi$  is a polynomial of weighted degree  $k$  in the lower order derivatives. This shows that the transformation rule of the top derivative is linear and, as a consequence, the partial degree in  $f^{(k)}$  of the polynomial  $P(f; f', \dots, f^{(k)})$  is intrinsically defined. By taking the corresponding filtration and factorizing the monomials  $(f^{(k)})^{\alpha_k}$  with polynomials in  $f', f'', \dots, f^{(k-1)}$ , we get graded pieces

$$G^\bullet(E_{k,m} V^*) = \bigoplus_{\ell_k \in \mathbb{N}} E_{k-1, m-k\ell_k} V^* \otimes S^{\ell_k} V^*.$$

By considering successively the partial degrees with respect to  $f^{(k)}, f^{(k-1)}, \dots, f'', f'$  and merging inductively the resulting filtrations, we get a multi-filtration such that

$$(1.15) \quad G^\bullet(E_{k,m} V^*) = \bigoplus_{\ell_1, \dots, \ell_k \in \mathbb{N}, \ell_1 + 2\ell_2 + \dots + k\ell_k = m} S^{\ell_1} V^* \otimes S^{\ell_2} V^* \otimes \dots \otimes S^{\ell_k} V^*.$$

### 1.B. Logarithmic directed varieties

We now turn ourselves to the logarithmic case. Let  $(X, V, D)$  be a non singular logarithmic variety, where  $D = \sum \Delta_j$  is a simple normal crossing divisor. Fix a point  $x_0 \in X$ . By the assumption that  $D$  is transverse to  $V$ , we can then select holomorphic coordinates  $(z_1, \dots, z_n)$  centered at  $x_0$  such that  $V_{x_0} = \text{Span}(\frac{\partial}{\partial z_j})_{1 \leq j \leq r}$  and  $\Delta_j = \{z_j = 0\}$ ,  $1 \leq j \leq p$ , are the components of  $D$  that contain  $x_0$  (here  $p \leq r$  and we can have  $p = 0$  if  $x_0 \notin |D|$ ). What we want is to introduce an algebra of differential operators, defined locally near  $x_0$  as the weighted polynomial ring

$$(1.16) \quad \mathcal{O}_X[(\log f_j)^{(s)}_{1 \leq j \leq p}, (f_j^{(s)})_{p+1 \leq j \leq r}]_{1 \leq s \leq k}, \quad \deg f_j^{(s)} = \deg(\log f_j)^{(s)} = s,$$

or equivalently

$$(1.16') \quad \mathcal{O}_X[(f_j^{-1} f_j^{(s)})_{1 \leq j \leq p}, (f_j^{(s)})_{p+1 \leq j \leq r}]_{1 \leq s \leq k}, \quad \deg f_j^{(s)} = s, \deg f_j^{-1} = 0.$$

For this we notice that

$$\begin{aligned} (\log f_1)'' &= (f_1^{-1} f_1')' = f_1^{-1} f_1'' - (f_1^{-1} f_1')^2, \\ (\log f_1)''' &= f_1^{-1} f_1''' - 3(f_1^{-1} f_1')(f_1^{-1} f_1'') + 2(f_1^{-1} f_1')^3, \dots \end{aligned}$$

A similar argument easily shows that the above graded rings do not depend on the particular choice of coordinates made, as soon as they satisfy  $\Delta_j = \{z_j = 0\}$ .

Now (as is well known in the absolute case  $V = T_X$ ), we have a corresponding logarithmic directed structure  $V\langle D \rangle$  and its dual  $V^*\langle D \rangle$ . If the coordinates  $(z_1, \dots, z_n)$  are chosen so that  $V_{x_0} = \{dz_{r+1} = \dots = dz_n = 0\}$ , then the fiber  $V\langle D \rangle_{x_0}$  is spanned by the derivations

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_p \frac{\partial}{\partial z_p}, \frac{\partial}{\partial z_{p+1}}, \dots, \frac{\partial}{\partial z_r}.$$

The dual sheaf  $\mathcal{O}_X(V^*\langle D \rangle)$  is the locally free sheaf generated by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_p}{z_p}, dz_{p+1}, \dots, dz_r$$

[where the 1-forms are considered in restriction to  $\mathcal{O}_X(V\langle D \rangle) \subset \mathcal{O}_X(V)$ ]. It follows from this that  $\mathcal{O}_X(V\langle D \rangle)$  and  $\mathcal{O}_X(V^*\langle D \rangle)$  are locally free sheaves of rank  $r$ . By taking  $\det(V^*\langle D \rangle)$  and using the above generators, we find

$$(1.17) \quad \det(V^*\langle D \rangle) = \det(V^*) \otimes \mathcal{O}_X(D) = K_V + D$$

in additive notation. Quite similarly to 1.13 and 1.15, we have :

**1.18. Proposition.** *Let  $\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle D \rangle)$  be the graded algebra defined in coordinates by (1.16) or (1.16'). We define the logarithmic  $k$ -jet bundle to be*

$$(a) \quad X_k(V\langle D \rangle) := \text{Proj} \left( \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle D \rangle) \right) \rightarrow X.$$

If  $\mathcal{O}_{X_k(V\langle D \rangle)}(m)$  denote the associated tautological sheaves, we get the direct image formula

$$(b) \quad (\pi_k)_* \mathcal{O}_{X_k(V\langle D \rangle)}(m) = \mathcal{O}_X(E_{k,m} V^*\langle D \rangle).$$

Moreover, the mult-filtration by the partial degrees in the derivatives  $f_j^{(s)}$  has graded pieces

$$(c) \quad G^\bullet(E_{k,m} V^*\langle D \rangle) = \bigoplus_{\ell_1, \dots, \ell_k \in \mathbb{N}, \ell_1 + 2\ell_2 + \dots + k\ell_k = m} S^{\ell_1} V^*\langle D \rangle \otimes S^{\ell_2} V^*\langle D \rangle \otimes \dots \otimes S^{\ell_k} V^*\langle D \rangle.$$

### 1.C. Orbifold directed varieties

We finally consider a non singular directed orbifold  $(X, V, D)$ , where  $D = \sum(1 - \frac{1}{\rho_j})\Delta_j$  is a simple normal crossing divisor transverse to  $V$ . Let  $[D] = \sum \Delta_j$  be the corresponding reduced divisor. By §1.B, we have associated logarithmic sheaves  $\mathcal{O}_X(E_{k,m}V^*\langle [D] \rangle)$ . We want to introduce a graded subalgebra

$$(1.19) \quad \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle D \rangle) \subset \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle [D] \rangle)$$

in such a way that for every germ  $P \in \mathcal{O}_X(E_{k,m}V^*\langle D \rangle)$  and every germ of orbifold curve  $f : (\mathbb{C}, 0) \rightarrow (X, V, D)$  the germ of meromorphic function  $P(f)(t)$  is bounded at  $t = 0$  (hence holomorphic). Assume that  $\Delta_1 = \{z_1 = 0\}$  and that  $f$  has multiplicity  $q \geq \rho_1 > 1$  along  $\Delta_1$  at  $t = 0$ . Then  $f_1^{(s)}$  still vanishes at order  $\geq (q - s)_+$ , thus  $(f_1)^{-\beta} f_1^{(s)}$  is bounded as soon as  $\beta q \leq (q - s)_+$ , i.e.  $\beta \leq (1 - \frac{s}{q})_+$ . Thus, it is sufficient to ask that  $\beta \leq (1 - \frac{s}{\rho_1})_+$ . At a point  $x_0 \in |\Delta_1| \cap \dots \cap |\Delta_p|$ , a sufficient condition for a monomial of the form

$$(1.20) \quad f_1^{-\beta_1} \dots f_p^{-\beta_p} \prod_{s=1}^k \prod_{j=1}^r (f_j^{(s)})^{\alpha_{s,j}}, \quad \alpha_s = (\alpha_{s,j}) \in \mathbb{N}^r, \beta_1, \dots, \beta_p \in \mathbb{N}$$

to be bounded is to require that the multiplicities of poles satisfy

$$(1.20') \quad \beta_j \leq \sum_{s=1}^k \alpha_{s,j} \left(1 - \frac{s}{\rho_j}\right)_+, \quad 1 \leq j \leq p.$$

**1.21. Definition.** *The subalgebra  $\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle D \rangle)$  is taken to be the graded ring generated by monomials (1.20) of degree  $\sum s|\alpha_s| = m$ , satisfying the pole multiplicity conditions (1.20'). These conditions do not depend on the choice of coordinates, hence we get a globally and intrinsically defined sheaf of algebras on  $X$ .*

*Proof.* We only have to prove the last assertion. Consider a change of variables  $w = \psi(z)$  such that  $\Delta_j$  can still be expressed as  $\Delta_j = \{w_j = 0\}$ . Then, for  $j = 1, \dots, p$ , we can write  $w_j = z_j u_j(z)$  with an invertible holomorphic factor  $u_j$ . We need to check that the monomials (1.20) computed with  $g = \psi \circ f$  are holomorphic combinations of those associated with  $f$ . However, we have  $g_j = f_j u_j(f)$ , hence  $g_j^{(s)} = \sum_{0 \leq \ell \leq s} \binom{s}{\ell} f_j^{(\ell)} (u_j(f))^{(s-\ell)}$  by the Leibniz formula, and we see that

$$g_1^{-\beta_1} \dots g_p^{-\beta_p} \prod_{s=1}^k \prod_{j=1}^r (g_j^{(s)})^{\alpha_{s,j}}$$

expands as a linear combination of monomials

$$f_1^{-\beta_1} \dots f_p^{-\beta_p} \prod_{s=1}^k \prod_{j=1}^r \prod_{m=1}^{\alpha_{s,j}} f_j^{(\ell_{s,j,m})}, \quad \ell_{s,j,m} \leq s,$$

multiplied by holomorphic factors of the form

$$\prod_{j=1}^p u_j(f)^{-\beta_j} \times \prod_{s=1}^k \prod_{j=1}^r \prod_{m=1}^{\alpha_{s,j}} (u_j(f))^{(s-\ell_{s,j,m})}.$$

However, we have

$$\beta_j \leq \sum_{s=1}^k \alpha_{s,j} \left(1 - \frac{s}{\rho_j}\right)_+ \leq \sum_{s=1}^k \sum_{m=1}^{\alpha_{s,j}} \left(1 - \frac{\ell_{s,j,m}}{\rho_j}\right)_+,$$

so the  $f$ -monomials satisfy again the required multiplicity conditions for the poles  $f_j^{-\beta_j}$ .  $\square$

The above conditions (1.20') suggest to introduce a sequence of “differentiated” orbifold divisors

$$(1.22) \quad D^{(s)} = \sum_j \left(1 - \frac{s}{\rho_j}\right)_+ \Delta_j.$$

We say that  $D^{(s)}$  is the order  $s$  orbifold divisor associated to  $D$ ; its ramification numbers are  $\rho_j^{(s)} = \max(\rho_j/s, 1)$ . By definition, the logarithmic components ( $\rho_j = \infty$ ) of  $D$  remain logarithmic in  $D^{(s)}$ , while all others eventually disappear when  $s$  is large.

Now, we introduce (in a purely formal way) a sheaf of rings  $\tilde{\mathcal{O}}_X = \mathcal{O}_X[z_j^\bullet]$  by adjoining all positive real powers of coordinates  $z_j$  such that  $\Delta_j = \{z_j = 0\}$  is locally a component of  $D$ . Locally over  $X$ , this can be done by taking the universal cover  $Y$  of a punctured polydisk

$$\mathbb{D}^*(0, r) := \prod_{1 \leq j \leq p} \mathbb{D}^*(0, r_j) \times \prod_{p+1 \leq j \leq n} \mathbb{D}(0, r_j) \subset \mathbb{D}(0, r) := \prod_{1 \leq j \leq n} \mathbb{D}(0, r_j)$$

in the local coordinates  $z_j$  on  $X$ . If  $\gamma : Y \rightarrow \mathbb{D}^*(0, r) \hookrightarrow X$  is the covering map and  $U \subset \mathbb{D}(0, r)$  is an open subset, we can then consider the functions of  $\tilde{\mathcal{O}}_X(U)$  as being defined on  $\gamma^{-1}(U \cap \mathbb{D}^*(0, r))$ . In case  $X$  is projective, one can even achieve such a construction “globally”, by taking  $Y$  to be the universal cover of a complement  $X \setminus (|D| \cup |A|)$ , where  $A = \sum A_j$  is a very ample normal crossing divisor transverse to  $D$ , such that  $\mathcal{O}_X(\Delta_j)|_{X \setminus |A|}$  is trivial for every  $j$ .

In this setting, the subalgebra  $\bigoplus_m \mathcal{O}_X(E_{k,m} V^*\langle D \rangle)$  still has a multi-filtration induced by the one on  $\bigoplus_m \mathcal{O}_X(E_{k,m} V^*\langle [D] \rangle)$ , and by extending the structure sheaf  $\mathcal{O}_X$  into  $\tilde{\mathcal{O}}_X$ , we get an inclusion

$$(1.23) \quad \tilde{\mathcal{O}}_X(G^\bullet E_{k,m} V^*\langle D \rangle) \subset \bigoplus_{\ell_1 + 2\ell_2 + \dots + k\ell_k = m} \tilde{\mathcal{O}}_X(S^{\ell_1} V^*\langle D^{(1)} \rangle) \otimes \dots \otimes \tilde{\mathcal{O}}_X(S^{\ell_k} V^*\langle D^{(k)} \rangle),$$

$\tilde{\mathcal{O}}_X(V^*\langle D^{(s)} \rangle)$  is the “ $s$ -th orbifold (dual) directed structure”, generated by the order  $s$  differentials

$$(1.24) \quad z_j^{-(1-s/\rho_j)_+} d^{(s)} z_j, \quad 1 \leq j \leq p, \quad d^{(s)} z_j, \quad p+1 \leq j \leq r.$$

By construction, we have

$$(1.25) \quad \det(\tilde{\mathcal{O}}_X(V^*\langle D^{(s)} \rangle)) = \tilde{\mathcal{O}}_X(K_V + D^{(s)}).$$

**1.26. Remark.** When  $\rho_j = a_j/b_j \in \mathbb{Q}_+$ , one can find a finite ramified Galois cover  $g : Y \rightarrow X$  from a smooth projective variety  $Y$  onto  $X$ , such that the compositions  $(z_j \circ g)^{1/a_j}$



become single-valued functions  $w_j$  on  $Y$ . In this way, the pull-back  $\mathcal{O}_Y(g^*V^*\langle D^{(s)} \rangle)$  is actually a locally free  $\mathcal{O}_Y$ -module. One can also introduce a sheaf of algebras which we will denote by  $\bigoplus \mathcal{O}_Y(E_{k,m}\tilde{V}^*\langle D \rangle)$ , generated, according to the notation of §1.B, by the elements  $g^*(z_j^{(1-s/\rho_j)+d^{(s)}z_j})$ ,  $1 \leq j \leq p$ , and  $g^*(d^{(s)}z_j)$ ,  $p+1 \leq j \leq r$ . Then there is indeed a multifiltration on  $\mathcal{O}_Y(E_{k,m}\tilde{V}^*\langle D \rangle)$  whose graded pieces are

$$(1.27) \quad \mathcal{O}_Y(G^\bullet E_{k,m}\tilde{V}^*\langle D \rangle) = \bigoplus_{\ell_1+2\ell_2+\dots+k\ell_k=m} \mathcal{O}_Y(S^{\ell_1}\tilde{V}^*\langle D^{(1)} \rangle) \otimes \dots \otimes \mathcal{O}_Y(S^{\ell_k}\tilde{V}^*\langle D^{(k)} \rangle).$$

However, we will adopt here an alternative viewpoint that avoids the introduction of finite or infinite covers, and suits better our approach. Using the general philosophy of [Laz??], the idea is to consider a “jet orbifold directed structure”  $X_k(V\langle D \rangle)$  as the underlying “jet logarithmic directed structure”  $X_k(V\langle [D] \rangle)$ , equipped additionally with a submultiplicative sequence of ideal sheaves  $\mathcal{J}_m\langle D \rangle \subset \mathcal{O}_{X_k(V\langle [D] \rangle)}$ . These are precisely defined as the base loci ideals of the local sections defined by (1.20) and (1.20'), when these are seen as sections of the logarithmic tautological sheaves  $\mathcal{O}_{X_k(V\langle [D] \rangle)}(m)$ . The corresponding analytic viewpoint is to consider ad hoc singular hermitian metrics on  $\mathcal{O}_{X_k(V\langle [D] \rangle)}(1)$  whose singularities are asymptotically described by the limit of the formal  $m$ -th root of  $\mathcal{J}_m\langle D \rangle$ , see §3.B. It then becomes possible to deal without trouble with real coefficients  $\rho_j \in ]1, \infty]$ , and since we no longer have to worry about the existence of Galois covers, the projectivity assumption on  $X$  can be dropped as well.

## 2. Preliminaries on holomorphic Morse inequalities

### 2.A. Basic results

We first recall the basic results concerning holomorphic Morse inequalities for smooth hermitian line bundles, first proved in [Dem85].

**2.1. Theorem.** *Let  $X$  be a compact complex manifold,  $E \rightarrow X$  a holomorphic vector bundle of rank  $r$ , and  $(L, h)$  a hermitian line bundle. We denote by  $\Theta_{L,h} = \frac{i}{2\pi} \nabla_h^2 = -\frac{i}{2\pi} \partial \bar{\partial} \log h$  the curvature form of  $(L, h)$  and introduce the open subsets of  $X$*

$$(*) \quad \begin{cases} X(L, h, q) = \{x \in X; \Theta_{L,h}(x) \text{ has signature } (n-q, q)\}, \\ X(L, h, S) = \bigcup_{q \in S} X(L, h, q), \quad \forall S \subset \{0, 1, \dots, n\}. \end{cases}$$

*Then, for all  $q = 0, 1, \dots, n$ , the dimensions  $h^q(X, E \otimes L^m)$  of cohomology groups of the tensor powers  $E \otimes L^m$  satisfy the following “Strong Morse inequalities” as  $m \rightarrow +\infty$ :*

$$\text{SM}(q) : \quad \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, E \otimes L^m) \leq r \frac{m^n}{n!} \int_{X(L,h,\leq q)} (-1)^q \Theta_{L,h}^n + o(m^n),$$

*with equality  $\chi(X, E \otimes L^m) = r \frac{m^n}{n!} \int_X \Theta_{L,h}^n + o(m^n)$  for the Euler characteristic ( $q = n$ ).*

As a consequence, one gets upper and lower bounds for all cohomology groups, and especially a very useful criterion for the existence of sections of large multiples of  $L$ .

**2.2. Corollary.** *Under the above hypotheses, we have*

(a) *Upper bound for  $h^q$  (Weak Morse inequalities) :*

$$h^q(X, E \otimes L^m) \leq r \frac{m^n}{n!} \int_{X(L, h, q)} (-1)^q \Theta_{L, h}^n + o(m^n) .$$

(b) *Lower bound for  $h^0$  :*

$$h^0(X, E \otimes L^m) \geq h^0 - h^1 \geq r \frac{m^n}{n!} \int_{X(L, h, \leq 1)} \Theta_{L, h}^n - o(m^n) .$$

*Especially  $L$  is big as soon as  $\int_{X(L, h, \leq 1)} \Theta_{L, h}^n > 0$  for some hermitian metric  $h$  on  $L$ .*

(c) *Lower bound for  $h^q$  :*

$$h^q(X, E \otimes L^m) \geq h^q - h^{q-1} - h^{q+1} \geq r \frac{m^n}{n!} \int_{X(L, h, \{q, q \pm 1\})} (-1)^q \Theta_{L, h}^n + o(m^n) .$$

*Proof.* (a) is obtained by taking  $\text{SM}(q) + \text{SM}(q-1)$ , (b) is equivalent to  $-\text{SM}(1)$  and (c) is equivalent to  $-(\text{SM}(q+1) + \text{SM}(q-2))$ .  $\square$

The following simple lemma is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

**2.3. Lemma.** *Let  $\eta = \alpha - \beta$  be a difference of semipositive  $(1, 1)$ -forms on an  $n$ -dimensional complex manifold  $X$ , and let  $\mathbb{1}_{\eta, \leq q}$  be the characteristic function of the open set where  $\eta$  is non degenerate with a number of negative eigenvalues at most equal to  $q$ . Then*

$$(-1)^q \mathbb{1}_{\eta, \leq q} \eta^n \leq \sum_{0 \leq j \leq q} (-1)^{q-j} \binom{n}{j} \alpha^{n-j} \wedge \beta^j ,$$

*in particular*

$$\mathbb{1}_{\eta, \leq 1} \eta^n \geq \alpha^n - n\alpha^{n-1} \wedge \beta \quad \text{for } q = 1.$$

*Proof.* Without loss of generality, we can assume  $\alpha > 0$  positive definite, so that  $\alpha$  can be taken as the base hermitian metric on  $X$ . Let us denote by

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

the eigenvalues of  $\beta$  with respect to  $\alpha$ . The eigenvalues of  $\eta = \alpha - \beta$  are then given by

$$1 - \lambda_1 \leq \dots \leq 1 - \lambda_q \leq 1 - \lambda_{q+1} \leq \dots \leq 1 - \lambda_n ,$$

hence the open set  $\{\lambda_{q+1} < 1\}$  coincides with the support of  $\mathbb{1}_{\eta, \leq q}$ , except that it may also contain a part of the degeneration set  $\eta^n = 0$ . On the other hand we have

$$\binom{n}{j} \alpha^{n-j} \wedge \beta^j = \sigma_n^j(\lambda) \alpha^n ,$$

where  $\sigma_n^j(\lambda)$  is the  $j$ -th elementary symmetric function in the  $\lambda_j$ 's. Thus, to prove the lemma, we only have to check that

$$\sum_{0 \leq j \leq q} (-1)^{q-j} \sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1} < 1\}} (-1)^q \prod_{1 \leq j \leq n} (1 - \lambda_j) \geq 0 .$$

This is easily done by induction on  $n$  (just split apart the parameter  $\lambda_n$  and write  $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda) \lambda_n$ ).  $\square$

**2.4. Corollary.** *Assume that  $\eta = \Theta_{L,h}$  can be expressed as a difference  $\eta = \alpha - \beta$  of smooth  $(1,1)$ -forms  $\alpha, \beta \geq 0$ . Then we have*

$$\text{SM}(q) : \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, E \otimes L^m) \leq r \frac{m^n}{n!} \int_X \sum_{0 \leq j \leq q} (-1)^{q-j} \binom{n}{j} \alpha^{n-j} \wedge \beta^j + o(m^n),$$

and in particular, for  $q = 1$ ,

$$h^0(X, E \otimes L^m) \geq h^0 - h^1 \geq r \frac{m^n}{n!} \int_X \alpha^n - n \alpha^{n-1} \wedge \beta + o(m^n).$$

**2.5. Remark.** These estimates are consequences of Theorem 2.1 and Lemma 2.3, by taking the integral over  $X$ . The estimate for  $h^0$  was stated and studied by Trapani [Tra93]. In the special case  $\alpha = \Theta_{A,h_A} > 0$ ,  $\beta = \Theta_{B,h_B} > 0$  where  $A, B$  are ample line bundles, a direct proof can be obtained by purely algebraic means, via the Riemann-Roch formula. However, we will later have to use Corollary 2.4 in case  $\alpha$  and  $\beta$  are not closed, a situation in which no algebraic proof seems to exist.

## 2.B. Singular holomorphic Morse inequalities

The case of singular hermitian metrics has been considered in Bonavero's PhD thesis [Bon93] and will be important for us. We assume that  $L$  is equipped with a singular hermitian metric  $h = h_\infty e^{-\varphi}$  with analytic singularities, i.e.,  $h_\infty$  is a smooth metric, and on an neighborhood  $V \ni x_0$  of an arbitrary point  $x_0 \in X$ , the weight  $\varphi$  is of the form

$$(2.6) \quad \varphi(z) = c \log \sum_{1 \leq j \leq N} |g_j|^2 + u(z)$$

where  $g_j \in \mathcal{O}_X(V)$  and  $u \in C^\infty(V)$ . We then have  $\Theta_{L,h} = \alpha + \frac{i}{2\pi} \partial \bar{\partial} \varphi$  where  $\alpha = \Theta_{L,h_\infty}$  is a smooth closed  $(1,1)$ -form on  $X$ . In this situation, the multiplier ideal sheaves

$$(2.7) \quad \mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \left\{ f \in \mathcal{O}_{X,x}, \exists V \ni x, \int_V |f(z)|^2 e^{-m\varphi(z)} d\lambda(z) < +\infty \right\}$$

play an important role. We define the singularity set of  $h$  by  $\text{Sing}(h) = \text{Sing}(\varphi) = \varphi^{-1}(-\infty)$  which, by definition, is an analytic subset of  $X$ . The associated  $q$ -index sets are

$$(2.8) \quad X(L, h, q) = \{x \in X \setminus \text{Sing}(h); \Theta_{L,h}(x) \text{ has signature } (n - q, q)\}.$$

We can then state:

**2.9. Theorem** ([Bon93]). *Morse inequalities still hold in the context of singular hermitian metric with analytic singularities, provided the cohomology groups under consideration are twisted by the appropriate multiplier ideal sheaves, i.e. replaced by  $H^q(X, E \otimes L^m \otimes \mathcal{I}(h^m))$ .*

**2.10. Remark.** The assumption (2.6) guarantees that the measure  $\mathbb{1}_{X \setminus \text{Sing}(h)} (\Theta_{L,h})^n$  is locally integrable on  $X$ , as is easily seen by using the Hironaka desingularization theorem

and by taking a log resolution  $\mu : \tilde{X} \rightarrow X$  such that  $\mu^*(g_j) = (\gamma) \subset \mathcal{O}_{\tilde{X}}$  becomes a principal ideal associated with a simple normal crossing divisor  $E = \text{div}(\gamma)$ . Then  $\mu^*\Theta_{L,h} = c[E] + \beta$  where  $\beta$  is a smooth closed  $(1,1)$ -form on  $\tilde{X}$ , hence

$$\mu^*(\mathbb{1}_{X \setminus \text{Sing}(h)} \Theta_{L,h}^n) = \beta^n \Rightarrow \int_{X \setminus \text{Sing}(h)} \Theta_{L,h}^n = \int_{\tilde{X}} \beta^n.$$

It should be observed that the multiplier ideal sheaves  $\mathcal{I}(h^m)$  and the integral  $\int_{X \setminus \text{Sing}(h)} \Theta_{L,h}^n$  only depend on the equivalence class of singularities of  $h$ : if we have two metrics with analytic singularities  $h_j = h_\infty e^{-\varphi_j}$ ,  $j = 1, 2$ , such that  $\psi = \varphi_2 - \varphi_1$  is bounded, then, with the above notation, we have  $\mu^*\Theta_{L,h_j} = c[E] + \beta_j$  and  $\beta_2 = \beta_1 + \frac{i}{2\pi} \partial \bar{\partial} \psi$ , therefore  $\int_{\tilde{X}} \beta_2^n = \int_{\tilde{X}} \beta_1^n$  by Stokes theorem. By using Monge-Ampère operators in the sense of Bedford-Taylor [BT76], it is in fact enough to assume  $u \in L_{\text{loc}}^\infty(X)$  in (2.6), and  $\psi \in L^\infty(X)$  here. In general, however, the Morse integrals  $\int_{X(L,h_j,q)} (-1)^q \Theta_{L,h_j}^n$ ,  $j = 1, 2$ , will differ.

## 2.C. Morse inequalities and semi-continuity

Let  $\mathcal{X} \rightarrow S$  be a proper and flat morphism of reduced complex spaces, and let  $(X_t)_{t \in S}$  be the fibers. Given a sheaf  $\mathcal{E}$  over  $\mathcal{X}$  of locally free  $\mathcal{O}_{\mathcal{X}}$ -modules of rank  $r$ , inducing on the fibres a family of sheaves  $(E_t \rightarrow X_t)_{t \in S}$ , the following semicontinuity property holds ([CRAS]):

**2.11. Proposition.** *For every  $q \geq 0$ , the alternate sum*

$$t \mapsto h^q(X_t, E_t) - h^{q-1}(X_t, E_t) + \dots + (-1)^q h^0(X_t, E_t)$$

*is upper semicontinuous with respect to the (analytic) Zariski topology on  $S$ .*

Now, if  $\mathcal{L} \rightarrow \mathcal{X}$  is an invertible sheaf equipped with a smooth hermitian metric  $h$ , and if  $(h_t)$  are the fiberwise metrics on the family  $(L_t \rightarrow X_t)_{t \in S}$ , we get

$$(2.12) \quad \sum_{j=0}^q (-1)^{q-j} h^j(X_t, E_t \otimes L_t^{\otimes m}) \leq r \frac{m^n}{n!} \int_{X(L_0, h_0, \leq q)} (-1)^q \Theta_{L_0, h_0}^n + \delta(t) m^n,$$

where  $\delta(t) \rightarrow 0$  as  $t \rightarrow 0$ . In fact, the proof of holomorphic Morse inequalities shows that the inequality holds uniformly on every relatively compact  $S' \Subset S$ , with

$$I(t) = \int_{X(L_t, h_t, \leq q)} (-1)^q \Theta_{L_t, h_t}^n = \int_X (-1)^q \mathbb{1}_{X(L_t, h_t, \leq q)} \Theta_{L_t, h_t}^n$$

in the right hand side, and  $t \mapsto I(t)$  is clearly continuous with respect to the ordinary topology. In other words, the Morse integral computed on the central fibers provides uniform upper bounds for cohomology groups of  $E_t \otimes L_t^{\otimes m}$  when  $t$  is close to 0 in ordinary topology (and also, as a consequence, for  $t$  in a complement  $S \setminus \bigcup S_m$  of at most countably many analytic strata  $S_m \subsetneq S$ ).

**2.13. Remark.** Similar results would hold when  $h$  is a singular hermitian metric with analytic singularities on  $\mathcal{L} \rightarrow \mathcal{X}$ , under the restriction that the families of multiplier ideal sheaves  $(\mathcal{I}(h_t^m))_{t \in S}$  “never jump”.

## 2.D. Case of filtered bundles

Let  $E \rightarrow X$  be a vector bundle over a variety, equipped with a filtration (or multi-filtration)  $F^p(E)$ , and let  $G = \bigoplus G^p(E) \rightarrow X$  be the graded bundle associated to this filtration.

**2.14. Lemma.** *In the above setting, one has for every  $q \geq 0$*

$$\sum_{j=0}^q (-1)^{q-j} h^j(X, E) \leq \sum_{j=0}^q (-1)^{q-j} h^j(X, G).$$

*Proof.* One possible argument is to use the well known fact that there is a family of filtered bundles  $(E_t \rightarrow X)_{t \in \mathbb{C}}$  (with the same graded pieces  $G^p(E_t) = G^p(E)$ ), such that  $E_t \simeq E$  for all  $t \neq 0$  and  $E_0 \simeq G$ . The result is then an immediate consequence of the semi-continuity result 2.11. A more direct very elementary argument can be given as follows: by transitivity of inequalities, it is sufficient to prove the result for simple filtrations; then, by induction on the length of filtrations, it is sufficient to prove the result for exact sequences  $0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$  of vector bundles on  $X$ . Consider the associated (truncated) long exact sequence in cohomology:

$$\begin{aligned} 0 \rightarrow H^0(S) \rightarrow H^0(E) \rightarrow H^0(Q) \xrightarrow{\delta_1} \dots \\ \xrightarrow{\delta_{q-1}} H^q(S) \rightarrow H^q(E) \rightarrow H^q(Q) \xrightarrow{\delta_q} \text{Im}(\delta_q) \rightarrow 0. \end{aligned}$$

By the rank theorem of linear algebra,

$$0 \leq \text{rank}(\delta_q) = (-1)^q \sum_{j=0}^q (-1)^j (h^j(X, Q) - h^j(X, E) + h^j(X, S)).$$

The result follows, since here  $h^j(X, G) = h^j(X, Q) + h^j(X, S)$ . □

## 2.E. Rees deformation construction (after Cadorel)

In this short paragraph, we outline a nice algebraic interpretation by Benoît Cadorel of certain semi-continuity arguments for cohomology group dimensions that underline the analytic approach of [Dem11, Lemma 2.12 and Prop. 2.13] and [Dem12, Prop. 9.28] (we will anyway explain again its essential points in §3, since we have to deal here with a more general situation). Recall after [Cad17, Prop. 4.2, Prop. 4.5], that the Rees deformation construction allows one to construct natural deformations of Green-Griffiths jets spaces to weighted projectivized bundles.

Let  $(X, V, D)$  be a non singular directed orbifold, and let  $g : Y \rightarrow (X, D)$  be an adapted Galois cover, as briefly described in remark 1.26, see also [CDR18, §2.1] for more details. We then get a Green-Griffiths jet bundle of graded algebras  $E_{k, \bullet} \tilde{V}^* \langle D \rangle \rightarrow Y$  which admits a multifiltration of associated graded algebra

$$G^\bullet E_{k, \bullet} \tilde{V}^* \langle D \rangle = \bigoplus_{m \in \mathbb{N}} \bigoplus_{\ell_1 + 2\ell_2 + \dots + k\ell_k = m} S^{\ell_1} \tilde{V}^* \langle D^{(1)} \rangle \otimes \dots \otimes S^{\ell_k} \tilde{V}^* \langle D^{(k)} \rangle.$$

where the tilde means taking pull-backs by  $g^*$ . Applying the Proj functor, one gets a weighted projective bundle:

$$\mathbb{P}_{(1, \dots, k)} \left( \tilde{V}^* \langle D^{(1)} \rangle \oplus \dots \oplus \tilde{V}^* \langle D^{(k)} \rangle \right) = \text{Proj} \left( G^\bullet E_{k, \bullet} \tilde{V}^* \langle D \rangle \right) \xrightarrow{\rho_k} Y,$$

Then, following mutadis mutandus the arguments of Cadorel, one constructs a family  $Y \xleftarrow{p_k} \mathcal{Y}_k \rightarrow \mathbb{C}$  parametrized by  $\mathbb{C}$ , with a canonical line bundle  $\mathcal{O}_{\mathcal{Y}_k}(1)$  such that:

- the central fiber  $\mathcal{Y}_{k,0}$  is  $\mathbb{P}_{(1,\dots,k)} \left( \tilde{V}^* \langle D^{(1)} \rangle \oplus \dots \oplus \tilde{V}^* \langle D^{(k)} \rangle \right)$  and the restriction of  $\mathcal{O}_{\mathcal{Y}_k}(1)$  coincide with the canonical line bundle of this weighted projective bundle. Hence  $(\pi_k)_* \mathcal{O}_{\mathcal{Y}_{k,0}}(m) = G^\bullet E_{k,m} \tilde{V}^* \langle D \rangle$ .
- the other fibers  $\mathcal{Y}_{k,t}$  are isomorphic to the singular quotient  $J^k(Y, \tilde{V}, D)/\mathbb{C}^*$  for the natural  $\mathbb{C}^*$ -action by homotheties, where  $J^k(Y, \tilde{V}, D)$  is the affine algebraic bundle associated with the sheaf of algebras, and  $(\pi_k)_* \mathcal{O}_{\mathcal{Y}_{k,t}}(m) \simeq E_{k,m} \tilde{V}^* \langle D \rangle$ .

Applying the semicontinuity result of [Dem95], and working with holomorphic inequalities, we obtain a control about dimensions of cohomology spaces of  $E_{k,m} \tilde{V}^* \langle D \rangle$  in terms of dimensions of cohomology spaces of the a priori simpler graded pieces  $G^\bullet E_{k,m} \tilde{V}^* \langle D \rangle$ . This reduces the study of higher order jet differentials to sections of the tautological sheaves on the weighted projective space associated with a direct sum combination of symmetric differentials. In particular, we have

**2.15. Lemma.** *For every  $q \in \mathbb{N}$*

$$\sum_{j=0}^q (-1)^{q-j} h^j(Y, E_{k,m} \tilde{V}^* \langle D \rangle) \geq \sum_{j=0}^q (-1)^{q-j} h^j(Y, G^\bullet E_{k,m} \tilde{V}^* \langle D \rangle).$$

*Especially, for  $q = 1$ , we get*

$$\begin{aligned} h^0(Y, E_{k,m} \tilde{V}^* \langle D \rangle) &\geq h^0(Y, E_{k,m} \tilde{V}^* \langle D \rangle) - h^1(Y, E_{k,m} \tilde{V}^* \langle D \rangle) \\ &\geq h^0(Y, G^\bullet E_{k,m} \tilde{V}^* \langle D \rangle) - h^1(Y, G^\bullet E_{k,m} \tilde{V}^* \langle D \rangle). \end{aligned}$$

### 3. Construction of jet metrics and orbifold jet metrics

#### 3.A. Jet metrics and curvature tensor of jet bundles

Let  $(X, V)$  be a non singular directed variety and  $h$  a hermitian metric on  $V$ . We assume that  $h$  is smooth at this point (but will later relax a little bit this assumption and allow certain singularities). Near any given point  $z_0 \in X$ , we can choose local coordinates  $z = (z_1, \dots, z_n)$  centered at  $z_0$  and a local holomorphic coordinate frame  $(e_\lambda(z))_{1 \leq \lambda \leq r}$  of  $V$  on an open set  $U \ni z_0$ , such that

$$(3.1) \quad \langle e_\lambda(z), e_\mu(z) \rangle_{h(z)} = \delta_{\lambda\mu} + \sum_{1 \leq i, j \leq n, 1 \leq \lambda, \mu \leq r} c_{ij\lambda\mu} z_i \bar{z}_j + O(|z|^3)$$

for suitable complex coefficients  $(c_{ij\lambda\mu})$ . It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor  $\frac{i}{2\pi} \nabla_{V,h}^2$  of  $(V, h)$  at  $z_0$  is given by

$$(3.2) \quad \Theta_{V,h}(z_0) = -\frac{i}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} dz_i \wedge d\bar{z}_j \otimes e_\lambda^* \otimes e_\mu.$$

Therefore,  $(\frac{i}{2\pi} c_{ij\lambda\mu})$  are the coefficients of  $-\Theta_{V,h}$ . Up to taking the transposed tensor with respect to  $\lambda, \mu$ , these coefficients are also the components of the curvature tensor

$\Theta_{V^*, h^*} = -{}^t\Theta_{V, h}$  of the dual bundle  $(V^*, h^*)$ . By (1.5), the connection  $\nabla = \nabla_h$  yields a  $C^\infty$  isomorphism  $J_k V \rightarrow V^{\oplus k}$ . Let us fix an integer  $b \in \mathbb{N}^*$  that is a multiple of  $\text{lcm}(1, 2, \dots, k)$ , and positive numbers  $1 = \varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k > 0$ . Following [Dem11], we define a global weighted Finsler metric on  $J^k V$  by putting for any  $k$ -jet  $f \in J^k V_z$

$$(3.3) \quad \Psi_{h, b, \varepsilon}(f) := \left( \sum_{1 \leq s \leq k} \varepsilon_s^{2b} \|\nabla^s f(0)\|_{h(z)}^{2b/s} \right)^{1/b},$$

where  $\|\cdot\|_{h(z)}$  is the hermitian metric  $h$  of  $V$  evaluated on the fiber  $V_z$ ,  $z = f(0)$ . The function  $\Psi_{h, b, \varepsilon}$  satisfies the fundamental homogeneity property

$$(3.4) \quad \Psi_{h, b, \varepsilon}(\alpha \cdot f) = |\alpha|^2 \Psi_{h, b, \varepsilon}(f)$$

with respect to the  $\mathbb{C}^*$  action on  $J^k V$ , in other words, it induces a hermitian metric on the dual  $L_k^*$  of the tautological  $\mathbb{Q}$ -line bundle  $L_k = \mathcal{O}_{X_k(V)}(1)$  over  $X_k(V)$ . The curvature of  $L_k$  is given by

$$(3.5) \quad \pi_k^* \Theta_{L_k, \Psi_{h, b, \varepsilon}^*} = \frac{i}{2\pi} \partial \bar{\partial} \log \Psi_{h, b, \varepsilon}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to  $L \rightarrow X_k(V)$  with the above metric. This might look a priori like an untractable problem, since the definition of  $\Psi_{h, b, \varepsilon}$  is a rather complicated one, involving the hermitian metric in an intricate manner. However, the “miracle” is that the asymptotic behavior of  $\Psi_{h, b, \varepsilon}$  as  $\varepsilon_s/\varepsilon_{s-1} \rightarrow 0$  is in some sense uniquely defined, and “splits” according to the natural multifiltration on jet differentials (as already hinted in §2.E). This leads to a computable asymptotic formula, which is moreover simple enough to produce useful results.

**3.6. Lemma.** *Let us consider the global  $C^\infty$  bundle isomorphism  $J^k V \rightarrow V^{\oplus k}$  associated with an arbitrary global  $C^\infty$  connection  $\nabla$  on  $V \rightarrow X$ , and let us introduce the rescaling transformation*

$$\rho_{\nabla, \varepsilon}(\xi_1, \xi_2, \dots, \xi_k) = (\varepsilon_1^1 \xi_1, \varepsilon_2^2 \xi_2, \dots, \varepsilon_k^k \xi_k) \quad \text{on fibers } J^k V_z, \ z \in X.$$

*Such a rescaling commutes with the  $\mathbb{C}^*$ -action. Moreover, if  $p$  is a multiple of  $\text{lcm}(1, 2, \dots, k)$  and the ratios  $\varepsilon_s/\varepsilon_{s-1}$  tend to 0 for all  $s = 2, \dots, k$ , the rescaled Finsler metric  $\Psi_{h, b, \varepsilon} \circ \rho_{\nabla, \varepsilon}^{-1}(\xi_1, \dots, \xi_k)$  converges towards the limit*

$$\left( \sum_{1 \leq s \leq k} \|\xi_s\|_h^{2b/s} \right)^{1/b}$$

*on every compact subset of  $V^{\oplus k} \setminus \{0\}$ , uniformly in  $C^\infty$  topology, and the limit is independent of the connection  $\nabla$ . The error is measured by a multiplicative factor  $1 \pm O(\max_{2 \leq s \leq k} (\varepsilon_s/\varepsilon_{s-1})^s)$ .*

*Proof.* Let us pick another  $C^\infty$  connection  $\tilde{\nabla} = \nabla + \Gamma$  where  $\Gamma \in C^\infty(U, T_X^* \otimes \text{Hom}(V, V))$ . Then  $\tilde{\nabla}^2 f = \nabla^2 f + \Gamma(f)(f') \cdot f'$ , and inductively we get

$$\tilde{\nabla}^s f = \nabla^s f + P_s(f; \nabla^1 f, \dots, \nabla^{s-1} f)$$

where  $P(z; \xi_1, \dots, \xi_{s-1})$  is a polynomial with  $C^\infty$  coefficients in  $z \in U$ , which is of weighted homogeneous degree  $s$  in  $(\xi_1, \dots, \xi_{s-1})$ . In other words, the corresponding isomorphisms  $J^k V \simeq V^{\oplus k}$  correspond to each other by a  $\mathbb{C}^*$ -homogeneous transformation  $(\xi_1, \dots, \xi_k) \mapsto (\tilde{\xi}_1, \dots, \tilde{\xi}_k)$  such that

$$\tilde{\xi}_s = \xi_s + P_s(z; \xi_1, \dots, \xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon}, \dots, \xi_{k,\varepsilon}) = (\varepsilon_1^1 \xi_1, \dots, \varepsilon_k^k \xi_k), \quad (\tilde{\xi}_{1,\varepsilon}, \dots, \tilde{\xi}_{k,\varepsilon}) = (\varepsilon_1^1 \tilde{\xi}_1, \dots, \varepsilon_k^k \tilde{\xi}_k).$$

Then

$$\begin{aligned} \tilde{\xi}_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s P_s(x; \varepsilon_1^{-1} \xi_1, \dots, \varepsilon_{s-1}^{-(s-1)} \xi_{s-1,\varepsilon}) \\ &= \xi_{s,\varepsilon} + O(\varepsilon_s / \varepsilon_{s-1})^s O(\|\xi_{1,\varepsilon}\| + \dots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s \end{aligned}$$

and it is easily seen, as a simple consequence of the mean value inequality  $\|x\|^\gamma - \|y\|^\gamma \leq \gamma \sup_{z \in [x,y]} \|z\|^{\gamma-1} \|x - y\|$ , that the “error term” in the difference  $\|\tilde{\xi}_{s,\varepsilon}\|^{2b/s} - \|\xi_{s,\varepsilon}\|^{2b/s}$  is bounded by

$$(\varepsilon_s / \varepsilon_{s-1})^s (\|\xi_{1,\varepsilon}\| + \dots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)} + \|\xi_{s,\varepsilon}\|^{1/s})^{2b}.$$

When  $b/s$  is an integer, similar bounds hold for all derivatives  $D_{z,\xi}^\beta (\|\tilde{\xi}_{s,\varepsilon}\|^{2b/s} - \|\xi_{s,\varepsilon}\|^{2b/s})$  and the lemma follows.  $\square$

Now, we fix a point  $z_0 \in X$ , a local holomorphic frame  $(e_\lambda(z))_{1 \leq \lambda \leq r}$  satisfying (3.1) on a neighborhood  $U$  of  $z_0$ , and the *holomorphic* connection  $\nabla$  on  $V|_U$  such that  $\nabla e_\lambda = 0$ . Since the uniform estimates of Lemma 3.6 also apply locally (provided they are applied on a relatively compact open subset  $U' \Subset U$ ), we can use the corresponding holomorphic trivialization  $J^k V|_U \simeq V|_U^{\oplus k} \simeq U \times (\mathbb{C}^r)^{\oplus k}$  to make our calculations. We do this in terms of the rescaled components  $\xi_s = \varepsilon_s^s \nabla^s f(0)$ . Then, uniformly on compact subsets of  $J^k V|_U \setminus \{0\}$ , we have

$$\Psi_{h,b,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(z; \xi_1, \dots, \xi_k) = \left( \sum_{1 \leq s \leq k} \|\xi_s\|_{h(z)}^{2b/s} \right)^{1/b} + O(\max((\varepsilon_s / \varepsilon_{s-1})^{1/b})),$$

and the error term remains of the same magnitude when we take any derivative  $D_{z,\xi}^\beta$ . By (3.1) we find

$$\|\xi_s\|_{h(z)}^2 = \sum_\lambda |\xi_{s,\lambda}|^2 + \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} z_i \bar{z}_j \xi_{s,\lambda} \bar{\xi}_{s,\mu} + O(|z|^3 |\xi|^2).$$

The question is thus reduced to evaluating the curvature of the weighted Finsler metric on  $V^{\oplus k}$  defined by

$$\begin{aligned} \Psi(z; \xi_1, \dots, \xi_k) &= \left( \sum_{1 \leq s \leq k} \|\xi_s\|_{h(z)}^{2b/s} \right)^{1/b} \\ &= \left( \sum_{1 \leq s \leq k} \left( \sum_\lambda |\xi_{s,\lambda}|^2 + \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} z_i \bar{z}_j \xi_{s,\lambda} \bar{\xi}_{s,\mu} \right)^{b/s} \right)^{1/b} + O(|z|^3). \end{aligned}$$



We set  $|\xi_s|^2 = \sum_\lambda |\xi_{s,\lambda}|^2$ . A straightforward calculation yields the Taylor expansion

$$\begin{aligned} \log \Psi(z; \xi_1, \dots, \xi_k) \\ = \frac{1}{b} \log \sum_{1 \leq s \leq k} |\xi_s|^{2b/s} + \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2b/s}}{\sum_t |\xi_t|^{2b/t}} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} z_i \bar{z}_j \frac{\xi_{s,\lambda} \bar{\xi}_{s,\mu}}{|\xi_s|^2} + O(|z|^3). \end{aligned}$$

By (3.5), the curvature form of  $L_k = \mathcal{O}_{X_k(V)}(1)$  is given at the central point  $z_0$  by the formula

$$(3.7) \quad \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}^*(z_0, [\xi]) \simeq \omega_{r,k,b}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2b/s}}{\sum_t |\xi_t|^{2b/t}} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} \frac{\xi_{s,\lambda} \bar{\xi}_{s,\mu}}{|\xi_s|^2} dz_i \wedge d\bar{z}_j$$

where  $[\xi] = [\xi_1, \dots, \xi_k] \in \mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})$  and  $\omega_{r,k,b}(\xi) = \frac{i}{2\pi} \partial \bar{\partial} (\frac{1}{b} \log \sum_{1 \leq s \leq k} |\xi_s|^{2b/s})$ . The fibers  $\mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})$  of  $X_k(V) \rightarrow X$  can be represented as a quotient of the “weighted ellipsoid”  $\sum_{s=1}^k |\xi_s|^{2b/s} = 1$  by the  $\mathbb{S}^1$ -action induced by the weighted  $\mathbb{C}^*$ -action. This suggests to make use of polar coordinates and to set

$$(3.8) \quad x_s = |\xi_s|^{2b/s}, \quad x = (x_1, \dots, x_k) \in \mathbb{R}^k,$$

$$(3.8') \quad u_s = \frac{\xi_s}{|\xi_s|} \in \mathbb{S}^{2r-1} \subset \mathbb{C}^r, \quad u = (u_1, \dots, u_k) \in (\mathbb{S}^{2r-1})^k,$$

so that

$$(3.8'') \quad \sum_{s=1}^k x_s = 1 \quad \text{and} \quad \xi_s = x_s^{s/2b} u_s.$$

The Morse integrals will then have to be computed for  $(x, u) \in \Delta^{k-1} \times (\mathbb{S}^{2r-1})^k$ , where  $\Delta^{k-1} \subset \mathbb{R}^k$  is the  $(k-1)$ -dimensional simplex.

**3.9. Proposition.** *With respect to the rescaled components  $\xi_s = \varepsilon_s^s \nabla^s f(0)$  at  $z = f(0) \in X$  and the above choice of coordinates (3.8\*), the curvature of the tautological sheaf  $L_k = \mathcal{O}_{X_k(V)}(1)$  admits an approximate expression*

$$(a) \quad \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}^*(z, [\xi]) = \omega_{r,k,b}(\xi) + g_{V,k}(z, x, u) + (\text{error terms}),$$

where  $(x, u) \in \Delta^{k-1} \times (\mathbb{S}^{2r-1})^k$ ,  $\xi_s = x_s^{s/2b} u_s \in \mathbb{C}^r$ ,

$$(b) \quad \omega_{r,k,b}(\xi) = \frac{i}{2\pi} \partial \bar{\partial} \left( \frac{1}{b} \sum_{1 \leq s \leq k} |\xi_s|^{2b/s} \right)$$

is a Fubini-Study type Kähler metric on  $\mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})$ , associated with the canonical  $\mathbb{C}^*$  action on  $J^k V$  of weight  $a = (1^{[r]}, 2^{[r]}, \dots, k^{[r]})$ , and

$$(c) \quad g_{V,k}(z, x, u) = \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{x_s}{s} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}(z) u_{s,\lambda} \bar{u}_{s,\mu} dz_i \wedge d\bar{z}_j.$$

Here  $(\frac{i}{2\pi} c_{ij\lambda\mu})$  are the coefficients of  $-\Theta_{V,h}$ , and the error terms admit an upper bound

$$(d) \quad (\text{error terms}) \leq O\left(\max_{2 \leq s \leq k} (\varepsilon_s / \varepsilon_{s-1})^s\right) \quad \text{uniformly on the compact variety } X_k(V).$$

*Proof.* The error terms on  $\Theta_{L_k}$  come from the differentiation of the error terms on the Finsler metric, found in Lemma 3.6. They can indeed be differentiated if  $b$  is a multiple of  $\text{lcm}(1, 2, \dots, k)$ , since  $2b/s$  is then an even integer.  $\square$

For the calculation of Morse integrals, it is useful to find the expression of the volume form  $\omega_{r,k,b}^{kr-1}$  on  $\mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]}) = (\Delta^{k-1} \times (\mathbb{S}^{2r-1})^k) / \mathbb{S}^1$  in terms of the coordinates  $(x, u)$ . We refer to [Dem11, Prop. 1.13] for the proof.

### 3.10. Proposition.

(a) *The volume form  $\omega_{r,k,b}^{kr-1}$  is the quotient of the measure  $\frac{1}{k!^r} \nu_{k,r} \otimes \mu$  on  $\Delta^{k-1} \times (\mathbb{S}^{2r-1})^k$ , where*

$$d\nu_{k,r}(x) = (kr - 1)! \frac{(x_1 \dots x_k)^{r-1}}{(r-1)!^k} dx_1 \wedge \dots \wedge dx_{k-1}, \quad d\mu(u) = d\mu_1(u_1) \dots d\mu_k(u_k)$$

*are probability measures on  $\Delta^{k-1}$  and  $(\mathbb{S}^{2r-1})^k$  respectively ( $\mu$  being the rotation invariant one).*

(b) *We have the equality  $\int_{\mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})} \omega_{r,k,b}^{kr-1} = \frac{1}{k!^r}$  (independent of  $b$ ).*

### §3.B. Logarithmic and orbifold jet metrics

Consider now an orbifold directed structure  $(X, V, D)$ , where  $V \subset T_X$  is a subbundle,  $r = \text{rank}(V)$ , and  $D = \sum (1 - \frac{1}{\rho_j}) \Delta_j$  is a normal crossing divisor that is assumed to intersect  $V$  transversally everywhere. One then performs very similar calculations to what we did in §3.A, but with adapted Finsler metrics. Fix a point  $z_0$  at which  $p$  components  $\Delta_j$  meet, and use coordinates  $(z_1, \dots, z_n)$  such that  $V_{z_0}$  is spanned by  $(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_r})$  and  $\Delta_j$  is defined by  $z_j = 0$ ,  $1 \leq j \leq p \leq r$ . In the logarithmic case  $\rho_j = \infty$ , the logarithmic dual bundle  $\mathcal{O}(V^*\langle D \rangle)$  is spanned by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_p}{z_p}, dz_{p+1}, \dots, dz_n.$$

The logarithmic jet differentials are just polynomials in

$$\frac{d^s z_1}{z_1}, \dots, \frac{d^s z_p}{z_p}, d^s z_{p+1}, \dots, d^s z_n, \quad 1 \leq s \leq k,$$

and the corresponding  $(\varepsilon_1, \dots, \varepsilon_k)$ -rescaled Finsler metric is

$$(3.11) \quad \left( \sum_{s=1}^k \varepsilon_s^{2b} \left( \sum_{j=1}^p |f_j|^{-2} |f_j^{(s)}|^2 + \sum_{j=p+1}^r |f_j^{(s)}|^2 \right)^{2b/s} \right)^{1/b}.$$

Alternatively, we could replace  $|f_j|^{-2} |f_j^{(s)}|^2$  by  $|(\log f_j)^{(s)}|^2$  which has the same leading term and differs by a weighted degree  $s$  polynomial in the  $f_j^{-1} f_j^{(\ell)}$ ,  $\ell < s$ ; an argument very similar to the one used in the proof of Lemma 3.6 then shows that the difference is negligible when  $\varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k$ . However (3.11) is just the case of the model metric, in fact we get  $r$ -tuples  $\xi_s = (\xi_{s,j})_{1 \leq j \leq r}$  of components produced by the trivialization of the logarithmic bundle  $\mathcal{O}(V\langle D \rangle)$ , such that

$$(3.12) \quad \xi_{s,j} = f_j^{-1} f_j^{(s)} \quad \text{for } 1 \leq s \leq p \text{ and } \quad \xi_{s,j} = f_j^{(s)} \quad \text{for } p+1 \leq s \leq r.$$

In general, we are led to consider Finsler metrics of the form

$$(3.13) \quad \left( \sum_{s=1}^k \varepsilon_s^{2b} \|\xi_s\|_{h(z)}^{2b/s} \right)^{1/b}, \quad \xi_s = (\xi_{s,j})_{1 \leq j \leq r},$$

where  $h(z)$  is a variable hermitian metric on the logarithmic bundle  $V\langle D \rangle$ . In the orbifold case, the appropriate “model” Finsler metric is

$$(3.14) \quad \left( \sum_{s=1}^k \varepsilon_s^{2b} \left( \sum_{j=1}^p |f_j|^{-2(1-s/\rho_j)+} |f_j^{(s)}|^2 + \sum_{j=p+1}^r |f_j^{(s)}|^2 \right)^{2b/s} \right)^{1/b}.$$

As a consequence of Remark 2.10, we would get a metric with equivalent singularities on the dual  $L_k^*$  of the tautological sheaf  $L_k = \mathcal{O}_{X_k(V\langle D \rangle)}(1)$  by replacing  $\sum_{j=p+1}^r |f_j^{(s)}|^2$  with  $\sum_{j=1}^r |f_j^{(s)}|^2$  (or by any smooth hermitian norm  $h$  on  $V$ ), since the extra terms  $\sum_{j=1}^p |f_j^{(s)}|^2$  are anyway controlled by the “orbifold part” of the summation. Of course, we need to find a suitable Finsler metric that is globally defined on  $X$ . This can be done by taking smooth metrics  $h_{V,s}$  on  $V$  and  $h_j$  on  $\mathcal{O}_X(\Delta_j)$  respectively, as well as smooth connections  $\nabla$  and  $\nabla_j$ . One can then consider the globally defined metric

$$(3.15) \quad \left( \sum_{s=1}^k \varepsilon_s^{2b} \left( \|\nabla^{(s)} f\|_{h_{V,s}}^2 + \sum_j \|\sigma_j(f)\|_{h_j}^{-2(1-s/\rho_j)+} \|\nabla_j^{(s)}(\sigma_j \circ f)\|_{h_j}^2 \right)^{2b/s} \right)^{1/b}$$

where  $D = \sum(1 - \frac{1}{\rho_j})\Delta_j$  and  $\sigma_j \in H^0(X, \mathcal{O}_X(\Delta_j))$  are the tautological sections; here, we want the flexibility of not necessarily taking the same hermitian metrics on  $V$  to evaluate the various norms  $\|\nabla^{(s)} f\|_{h_{V,s}}$ . We obtain Finsler metrics with equivalent singularities by just changing the  $h_{V,s}$  and  $h_j$  (and keeping  $\nabla, \nabla_j$  unchanged). If we also change the connections, then an argument very similar to the one used in the proof of Lemma 3.6 shows that the ratio of the corresponding metrics is  $1 \pm O(\max(\varepsilon_s/\varepsilon_{s-1}))$ , and therefore arbitrary close to 1 whenever  $\varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k$ ; in any case, we get metrics with equivalent singularities. Fix  $z_0 \in X$  and use coordinates  $(z_1, \dots, z_n)$  as described at the beginning of §3.B, so that  $\sigma_j(z) = z_j$ ,  $1 \leq j \leq p$ , in a suitable trivialization of  $\mathcal{O}_X(\Delta_j)$ . Let  $f$  be a  $k$ -jet of curve such that  $f(0) = z \in X \setminus |D|$  is in a sufficiently small neighborhood of  $z_0$ . By employing the trivial connections associated with the above coordinates, the derivative  $f^{(s)}$  is described by components

$$\xi_{s,j} = f_j^{(s)}, \quad 1 \leq j \leq r, \quad \xi_{s,j}^{\log} = f_j^{-1} f_j^{(s)}, \quad \xi_{s,j}^{\text{orb}} = f_j^{-(1-s/\rho_j)+} f_j^{(s)}, \quad 1 \leq j \leq p,$$

and  $\xi_{s,j}^{\text{orb}} = \xi_{s,j}^{\log} = \xi_{s,j}$  for  $p+1 \leq j \leq r$ . Here  $\xi_{s,j}^{\text{orb}}$  are to be thought of as the components of  $f^{(s)}$  in the “virtual” vector bundle  $V\langle D^{(s)} \rangle$ , and the fact that the argument of these complex numbers is not uniquely defined is irrelevant, because the only thing we need to compute the norms is  $|\xi_{s,j}^{\text{orb}}|$ . Accordingly, for  $v \in V_z$ ,  $v \simeq (v_j)_{1 \leq j \leq r} \in \mathbb{C}^r$ , we put

$$v_j^{\log} = z_j^{-1} v_j = \sigma_j(z)^{-1} \nabla_j \sigma_j(v) \quad \text{and} \quad v_j^{\text{orb}} = z_j^{-(1-s/\rho_j)+} v_j, \quad 1 \leq j \leq p,$$

and define the orbifold hermitian norm on  $V\langle D^{(s)} \rangle$  associated with  $h_{V,s}$  and  $h_j$  by

$$(3.16) \quad \|v^{\text{orb}}\|_{h_s}^2 = \|v\|_{h_{V,s}}^2 + \sum_{j=1}^p \|\sigma_j(z)\|_{h_j}^{-2(1-s/\rho_j)+} \|\nabla_j \sigma_j(v)\|_{h_j}^2$$

$$(3.16') \quad = \|v\|_{h_{V,s}}^2 + \sum_{j=1}^p \|\sigma_j(z)\|_{h_j}^{2(1-(1-s/\rho_j)_+)} |v_j^{\log}|^2$$

$$(3.16'') \quad = \|v\|_{h_{V,s}}^2 + \sum_{j=1}^p \|v_j^{\text{orb}}\|_{h_j}^{2(1-(1-s/\rho_j)_+)}.$$

With this notation, the orbifold Finsler metric (3.15) on  $k$ -jets is reduced to an expression

$$(3.17) \quad \|\xi^{\text{orb}}\|_{\Psi_{h,b,\varepsilon}}^2 = \left( \sum_{s=1}^k \varepsilon_s^{2b} \|\xi_s^{\text{orb}}\|_{h_s}^{2b/s} \right)^{1/b}, \quad \xi_s^{\text{orb}} = (\xi_{s,j}^{\text{orb}})_{1 \leq j \leq r}, \quad \xi^{\text{orb}} = (\xi_s^{\text{orb}})_{1 \leq s \leq k},$$

formally identical to what we had in the compact or logarithmic cases. If  $v$  is a local holomorphic section of  $\mathcal{O}_X(V)$ , formula (3.16) shows that the norm  $\|v^{\text{orb}}\|_{h_s}$  can take infinite values when  $z \in |D|$ , while, by (3.16'), the norm is always bounded (but slightly degenerate along  $|D|$ ) if  $v$  is a section of the logarithmic sheaf  $\mathcal{O}_X(V\langle[D]\rangle)$ ; we think intuitively of the orbifold total space  $V\langle D^{(s)} \rangle$  as the subspace of  $V$  in which the tubular neighborhoods of the zero section are defined by  $\|v^{\text{orb}}\|_{h_s} < \varepsilon$  for  $\varepsilon > 0$ .

**3.18. Remark.** When  $\rho_j \in \mathbb{Q}$ , we can take an adapted Galois cover  $g : Y \rightarrow X$  such that  $(z_j \circ g)^{1-(1-s/\rho_j)_+}$  is univalent on  $Y$  for all components  $\Delta_j$  involved, and we then get a well defined locally free sheaf  $\mathcal{O}_Y(g^*V\langle D^{(s)} \rangle)$  such that

$$g^*(\mathcal{O}_X(V\langle[D]\rangle)) \subset \mathcal{O}_Y(g^*V\langle D^{(s)} \rangle) \subset g^*(\mathcal{O}_X(V)).$$

However, as already stressed in Remark 1.26, this viewpoint is not needed in our analytic approach.

### 3.C. Orbifold tautological sheaves and their curvature

In this context, we define the orbifold tautological sheaves

$$(3.19) \quad \mathcal{O}_{X_k(V\langle D \rangle)}(m) := \mathcal{O}_{X_k(V\langle[D]\rangle)}(m) \otimes \mathcal{I}((\Psi_{k,b,\varepsilon}^*)^m)$$

to be the logarithmic tautological sheaves  $\mathcal{O}_{X_k(V\langle[D]\rangle)}(m)$  twisted by the multiplier ideal sheaves associated with the dual metric  $\Psi_{k,b,\varepsilon}^*$  (cf. (3.17)), when these are viewed as singular hermitian metrics over the logarithmic  $k$ -jet bundle  $X_k(V\langle[D]\rangle)$ . In accordance with this viewpoint, we simply define the orbifold  $k$ -jet bundle to be  $X_k(V\langle D \rangle) = X_k(V\langle[D]\rangle)$ . The calculation of the curvature tensor is formally the same as in the case  $D = 0$ , and we obtain :

**3.20. Proposition.** *With respect to the (rescaled) orbifold  $k$ -jet components*

$$\xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(1-(1-\rho_\lambda/s)_+)} f_\lambda^{(s)}(0), \quad 1 \leq \lambda \leq p, \quad \text{and} \quad \xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(s)}(0), \quad p+1 \leq \lambda \leq r,$$

*and of the dual metric  $\Psi_{h,b,\varepsilon}^*$ , the curvature form of the tautological sheaf  $L_k = \mathcal{O}_{X_k(V\langle D \rangle)}(1)$  admits at any point  $(z, [\xi]) \in X_k(V\langle D \rangle)$  an approximate expression*

$$(a) \quad \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}(z, [\xi]) \simeq \omega_{r,k,b}(\xi) + g_{V,D,k}(z, x, u),$$

*where  $x_s = |\xi_s|^{2b/s}$ ,  $u_s = \frac{\xi_s}{|\xi_s|} \in \mathbb{S}^{2r-1}$  are polar coordinates associated with  $\xi_s = (\xi_{s,\lambda})_{1 \leq \lambda \leq k}$  in  $\mathbb{C}^r$ ,  $x = (x_1, \dots, x_k) \in \Delta^{k-1}$ ,  $[\xi] = [\xi_1, \dots, \xi_k] \in \mathbb{P}(1^{[r]}, 2^{[r]}, \dots, k^{[r]})$  and*

$$(b) \quad g_{V,D,k}(z, x, u) = \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{x_s}{s} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}^{(s)}(z) u_{s,\lambda} \bar{u}_{s,\mu} dz_i \wedge d\bar{z}_j.$$

Here  $(\frac{i}{2\pi} c_{ij\lambda\mu}^{(s)})$  are the coefficients of the curvature tensor  $-\Theta_{V\langle D^{(s)} \rangle, \tilde{h}_s}$ , and the error terms are  $O(\max_{2 \leq s \leq k} (\varepsilon_s / \varepsilon_{s-1})^s)$ , uniformly on the projectivized orbifold variety  $X_k(V\langle D \rangle)$ .

Notice, as is clear from the expressions (3.16''), (3.17) and the fact that  $v_j = z_j v_j^{\text{orb}}$ , that our orbifold Finsler metrics always have fiberwise positive curvature, equal to  $\omega_{k,r,b}(\xi)$ , along the fibers of  $X_k(V\langle D \rangle) \rightarrow X$  (even after taking into account the so-called error terms, because fiberwise, the functions under consideration are just sums of even powers  $|\tilde{\xi}_s^{\text{orb}}|^{2b/s}$  in suitable  $k$ -jet components, and are therefore plurisubharmonic.)

## 4. Existence theorems for jet differentials

### 4.A. Expression of the Morse integral

Thanks to the uniform approximation provided by proposition 3.20, we can (and will) neglect the  $O(\varepsilon_s / \varepsilon_{s-1})$  error terms in our calculations. Since  $\omega_{r,k,b}$  is positive definite on the fibers of  $X_k(V\langle D \rangle) \rightarrow X$  (at least outside of the axes  $\xi_s = 0$ ), the index of the  $(1, 1)$  curvature form  $\Theta_{L_k, \Psi_{h,b,\varepsilon}^*}(z, [\xi])$  is equal to the index of the  $(1, 1)$ -form  $g_{V,D,k}(z, x, u)$ . By the binomial formula, the  $q$ -index integral of  $(L_k, \Psi_{h,b,\varepsilon}^*)$  on  $X_k(V\langle D \rangle)$  is therefore equal to

$$(4.1) \quad \int_{X_k(V\langle D \rangle)(L_k, q)} \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}^{n+kr-1} = \frac{(n+kr-1)!}{n!(kr-1)!} \int_{z \in X} \int_{\xi \in \mathbb{P}(1^{[r]}, \dots, k^{[r]})} \omega_{r,k,b}^{kr-1}(\xi) \wedge \mathbb{1}_{g_{V,D,k},q}(z, x, u) g_{V,D,k}(z, x, u)^n$$

where  $\mathbb{1}_{g_{V,D,k},q}(z, x, u)$  is the characteristic function of the open set of points where  $g_{V,D,k}(z, x, u)$  has signature  $(n-q, q)$  in terms of the  $dz_j$ 's. Notice that since  $g_{V,D,k}(z, x, u)^n$  is a determinant, the product  $\mathbb{1}_{g_{V,D,k},q}(z, x, u) g_{V,D,k}(z, x, u)^n$  gives rise to a continuous function on  $X_k(V\langle D \rangle)$ . By Formula 3.10 (b), we get

$$(4.2) \quad \int_{X_k(V\langle D \rangle)(L_k, q)} \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}^{n+kr-1} = \frac{(n+kr-1)!}{n! k!^r (kr-1)!} \times \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times (\mathbb{S}^{2r-1})^k} \mathbb{1}_{g_{V,D,k},q}(z, x, u) g_{V,D,k}(z, x, u)^n d\nu_{k,r}(x) d\mu(u).$$

### 4.B. Probabilistic estimate of cohomology groups

We assume here that we are either in the “compact” case ( $D = 0$ ), or in the logarithmic case ( $\rho_j = \infty$ ). Then the curvature coefficients  $c_{ij\lambda\mu}^{(s)} = c_{ij\lambda\mu}$  do not depend on  $s$  and are those of the dual bundle  $V^*$  (resp.  $V^*\langle D \rangle$ ). In this situation, formula 3.20 (b) for  $g_{V,D,k}(z, x, u)$  can be thought of as a “Monte Carlo” evaluation of the curvature tensor, obtained by averaging the curvature at random points  $u_s \in \mathbb{S}^{2r-1}$  with certain positive weights  $x_s/s$ ; we then think of the  $k$ -jet  $f$  as some sort of random variable such that the derivatives  $\nabla^k f(0)$  (resp. logarithmic derivatives) are uniformly distributed in all directions. Let us compute the expected value of  $(x, u) \mapsto g_{V,D,k}(z, x, u)$  with respect to the probability measure  $d\nu_{k,r}(x) d\mu(u)$ . Since  $\int_{\mathbb{S}^{2r-1}} u_{s,\lambda} \bar{u}_{s,\mu} d\mu(u_s) = \frac{1}{r} \delta_{\lambda\mu}$  and  $\int_{\Delta^{k-1}} x_s d\nu_{k,r}(x) = \frac{1}{k}$ , we find

$$\mathbf{E}(g_{V,D,k}(z, \bullet, \bullet)) = \frac{1}{kr} \sum_{1 \leq s \leq k} \frac{1}{s} \cdot \frac{i}{2\pi} \sum_{i,j,\lambda} c_{ij\lambda\lambda}(z) dz_i \wedge d\bar{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$(4.3) \quad \mathbf{E}(g_{V,D,k}(z, \bullet, \bullet)) = \frac{1}{kr} \left( 1 + \frac{1}{2} + \dots + \frac{1}{k} \right) \Theta_{\det(V^*\langle D \rangle), \det h^*},$$

where  $\Theta_{\det(V^*\langle D \rangle), \det h^*}$  is the  $(1, 1)$ -curvature form of  $\det(V^*\langle D \rangle)$  with the metric induced by  $h$ . It is natural to guess that  $g_{V,D,k}(z, x, u)$  behaves asymptotically as its expected value  $\mathbf{E}(g_{V,D,k}(z, \bullet, \bullet))$  when  $k$  tends to infinity. If we replace brutally  $g_{V,D,k}$  by its expected value in (4.2), we get the integral

$$\frac{(n + kr - 1)!}{n! k!^r (kr - 1)!} \frac{1}{(kr)^n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^n \int_X \mathbb{1}_{\eta,q} \eta^n,$$

where  $\eta := \Theta_{\det(V^*\langle D \rangle), \det h^*}$  and  $\mathbb{1}_{\eta,q}$  is the characteristic function of its  $q$ -index set in  $X$ . The leading constant is equivalent to  $(\log k)^n / n! k!^r$  modulo a multiplicative factor  $1 + O(1/\log k)$ . By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] in the compact case; the more general logarithmic case can be treated without any change, so we state the result in this situation by just transposing the results of [Dem11].

**4.4. Probabilistic estimate.** *Let  $(X, V, D)$  be a non singular logarithmic directed variety. Fix smooth hermitian metrics  $\omega$  on  $T_X$ ,  $h$  on  $V\langle D \rangle$ , and write  $\omega = \frac{i}{2\pi} \sum \omega_{ij} dz_i \wedge d\bar{z}_j$  on  $X$ . Denote by  $\Theta_{V\langle D \rangle, h} = -\frac{i}{2\pi} \sum c_{ij\lambda\mu} dz_i \wedge d\bar{z}_j \otimes e_\lambda^* \otimes e_\mu$  the curvature tensor of  $V\langle D \rangle$  with respect to an  $h$ -orthonormal frame  $(e_\lambda)$ , and put*

$$\eta(z) := \Theta_{\det(V^*\langle D \rangle), \det h^*} = \frac{i}{2\pi} \sum_{1 \leq i, j \leq n} \eta_{ij} dz_i \wedge d\bar{z}_j, \quad \eta_{ij} := \sum_{1 \leq \lambda \leq r} c_{ij\lambda\lambda}.$$

Finally consider the  $k$ -jet line bundle  $L_k = \mathcal{O}_{X_k(V\langle D \rangle)}(1) \rightarrow X_k(V\langle D \rangle)$  equipped with the induced metric  $\Psi_{h,b,\varepsilon}^*$  (as defined above, with  $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k > 0$ ). When  $k$  tends to infinity, the integral of the top power of the curvature of  $L_k$  on its  $q$ -index set  $X_k(V\langle D \rangle)(L_k, q)$  is given by

$$\int_{X_k(V\langle D \rangle)(L_k, q)} \Theta_{L_k, \Psi_{h,b,\varepsilon}^*}^{n+kr-1} = \frac{(\log k)^n}{n! k!^r} \left( \int_X \mathbb{1}_{\eta,q} \eta^n + O((\log k)^{-1}) \right)$$

for all  $q = 0, 1, \dots, n$ , and the error term  $O((\log k)^{-1})$  can be bounded explicitly in terms of  $\Theta_{V\langle D \rangle}$ ,  $\eta$  and  $\omega$ . Moreover, the left hand side is identically zero for  $q > n$ .

The final statement follows from the observation that the curvature of  $L_k$  is positive along the fibers of  $X_k(V\langle D \rangle) \rightarrow X$ , by the plurisubharmonicity of the weight (this is true even when the error terms are taken into account, since they depend only on the base); therefore the  $q$ -index sets are empty for  $q > n$ . It will be useful to extend the above estimates to the case of sections of

$$(4.5) \quad L_{F,k} = \mathcal{O}_{X_k(V\langle D \rangle)}(1) \otimes \pi_k^* \mathcal{O} \left( -\frac{1}{kr} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) F \right)$$

where  $F \in \text{Pic}_{\mathbb{Q}}(X)$  is an arbitrary  $\mathbb{Q}$ -line bundle on  $X$  and  $\pi_k : X_k(V\langle D \rangle) \rightarrow X$  is the natural projection. We assume here that  $F$  is also equipped with a smooth hermitian metric  $h_F$ . In formulas (4.2–4.4), the curvature  $\Theta_{L_{F,k}}$  of  $L_{F,k}$  takes the form  $\Theta_{L_{F,k}} = \omega_{r,k,b}(\xi) + g_{V,D,F,k}(z, x, u)$  where

$$(4.6) \quad g_{V,D,F,k}(z, x, u) = g_{V,D,k}(z, x, u) - \frac{1}{kr} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \Theta_{F, h_F}(z),$$

and by the same calculations its normalized expected value is

$$(4.7) \quad \eta_F(z) := \frac{1}{\frac{1}{kr}(1 + \frac{1}{2} + \dots + \frac{1}{k})} \mathbf{E}(g_{V,D,F,k}(z, \bullet, \bullet)) = \Theta_{\det V^*\langle D \rangle, \det h^*}(z) - \Theta_{F, h_F}(z).$$

Then the variance estimate for  $g_{V,D,F,k}$  is the same as the variance estimate for  $g_{V,D,k}$ , and the recentered  $L^p$  bounds are still valid, since our forms are just shifted by adding the constant smooth term  $\Theta_{F, h_F}(z)$ . The probabilistic estimate 4.4 is therefore still true in exactly the same form for  $L_{F,k}$ , provided we use  $g_{V,D,F,k}$  and  $\eta_F$  instead of  $g_{V,D,k}$  and  $\eta$ . An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q(X, E_{k,m} V^*\langle D \rangle \otimes \mathcal{O}\left(-\frac{m}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right)) \\ = h^q(X_k(V\langle D \rangle), \mathcal{O}_{X_k(V\langle D \rangle)}(m) \otimes \pi_k^* \mathcal{O}\left(-\frac{m}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right)), \end{aligned}$$

provided  $m$  is sufficiently divisible to give a multiple of  $F$  which is a  $\mathbb{Z}$ -line bundle.

**4.8. Theorem.** *Let  $(X, V\langle D \rangle)$  be a non singular logarithmic directed variety,  $F \rightarrow X$  a  $\mathbb{Q}$ -line bundle,  $(V\langle D \rangle, h)$  and  $(F, h_F)$  smooth hermitian structure on  $V\langle D \rangle$  and  $F$  respectively. We define*

$$\begin{aligned} L_{F,k} &= \mathcal{O}_{X_k(V\langle D \rangle)}(1) \otimes \pi_k^* \mathcal{O}\left(-\frac{1}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right), \\ \eta_F &= \Theta_{\det V^*\langle D \rangle, \det h^*} - \Theta_{F, h_F} = \Theta_{\det V^*\langle D \rangle \otimes F^{-1}, \det h^*}. \end{aligned}$$

Then for all  $q \geq 0$  and all  $m \gg k \gg 1$  such that  $m$  is sufficiently divisible, we have

$$\begin{aligned} (a) \quad h^q(X_k(V\langle D \rangle), \mathcal{O}(L_{F,k}^{\otimes m})) &\leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! k!^r} \left( \int_{X(\eta_F, q)} (-1)^q \eta_F^n + O((\log k)^{-1}) \right), \\ (b) \quad h^0(X_k(V\langle D \rangle), \mathcal{O}(L_{F,k}^{\otimes m})) &\geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! k!^r} \left( \int_{X(\eta_F, \leq 1)} \eta_F^n - O((\log k)^{-1}) \right), \\ (c) \quad \chi(X_k(V\langle D \rangle), \mathcal{O}(L_{F,k}^{\otimes m})) &= \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! k!^r} (c_1(V^*\langle D \rangle \otimes F)^n + O((\log k)^{-1})). \end{aligned}$$

Green and Griffiths [GrGr80] already checked the Riemann-Roch calculation (4.8c) in the special case  $D = 0$ ,  $V = T_X^*$  and  $F = \mathcal{O}_X$ . Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies  $\chi = h^0 - h^1 + h^2 \leq h^0 + h^2$ , hence it is enough to get the vanishing of the top cohomology group  $H^2$  to infer  $h^0 \geq \chi$ ; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\left(X, E_{k,m} T_X^* \otimes \mathcal{O}\left(-\frac{m}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right)\right) = 0$$

as soon as  $K_X \otimes F^{-1}$  is big and  $m \gg 1$ .

In fact, thanks to Bonavero's singular holomorphic Morse inequalities (Theorem 2.9, cf. [Bon93]), everything works almost unchanged in the case where the metric  $h$  on  $V$  is taken to

a product  $h = h_\infty e^\varphi$  of a smooth metric  $h_\infty$  by the exponential of a quasi-plurisubharmonic weight  $\varphi$  with analytic singularities (so that  $\det(h^*) = \det(h_\infty^*) e^{-r\varphi}$ ). Then  $\eta$  is a  $(1,1)$ -current with logarithmic poles, and we just have to twist our cohomology groups by the appropriate multiplier ideal sheaves  $\mathcal{I}_{k,m}$  associated with the weight  $\frac{1}{k}(1 + \frac{1}{2} + \dots + \frac{1}{k})m\varphi$ , since this is the multiple of  $\det V^*$  that occurs in the calculation, up to the factor  $\frac{1}{r} \times r\varphi$ . The corresponding Morse integrals need only be evaluated in the complement of the poles, i.e., on  $X(\eta, q) \setminus S$  where  $S = \text{Sing}(\varphi)$ . Since

$$(\pi_k)_*(\mathcal{O}(L_{F,k}^{\otimes m}) \otimes \mathcal{I}_{k,m}) \subset E_{k,m} V^* \otimes \mathcal{O}\left(-\frac{m}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right)$$

we still get a lower bound for the  $H^0$  of the latter sheaf (or for the  $H^0$  of the un-twisted line bundle  $\mathcal{O}(L_k^{\otimes m})$  on  $X_k(V)$ ). If we assume that  $K_V \otimes F^{-1}$  is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of  $X$ .

**4.9. Corollary.** *If  $F$  is an arbitrary  $\mathbb{Q}$ -line bundle over  $X$ , one has*

$$\begin{aligned} h^0\left(X_k(V), \mathcal{O}_{X_k(V)}(m) \otimes \pi_k^* \mathcal{O}\left(-\frac{m}{kr}\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)F\right)\right) \\ \geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! k!^r} \left(\text{Vol}(K_V \otimes F^{-1}) - O((\log k)^{-1})\right) - o(m^{n+kr-1}), \end{aligned}$$

when  $m \gg k \gg 1$ , in particular there are many sections of the  $k$ -jet differentials of degree  $m$  twisted by the appropriate power of  $F$  if  $K_V \otimes F^{-1}$  is big.

*Proof.* The volume is computed here as usual, i.e. after performing a suitable modification  $\mu : \tilde{X} \rightarrow X$  which converts  $K_V$  into an invertible sheaf. There is of course nothing to prove if  $K_V \otimes F^{-1}$  is not big, so we can assume  $\text{Vol}(K_V \otimes F^{-1}) > 0$ . Let us fix smooth hermitian metrics  $h_0$  on  $T_X$  and  $h_F$  on  $F$ . They induce a metric  $\mu^*(\det h_0^{-1} \otimes h_F^{-1})$  on  $\mu^*(K_V \otimes F^{-1})$  which, by our definition of  $K_V$ , is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every  $\delta > 0$ , one can find a modification  $\mu_\delta : \tilde{X}_\delta \rightarrow X$  dominating  $\mu$  such that

$$\mu_\delta^*(K_V \otimes F^{-1}) = \mathcal{O}_{\tilde{X}_\delta}(A + E)$$

where  $A$  and  $E$  are  $\mathbb{Q}$ -divisors,  $A$  ample and  $E$  effective, with

$$\text{Vol}(A) = A^n \geq \text{Vol}(K_V \otimes F^{-1}) - \delta.$$

If we take a smooth metric  $h_A$  with positive definite curvature form  $\Theta_{A,h_A}$ , then we get a singular hermitian metric  $h_A h_E$  on  $\mu_\delta^*(K_V \otimes F)$  with poles along  $E$ , i.e. the quotient  $h_A h_E / \mu^*(\det h_0^{-1} \otimes h_F)$  is of the form  $e^{-\varphi}$  where  $\varphi$  is quasi-psh with log poles  $\log |\sigma_E|^2 \pmod{C^\infty(\tilde{X}_\delta)}$  precisely given by the divisor  $E$ . We then only need to take the singular metric  $h$  on  $T_X$  defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor  $\frac{1}{r}$  is there to correct adequately the metric on  $\det V$ ). By construction  $h$  induces an admissible metric on  $V$  and the resulting curvature current  $\eta_F = \Theta_{K_V, \det h^*} - \Theta_{F, h_F}$  is such that

$$\mu_\delta^* \eta_F = \Theta_{A, h_A} + [E], \quad [E] = \text{current of integration on } E.$$



Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0) \setminus S} \eta_F^n = \int_{\tilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geq \text{Vol}(K_V \otimes F^{-1}) - \delta$$

and Corollary 4.9 follows from the fact that  $\delta$  can be taken arbitrary small.  $\square$

**4.10. Remark.** Since the probability estimate requires  $k$  to be very large, and since all non logarithmic components disappear from  $D^{(s)}$  when  $s$  is large, the above lower bound does not work in the general orbifold case. In that case, one can only hope to get an interesting result when  $k$  is fixed and not too large. This is what we will do in § 6.

## 5. Curvature of orbifold tangent bundles

### 5.A. Positivity concepts for vector bundles

Let  $E \rightarrow X$  be a holomorphic vector bundle equipped with a hermitian metric. Then  $E$  possesses a uniquely defined Chern connection  $\nabla_h$  compatible with  $h$  and such that  $\nabla_h^{0,1} = \bar{\partial}$ . The curvature tensor of  $(E, h)$  is defined to be

$$(5.1) \quad \Theta_{E,h} := \frac{i}{2\pi} \partial \bar{\partial} \nabla_h^2 \in C^\infty(X, \Lambda^{1,1} T_X^* \otimes \text{Hom}(E, E)).$$

One can then associate bijectively to  $\Theta_{E,h}$  a hermitian form  $\tilde{\Theta}_{E,h}$  on  $TX \otimes E$ , such that

$$(5.2) \quad \tilde{\Theta}_{E,h}(\xi \otimes u, \xi \otimes u) = \langle \Theta_{E,h}(\xi, \xi) \cdot u, u \rangle_h.$$

and can be written

$$\Theta_{E,h} = \frac{i}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} dz_i \wedge d\bar{z}_j \otimes e_\lambda^* \otimes e_\mu$$

Let  $(z_1, \dots, z_n)$  be a holomorphic coordinate system and  $(e_\lambda)_{1 \leq \lambda \leq r}$  a smooth frame of  $E$ . If  $(e_\lambda)$  is chosen to be orthonormal, then we can write

$$(5.3) \quad \Theta_{E,h} = \frac{i}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} dz_i \wedge d\bar{z}_j \otimes e_\lambda^* \otimes e_\mu,$$

$$(5.3') \quad \tilde{\Theta}_{E,h}(\xi \otimes u, \xi \otimes u) = \frac{1}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} \xi_i \bar{\xi}_j u_\lambda \bar{u}_\mu,$$

and more generally  $\tilde{\Theta}_{E,h}(\tau, \tau) = \frac{1}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu} \tau_{i\lambda} \bar{\tau}_{j\mu}$  for every tensor  $\tau \in T_X \otimes E$ . We now consider three concepts of (semi-)positivity, the first two being very classical.

**5.4. Definition.** Let  $\theta$  be a hermitian form on a tensor product  $T \otimes E$  of complex vector spaces. We say that

- (a)  $\theta$  is Griffiths semi-positive if  $\theta(\xi \otimes u, \xi \otimes u) \geq 0$  for every  $\xi \in T$  and every  $u \in E$ ;
- (b)  $\theta$  is Nakano semi-positive if  $\theta(\tau, \tau) \geq 0$  for every  $\tau \in T \otimes E$ ;
- (c)  $\theta$  is strongly semi-positive if there exist a finite collection of linear forms  $\alpha_j \in T^*$ ,  $\psi_j \in E^*$  such that  $\theta = \sum_j |\alpha_j \otimes \psi_j|^2$ , i.e.

$$\theta(\tau, \tau) = \sum_j |(\alpha_j \otimes \psi_j) \cdot \tau|^2, \quad \forall \tau \in T \otimes E.$$

*Semi-negativity concepts are introduced in a similar way.*

- (d) *We say that the hermitian bundle  $(E, h)$  is Griffiths semi-positive, resp. Nakano semi-positive, resp. strongly semi-positive, if  $\tilde{\Theta}_{E,h}(x) \in \text{Herm}(T_{X,x} \otimes E_x)$  satisfies the corresponding property for every point  $x \in X$ .*
- (e) *(Strict) Griffiths positivity means that  $\tilde{\Theta}_{E,h}(\xi \otimes u, \xi \otimes u) > 0$  for every non zero vectors  $\xi \in T_{X,x}$ ,  $u \in E_x$ .*
- (f) *(Strict) strong positivity means that at every point  $x \in X$  we can decompose  $\tilde{\Theta}_{E,h}$  as  $\tilde{\Theta}_{E,h} = \sum_j |\alpha_j \otimes \psi_j|^2$  where  $\text{Span}(\alpha_j \otimes \psi_j) = T_{X,x}^* \otimes E_x^*$ .*

We will denote respectively by  $\geq_G, \geq_N, \geq_S$  (and  $>_G, >_N, >_S$ ) the Griffiths, Nakano, strong (semi-)positivity relations. It is obvious that

$$\theta \geq_S 0 \Rightarrow \theta \geq_N 0 \Rightarrow \theta \geq_G 0,$$

and one can show that the reverse implications do not hold when  $\dim T > 1$  and  $\dim E > 1$ . The following result from [Dem80] will be useful.

**5.5. Proposition.** *Let  $\theta \in \text{Herm}(T \otimes E)$ , where  $(E, h)$  is a hermitian vector space. We define  $\text{Tr}_E(\theta) \in \text{Herm}(T)$  to be the hermitian form such that*

$$\text{Tr}_E(\theta)(\xi, \xi) = \sum_{1 \leq \lambda \leq r} \theta(\xi \otimes e_\lambda, \xi \otimes e_\lambda)$$

where  $(e_\lambda)_{1 \leq \lambda \leq r}$  is an arbitrary orthonormal basis of  $E$ . Then

$$\theta \geq_G 0 \implies \theta + \text{Tr}_E(\theta) \otimes h \geq_S 0.$$

As a consequence, if  $(E, h)$  is a Griffiths (semi-)positive vector bundle, then the tensor product  $(E \otimes \det E, h \otimes \det(h))$  is strongly (semi-)positive.

*Proof.* Since [Dem80] is written in French and perhaps not so easy to find, we repeat here briefly the arguments. They are based on a Fourier inversion formula for discrete Fourier transforms.

**5.6. Lemma.** *Let  $q$  be an integer  $\geq 3$ , and  $x_\alpha, y_\beta$ ,  $1 \leq \alpha, \beta \leq r$ , be complex numbers. Let  $\chi$  describe the set  $U_q^r$  of  $r$ -tuples of  $q$ -th roots of unity and put*

$$\hat{x}(\chi) = \sum_{1 \leq \alpha \leq r} x_\alpha \bar{\chi}_\alpha, \quad \hat{y}(\chi) = \sum_{1 \leq \beta \leq r} y_\beta \bar{\chi}_\beta, \quad \chi \in U_q^r.$$

Then for every pair  $(\lambda, \mu)$ ,  $1 \leq \lambda, \mu \leq r$ , the following identity holds:

$$q^{-r} \sum_{\chi \in U_q^r} \hat{x}(\chi) \overline{\hat{y}(\chi)} \chi_\lambda \bar{\chi}_\mu = \begin{cases} x_\lambda \bar{y}_\mu & \text{if } \lambda \neq \mu, \\ \sum_{1 \leq \alpha \leq r} x_\alpha \bar{y}_\alpha & \text{if } \lambda = \mu. \end{cases}$$

In fact, the coefficient of  $x_\alpha \bar{y}_\beta$  in the summation  $q^{-r} \sum_{\chi \in U_q^r} \hat{x}(\chi) \overline{\hat{y}(\chi)} \chi_\lambda \bar{\chi}_\mu$  is given by

$$q^{-r} \sum_{\chi \in U_q^r} \chi_\alpha \bar{\chi}_\beta \bar{\chi}_\lambda \chi_\mu,$$

so it is equal to 1 when the pairs  $\{\alpha, \mu\}$  and  $\{\beta, \lambda\}$  coincide, and is equal to 0 otherwise. The identity stated in Lemma 5.6 follows immediately.  $\square$

Now, let  $(t_j)_{1 \leq j \leq n}$  be a basis of  $T$ ,  $(e_\lambda)_{1 \leq \lambda \leq r}$  an orthonormal basis of  $E$  and  $\xi = \sum_j \xi_j t_j \in T$ ,  $w = \sum_{j,\lambda} w_{j\lambda} t_j \otimes e_\lambda \in T \otimes E$ . The coefficients  $c_{jk\lambda\mu}$  of  $\theta$  with respect to the basis  $t_j \otimes e_\lambda$  satisfy the symmetry relation  $\bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$ , and we have the formulas

$$\begin{aligned} \theta(w, w) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} w_{j\lambda} \bar{w}_{k\mu}, \quad \text{Tr}_E \theta(\xi, \xi) = \sum_{j,k,\lambda} c_{jk\lambda\lambda} \xi_j \bar{\xi}_k, \\ (\theta + \text{Tr}_E \theta \otimes h)(w, w) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} w_{j\lambda} \bar{w}_{k\mu} + c_{jk\lambda\lambda} w_{j\mu} \bar{w}_{k\mu}. \end{aligned}$$

For every  $\chi \in U_q^r$ , let us put

$$\hat{w}_j(\chi) = \sum_\alpha w_{j\alpha} \bar{\chi}_\alpha, \quad \hat{w}(\chi) = \sum_j \hat{w}_j(\chi) t_j \in T, \quad \hat{e}_\chi = \sum_\lambda \chi_\lambda e_\lambda \in E.$$

Lemma 5.6 implies

$$\begin{aligned} q^{-r} \sum_{\chi \in U_q^r} \theta(\hat{w}(\chi) \otimes \hat{e}_\chi, \hat{w}(\chi) \otimes \hat{e}_\chi) &= q^{-r} \sum_{\chi \in U_q^r} \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} \hat{w}_j(\chi) \overline{\hat{w}_k(\chi)} \chi_\lambda \bar{\chi}_\mu \\ &= \sum_{j,k,\lambda \neq \mu} c_{jk\lambda\mu} w_{j\lambda} \bar{w}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} w_{j\mu} \bar{w}_{k\mu}. \end{aligned}$$

The Griffiths positivity assumption  $\theta_G \geq 0$  shows that  $\xi \mapsto q^{-r} \theta(\xi \otimes \hat{e}_\chi, \xi \otimes \hat{e}_\chi)$  is a semi-positive hermitian form on  $T$ , hence there are linear forms  $\ell_{\chi,j} \in T^*$  such that  $q^{-r} \theta(\xi \otimes \hat{e}_\chi, \xi \otimes \hat{e}_\chi) = \sum_j |\ell_{\chi,j}(\xi)|^2$  for all  $\xi \in T$ . Similarly, there are  $\ell'_{\lambda,j} \in T^*$  such that

$$\sum_{j,k} c_{jk\lambda\lambda} \xi_j \bar{\xi}_k = \sum_j |\ell'_{\lambda,j}(\xi)|^2, \quad \text{for all } \lambda = 1, \dots, r.$$

Our final Fourier identity can be rewritten

$$\begin{aligned} (\theta + \text{Tr}_E \theta \otimes h)(w, w) &= \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} w_{j\lambda} \bar{w}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} w_{j\mu} \bar{w}_{k\mu} \\ &= q^{-r} \sum_{\chi \in U_q^r} \theta(\hat{w}(\chi) \otimes \hat{e}_\chi, \hat{w}(\chi) \otimes \hat{e}_\chi) + \sum_{j,k,\lambda} c_{jk\lambda\lambda} w_{j\lambda} \bar{w}_{j\lambda} \\ &= \sum_{\chi \in U_q^r} \sum_j |\ell_{\chi,j}(\hat{w}(\chi))|^2 + \sum_{j,\lambda} |\ell'_{\lambda,j}(w_{\bullet,\lambda})|^2 \\ &= \sum_{\chi \in U_q^r} \sum_j |\ell_{\chi,j} \otimes \chi^*(w)|^2 + \sum_{j,\lambda} |\ell'_{\lambda,j} \otimes e_\lambda^*(w)|^2 \end{aligned}$$

where  $\chi^* = \langle \bullet, \chi \rangle \in E^*$ , thus  $\theta + \text{Tr}_E \theta \otimes h \geq_S 0$ .  $\square$

**5.7. Corollary.** *Let  $r = \dim E$  and  $\Theta \in \text{Herm}(T \otimes E)$ .*

(a) *If  $\theta \geq_G 0$ , then  $-\text{Tr}_E \theta \otimes h \leq_S \theta \leq_S r \text{Tr}_E \theta \otimes h$ .*

(b) *If  $\theta \leq_G 0$ , then  $-r \text{Tr}_E(-\theta) \otimes h \leq_S \theta \leq_S \text{Tr}_E(-\theta) \otimes h$ .*

(c) If  $\pm\theta \leqslant_G \tau \otimes h$  where  $\tau \in \text{Herm}(T)$  is semi-positive, then

$$-(2r+1)\tau \otimes h \leqslant_S \theta \leqslant_S (2r+1)\tau \otimes h.$$

*Proof.* (a) It is easy to check that  $\theta' = \text{Tr}_E \theta \otimes h - \theta$  satisfies  $\theta' \geqslant_G 0$  and that we have  $\text{Tr}_E \theta' = (r-1)\text{Tr}_E \theta$ . Lemma 5.6 implies

$$\theta' + \text{Tr}_E \theta' \otimes h = r \text{Tr}_E \theta \otimes h - \theta \geqslant_S 0.$$

(b) follows from (a), after replacing  $\theta$  with  $-\theta$ .

(c) also follows from Lemma 5.6 by taking  $\theta' = \tau \otimes h + \theta$  (resp.  $\theta' = \tau \otimes h - \theta$ ), since  $\text{Tr}_E \theta \leqslant r\tau$  and we have e.g.

$$0 \leqslant_S \theta' + \text{Tr}_E \theta' \otimes h = \theta + \text{Tr}_E \theta \otimes h + (r+1)\tau \otimes h \leqslant_S \theta + (2r+1)\tau \otimes h. \quad \square$$

## 5.B. Estimate of the curvature tensor in the orbifold setting

The main qualitative result is summarized in the following statement.

**5.8. Proposition.** *Let  $X$  be a projective variety,  $A$  an ample line bundle, and  $(X, V, D)$  an orbifold directed structure where  $D = \sum (1 - \frac{1}{\rho_j})\Delta_j$  is a normal crossing divisor transverse to  $V$  in  $X$ . Let  $d_j$  be the infimum of numbers  $\lambda \in \mathbb{R}_+$  such that  $\lambda A - \Delta_j$  is nef, and  $\gamma_V$  be the infimum of numbers  $\gamma \geqslant 0$  such that  $\gamma \Theta_{A, h_A} \otimes \text{Id}_V - \Theta_{V, h_V} \geqslant_G 0$  for suitable hermitian metrics  $h_V$  on  $V$ . Then for every  $\gamma > \gamma_{V, D} := \max(\max_j(d_j/\rho_j), \gamma_V)$ , the orbifold vector bundle  $V\langle D \rangle$  possesses a hermitian metric  $h_{V\langle D \rangle, \gamma, \varepsilon}$  such that*

- (a)  $h_{V\langle D \rangle, \gamma, \varepsilon}$  is smooth on  $X \setminus |D|$ ,
- (b)  $h_{V\langle D \rangle, \gamma, \varepsilon}$  has the appropriate orbifold singularities along  $D$ ,
- (c) we have  $\gamma \Theta_{A, h_A} \otimes \text{Id} - \Theta_{V\langle D \rangle, h_{V\langle D \rangle, \gamma, \varepsilon}} \geqslant_G 0$  on  $X \setminus |D|$ .

*Proof.* Let  $h_A$  be a metric on  $A$  such that  $\Theta_{A, h_A} > 0$ , written locally as  $h_A = e^{-\psi}$ , and take  $\gamma > \max(\max_j(d_j/\rho_j), \gamma_V)$ . Consider the tautological sections  $\sigma_j \in H^0(X, \mathcal{O}_X(\Delta_j))$  defining  $\Delta_j = \sigma_j^{-1}(0)$ , and let  $h_j$  be a smooth hermitian metric on  $\mathcal{O}_X(\Delta_j)$  for which

$$(5.9) \quad \gamma \Theta_A - \frac{1}{\rho_j} \Theta_{\mathcal{O}_X(\Delta_j), h_j} > 0,$$

as is possible by our choice of constants  $d_j$  and  $\gamma$ . Finally, denote by  $\nabla_j$  the associated Chern connection on  $\mathcal{O}_X(\Delta_j)$ . If we write  $h_j = e^{-\varphi_j}$  in some local trivialization, then  $\nabla_j \sigma_j = \nabla_j^{1,0} \sigma_j = \partial \sigma_j - \sigma_j \partial \varphi_j$ . We are going to estimate the curvature of the orbifold metric  $h_{V\langle D \rangle, \varepsilon}$  on  $V\langle D \rangle$  defined by

$$(5.10) \quad \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 = |u|_{h_V}^2 + \sum_j \varepsilon_j |\sigma_j|_{h_j}^{-2(1-1/\rho_j)} |\nabla_j \sigma_j(u)|_{h_j}^2, \quad \varepsilon_j \ll 1,$$

where the metric  $h_V$  is chosen so that  $\gamma \Theta_{A, h_A} \otimes \text{Id}_V - \Theta_{V, h_V} \geqslant_G 0$  (resp.  $\geqslant_S 0$ ). We will later to slightly perturb the metric as  $\|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 e^{K\chi_\varepsilon}$  with an extra weight  $\chi_\varepsilon$ , but we ignore this minor twist for the time being. Since

$$\mathbf{i} \partial \bar{\partial} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 = \mathbf{i} \langle \nabla u, \nabla u \rangle_{h_{V\langle D \rangle, \varepsilon}} - 2\pi \langle \Theta_{V\langle D \rangle, h_{V\langle D \rangle, \varepsilon}}(u), u \rangle_{h_{V\langle D \rangle, \varepsilon}}$$

where  $\nabla u = du + \Gamma(dz) \cdot u$  is the Chern connection of  $(V\langle D \rangle, h_{V\langle D \rangle, \varepsilon})$ , what we need to prove is that on the total space of  $V$  over  $X \setminus |D|$ , the  $(1, 1)$ -form

$$(5.11) \quad Q_{\gamma, \varepsilon}(z, u) := i \partial \bar{\partial} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 + 2\pi \gamma \Theta_{A, h_A} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2,$$

is non negative. For this, we calculate the associated hermitian quadratic form on  $T_V$

$$\tilde{Q}_{\gamma, \varepsilon}(z, u)(\xi, \eta)^2, \quad (\xi, \eta) \in T_{V, (z, u)}, \quad \xi = \sum_{j=1}^n \xi \frac{\partial}{\partial z}, \quad \eta = \sum_{\lambda=1}^r \eta_\lambda \frac{\partial}{\partial u_\lambda},$$

and observe that the curvature tensor is obtained by taking the restriction to the “parallel” directions  $\nabla u = 0$ , that is, by substituting  $du = -\Gamma(dz) \cdot u$ , i.e.  $\eta = -\Gamma(\xi) \cdot u$ . Let us fix an arbitrary point  $z_0 \in X \setminus |D|$ . We take local holomorphic coordinates  $(z_1, \dots, z_n)$  centered at  $z_0$ , and let  $(e_1, \dots, e_r)$  be a local holomorphic frame of  $V$  such that

$$\langle e_\lambda, e_\mu \rangle_{h_V} = \delta_{\lambda\mu} + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} z_\ell \bar{z}_m + O(|z|^3),$$

where the  $\frac{i}{2\pi} c_{\ell m \lambda \mu}$  are the coefficients of  $-\Theta_{V, h_V}$ . Let us write  $u = \sum_{\lambda=1}^r u_\lambda e_\lambda$  and denote by  $\langle u, v \rangle = \sum_{1 \leq \lambda \leq r} u_\lambda \bar{v}_\lambda$  the standard hermitian form,  $|u|$  the associated norm. We find

$$(5.12) \quad \begin{aligned} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 &= |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} z_\ell \bar{z}_m u_\lambda \bar{u}_\mu + O(|z|^3) \\ &+ \sum_j \varepsilon_j (|\sigma_j|^2 e^{-\varphi_j})^{-1+1/\rho_j} |\partial \sigma_j(u) - \sigma_j \partial \varphi_j(u)|^2 e^{-\varphi_j}, \end{aligned}$$

since  $\bar{\partial} \sigma_j = 0$ . In order to simplify the calculation, we set formally

$$(5.13) \quad \begin{aligned} \tilde{\sigma}_j &= \sigma_j^{1/\rho_j}, \quad \tilde{\varepsilon}_j = \rho_j^2 \varepsilon_j, \quad \tilde{\varphi}_j = \rho_j^{-1} \varphi_j, & \text{if } \rho_j < \infty, \\ \tilde{\sigma}_j &= \log \sigma_j, \quad \tilde{\varepsilon}_j = \varepsilon_j, \quad \tilde{\varphi}_j = \varphi_j, & \text{if } \rho_j = \infty. \end{aligned}$$

Respectively to the non logarithmic and logarithmic situations, we then get the more tractable expression

$$(5.14) \quad \begin{aligned} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 &= |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} z_\ell \bar{z}_m u_\lambda \bar{u}_\mu + O(|z|^3) + \sum_j \tilde{\varepsilon}_j |\partial \tilde{\sigma}_j(u) - \sigma_j \partial \tilde{\varphi}_j(u)|^2 e^{-\tilde{\varphi}_j}, \\ \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 &= |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} z_\ell \bar{z}_m u_\lambda \bar{u}_\mu + O(|z|^3) + \sum_j \tilde{\varepsilon}_j |\partial \tilde{\sigma}_j(u) - \partial \tilde{\varphi}_j(u)|^2. \end{aligned}$$

In what follows, for the sake of simplicity, we remove the tildes in the notation, and conduct the calculation only in the non logarithmic situation ( $\rho_j < \infty$ ), since the logarithmic case can be recovered by taking  $\rho_j$  very large; this actually amounts to using a ramified change of variable  $\tilde{z}'_\ell = z_\ell^{1/\rho_\ell}$  in suitable coordinates, allowing us in this way to take  $\rho_j = 1$  in (5.12). We then obtain

$$(5.15) \quad \begin{aligned} \|u\|_{h_{V\langle D \rangle, \varepsilon}}^2 &= |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} z_\ell \bar{z}_m u_\lambda \bar{u}_\mu + O(|z|^3) \\ &+ \sum_j \left( |\partial \sigma_j(u)|^2 - 2 \operatorname{Re} (\partial \sigma_j(u) \bar{\sigma}_j \bar{\partial} \varphi_j(u)) + |\sigma_j|^2 |\partial \varphi_j(u)|^2 \right) e^{-\varphi_j}. \end{aligned}$$

We also take holomorphic trivializations of the line bundles  $\mathcal{O}_X(\Delta_j)$  so that the associated weight  $\varphi_j$  satisfies  $\varphi_j(z) = \sum_{\ell,m} \alpha_{j\ell m} z_\ell \bar{z}_m + O(|z|^3)$  near  $z_0 = 0$ . Then

$$\partial\varphi_j = \sum \alpha_{j\ell m} \bar{z}_m dz_\ell + O(|z|^2), \quad \bar{\partial}\varphi_j = \sum \alpha_{j\ell m} z_\ell d\bar{z}_m + O(|z|^2).$$

At the point  $z = z_0$ , we have  $\partial\varphi(z_0) = \bar{\partial}\varphi_j(z_0) = 0$ ,  $\nabla_j\sigma_j = \partial\sigma_j$ , and our norm admits the expression

$$(5.16) \quad \|u\|_{h_{V\langle D \rangle}, \varepsilon}^2 = |u|^2 + \sum_j \varepsilon_j |\partial\sigma_j(u)|^2.$$

Let  $u, v$  be arbitrary local holomorphic sections of  $V$ , and denote by  $\nabla_\xi$  the Chern covariant differentiation of  $(V\langle D \rangle, h_{V\langle D \rangle}, \varepsilon)$  in the direction  $\xi \in T_X$ . By polarizing the quadratic form  $\|u\|_{h_{V\langle D \rangle}, \varepsilon}^2$  into a hermitian inner product  $\partial_\xi \langle u, v \rangle_{h_{V\langle D \rangle}, \varepsilon}$  and setting  $\nabla_\xi u = \nabla_\xi^{1,0} u = \partial_\xi u + \Gamma(\xi) \cdot u$ , a differentiation of (5.15) at  $z = z_0$  yields

$$\begin{aligned} \partial_\xi \langle u, v \rangle_{h_{V\langle D \rangle}, \varepsilon} &= \langle \nabla_\xi u, v \rangle + \sum_j \varepsilon_j \partial\sigma_j(\nabla_\xi u) \overline{\partial\sigma_j(v)} \\ &= \langle \partial_\xi u, v \rangle + \sum_j \varepsilon_j \partial\sigma_j(\partial_\xi u) \overline{\partial\sigma_j(v)} \\ &\quad + \varepsilon_j \partial^2\sigma_j(\xi, u) \overline{\partial\sigma_j(v)} - \varepsilon_j \partial\sigma_j(u) \bar{\sigma}_j \partial\bar{\partial}\varphi_j(\xi, v), \end{aligned}$$

where  $\partial^2\sigma_j(\xi, u) := \sum_\lambda \partial_\xi(\partial\sigma_j(e_\lambda)) u_\lambda$  is viewed as an element of  $(T_X^* \otimes V^*)_{z_0}$  and  $\partial\bar{\partial}\varphi_j$  as a hermitian form on  $T_X$ , operating on  $T_X \otimes \bar{V} \subset T_X \otimes \bar{T}_X$ . In fact,  $u \mapsto \partial\sigma_j(u)$  and  $(\xi, u) \mapsto \partial^2\sigma_j(\xi, u)$  can be intrinsically defined as  $\nabla_j^{1,0}\sigma_j|_V$  and  $\nabla_{V^* \otimes A_j}^{1,0}(\nabla_j^{1,0}\sigma_j|_V)$  at  $z_0$ , and we will denote them by  $\nabla_j\sigma_j$  and  $\nabla_j^2\sigma_j$ . In this setting, the  $(1,0)$ -form  $\Gamma$  of the connection of  $(V\langle D \rangle, h_{V\langle D \rangle})$  is given at  $z_0$  by the formula

$$\begin{aligned} \langle \Gamma(\xi) \cdot u, v \rangle + \sum_j \varepsilon_j \nabla_j\sigma_j(\Gamma(\xi) \cdot u) \overline{\nabla_j\sigma_j(v)} \\ (5.17) \quad = \sum_j \varepsilon_j \nabla_j^2\sigma_j(\xi, u) \overline{\nabla_j\sigma_j(v)} - \varepsilon_j \nabla_j\sigma_j(u) \bar{\sigma}_j \partial\bar{\partial}\varphi_j(\xi, v). \end{aligned}$$

This equality is valid pointwise for any  $u, v \in V_{z_0}$ . As a consequence

$$\begin{aligned} \Gamma(\xi) \cdot u + \sum_j \varepsilon_j \nabla_j\sigma_j(\Gamma(\xi) \cdot u) (\nabla\sigma_j)^* \\ (5.18) \quad = \sum_j \varepsilon_j \nabla_j^2\sigma_j(\xi, u) (\nabla\sigma_j)^* - \varepsilon_j \nabla_j\sigma_j(u) \bar{\sigma}_j (\partial\bar{\partial}\varphi_j(\bullet, \xi))^* \end{aligned}$$

where  $\alpha^* \in V$  is the dual vector to a 1-form  $\alpha \in V^*$ , such that  $\langle \alpha^*, \bullet \rangle_{h_V} = \bar{\alpha}$ . The special choice  $w = \Gamma(\xi) \cdot u$  yields a (non negative) real value in the left hand side of (5.17), and by taking the real part of the right hand side, we obtain

$$\begin{aligned} |\Gamma(\xi) \cdot u|^2 + \sum_j \varepsilon_j |\nabla_j\sigma_j(\Gamma(\xi) \cdot u)|^2 \\ (5.19_0) \quad = \sum_j \varepsilon_j \operatorname{Re} (\nabla_j^2\sigma_j(\xi, u) \overline{\nabla_j\sigma_j(\Gamma(\xi) \cdot u)}) - \varepsilon_j \operatorname{Re} (\nabla_j\sigma_j(u) \bar{\sigma}_j \partial\bar{\partial}\varphi_j(\xi, \Gamma(\xi) \cdot u)). \end{aligned}$$

Also, by applying  $\nabla_j \sigma_j$  to (5.18), we obtain

$$(5.19_1) \quad \begin{aligned} & \nabla \sigma_j(\Gamma(\xi) \cdot u) + \sum_{\ell} \varepsilon_{\ell} \nabla_{\ell} \sigma_{\ell}(\Gamma(\xi) \cdot u) \langle \nabla_j \sigma_j, \nabla_{\ell} \sigma_{\ell} \rangle \\ &= \sum_{\ell} \varepsilon_{\ell} \nabla_{\ell}^2 \sigma_{\ell}(\xi, u) \langle \nabla_j \sigma_j, \nabla_{\ell} \sigma_{\ell} \rangle - \varepsilon_{\ell} \nabla_{\ell} \sigma_{\ell}(u) \bar{\sigma}_{\ell} \langle \nabla_j \sigma_j, \partial \bar{\partial} \varphi_{\ell}(\bullet, \xi) \rangle, \end{aligned}$$

in other terms,

$$(5.19_2) \quad (I + M_{\varepsilon})p_{\varepsilon} = M_{\varepsilon}p'_{\varepsilon} - p''_{\varepsilon} \quad \text{where}$$

$$(5.19_3) \quad p_{\varepsilon} = \left( \varepsilon_j^{1/2} \nabla_j \sigma_j(\Gamma(\xi) \cdot u) \right)_j, \quad p'_{\varepsilon} = \left( \varepsilon_j^{1/2} \nabla_j^2 \sigma_j(\xi, u) \right)_j,$$

$$(5.19_4) \quad p''_{\varepsilon} = \left( \sum_{\ell} \varepsilon_j^{1/2} \varepsilon_{\ell} \nabla_{\ell} \sigma_{\ell}(u) \bar{\sigma}_{\ell} \langle \nabla_j \sigma_j, \partial \bar{\partial} \varphi_{\ell}(\bullet, \xi) \rangle \right)_j, \quad M_{\varepsilon} = \left( \varepsilon_j^{1/2} \varepsilon_{\ell}^{1/2} \langle \nabla_j \sigma_j, \nabla_{\ell} \sigma_{\ell} \rangle \right)_{j,\ell}.$$

It will be useful to observe that  $M_{\varepsilon}$  is a semi-positive hermitian matrix. As  $2\pi \Theta_{A, h_A} = i \partial \bar{\partial} \psi$ , we infer by a brute force calculation from (5.15) that

$$\tilde{Q}_{\gamma, \varepsilon}(z, u)(\xi, \eta)^2 = \partial \bar{\partial} \|u\|_{h_{V\langle D \rangle, \varepsilon_j}}^2 \cdot (\xi, \eta)^2 + \gamma \partial \bar{\partial} \psi(\xi, \xi) \|u\|_{h_{V\langle D \rangle, \varepsilon_j}}^2$$

$$(5.20_1) \quad = \gamma \partial \bar{\partial} \psi(\xi, \xi) |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} \xi_{\ell} \bar{\xi}_m u_{\lambda} \bar{u}_{\mu}$$

$$(5.20_2) \quad + \sum_j \varepsilon_j (\gamma \partial \bar{\partial} \psi(\xi, \xi) - \partial \bar{\partial} \varphi_j(\xi, \xi)) |\nabla_j \sigma_j(u)|^2$$

$$(5.20_3) \quad + |\eta|^2 + \sum_j \varepsilon_j |\nabla_j \sigma_j(\eta) + \nabla_j^2 \sigma_j(\xi, u)|^2$$

$$(5.20_4) \quad - 2\varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \bar{\sigma}_j \partial \bar{\partial} \varphi_j(\xi, \eta))$$

$$(5.20_5) \quad - 2\varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \partial \bar{\partial} \varphi_j(\xi, u) \overline{\nabla_j \sigma_j(\xi)})$$

$$(5.20_6) \quad - 2\varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \bar{\sigma}_j \partial \bar{\partial}^2 \varphi_j(\xi, \xi, u))$$

$$(5.20_7) \quad + \varepsilon_j |\sigma_j|^2 |\partial \bar{\partial} \varphi_j(u, \xi)|^2,$$

where we identify a  $(1, 1)$ -form such as  $\partial \bar{\partial} \varphi_j$  with a hermitian form, and take  $\eta = -\Gamma(\xi) \cdot u$ . The second term in (5.20<sub>2</sub>) is obtained by differentiating  $\varepsilon_j |\nabla_j \sigma_j(u)|^2$ , while (5.20<sub>3</sub>), (5.20<sub>4</sub>) and (5.20<sub>5</sub>) actually come from the differentiation of the term  $\dots \operatorname{Re}(\dots)$  in (5.15). By our assumptions, the first two terms (5.20<sub>1</sub>), (5.20<sub>2</sub>) are positive (in the sense of Griffiths, at least), and such that

$$(5.20_1) \geq c |\xi|^2 |u|^2, \quad (5.20_2) \geq c |\xi|^2 \sum_j \varepsilon_j |\nabla_j \sigma_j(u)|^2, \quad c > 0.$$

In order to improve the positivity of  $\tilde{Q}_{\gamma, \varepsilon}$ , we actually replace the metric  $|u|_{h_{V\langle D \rangle, \varepsilon}}^2$  by  $|u|_{h_{V\langle D \rangle, \varepsilon, K}}^2 = |u|_{h_{V\langle D \rangle, \varepsilon}}^2 e^{K\chi_{\varepsilon}(z)}$  with

$$\chi_{\varepsilon}(z) = \sum_j \varepsilon_j |\sigma_j(z)|_{h_j}^2 = \sum_j \varepsilon_j |\sigma_j(z)|^2 e^{-\varphi_j(z)}, \quad K \gg 1.$$

At  $z = z_0$ , we then get in  $\tilde{Q}_{\gamma,\varepsilon}$  an additional term

$$\begin{aligned}
 (5.20_0) \quad & K \partial \bar{\partial} \chi_\varepsilon(\xi, \xi) |u|_{h_{V\langle D \rangle, \varepsilon}}^2 \\
 &= K \left( \sum_j \varepsilon_j |\nabla \sigma_j(\xi)|^2 - \sum_j \varepsilon_j |\sigma_j|^2 \partial \bar{\partial} \varphi_j(\xi, \xi) \right) \left( |u|^2 + \sum_j \varepsilon_j |\nabla \sigma_j(u)|^2 \right), \\
 &\geq K \sum_j \varepsilon_j |\nabla \sigma_j(\xi)|^2 |u|^2 - CK \max_j \varepsilon_j |\xi|^2 \left( |u|^2 + \sum_j \varepsilon_j |\nabla \sigma_j(u)|^2 \right),
 \end{aligned}$$

and, as a consequence, for  $\varepsilon_j \leq c/2CK$ , we have

$$\sum_{j=0,1,2} (5.20_j) \geq \frac{c}{2} \left( |\xi|^2 |u|^2 + \sum_j \varepsilon_j |\xi|^2 |\nabla \sigma_j(u)|^2 \right) + K \sum_j \varepsilon_j |\nabla \sigma_j(\xi)|^2 |u|^2.$$

(Here the last two summations are significant, because we will later replace  $\sigma_j$  by  $\sigma_j^{1/\rho_j}$  in the orbifold case, and then  $\nabla_j \sigma_j^{1/\rho_j}$  is unbounded). The third term (5.20<sub>3</sub>) is semi-positive. We claim that the terms (5.20<sub>4</sub>...5.20<sub>7</sub>) are negligible for  $\varepsilon_j \ll 1$ , in the sense that  $\tilde{Q}_{\gamma,\varepsilon}(z, u)(\xi, \eta)^2$  is comprised between  $(1 \pm \delta) \sum_{j=0,1,2,3} (5.20_j)$ , with  $\delta > 0$  as small as we want when  $\varepsilon_j \leq \varepsilon_0(\delta)$ . In fact, as  $\partial \bar{\partial} \varphi_j$  is smooth, there exists  $C > 0$  such that

$$\begin{aligned}
 |(5.20_4)| &\leq C \varepsilon_j |\sigma_j| |\nabla_j \sigma_j(u)| |\xi| |\eta| \\
 &\leq \varepsilon_j^{3/2} |\xi|^2 |\nabla_j \sigma_j(u)|^2 + C^2 \varepsilon_j^{1/2} |\sigma_j|^2 |\eta|^2 \ll (5.20_2) + (5.20_3).
 \end{aligned}$$

Similarly

$$\begin{aligned}
 |(5.20_5)| &\leq C \varepsilon_j |\xi| |u| |\nabla_j \sigma_j(\xi)| |\nabla_j \sigma_j(u)| \\
 &\leq K^{-1/2} \varepsilon_j |\xi|^2 |\nabla_j \sigma_j(u)|^2 + C^2 K^{1/2} \varepsilon_j |\nabla_j \sigma_j(\xi)|^2 |u|^2 \ll \sum_{j=0,1,2} (5.20_j)
 \end{aligned}$$

for  $K \gg 1$ . The last two terms (5.20<sub>6,7</sub>) are even easier, since

$$\begin{aligned}
 |(5.20_6)| &\leq C \varepsilon_j |\sigma_j| |\xi|^2 |u| |\nabla_j \sigma_j(u)| \leq \varepsilon_j^{1/2} |\xi|^2 |u|^2 + C^2 \varepsilon_j^{3/2} |\sigma_j|^2 |\xi|^2 |\nabla_j \sigma_j(u)|^2 \\
 &\ll (5.20_1) + (5.20_2), \\
 |(5.20_7)| &\leq C \varepsilon_j |\xi|^2 |u|^2 \ll (5.20_1).
 \end{aligned}$$

Finally, by replacing  $\eta$  with  $-\Gamma(\xi) \cdot u$  and using (5.19<sub>0</sub>), we find

$$\begin{aligned}
 (5.20_3) + (5.20_4) &= |\Gamma(\xi) \cdot u|^2 \\
 &\quad + \sum_j \varepsilon_j |\nabla_j \sigma_j(\Gamma(\xi) \cdot u) - \nabla_j^2 \sigma_j(\xi, u)|^2 + 2\varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \bar{\sigma}_j \partial \bar{\partial} \varphi_j(\xi, \Gamma(\xi) \cdot u)) \\
 &= (5.19_0) + \sum_j \varepsilon_j |\nabla_j^2 \sigma_j(\xi, u)|^2 - 2\varepsilon_j \operatorname{Re} (\nabla_j^2 \sigma_j(\xi, u) \overline{\nabla_j \sigma_j(\Gamma(\xi) \cdot u)}) \\
 &\quad + 2\varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \bar{\sigma}_j \partial \bar{\partial} \varphi_j(\xi, \Gamma(\xi) \cdot u)) \\
 &= \sum_j \varepsilon_j |\nabla_j^2 \sigma_j(\xi, u)|^2 - \varepsilon_j \operatorname{Re} (\nabla_j^2 \sigma_j(\xi, u) \overline{\nabla_j \sigma_j(\Gamma(\xi) \cdot u)}) \\
 &\quad + \varepsilon_j \operatorname{Re} (\nabla_j \sigma_j(u) \bar{\sigma}_j \partial \bar{\partial} \varphi_j(\xi, \Gamma(\xi) \cdot u)).
 \end{aligned}$$



The last term equals  $\frac{1}{2}(5.20_4)$ , thus it is negligible, and by (5.19<sub>2,3,4</sub>) we get

$$\begin{aligned} (5.20_3) + (5.20_4) &\simeq |p'_\varepsilon|^2 - \operatorname{Re}\langle p_\varepsilon, p'_\varepsilon \rangle \\ &= |p'_\varepsilon|^2 - \operatorname{Re}\langle (I + M_\varepsilon)^{-1}(M_\varepsilon p'_\varepsilon - p''_\varepsilon), p'_\varepsilon \rangle = \operatorname{Re}\langle (I + M_\varepsilon)^{-1}p'_\varepsilon, p'_\varepsilon + p''_\varepsilon \rangle. \\ &= |(I + M_\varepsilon)^{-1/2}p'_\varepsilon|^2 + \operatorname{Re}\langle (I + M_\varepsilon)^{-1}p'_\varepsilon, p''_\varepsilon \rangle. \end{aligned}$$

However, with  $\varepsilon = \max \varepsilon_j$ , we have

$$\begin{aligned} |\operatorname{Re}\langle (I + M_\varepsilon)^{-1}p'_\varepsilon, p''_\varepsilon \rangle| &\leq \varepsilon^{1/2} |(I + M_\varepsilon)^{-1/2}p'_\varepsilon|^2 + \varepsilon^{-1/2} |(I + M_\varepsilon)^{-1/2}p''_\varepsilon|^2 \\ &\leq \varepsilon^{1/2} |(I + M_\varepsilon)^{-1/2}p'_\varepsilon|^2 + \varepsilon^{-1/2} |M_\varepsilon^{-1/2}p''_\varepsilon|^2, \end{aligned}$$

and, for any  $t = (t_j)$ ,

$$\begin{aligned} |\langle p''_\varepsilon, t \rangle| &= \left| \sum_\ell \varepsilon_\ell \nabla_\ell \sigma_\ell(u) \bar{\sigma}_\ell \left\langle \sum_j \bar{t}_j \varepsilon_j^{1/2} \nabla_j \sigma_j, \partial \bar{\partial} \varphi_\ell(\bullet, \xi) \right\rangle \right| \\ &\leq \left| \sum_j \bar{t}_j \varepsilon_j^{1/2} \nabla_j \sigma_j \right| \sum_\ell \varepsilon_\ell |\nabla_\ell \sigma_\ell(u)| |\sigma_\ell| |\partial \bar{\partial} \varphi_\ell(\bullet, \xi)| \\ &\leq C \varepsilon^{1/2} \langle M_\varepsilon t, t \rangle^{1/2} \left( \sum_\ell \varepsilon_\ell |\xi|^2 |\nabla_\ell \sigma_\ell(u)|^2 \right)^{1/2}, \end{aligned}$$

whence

$$\varepsilon^{-1/2} |M_\varepsilon^{-1/2}p''_\varepsilon|^2 \leq C^2 \varepsilon^{1/2} \sum_\ell \varepsilon_\ell |\xi|^2 |\nabla_\ell \sigma_\ell(u)|^2 \ll (5.20_1),$$

and

$$(5.20_3) + (5.20_4) \simeq |(I + M_\varepsilon)^{-1/2}p'_\varepsilon|^2, \quad p'_\varepsilon = \left( \varepsilon_j^{1/2} \nabla_j^2 \sigma_j(\xi, u) \right)_j.$$

At this point, we come back to the orbifold situation, and thus replace  $\sigma_j$  by  $\sigma_j^{1/\rho_j}$ ,  $\varphi_j$  by  $\rho_j^{-1} \varphi_j$  and  $\varepsilon_j$  by  $\rho_j^2 \varepsilon_j$ . The vector  $p'_\varepsilon$  becomes

$$p'_\varepsilon = \left( \varepsilon_j^{1/2} \sigma_j^{-1+1/\rho_j} \nabla_j^2 \sigma_j(\xi, u) - \varepsilon_j^{1/2} (1 - 1/\rho_j) \sigma_j^{-2+1/\rho_j} \nabla_j \sigma_j(\xi) \nabla_j \sigma_j(u) \right)_j.$$

By collecting all non negligible terms (5.20<sub>i</sub>),  $i = 0, 1, 2, 3$ , we obtain

**5.21. Corollary.** *With a choice of  $\gamma > \gamma_{V,D} := \max(\max_j(\delta_j/\rho_j), \gamma_V) \geq 0$  determined by the curvature assumptions of Proposition 5.8 and the hermitian metric on  $(V, D)$  defined as above, i.e.*

$$(a) \quad |u|_{h_{V\langle D \rangle, \varepsilon, K}} := e^{K\chi_\varepsilon} \left( |u|_{h_V}^2 + \sum_j \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j(u)|_{h_j}^2 \right), \quad \chi_\varepsilon = \sum_j \varepsilon_j \rho_j^2 |\sigma_j|^{2/\rho_j},$$

with  $K \gg 1$  and  $\varepsilon_j \ll K^{-1}$ , the hermitian quadratic form associated with the curvature tensor  $\gamma \Theta_{A, h_A} \otimes \operatorname{Id} - \Theta_{V\langle D \rangle, h_{V\langle D \rangle}, \gamma}$  satisfies  $\tilde{Q}_{\gamma, \varepsilon, K}(z)(\xi \otimes u)^2 \simeq Q_{\gamma, \varepsilon, K}(z)(\xi \otimes u)^2$  where

$$\begin{aligned} (b) \quad Q_{\gamma, \varepsilon, K}(z)(\xi \otimes u)^2 &= \gamma \partial \bar{\partial} \psi(\xi, \xi) |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} \xi_\ell \bar{\xi}_m u_\lambda \bar{u}_\mu \\ &\quad + \sum_j \varepsilon_j |\sigma_j|^{-2+2/\rho_j} (\gamma \partial \bar{\partial} \psi(\xi, \xi) - \rho_j^{-1} \partial \bar{\partial} \varphi_j(\xi, \xi)) |\nabla_j \sigma_j(u)|^2 \\ &\quad + K \left( \sum_j \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla \sigma_j(\xi)|^2 \right) \left( |u|^2 + \sum_j \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla \sigma_j(u)|^2 \right) \\ &\quad + \left| (I + M_{\rho, \sigma, \varepsilon})^{-1/2} (\varepsilon^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u) - \varepsilon^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}} \sigma(u)) \right|^2, \end{aligned}$$

and

$$\begin{aligned}\nabla_{A,h_A}^2 &= \partial\bar{\partial}\psi, \quad \nabla_{\Delta_j,h_j}^2 = \partial\bar{\partial}\varphi_j, \quad (c_{\ell m \lambda \mu}) = \text{coefficients of } -2\pi \Theta_{V,h_V}, \\ M_{\rho,\sigma,\varepsilon} &= \left( \varepsilon_j^{1/2} \varepsilon_\ell^{1/2} \sigma_j^{-1+1/\rho_j} \bar{\sigma}_\ell^{-1+1/\rho_\ell} \langle \nabla_j \sigma_j, \nabla_\ell \sigma_\ell \rangle \right)_{j,\ell}, \\ \varepsilon_j^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u) &= \left( \varepsilon_j^{1/2} \sigma_j^{-1+1/\rho_j} \nabla_j^2 \sigma_j(\xi, u) \right)_j, \\ \varepsilon_j^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}}(u) &= \left( \varepsilon_j^{1/2} (1 - 1/\rho_j) \sigma_j^{-2+1/\rho_j} \nabla_j \sigma_j(\xi) \nabla_j(u) \right)_j.\end{aligned}$$

Here, the symbol  $\simeq$  means that the ratio  $\tilde{Q}_{\gamma,\varepsilon,K}/Q_{\gamma,\varepsilon,K}$  is in  $[1 - \delta, 1 + \delta]$  as soon as  $K \geq K_0(\gamma, \delta)$  and  $\varepsilon_j \leq \varepsilon_0(\gamma, \delta, K)$ .

**5.22. Remark.** If  $c$  is a lower bound for the curvature coefficients of  $\gamma \Theta_{A,h_A} - \frac{1}{\rho_j} \Theta_{\Delta_j,h_j}$  and  $\gamma \Theta_{A,h_A} \otimes \text{Id} - \Theta_{V,h_V}$  with respect to  $\omega = \Theta_{A,h_A}$ , an examination of the estimates shows that  $\tilde{Q}_{\gamma,\varepsilon,K}/Q_{\gamma,\varepsilon,K}$  as soon as  $K \geq C_0 c^{-2} \delta^{-2}$  and  $\varepsilon_j \leq c(C_0 K \rho_j^2)^{-1}$  with  $C_0 > 1$  large enough.

**5.23. Proposition.** The term  $\left| (I + M_{\rho,\sigma,\varepsilon})^{-1/2} (\dots) \right|^2$  can be estimated as follows.

(a) In case there is only one component  $\Delta_j$ , its expression becomes

$$\frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} \left| \nabla_j^2 \sigma_j(\xi, u) - (1 - 1/\rho_j) \sigma_j^{-1} \nabla_j \sigma_j(\xi) \nabla_j \sigma_j(u) \right|^2.$$

(b) In general, we have a uniform upper bound

$$\left| (I + M_{\rho,\sigma,\varepsilon})^{-1/2} (\varepsilon_j^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u)) \right|^2 \leq C \sum_j \frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} |\xi|^2 |u|^2,$$

where the sum is bounded and converges pointwise to 0 on  $X \setminus \text{Supp}(D)$ .

(c) With the same constant  $C$ , we have for every  $\delta > 0$

$$\begin{aligned} & \left| (I + M_{\rho,\sigma,\varepsilon})^{-1/2} (\varepsilon_j^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u) - \varepsilon_j^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}} \sigma(u)) \right|^2 \\ & \left\{ \begin{aligned} & \leq (1 + \delta) \left| (I + M_{\rho,\sigma,\varepsilon})^{-1/2} (\varepsilon_j^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}} \sigma(u)) \right|^2 \\ & \quad + C (1 + \delta^{-1}) \sum_j \frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} |\xi|^2 |u|^2 \\ & \geq (1 - \delta) \left| (I + M_{\rho,\sigma,\varepsilon})^{-1/2} (\varepsilon_j^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}} \sigma(u)) \right|^2 \\ & \quad - C \delta^{-1} \sum_j \frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} |\xi|^2 |u|^2. \end{aligned} \right. \end{aligned}$$

*Proof.* Formula (a) is immediate from 5.21 (b). For estimate (b), let us fix  $\eta > 0$  such that  $(\nabla_j \sigma_j)_{j \in J}$  are transverse whenever we are at a point  $z \in X$  such that  $|\sigma_j(z)| < \eta$ ,

$\forall j \in J \subset \{1, 2, \dots, N\}$ . Then  $\sum_{j \in J} \langle \nabla_j \sigma_j, \nabla_\ell \sigma_\ell \rangle t_j \bar{t}_\ell \geq c \sum_{j \in J} |t_j|^2$  for some  $c > 0$  (uniformly with respect to  $z \in X$ ). Then, taking  $J = \{j; |\sigma_j(z)| < \eta\}$ , we obtain the existence of constants  $C, C'$  such that

$$\begin{aligned} & \sum_j \varepsilon_j^{1/2} \varepsilon_\ell^{1/2} \sigma_j^{-1+1/\rho_j} \bar{\sigma}_\ell^{-1+1/\rho_\ell} \langle \nabla_j \sigma_j, \nabla_\ell \sigma_\ell \rangle t_j \bar{t}_\ell \\ & \geq \frac{c}{2} \sum_{j \in J} \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2 |t_j|^2 - C \sum_{\ell \notin J} \varepsilon_\ell |\sigma_\ell|^{-2+2/\rho_\ell} |\nabla_\ell \sigma_\ell|^2 |t_\ell|^2 \\ & \geq \frac{c}{2} \sum_{j \in J} \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2 |t_j|^2 - C' \sum_{j \notin J} |t_j|^2, \end{aligned}$$

where

$$C' = C \sum_{j \notin J} \varepsilon_j \eta^{-2+2/\rho_j} \sup |\nabla_j \sigma_j|^2.$$

For  $\delta = \min(1, \frac{2}{c}, \frac{1}{2C'})$ , this implies

$$\begin{aligned} |t|^2 + \sum_j \varepsilon_j^{1/2} \varepsilon_\ell^{1/2} \sigma_j^{-1+1/\rho_j} \bar{\sigma}_\ell^{-1+1/\rho_\ell} \langle \nabla_j \sigma_j, \nabla_\ell \sigma_\ell \rangle t_j \bar{t}_\ell \\ \geq |t|^2 + \delta \sum_j \varepsilon_j^{1/2} \varepsilon_\ell^{1/2} \sigma_j^{-1+1/\rho_j} \bar{\sigma}_\ell^{-1+1/\rho_\ell} \langle \nabla_j \sigma_j, \nabla_\ell \sigma_\ell \rangle t_j \bar{t}_\ell \\ \geq \frac{\delta c}{2} \sum_{j \in J} (1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2) |t_j|^2 + (1 - \delta C') \sum_{j \notin J} |t_j|^2, \end{aligned}$$

and by inverting the matrices of these quadratic forms we get

$$\begin{aligned} \langle (I + M_{\rho, \sigma, \varepsilon})^{-1} t, t \rangle & \leq \frac{2}{\delta c} \sum_{j \in J} (1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2)^{-1} |t_j|^2 + 2 \sum_{j \notin J} |t_j|^2 \\ & \leq C'' \sum_{j \in J \cup \mathcal{C}J} (1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2)^{-1} |t_j|^2. \end{aligned}$$

Estimate (b) follows by taking  $t = \varepsilon^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u) = (\varepsilon_j^{1/2} \sigma_j^{-1+1/\rho_j} \nabla_j^2 \sigma(\xi, u))$ , since we have  $|\nabla_j^2 \sigma(\xi, u)| \leq C_3 |\xi| |u|$ , and (c) is an immediate consequence.  $\square$

In view of the estimates developed in section 6, we will have to evaluate integrals involving powers of curvature tensors, and the following basic inequalities will be useful.

**5.24. Lemma.** *Let  $\ell_j \in (\mathbb{C}^r)^*$ ,  $1 \leq j \leq p$ , be non zero complex linear forms on  $\mathbb{C}^r$ , where  $(\mathbb{C}^r)^* \simeq \mathbb{C}^r$  is equipped with its standard hermitian form, and let  $\mu$  the rotation invariant probability measure on  $\mathbb{S}^{2r-1} \subset \mathbb{C}^r$ . Then*

$$I(\ell_1, \dots, \ell_p) = \int_{\mathbb{S}^{2r-1}} |\ell_1(u)|^2 \dots |\ell_p(u)|^2 d\mu(u)$$

satisfies the following inequalities :

$$(a) \quad I(\ell_1, \dots, \ell_p) \leq \frac{p! (r-1)!}{(p+r-1)!} \prod_{j=1}^p |\ell_j|^2,$$

and the equality occurs if and only if the  $\ell_j$  are proportional;

$$(b) \ I(\ell_1, \dots, \ell_p) \geq \frac{(r-1)!}{(p+r-1)!} \prod_{j=1}^p |\ell_j|^2,$$

and the equality occurs if and only if  $p \leq r$  and the  $\ell_j$  are pairwise orthogonal.

*Proof.* Denote by  $d\lambda$  the Lebesgue measure on Euclidean space and by  $d\sigma$  the area measure of the sphere. One can easily check that the projection

$$\mathbb{S}^{2r-1} \rightarrow \mathbb{B}^{2r-2}, \quad u = (u_1, \dots, u_r) \mapsto v = (u_1, \dots, u_{r-1}),$$

yields  $d\sigma(u) = d\theta \wedge d\lambda(v)$  where  $u_r = |u_r| e^{i\theta}$  [just check that the wedge products of both sides with  $\frac{1}{2}d|u|^2$  are equal to  $d\lambda(u)$ , and use the fact that  $d\theta = \frac{1}{2i}(du_r/u_r - d\bar{u}_r/\bar{u}_r)$ ], thus, in terms of polar coordinates  $v = t u'$ ,  $u' \in \mathbb{S}^{2r-1}$ , we have  $d\sigma(u) = d\theta \wedge t^{2r-3} dt \wedge d\sigma'(u')$ , and going back to the invariant probability measures  $\mu$  on  $\mathbb{S}^{2r-1}$  and  $\mu'$  on  $\mathbb{S}^{2r-3}$ , we get  $|u_r|^2 = 1 - |v|^2 = 1 - t^2$  and an equality

$$(5.25) \quad d\mu(u) = \frac{2r-2}{2\pi} d\theta \wedge t^{2r-3} dt \wedge d\mu'(u').$$

If  $\ell_1, \dots, \ell_p$  are independent of  $u_r$ , (5.25) and the Fubini theorem imply by homogeneity

$$(5.26) \quad \int_{\mathbb{S}^{2r-1}} |\ell_1(u')|^2 \dots |\ell_p(u')|^2 d\mu(u) = \frac{r-1}{p+r-1} \int_{\mathbb{S}^{2r-3}} |\ell_1(u')|^2 \dots |\ell_p(u')|^2 d\mu'(u'),$$

$$(5.26') \quad \int_{\mathbb{S}^{2r-1}} |\ell_1(u')|^2 \dots |\ell_{p-1}(u')|^2 |u_r|^2 d\mu(u) = \frac{r-1}{(p+r-2)(p+r-1)} \int_{\mathbb{S}^{2r-3}} |\ell_1(u')|^2 \dots |\ell_{p-1}(u')|^2 d\mu'(u')$$

(for instance, in case (5.26'), we have to integrate  $t^{2p-2}(1-t^2) \times t^{2r-3} dt$ ). The formulas

$$\int_{\mathbb{S}^{2r-1}} |u_1|^{2p} d\mu(u) = \frac{p!(r-1)!}{(p+r-1)!}, \quad \int_{\mathbb{S}^{2r-1}} |u_1|^2 \dots |u_p|^2 d\mu(u) = \frac{(r-1)!}{(p+r-1)!} \quad (p \leq r),$$

are then obtained by induction on  $r$  and  $p$ .

(a) For any  $\ell \in (\mathbb{C}^r)^*$ , we can find orthonormal coordinates on  $\mathbb{C}^r$  such that  $\ell(u) = |\ell| u_1$  in the new coordinates. Hence

$$\int_{\mathbb{S}^{2r-1}} |\ell(u)|^{2p} d\mu(u) = m_{r,p} |\ell|^{2p} \quad \text{where} \quad m_{r,p} = \int_{\mathbb{S}^{2r-1}} |u_1|^{2p} d\mu(u) = \frac{p!(r-1)!}{(p+r-1)!}.$$

It follows from Hölder's inequality that

$$I(\ell_1, \dots, \ell_p) \leq \prod_{j=1}^p \left( \int_{\mathbb{S}^{2r-1}} |\ell_j|^{2p} d\mu(u) \right)^{1/p} = m_{r,p} \prod_{j=1}^p |\ell_j|^2,$$

and that the equality occurs if and only if all  $\ell_j$  are proportional.

(b) We prove the inequality

$$I(\ell_1, \dots, \ell_p) \geq \frac{(r-1)!}{(p+r-1)!} \prod_{j=1}^p |\ell_j|^2$$

by induction on  $p$ , the result being clear for  $p = 0$  or  $p = 1$ . If we choose an orthonormal basis  $(e_1, \dots, e_r) \in \mathbb{C}^r$  such that  $\ell_j(e_r) \neq 0$  for all  $j$  and replace  $\ell_j$  by  $(\ell_j(e_r))^{-1} \ell_j$ , we can assume  $\ell_j(e_r) = 1$ . We then write  $u = u' + u_r e_r$  with  $u' \in e_r^\perp \simeq \mathbb{C}^{r-1}$  and

$$\ell_j(u) = \ell'_j(u') + u_r, \quad 1 \leq j \leq p, \quad \ell'_j := \ell_j|_{e_r^\perp}.$$

Let  $s_k(\ell'_\bullet(u'))$  be the elementary symmetric functions in  $\ell'_j(u')$ ,  $1 \leq j \leq p$ , with  $s_0 := 1$ . We have

$$I(\ell_1, \dots, \ell_p) = \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^p |\ell'_j(u') + u_r|^2 d\mu(u) = \int_{\mathbb{S}^{2r-1}} \left| \sum_{k=0}^p s_k(\ell'_\bullet(u')) u_r^{p-k} \right|^2 d\mu(u).$$

We make a change of variable  $u_r \mapsto u_r e^{i\theta}$  and take the average over  $\theta \in [0, 2\pi]$ . Parseval's formula gives

$$I(\ell_1, \dots, \ell_p) = \int_{\mathbb{S}^{2r-1}} \sum_{k=0}^p |s_k(\ell'_\bullet(u'))|^2 |u_r|^{2(p-k)} d\mu(u),$$

and since

$$(2r-2) \int_0^1 t^{2k} (1-t^2)^{p-k} t^{2r-3} dt = \frac{(r-1)(k+r-2)!(p-k)!}{(p+r-1)!},$$

formula (5.25) implies

$$I(\ell_1, \dots, \ell_p) = \int_{\mathbb{S}^{2r-3}} \sum_{k=0}^p \frac{(r-1)(k+r-2)!(p-k)!}{(p+r-1)!} |s_k(\ell'_\bullet(u'))|^2 d\mu'(u').$$

As  $|\ell_j|^2 = 1 + |\ell'_j|^2$ , our inequality (5.24 (b)) is equivalent to

$$(5.27) \quad \int_{\mathbb{S}^{2r-3}} \sum_{k=0}^p \frac{(k+r-2)!(p-k)!}{(r-2)!} |s_k(\ell'_\bullet(u'))|^2 d\mu'(u') \geq \prod_{j=1}^p (1 + |\ell'_j|^2)$$

for all linear forms  $\ell'_j \in (\mathbb{C}^{r-1})^*$ . We actually prove (5.27) by induction on  $p$  (observing that the inequality is a trivial equality for  $p = 0, 1$ ). Assume that (5.27) (and hence (5.24 (b))) is known for any  $(p-1)$ -tuple of linear forms  $(\ell'_1, \dots, \ell'_{p-1})$ . As (5.24 (b)) is invariant under the action of  $U(r)$ , it is sufficient to consider the case when  $\ell_p(u) = u_r$ , i.e.  $\ell'_p = 0$ . The induction hypothesis tells us that

$$\int_{\mathbb{S}^{2r-3}} \sum_{k=0}^{p-1} \frac{(k+r-2)!(p-1-k)!}{(r-2)!} |s_k(\ell'_\bullet(u'))|^2 d\mu'(u') \geq \prod_{j=1}^{p-1} (1 + |\ell'_j|^2).$$

However, when we add the factor  $\ell_p$ , the elementary symmetric functions  $s_k(\ell'_\bullet(u'))$  are left unchanged for  $k \leq p-1$ , while  $s_p(\ell'_\bullet(u')) = 0$  and  $1 + |\ell'_p|^2 = 1$ . Therefore (5.27) holds true

for  $p$ , since  $(p-k)! \geq (p-1-k)!$  for all  $k = 0, 1, \dots, p-1$ . We have proved the inequality at order  $p$  whenever  $\ell_p = \alpha_p \langle \bullet, e_r \rangle$  and  $\ell_j(e_r) \neq 0$  for  $j \leq p-1$ . Since those  $(\ell_1, \dots, \ell_p)$  are dense in the space  $((\mathbb{C}^r)^*)^p$  of  $p$ -tuples of linear forms, the proof of the lower bound is complete.

(b, equality case) We argue by induction on  $r$ . For  $r = 1$ , we have in fact  $\ell_j(u) = \alpha_j u_1$ ,  $\alpha_j \in \mathbb{C}^*$ , and  $I(\ell_1, \dots, \ell_r) = \prod |\ell_j|^2$ , thus the coefficient  $\frac{1}{(p+r-1)!} = \frac{1}{p!}$  is reached if and only if  $p \leq 1$ . Now, assume  $r \geq 2$  and the equality case solved for dimension  $r-1$ . By rescaling and reordering the  $\ell_j$ , we can always assume that  $\ell_j(e_r) \neq 0$  (and hence  $\ell_j(e_r) = 1$ ) for  $q+1 \leq j \leq p$ , while  $\ell_j(e_r) = 0$  for  $1 \leq j \leq q$  (we can possibly have  $q = 0$  here). Then we write  $\ell_j(u) = \ell'_j(u')$  for  $1 \leq j \leq q$  and  $\ell_j(u) = \ell'_j(u') + u_r$  for  $q+1 \leq j \leq p$ . Therefore, if  $s_k(\ell'(u'))$  denotes the  $k$ -th elementary symmetric function in  $(\ell'_j(u'))_{q+1 \leq j \leq p}$ , we find

$$\begin{aligned}
I(\ell_1, \dots, \ell_p) &= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^q |\ell'_j(u')|^2 \prod_{j=q+1}^p |\ell'_j(u') + u_r|^2 d\mu(u) \\
&= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^q |\ell'_j(u')|^2 \left| \sum_{k=0}^{p-q} s_k(\ell'(u')) u_r^{p-q-k} \right|^2 d\mu(u) \\
&= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^q |\ell'_j(u')|^2 \sum_{k=0}^{p-q} |s_k(\ell'(u'))|^2 |u_r|^{2(p-q-k)} d\mu(u) \\
&= \int_{\mathbb{S}^{2r-3}} \prod_{j=1}^q |\ell'_j(u')|^2 \sum_{k=0}^{p-q} \frac{(r-1)(k+r-2)!(p-q-k)!}{(p-q+r-1)!} |s_k(\ell'(u'))|^2 d\mu'(u') \\
&\geq \frac{(r-1)!}{(p+r-1)!} \prod_{j=1}^q |\ell'_j|^2 \prod_{j=q+1}^p (1 + |\ell'_j|^2)
\end{aligned}$$

by what we have just proved. In an equivalent way, we get

$$\begin{aligned}
&\int_{\mathbb{S}^{2r-3}} \prod_{j=1}^q |\ell'_j(u')|^2 \sum_{k=0}^{p-q} \frac{(k+r-2)!(p-q-k)!(p+r-1)!}{(r-2)!(p-q+r-1)!} |s_k(\ell'(u'))|^2 d\mu'(u') \\
&\geq \prod_{j=1}^q |\ell'_j|^2 \prod_{j=q+1}^p (1 + |\ell'_j|^2)
\end{aligned}$$

for all  $0 \leq q \leq p-1$  and all choices of the forms  $\ell'_j \in (\mathbb{C}^{r-1})^*$ . In general, we can rotate coordinates in such a way that  $\ell_p(u) = u_r$  and  $\ell'_p = 0$ , and we see that the above inequality holds when  $p$  is replaced by  $p-1$ , as soon as  $q \leq p-2$ . Then the corresponding coefficients  $k = 0$  for  $p, p-1$  are

$$\frac{(p-q)!(p+r-1)!}{(p-q+r-1)!} > \frac{(p-1-q)!(p-1+r-1)!}{(p-1-q+r-1)!},$$

and since  $s_0 = 1$ , we infer that the inequality is strict. The only possibility for the equality case is  $q = p-1$ , but then

$$I(\ell_1, \dots, \ell_p) = \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{p-1} |\ell'_j(u')|^2 |u_r|^2 d\mu(u) = \frac{r-1}{p+r-1} \int_{\mathbb{S}^{2r-3}} \prod_{j=1}^{p-1} |\ell'_j(u')|^2 d\mu'(u'),$$

and we see that we must have equality in the case  $(r-1, p-1)$ . By induction, we conclude that  $p-1 \leq r-1$  and that the  $\ell_j(u) = \ell'_j(u')$  are orthogonal for  $j \leq p-1$ , as desired.  $\square$

**5.28. Remark.** When  $r = 2$ , our inequality (5.27) is equivalent to the “elementary” inequality

$$(*) \quad \prod_{j=1}^p (1 + |a_j|^2) \leq \sum_{k=0}^p k! (p-k)! |s_k|^2,$$

relating a polynomial  $X^p - s_1 X^{p-1} + \dots + (-1)^p s_p$  and its complex roots  $a_j$  (just consider  $\ell'_j(u') = a_j u_1$  and  $\ell_j(u) = a_j u_1 + u_2$  on  $\mathbb{C}^2$  to get this). It should be observed that  $(*)$  is not optimal asymptotically when  $p \rightarrow +\infty$ ; in fact, Landau’s inequality [Land05] gives  $\prod \max(1, |a_j|) \leq (\sum |s_k|^2)^{1/2}$ , from which one can easily derive that  $\prod (1 + |a_j|^2) \leq 2^p \sum |s_k|^2$ , which improves  $(*)$  as soon as  $p \geq 7$  (observe that  $2^7 = 128$  and  $k!(7-k)! \geq 3!4! = 144$ ). Our discussion of the equality case shows that inequality (5.24 (b)) is never sharp when  $p > r$ . It would be interesting, but probably challenging, if not impossible, to compute the optimal constant for all pairs  $(r, p)$ ,  $p > r$ , since this is an optimization problem involving the distribution of a large number of points in projective space.

We finally state one of the main consequences of these estimates concerning the Chern curvature form of a hermitian holomorphic vector bundle.

**5.29. Proposition.** *Let  $T, E$  be complex vector spaces of respective dimensions  $\dim T = n$ ,  $\dim E = r$ . Assume that  $E$  is equipped with a hermitian structure  $h$  and denote by  $\mu$  the unitary invariant probability measure  $\mu$  on the unit sphere bundle  $S(E) = \{u \in E; |u|_h\} \subset E$ .*

(a) *If  $\theta_1, \dots, \theta_p \geq_S 0$  are strongly semi-positive hermitian tensors in  $\text{Herm}(T \otimes E) \simeq \Lambda_{\mathbb{R}}^{1,1} T^* \otimes_{\mathbb{R}} \text{Herm}(E, E)$  then*

$$\int_{v \in S(E)} \langle \theta_1(u), u \rangle_h \wedge \dots \wedge \langle \theta_p(u), u \rangle_h d\mu(u) \begin{cases} \geq \frac{(r-1)!}{(p+r-1)!} \text{Tr}_h \theta_1 \wedge \dots \wedge \text{Tr}_h \theta_p, \\ \leq \frac{p! (r-1)!}{(p+r-1)!} \text{Tr}_h \theta_1 \wedge \dots \wedge \text{Tr}_h \theta_p, \end{cases}$$

*as pointwise strong inequalities of  $(p, p)$ -forms.*

(b) *If  $\theta \geq_G 0$  in  $\Lambda_{\mathbb{R}}^{1,1} T^* \otimes_{\mathbb{R}} \text{Herm}(E, E)$  and  $\ell_j \in E^*$ , then*

$$\int_{v \in S(E)} |\ell_1(u)|^2 \dots |\ell_k(u)|^2 \langle \theta(u), u \rangle_h^{p-k} d\mu(u) \leq \frac{p! (r-1)!}{(p+r-1)!} \left( \prod_{j=1}^k |\ell_j|^2 \right) (\text{Tr}_h \theta)^{p-k}$$

*as a pointwise weak inequality of  $(p-k, p-k)$ -forms.*

*In particular, the above inequalities apply when  $(E, h)$  is a hermitian holomorphic vector bundle of rank  $r$  on a complex  $n$ -dimensional manifold  $X$ , and one takes  $\theta_j = \Theta_{E, h}$  to be the curvature tensor of  $E$ , so that  $\text{Tr}_h \theta_j = c_1(E, h)$  is the first Chern form of  $(E, h)$ .*

*Proof.* (a) The assumption  $\theta_q \geq_S 0$  means that at every point  $x \in X$  we can write  $\theta$  as

$$\theta_q = \sum_{1 \leq j \leq N_q} |\beta_{qj} \otimes \ell_{qj}|^2 \simeq \sum_{1 \leq j \leq N_q} i \beta_{qj} \wedge \bar{\beta}_{qj} \otimes \ell_{qj} \otimes \ell_{qj}^*, \quad \beta_{qj} \in T^*, \quad \ell_{qj} \in E^*$$

as an element of  $\Lambda_{\mathbb{R}}^{1,1} T^* \otimes_{\mathbb{R}} \text{Herm}(E, E)$ , hence

$$\langle \theta_q(u), u \rangle_h = \sum_{1 \leq j \leq N_q} i \beta_{qj} \wedge \bar{\beta}_{qj} |\ell_{qj}(u)|^2.$$

Without loss of generality, we can assume  $|\ell_{qj}|_{h^*} = 1$ . Then

$$\langle \theta_1(u), u \rangle_h \wedge \dots \wedge \langle \theta_p(u), u \rangle_h = \sum_{j_1, \dots, j_p} i \beta_{1j_1} \wedge \bar{\beta}_{1j_1} \wedge \dots \wedge i \beta_{pj_p} \wedge \bar{\beta}_{pj_p} |\ell_{1j_1}(u)|^2 \dots |\ell_{pj_p}(u)|^2,$$

and since  $|\ell_{qj}|_{h^*} = 1$ , Lemma 5.24 (b) implies

$$\begin{aligned} \int_{v \in S(E)} \langle \theta_1(u), u \rangle_h \wedge \dots \wedge \langle \theta_p(u), u \rangle_h d\mu(u) \\ \geq \frac{(r-1)!}{(p+r-1)!} \sum_{j_1, \dots, j_p} i \beta_{1j_1} \wedge \bar{\beta}_{1j_1} \wedge \dots \wedge i \beta_{pj_p} \wedge \bar{\beta}_{pj_p} \\ = \frac{(r-1)!}{(p+r-1)!} \text{Tr}_h \theta_1 \wedge \dots \wedge \text{Tr}_h \theta_p, \end{aligned}$$

where  $\geq$  is in the sense of the strong positivity of  $(p, p)$ -forms. The upper bound is obtained by the same argument, via 5.24 (a).

(b) By the definition of weak positivity of forms, it is enough to show the inequality in restriction to every  $(p-k)$ -dimensional subspace  $T' \subset T$ . Without loss of generality, we can assume that  $\dim T = p-k$  (and then take  $T' = T$ ), that  $|\ell_j| = 1$ , and also that  $\theta >_G 0$  (otherwise take a positive definite form  $\eta \in \Lambda_{\mathbb{R}}^{1,1} T^*$ , replace  $\theta$  with  $\theta_\varepsilon = \theta + \varepsilon \eta \otimes h$ , and let  $\varepsilon$  tend to 0). For any  $v \in S(E)$ , let

$$0 \leq \lambda_1(u) \leq \dots \leq \lambda_{p-k}(u)$$

be the eigenvalues of the hermitian form  $q_v(\bullet) = \langle \theta(u), u \rangle$  on  $T$  with respect to

$$\omega = \text{Tr}_h \theta = \sum_{j=1}^r \langle \theta(e_j), e_j \rangle \in \text{Herm}(T), \quad \omega > 0,$$

$(e_j)$  being any orthonormal frame of  $E$ . We have to show that

$$\int_{v \in S(E)} |\ell_1(u)|^2 \dots |\ell_k(u)|^2 \lambda_1(u) \dots \lambda_{p-k}(u) d\mu(u) \leq \frac{p!(r-1)!}{(p+r-1)!}.$$

However, the inequality between geometric and arithmetic means implies

$$\lambda_1(u) \dots \lambda_p(u) \leq \left( \frac{1}{p-k} \sum_{j=1}^{p-k} \lambda_j(u) \right)^{p-k},$$

thus, putting  $Q(u) = \frac{1}{p-k} \langle \text{Tr}_\omega \theta(u), u \rangle$ ,  $Q \in \text{Herm}(E)$ , it is enough to prove that

$$(5.30) \quad \int_{v \in S(E)} |\ell_1(u)|^2 \dots |\ell_k(u)|^2 Q(u)^{p-k} d\mu(u) \leq \frac{p!(r-1)!}{(p+r-1)!}.$$



Our assumption  $\theta >_G 0$  implies  $Q(u) = \sum_{1 \leq j \leq r} c_j |\ell'_{qj}(u)|^2$  for some  $c_j > 0$  and some orthonormal basis  $(\ell'_{qj})_{1 \leq j \leq r}$  of  $E^*$ , and

$$\sum_{j=1}^r c_j = \text{Tr}_h Q = \frac{1}{p-k} \text{Tr}_h(\text{Tr}_\omega \theta) = \frac{1}{p-k} \text{Tr}_\omega(\text{Tr}_h \theta) = \frac{1}{p-k} \text{Tr}_\omega(\omega) = 1.$$

Inequality (5.30) is a consequence of Lemma 5.24 (a), by Newton's multinomial expansion.  $\square$

**5.31. Remark.** For  $p = 1$ , the inequalities of Proposition 5.29 are identities, and no semi-positivity assumption is needed in that case. However, when  $p \geq 2$ , inequality 5.29 (a) does not hold under the assumption that  $E \geq_G 0$  (or even that  $E$  is dual Nakano semi-positive, i.e.  $E^*$  Nakano semi-negative). Let us take for instance  $E = T_{\mathbb{P}^n} \otimes \mathcal{O}(1)$ . It is well known that  $E$  is isomorphic to the tautological quotient vector bundle  $\mathbb{C}^{n+1}/\mathcal{O}(-1)$  over  $\mathbb{P}^n$ , and that its curvature tensor form for the Fubini-Study metric is given by

$$\theta_E(\xi \otimes u, \xi \otimes u) = |\langle \xi, u \rangle|^2 \geq 0$$

(where  $v$  is identified with a tangent vector via the choice of a unit element  $e \in \mathcal{O}(-1)$ ). Then  $\det E = \mathcal{O}(1)$  and thus  $c_1(E, h) = \omega_{\text{FS}} > 0$ , although  $\langle \Theta_{E,h}(u), u \rangle_h^p = 0$  for all  $p \geq 2$ , as one can easily check.

Our aim is to apply Proposition 5.29 to the curvature tensor  $\theta = \Theta_{V\langle D \rangle}$  of a directed orbifold  $(X, V, D)$ . Under ad hoc hypotheses, Proposition 5.8 implies Griffiths positivity, but we want to invoke strong positivity to be able to apply the lower bound of 5.29 (a). The main observation is that one can somehow separate the contribution of  $V$  and the contribution of  $D$  in the calculation. For the sake of generality (and the needs of §6), we introduce the possibility of combining different orbifold divisors  $D_s$  with the same support.

**5.32. Proposition.** *Let  $X$  be a projective variety,  $A$  an ample line bundle, and let  $(X, V, D_s)$ ,  $1 \leq s \leq k$ , be orbifold directed structures where  $D_s = \sum_{1 \leq j \leq N} (1 - \frac{1}{\rho_{s,j}}) \Delta_j$  are normal crossing divisors on  $X$  transverse to  $V$ , sharing the same components  $\Delta_j$ . Let  $d_j$  be the infimum of numbers  $\lambda \in \mathbb{R}_+$  such that  $\lambda A - \Delta_j$  is nef, and let  $\gamma_V$  (resp.  $\tilde{\gamma}_V$ ) be the infimum of numbers  $\gamma \geq 0$  such that  $\theta_{V,\gamma} := \gamma \Theta_{A,h_A} \otimes \text{Id}_V - \Theta_{V,h_V} \geq_G 0$  (resp.  $\geq_S 0$ ) for suitable hermitian metrics  $h_V$  on  $V$ . Take  $p_1, \dots, p_k \in \mathbb{N}$  such that  $q = n - (p_1 + \dots + p_k) \geq 0$  and a weakly positive smooth  $(q, q)$  form  $\beta \geq_W 0$  on  $X$ . Then for every*

$$\gamma_s > \tilde{\gamma}_{V,D_s} := \max(\max_j (d_j / \rho_{s,j}), \tilde{\gamma}_V)$$

*there exist hermitian metrics  $h_{V\langle D_s \rangle, \varepsilon_s}$  on the orbifold vector bundles  $V_s = V\langle D_s \rangle$  such that*

$$\theta_{V_s, \gamma_s, \varepsilon_s} := \gamma_s \Theta_{A, h_A} \otimes \text{Id}_V - \Theta_{V\langle D_s \rangle, h_{V\langle D_s \rangle, \varepsilon_s}} >_G 0, \quad \varepsilon_s = (\varepsilon_{s1}, \dots, \varepsilon_{sN})$$

*in the sense of Griffiths, and*

$$\begin{aligned} & \lim_{\varepsilon_{ij} \rightarrow 0} \int_X \int_{u_s \in S(V\langle D_s \rangle)} \langle \theta_{V_1, \gamma_1, \varepsilon_1}(u_1), u_1 \rangle^{p_1} \wedge \dots \wedge \langle \theta_{V_k, \gamma_k, \varepsilon_k}(u_k), u_k \rangle^{p_k} \wedge \beta d\mu(u_1) \dots d\mu(u_k) \\ & \geq \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}} \sum_{\ell_j \geq 1, \sum_{j \in J_s} \ell_j \leq p_s} \prod_{1 \leq s \leq k} \frac{(r-1)! p_s! \prod_{j \in J_s} (1 - 1/\rho_{s,j})}{(p_s + r - 1)! (p_s - \sum_{j \in J_s} \ell_j)! \prod_{j \in J_s} \ell_j!} \\ & \times \int_{\Delta_{J_1 \amalg \dots \amalg J_k}} \bigwedge_{1 \leq s \leq k} \left( (\text{Tr } \theta_{V, \gamma_s})^{p_s - \sum_{j \in J_s} \ell_j} \wedge \bigwedge_{j \in J_s} (\gamma_s \omega_A - \rho_{s,j}^{-1} \theta_{\Delta_j, h_j})^{\ell_j - 1} \right) \wedge \beta, \end{aligned}$$

where the limit  $\lim_{\varepsilon_{ij} \rightarrow 0}$  is an iterated limit  $\lim_{\varepsilon_{11} \rightarrow 0} \dots \lim_{\varepsilon_{kN} \rightarrow 0}$  with respect to the lexicographic order  $(i, j) < (i', j')$  if  $i < i'$  or  $i = i'$  and  $j < j'$ . The summation is taken over all disjoint subsets  $J_1, \dots, J_k \subset \{1, 2, \dots, N\}$ , and we have set here  $\Delta_J := \bigcap_{j \in J} \Delta_j$ . For  $\beta \geq_S 0$  and  $\gamma_s$  satisfying the Griffiths type condition

$$\gamma_s > \gamma_{V, D_s} := \max(\max_j (d_j / \rho_{s,j}), \gamma_V),$$

a similar upper bound holds with constants  $(r-1)! p_s!$  replaced by  $(r-1)! p_s!^2$ .

*Proof.* For the sake of simplicity, we first deal with the case where a single divisor  $D = \sum (1 - 1/\rho_j) \Delta_j$  is involved. We apply Corollary 5.21 (b) and compute the limit as  $\varepsilon \rightarrow 0$  when the curvature tensor is replaced by  $Q_{\gamma, \varepsilon, K}$ , since the other terms are negligible. The formula shows that the metric  $h_{V \langle D \rangle, \varepsilon}$  converges to  $h_V$ , and that its curvature tensor converges uniformly to  $\theta_{V, \gamma}$  in the complement of any fixed neighborhood of  $|D|$ . This contribution corresponds to taking powers of the terms in the first line of 5.21 (b), and yields the term  $J_1 = \dots = J_k = \emptyset$  in the right hand side of the limit. In order to evaluate the other terms, which produce contributions supported in  $|D|$ , we introduce the “orbifold” coordinates

$$t_j = \varepsilon_j^{-1/2} \sigma_j(z)^{1-1/\rho_j} |\nabla_j \sigma_j(z)|^{-1}, \quad j = j_1, \dots, j_s,$$

in a neighborhood of any point  $z_0 \in \Delta_{j_1} \cap \dots \cap \Delta_{j_s}$  (and complete those coordinates with  $n-s$  variables  $z_\ell$  that define coordinates on  $\Delta_{j_1} \cap \dots \cap \Delta_{j_s}$ ). These coordinates  $t_j$  are not single valued near  $z_0$ , but we can always (locally) make a “cut” in  $X$  along  $\Delta_j$  to exclude the negligible set of points where  $\sigma_j(z) \in \mathbb{R}_-$ , and take the argument in  $] -\pi, \pi[$ , so that  $\text{Arg}(t_j) \in ] - (1 - 1/\rho_j)\pi, (1 - 1/\rho_j)\pi[$ . If we integrate over complex numbers  $t_j$  without such a restriction on the argument, the integral will have to be multiplied by the factor  $(1 - 1/\rho_j)$  to get the correct value. Since  $|\nabla_j \sigma_j(z)|$  does not vanish near  $\Delta_j$ , the range of the absolute value  $|t_j|$  is an interval  $]0, C_j \varepsilon_j^{-1/2}[$ , thus  $t_j$  will cover asymptotically an entire angular sector in  $\mathbb{C}$ . With the above coordinates, we can write

$$\sigma_j^{1/\rho_j} = (\varepsilon_j^{1/2} t_j)^{1/(\rho_j-1)} |\nabla_j \sigma_j(z)|^{1/(\rho_j-1)},$$

thus by differentiation along a given tangent vector  $\xi \in T_{X,z}$

$$\varepsilon_j^{1/2} \sigma_j^{-1+1/\rho_j} \nabla \sigma_j(\xi) = O\left(\varepsilon_j (\varepsilon_j^{1/2} |t_j|)^{-1+1/(\rho_j-1)} |dt_j(\xi)| + \varepsilon_j^{1/2} |\xi|\right)$$

as  $\varepsilon_j^{1/2} |t_j| \leq C_j$  and  $d(|\nabla_j \sigma_j(z)|^{1/(\rho_j-1)})$  is bounded. By 5.21 (b), we find

$$\begin{aligned} |u|_{h_{V \langle D \rangle, \varepsilon}}^2 &= |u|^2 + \sum |t_j|^{-2} |e_j^*(u)|^2, \quad e_j^* = \frac{\nabla_j \sigma_j}{|\nabla_j \sigma_j|} \in S(V^*), \\ Q_{\gamma, \varepsilon, K}(z)(\xi \otimes u)^2 &= \gamma \partial \bar{\partial} \psi(\xi, \xi) |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} \xi_\ell \bar{\xi}_m u_\lambda \bar{u}_\mu \\ &\quad + \sum_j |t_j|^{-2} (\gamma \partial \bar{\partial} \psi(\xi, \xi) - \rho_j^{-1} \partial \bar{\partial} \varphi_j(\xi, \xi)) |e_j^*(u)|^2 \\ (5.33) \quad &+ O\left(\sum_j \varepsilon_j^2 (\varepsilon_j^{1/2} |t_j|)^{-2+2/(\rho_j-1)} |dt_j(\xi)|^2 + \varepsilon_j |\xi|^2\right) \left(|u|^2 + \sum_j |t_j|^{-2} |e_j^*(u)|^2\right) \\ &+ \left| (I + M_{\rho, \sigma, \varepsilon})^{-1/2} (\varepsilon^{1/2} \nabla_{\text{orb}}^2 \sigma(\xi, u) - \varepsilon^{1/2} (1 - 1/\rho) \sigma^{-1} \nabla \sigma(\xi) \nabla_{\text{orb}} \sigma(u)) \right|^2, \end{aligned}$$

and with respect to the variables  $u_\ell$ , we have to integrate on the sphere

$$S(V\langle D \rangle, h_{V\langle D \rangle, \varepsilon}) = \{u; |u|^2 + \sum |t_j|^{-2} |e_j^*(u)|^2 = 1\}.$$

We argue by induction on the number of components  $\Delta_j$  in  $D$ . When there is only one component  $\Delta_j$ , Proposition 5.23 (a) gives

$$\begin{aligned} & |(I + M_{\rho, \sigma, \varepsilon})^{-1/2}(\dots)|^2 \\ &= \frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} \left| \nabla_j^2 \sigma_j(\xi, u) - (1 - 1/\rho_j) \sigma_j^{-1} \nabla_j \sigma_j(\xi) \nabla_j \sigma_j(u) \right|^2 \\ (5.34) \quad &= \frac{|t_j|^{-2}}{1 + |t_j|^{-2}} \left| \frac{dt_j}{t_j} e_j^*(u) + O(|\xi| |u|) \right|^2 = \frac{1}{1 + |t_j|^2} \left| \frac{dt_j}{t_j} e_j^*(u) + O(|\xi| |u|) \right|^2. \end{aligned}$$

At this point, we have to make several observations. The most important one is that (5.34) can be viewed as a rank one  $(1, 1)$  form  $i\psi_u \wedge \bar{\psi}_u \in \Lambda^{1,1} T_X^*$  (the associated hermitian form is the square of a linear form), thus if we expand  $\langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p$  via its  $Q_{\gamma, \varepsilon, K}$  approximation, this term will only appear with an exponent equal to 0 or 1. The complicated term  $O(\varepsilon_j^2 \dots)$  in (5.33) will have a zero contribution in the limit (the right hand side factor  $|u|^2 + \sum_j |t_j|^{-2} |e_j^*(u)|^2$  is equal to 1 and there is a sufficiently large exponent in  $\varepsilon_j$  in the left hand side factor, so that even after integrating on a large disc  $|t_j| < C_j \varepsilon_j^{-1/2}$ , one factor  $\varepsilon_j$  is left). Finally, the bounded term  $O(|\xi| |u|)^2$  in (5.34) does not contribute, because it is multiplied by a factor

$$\frac{|t_j|^{-2}}{1 + |t_j|^{-2}} = \frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} \leq 1$$

that has the pleasant property of converging pointwise to 0 almost everywhere on  $X$  (namely on  $X \setminus \sigma_j^{-1}(0)$ ), and the other possible factors either come from (5.33), or are forms with uniformly bounded coefficients on  $X$ . Similarly the double product term  $2 \frac{dt_j}{t_j} e_j^*(u) O(|\xi| |u|)$  can be bounded by  $\delta \left| \frac{dt_j}{t_j} e_j^*(u) \right|^2 + \delta^{-1} O(|\xi| |u|)^2$ ,  $\delta \ll 1$ . Putting everything together,  $Q_{\gamma, \varepsilon, K}$  can be simplified and approximated as

$$\begin{aligned} Q_{\gamma, \varepsilon, K}(z)(\xi \otimes u)^2 &\simeq \gamma \partial \bar{\partial} \psi(\xi, \xi) |u|^2 + \sum_{\ell, m, \lambda, \mu} c_{\ell m \lambda \mu} \xi_\ell \bar{\xi}_m u_\lambda \bar{u}_\mu \\ &\quad + |t_j|^{-2} (\gamma \partial \bar{\partial} \psi(\xi, \xi) - \rho_j^{-1} \partial \bar{\partial} \varphi_j(\xi, \xi)) |e_j^*(u)|^2 + \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u)|^2 \\ (5.35) \quad &\simeq \langle \theta_{V, \gamma}(u), u \rangle + (\gamma \omega_A - \rho_j^{-1} \theta_{\Delta_j, h_j}) |t_j|^{-2} |e_j^*(u)|^2 + \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u)|^2. \end{aligned}$$

It is also important to notice that since  $|u|^2 \geq |e_j^*(u)|^2$ , we have  $|e_j^*(u)|^2 \leq (1 + |t_j|^{-2})^{-1}$  on the unit sphere  $S(V\langle D \rangle)$ , thus

$$|t_j|^{-2} |e_j^*(u)|^2 \leq \frac{1}{1 + |t_j|^2} \leq 1, \quad \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u)|^2 \leq \frac{1}{(1 + |t_j|^2)^2} |dt_j(\xi)|^2,$$

and all terms yield convergent integrals in the limit, even after integrating over the whole complex line  $t_j \in \mathbb{C}$ . When  $D$  has  $N$  components  $\Delta_j$ , we can argue by picking the

components one by one, starting e.g. with  $\Delta_N$ . When  $(\varepsilon_j)_{1 \leq j \leq N-1}$  are fixed and  $\varepsilon_N \rightarrow 0$ , the terms occurring in  $\langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle$  corresponding to indices  $j \neq N-1$  add up to make a  $(1, 1)$ -form that can be considered as smooth in the orbifold coordinates of  $(X, \Delta_j)_{1 \leq j \leq N-1}$ . Therefore, by replacing  $V$  with  $V' = V(\langle D' \rangle)$ ,  $D' = \sum_{1 \leq j \leq N-1} (1 - 1/\rho_j) \Delta_j$ , we can argue as if we had only one component  $\Delta_N$  to consider in addition to  $\langle \theta_{\gamma, V'}(u), u \rangle$ , and we take the weak limit as  $\varepsilon_N \rightarrow 0$ . As will be clear in a moment, the iterated limit  $\lim_{\varepsilon_1 \rightarrow 0} \lim_{\varepsilon_2 \rightarrow 0} \dots \lim_{\varepsilon_N \rightarrow 0}$  will be the same as the limit obtained for a split (i.e. perpendicular) decomposition

$$(5.35_{\perp}) \quad \langle \theta_{V, \gamma}(u), u \rangle + \sum_{1 \leq j \leq N} \left( (\gamma \omega_A - \rho_j^{-1} \theta_{\Delta_j, h_j}) |t_j|^{-2} |e_j^*(u)|^2 + \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u)|^2 \right).$$

We use the approximation (5.35 $_{\perp}$ ) to compute  $\langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p$ . The terms  $dt_j \wedge \bar{d}t_j$  can only be taken once in the wedge product, and we get in this way by a multinomial expansion

$$\begin{aligned} & \langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p \\ &= \sum_{J \subset \{1, \dots, N\}, |J| \leq p} \frac{p!}{(p - |J|)!} \sum_{\ell = (\ell_1, \dots, \ell_N), |\ell| \leq p - |J|} \frac{(p - |J|)!}{\ell! (p - |J| - |\ell|)!} \\ & \quad \langle \theta_{V, \gamma}(u), u \rangle^{p - |J| - |\ell|} \wedge \prod_{1 \leq s \leq N} (\gamma \omega_A - \rho_s^{-1} \theta_{\Delta_s, h_s})^{\ell_s} (|t_s|^{-2} |e_s^*(u)|^2)^{\ell_s} \\ & \quad \wedge \bigwedge_{j \in J} \frac{|t_j|^{-2} |e_j^*(u)|^2}{1 + |t_j|^2} \frac{i dt_j \wedge \bar{d}t_j}{2\pi} + \text{negligible terms,} \end{aligned}$$

where  $J = \emptyset$  produces the “main term”  $\langle \theta_{V, \gamma}(u), u \rangle^p$  to be integrated over  $X$ , and the wedge products  $\bigwedge_{j \in J} \dots \frac{i dt_j \wedge \bar{d}t_j}{2\pi}$ ,  $J \neq \emptyset$ , will lead in the limit as  $(\varepsilon_j)_{j \in J} \rightarrow 0$  to integrals calculated on the transverse intersections  $\Delta_J := \bigcap_{j \in J} \Delta_j$ . Since

$$(|t_s|^{-2} |e_s^*(u)|^2)^{\ell_s} \leq \left( \frac{|t_s|^{-2}}{1 + |t_s|^{-2}} \right)^{\ell_s} = \left( \frac{1}{(1 + \varepsilon_s^{-1} |\sigma_s|^{2-2/\rho_s} |\nabla_s \sigma_s|^2)} \right)^{\ell_s} \leq 1$$

converges to 0 pointwise to 0 on  $X \setminus \sigma_s^{-1}(0)$  when  $\ell_s > 0$ , it is clear that terms for which there is an index  $s \notin J$  such that  $\ell_s > 0$  will give in the limit a zero contribution, thanks to the bounded convergence theorem applied on  $X$  or on the  $\Delta_J$ 's. We can thus take  $\ell = (\ell_j)_{j \in J}$ . In the end, we get

$$\begin{aligned} & \langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p \\ &= \sum_{J \subset \{1, \dots, N\}, |J| \leq p} \sum_{\ell = (\ell_j)_{j \in J}, |\ell| \leq p - |J|} \frac{p!}{\ell! (p - |J| - |\ell|)!} \langle \theta_{V, \gamma}(u), u \rangle^{p - |J| - |\ell|} \\ (5.36) \quad & \wedge \bigwedge_{j \in J} \left( (\gamma \omega_A - \rho_j^{-1} \theta_{\Delta_j, h_j})^{\ell_j} \wedge \frac{(|t_j|^{-2} |e_j^*(u)|^2)^{\ell_j + 1}}{1 + |t_j|^2} \frac{i dt_j \wedge \bar{d}t_j}{2\pi} \right) \\ & + \text{asymptotically vanishing terms.} \end{aligned}$$

The square of the norm of the linear form  $u \mapsto t_j^{-1} e_j^*(u)$  is  $(1 + |t_j|^2)^{-1}$ , thus by Lemma 5.24, a partial integration  $\int_{S(V\langle D \rangle)} (\dots) d\mu(u)$  produces a factor  $(1 + |t_j|^2)^{-(\ell_j + 2)} \frac{i dt_j \wedge \bar{d}t_j}{2\pi}$ . An elementary calculation gives

$$\int_{t \in \mathbb{C}, \text{Arg } t \in ]-\eta, \eta[} \frac{1}{(1 + |t|^2)^{\ell + 2}} \frac{i dt \wedge \bar{d}t}{2\pi} = \frac{\eta}{(\ell + 1)\pi},$$

for every  $\ell \geq 0$ , and the current

$$\int_{t_j \in \mathbb{C}, \text{Arg } t_j \in ]-(1-1/\rho_j)\pi, (1-1/\rho_j)\pi[} \bigwedge_{j \in J} \frac{1}{(1+|t_j|^2)^{\ell_j+2}} \frac{i dt_j \wedge d\bar{t}_j}{2\pi}$$

converges weakly to the effective cycle  $\prod_{j \in J} (\ell_j + 1)^{-1} (1 - 1/\rho_j) [\Delta_J]$ . Therefore Proposition 5.29 (a) implies

$$\begin{aligned} & \lim_{\varepsilon_j \rightarrow 0} \int_X \int_{v \in S(V\langle D \rangle)} \langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p \wedge \beta d\mu(u) \\ & \geq \sum_{J \subset \{1, \dots, N\}} \sum_{\ell = (\ell_j)_{j \in J}, |\ell| \leq p - |J|} \frac{(r-1)!}{(p+r-1)!} \frac{p!}{\prod (\ell_j + 1)! (p - |J| - |\ell|)!} \\ & \quad \prod_{j \in J} \left(1 - \frac{1}{\rho_j}\right) \int_{\Delta_J} (\text{Tr } \theta_{V, \gamma})^{p-|J|-|\ell|} \wedge \bigwedge_{j \in J} (\gamma \omega_A - \rho_j^{-1} \theta_{\Delta_j, h_j})^{\ell_j} \wedge \beta. \end{aligned}$$

Now, we consider the case of a product of terms  $\langle \theta_{V_s, \gamma_s, \varepsilon_s}(u_s), u_s \rangle^{p_s}$  associated with orbifold divisors  $D_s = \sum_{1 \leq j \leq N} (1 - 1/\rho_{s,j}) \Delta_j$ , and metrics  $h_{V\langle D_s \rangle, \varepsilon_s}$ ,  $\varepsilon_s = (\varepsilon_{s1}, \dots, \varepsilon_{sN}) \rightarrow 0$ , for  $1 \leq s \leq k$ . We take the wedge product of the expansions given by (5.36) for each factor  $D_s$ , and replace  $\ell_j$  by  $\ell'_j - 1$ ,  $\ell'_j \geq 1$ . We obtain a similar more general expression, given by a summation on all disjoint subsets  $J_1, \dots, J_k \subset \{1, 2, \dots, N\}$  and all indices  $\ell = (\ell_j)_{j \in J_1 \sqcup \dots \sqcup J_k}$ ,  $\ell_j \geq 1$

$$\begin{aligned} & \langle \theta_{V_1, \gamma_1, \varepsilon_1}(u_1), u_1 \rangle^{p_1} \wedge \dots \wedge \langle \theta_{V_k, \gamma_k, \varepsilon_k}(u_k), u_k \rangle^{p_k} \\ & = \sum_{J_1 \sqcup \dots \sqcup J_k \subset \{1, \dots, N\}} \sum_{\ell_j \geq 1, \sum_{j \in J_s} \ell_j \leq p_s} \bigwedge_{1 \leq s \leq k} \frac{p_s!}{\prod_{j \in J_s} \ell_j! (p_s - \sum_{j \in J_s} \ell_j)!} \left( \right. \\ & \quad \left. \langle \theta_{V, \gamma_s}(u_s), u_s \rangle^{p_s - \sum_{j \in J_s} \ell_j} \wedge \bigwedge_{j \in J_s} \left( (\gamma_s \omega_A - \rho_{s,j}^{-1} \theta_{\Delta_j, h_j})^{\ell_j - 1} \wedge \frac{(|t_j|^{-2} |e_j^*(u_s)|^2)^{\ell_j}}{1 + |t_j|^2} \frac{i dt_j \wedge d\bar{t}_j}{2\pi} \right) \right) \\ & \quad + \text{asymptotically vanishing terms.} \end{aligned}$$

The arguments explained above for the case of a single term  $\langle \theta_{V\langle D \rangle, \gamma, \varepsilon}(u), u \rangle^p$  can be generalized to such products, and easily imply the lower bound stated in Proposition 5.32. The upper bound is proved in a similar way, except that we use 5.29 (b) instead of 5.29 (a), and merely need a Griffiths semi-positivity condition for  $\theta_{V, \gamma}$  instead of strong semi-positivity. One has to observe that this bound involves quantities of the form

$$\bigwedge_{s=1}^k \left( \langle \theta_s(u_s), u_s \rangle^{p_s - m_s} \right) \wedge \prod_{1 \leq j \leq m_s} |\ell'_{sj}(u_s)|^2 \beta', \quad \theta_s \geq_G 0, \quad \beta' \geq_S 0.$$

A priori, the inequality provided by 5.29 (b) for each integral  $\int_{u_s \in S(V\langle D_s \rangle)}$  is merely a weak inequality, but we are anyway integrating products of strongly semi-positive forms, so the expected inequalities still hold, and we can apply the Fubini theorem without trouble. By 5.29, the constants  $(r-1)! p_s!$  have to be replaced by  $(r-1)! p_s!^2$  for the upper bound.  $\square$

**5.37. Remark.** One of the technical difficulties is that, strictly speaking, the fourth line term of  $Q_{\gamma, \varepsilon, K}$  in (5.33) is not strongly semi-positive, while the other terms are, if we assume

$\gamma \Theta_{A,h_A} \otimes \text{Id} - \Theta_{V,h_V} >_S 0$ , as implied by the condition  $\gamma > \tilde{\gamma}_{V,D}$ . However, near any point of  $\Delta_j \setminus \bigcup_{\ell \neq j} \Delta_\ell$ , we have seen that the fourth line term has the same limit as the strongly semi-positive rank one tensor

$$\frac{\varepsilon_j |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j |\sigma_j|^{-2+2/\rho_j} |\nabla_j \sigma_j|^2} \left| (1 - 1/\rho_j) \sigma_j^{-1} \nabla_j \sigma_j(\xi) \nabla_j \sigma_j(u) \right|^2.$$

The latter converges weakly in the sense of currents to the integral on  $\Delta_j$  of a strongly semi-positive term; inductively, via the iterated limit process, the remaining terms combining several components  $\Delta_j$  also produce strongly semi-positive terms with support in higher codimensional strata  $\Delta_J = \bigcap_{j \in J} \Delta_j$ . It should be observed that the various components  $\Delta_j$  never produce any self-intersection  $(\Delta_j)^\ell$  in  $\bigwedge \langle \theta_{V_s, \gamma_s, \varepsilon_s}(u_s), u_s \rangle^{p_s}$ , as a consequence of the fact that the principal term of the  $\Delta_j$  contribution is a rank one  $(1, 1)$ -form.

What is a bit surprising in proposition 5.32 is that, in spite of the fact that we are integrating non closed and metric dependent forms, the limits of all integrals as  $\varepsilon \rightarrow 0$  admit lower and upper bounds that are purely cohomological, and can be expressed solely in terms of well understood Chern classes. This will also be true for the related Morse integrals in § 6. It would be desirable to have a more geometric explanation of this phenomenon, at least for the quest of an algebraic proof; at this stage, and for this very reason, our proof remains highly transcendental.

## 6. Non probabilistic estimates of the Morse integrals

### 6.A. Case of general directed orbifolds

The non probabilistic estimate uses more explicit curvature inequalities and has the advantage of producing results also in the general orbifold case. Let us fix an ample line bundle  $A$  on  $X$  equipped with a smooth hermitian metric  $h_A$  such that  $\omega_A := \Theta_{A,h_A} > 0$ , and let  $\gamma$  be the infimum of values  $\lambda \in \mathbb{R}_+$  such that

$$(6.1) \quad \lambda \omega_A \otimes \text{Id}_V - \Theta_{V,h_V} >_G 0 \quad (\text{resp. } >_S 0).$$

Instead of exploiting a Monte Carlo convergence process for the curvature tensor as was done in § 4.B, we are going to use a more precise lower bound of the curvature tensor  $\Theta_{L_{\tau,k,\varepsilon}}$  of the orbifold rank 1 sheaf

$$(6.2) \quad L_{\tau,k} := \mathcal{O}_{X_k(V\langle D \rangle)}(1) \otimes \pi_k^* \mathcal{O}_X(-\tau A).$$

Our formulas 3.20 (a,b) become

$$(6.3_1) \quad \Theta_{L_{\tau,k,\varepsilon}} = \omega_{r,k,b}(\xi) + g_{k,\varepsilon}(z, x, u) - \tau \omega_A(z),$$

where

$$(6.3_2) \quad g_{k,\varepsilon}(z, x, u) = \sum_{s=1}^k \frac{x_s}{s} \theta_{s,\varepsilon}(z, u_s),$$

$$(6.3_3) \quad \theta_{s,\varepsilon}(z, u_s) = \frac{i}{2\pi} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}^{(s,\varepsilon)}(z) u_{s,\lambda} \bar{u}_{s,\mu} dz_i \wedge d\bar{z}_j = -\langle \theta_{V\langle D^{(s)} \rangle, h_{V\langle D \rangle, \varepsilon, K}}(u_s), u_s \rangle.$$

By Proposition 5.21 and Remark ..., we have

$$\theta_{s,\varepsilon}(u_s) \simeq -\theta_V(u_s) + \sum_{1 \leq j \leq N} \left( -\frac{1}{\rho_j^{(s)}} \theta_{\Delta_j, h_j} |t_j|^{-2} |e_j^*(u_s)|^2 + \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u_s)|^2 \right).$$

As  $\sum_{1 \leq s \leq k} x_s = 1$ , we get

$$(6.4_0) \quad \Theta_{L_{\tau,k,\varepsilon}} \simeq \omega_{r,k,b} + \alpha_{\gamma,\varepsilon} - \beta_{\gamma,\varepsilon}$$

where

$$(6.4_1) \quad \alpha_{\gamma,\varepsilon}(z, x, u) = \sum_{s=1}^k \frac{x_s}{s} \alpha_{s,\gamma,\varepsilon}(z, u_s), \quad \beta_{\varepsilon}(z, x, u) = \sum_{s=1}^k \frac{x_s}{s} \beta_{s,\gamma,\varepsilon}(z, u_s),$$

$$(6.4_2) \quad \alpha_{s,\gamma,\varepsilon}(z, u_s) = \gamma \omega_A(z) - \theta_V(u_s) + \sum_{1 \leq j \leq N} \frac{|t_j|^{-2}}{1 + |t_j|^2} |dt_j(\xi)|^2 |e_j^*(u_s)|^2 \geq 0,$$

$$(6.4_3) \quad \beta_{s,\gamma,\varepsilon}(z, u_s) = \sum_{1 \leq j \leq N} \frac{1}{\rho_j^{(s)}} \theta_{\Delta_j, h_j} |t_j|^{-2} |e_j^*(u_s)|^2 + (s\tau + \gamma) \omega_A(z) \geq 0.$$

Then (6.4<sub>\*</sub>) and the inequalities used for (4.2), especially Lemma 2.3 and Proposition 3.10 (b), lead to

$$(6.5) \quad \begin{aligned} & \frac{1}{(n + kr - 1)!} \int_{X_k(V\langle D \rangle)(L_{\tau,k,\leq 1})} \Theta_{L_{\tau,k,\varepsilon}}^{n+kr-1} \\ & \sim \frac{1}{n! k!^r (kr - 1)!} \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} \mathbb{1}_{\alpha_{\varepsilon} - \beta_{\varepsilon} \leq 1} (\alpha_{\gamma,\varepsilon} - \beta_{\gamma,\varepsilon})^n d\nu_{k,r}(x) d\mu(u) \\ & \geq I_0(\varepsilon) - I_1(\varepsilon), \end{aligned}$$

where

$$(6.5_0) \quad I_0(\varepsilon) = \frac{1}{n! k!^r (kr - 1)!} \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} \alpha_{\gamma,\varepsilon}^n d\nu_{k,r}(x) d\mu(u),$$

$$(6.5_1) \quad I_1(\varepsilon) = \frac{1}{n! k!^r (kr - 1)!} \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} n \alpha_{\gamma,\varepsilon}^{n-1} \wedge \beta_{\gamma,\varepsilon} d\nu_{k,r}(x) d\mu(u).$$

As we will see, the limit of (6.5) as  $\varepsilon \rightarrow 0$  admits a natural lower bound that can be expressed solely in terms of geometric data: Chern class of  $V$ , Chern classes of the components  $\Delta_j$ , and multiplicities  $\rho_j$  of the orbifold divisor  $D$ . At this stage, we invoke two types of lower bounds for  $\alpha_{\varepsilon}^n$ , a sharper form that requires strong semi-positivity, and a weaker form for which Griffiths semi-positivity is sufficient. The sharper form uses a complete expansion of  $\alpha_{\varepsilon}^n$  by Newton's multinomial formula and is valid for every  $k \geq 1$ . We write

$$(6.6) \quad \alpha_{\gamma,\varepsilon}^n = \sum_{p \in \mathbb{N}^k, |p|=n} \frac{n!}{p_1! \dots p_k!} \prod_{s=1}^k \left( \frac{x_s}{s} \alpha_{s,\gamma,\varepsilon}(z, u_s) \right)^{p_s}.$$

The weaker bound consists of keeping only the terms for which  $p_s = 0$  or 1; the existence of such terms requires  $k \geq n$ . As  $\alpha_{s,\gamma,\varepsilon} \geq 0$ , we find in any case

$$(6.6') \quad \alpha_{\gamma,\varepsilon}^n \geq \sum_{1 \leq s_1 < \dots < s_n \leq k} n! \frac{x_{s_1} \dots x_{s_n}}{s_1 \dots s_n} \alpha_{s_1,\gamma,\varepsilon}(z, u_{s_1}) \wedge \dots \wedge \alpha_{s_n,\gamma,\varepsilon}(z, u_{s_n}).$$

The complete expansion (6.6) implies

$$\begin{aligned} & \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} \alpha_{\gamma,\varepsilon}(z, x, u)^n d\nu_{k,r}(x) d\mu(u_1) \dots d\mu(u_k) \\ & \geq \sum_{p \in \mathbb{N}^k, |p|=n} \int_{x \in \Delta^{k-1}} \frac{n! x_1^{p_1} \dots x_k^{p_k}}{\prod_{s=1}^k p_s! s^{p_s}} d\nu_{k,r}(x) \int_{z \in X} \int_{u \in \prod S(V\langle D^{(s)} \rangle)} \bigwedge_{s=1}^k \alpha_{s,\gamma,\varepsilon}(u_s)^{p_s} d\mu(u_s). \end{aligned}$$

By formula 3.10 (a) and an elementary calculation (cf. [Dem11, Prop. 1.13]), one gets for every  $(p_1, \dots, p_k) \in \mathbb{N}^k$

$$(6.7) \quad \int_{\Delta^{k-1}} x_1^{p_1} \dots x_k^{p_k} d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k} \frac{\prod_{1 \leq s \leq k} (p_s + r - 1)!}{(\sum_{1 \leq s \leq k} p_s + kr - 1)!},$$

and in particular, for  $k \geq n$

$$(6.7') \quad \int_{\Delta^{k-1}} x_1 \dots x_n d\nu_{k,r}(x) = \frac{(kr-1)! r^n}{(n + kr - 1)!}.$$

(Here only the case  $\sum_{1 \leq s \leq k} p_s = n$  is needed). By (6.4<sub>2</sub>), we have

$$\alpha_{s,\gamma,\varepsilon}(u_s)^p = \sum_{\substack{J \subset \{1, \dots, N\} \\ |J| \leq p}} \frac{p!}{(p - |J|)!} (\gamma \omega_A - \theta_V(u_s))^{p - |J|} \wedge \bigwedge_{j \in J} |t_j|^{-2} |e_j^*(u_s)|^2 \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{1 + |t_j|^2},$$

therefore, for any  $k$ -tuple  $p = (p_1, \dots, p_k)$ ,

$$\begin{aligned} & \bigwedge_{s=1}^k \langle \alpha_{s,\gamma,\varepsilon}(u_s), u_s \rangle^{p_s} = \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \prod_{1 \leq s \leq k} \frac{p_s!}{(p_s - |J_s|)!} (\gamma \omega_A - \theta_V(u_s))^{p_s - |J_s|} \wedge \bigwedge_{j \in J_s} |t_j|^{-2} |e_j^*(u_s)|^2 \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{1 + |t_j|^2}. \end{aligned}$$

Under the assumption  $\gamma \omega_A \otimes \text{Id} - \Theta_V >_S 0$ , Proposition ... and (6.7) imply

$$\begin{aligned} & \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} \alpha_{\gamma,\varepsilon}(z, x, u)^n d\nu_{k,r}(x) d\mu(u_1) \dots d\mu(u_k) \\ & \geq \sum_{p \in \mathbb{N}^k, |p|=n} \int_{\Delta^{k-1}} \frac{n! x_1^{p_1} \dots x_k^{p_k}}{\prod_{s=1}^k p_s! s^{p_s}} d\nu_{k,r}(x) \int_X \int_{\prod S(V\langle D^{(s)} \rangle)} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \bigwedge_{1 \leq s \leq k} \frac{p_s!}{(p_s - |J_s|)!} (\gamma \omega_A |u_s|^2 - \theta_V(u_s))^{p_s - |J_s|} \wedge \bigwedge_{j \in J_s} |t_j|^{-2} |e_j^*(u_s)|^2 \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{1 + |t_j|^2} d\mu(u_s) \\ & \geq \sum_{p \in \mathbb{N}^k, |p|=n} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \frac{n!}{\prod_{s=1}^k p_s! s^{p_s}} \frac{(kr-1)! \prod_{1 \leq s \leq k} (p_s + r - 1)!}{(r-1)!^k (n + kr - 1)!} \times \\ & \quad \int_X \bigwedge_{1 \leq s \leq k} \frac{(r-1)!}{(p_s + r - 1)!} \frac{p_s!}{(p_s - |J_s|)!} (r\gamma \omega_A - \text{Tr } \theta_V)^{p_s - |J_s|} \wedge \bigwedge_{j \in J_s} \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{(1 + |t_j|^2)^2} d\mu(u_s) \\ & = \sum_{p \in \mathbb{N}^k, |p|=n} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \frac{n!}{\prod_{s=1}^k (p_s - |J_s|)! s^{p_s}} \frac{(kr-1)!}{(n + kr - 1)!} \times \\ & \quad \int_X \bigwedge_{1 \leq s \leq k} (r\gamma \omega_A - \text{Tr } \theta_V)^{p_s - |J_s|} \wedge \bigwedge_{j \in J_s} \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{(1 + |t_j|^2)^2}. \end{aligned}$$



Finally, we take the iterated limit  $I_0 = \lim_{\varepsilon \rightarrow 0} I_0(\varepsilon)$  and find

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{n! k!^r (kr - 1)!} \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times \prod S(V\langle D^{(s)} \rangle)} \alpha_{\gamma, \varepsilon}(z, x, u)^n d\nu_{k,r}(x) d\mu(u_1) \dots d\mu(u_k) \\ & \geq \frac{1}{n! k!^r (n + kr - 1)!} \sum_{p \in \mathbb{N}^k, |p|=n} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \frac{n!}{\prod_{s=1}^k (p_s - |J_s|)! s^{p_s}} \times \\ & \quad \prod_{s=1}^k \prod_{j \in J_s} \left(1 - \frac{1}{\rho_j^{(s)}}\right) \int_{\Delta_{J_1 \amalg \dots \amalg J_k}} (r\gamma c_1(A) + c_1(V^*))^{n - \sum |J_s|}. \end{aligned}$$

This can be rewritten

$$(6.8) \quad I_0 \geq \frac{1}{n! k!^r (n + kr - 1)!} \left( \sum_{s=1}^k \frac{1}{s} \left( r\gamma c_1(A) + c_1(V^*) + \sum_{j=1}^N \left(1 - \frac{1}{\rho_j^{(s)}}\right) \Delta_j \right) \right)^{(n)},$$

where the exponent  $\bullet^{(n)}$  means that in the expansion of the  $n$ -th power, all self-intersections of the components  $\Delta_j$  are omitted, i.e., one formally sets  $\Delta_j^2 = 0$ . In the case we only assume  $\gamma \omega_A \otimes \text{Id} - \Theta_V >_G 0$ , we can only keep products of terms corresponding to  $n$  distinct values of the indices  $s$ , and get for  $k \geq n$  a lower bound

$$(6.8') \quad I_0 \geq \frac{1}{k!^r (n + kr - 1)!} \sum_{1 \leq s_1 < \dots < s_n \leq k} \frac{1}{s_1 \dots s_n} \sum_{J \subset \{1, \dots, N\}} \frac{n!}{(n - |J|)!} \times \\ (r\gamma c_1(A) + c_1(V^*))^{n - |J|} \cdot \prod_{j \in J} \left(1 - \frac{1}{\rho_j^{(s)}}\right) \Delta_j.$$

The evaluation of

$$I_1(\varepsilon) = \frac{1}{n! k!^r (kr - 1)!} \int_{z \in X} \int_{(x,u) \in \Delta^{k-1} \times (\mathbb{S}^{2r-1})^k} n \alpha_{\gamma, \varepsilon}^{n-1} \wedge \beta_{\gamma, \varepsilon} d\nu_{k,r}(x) d\mu(u_1) \dots d\mu(u_k)$$

is very similar. One change is that we raise  $\alpha_{\gamma, \varepsilon}$  to power  $n - 1$ , so the summation in  $p$  has to be taken over multi-indices  $p$  with  $|p| = n - 1$ ; this leads to a multinomial coefficient  $(n - 1)! / \prod p_s!$ , but since we have an initial factor  $n$  anyway, we still get a formal product  $n! / \prod p_s!$ . Next, by (6.4<sub>3</sub>), the product with  $\beta_{\gamma, \varepsilon} = \sum \frac{x_q}{q} \beta_{q, \gamma, \varepsilon}$  leads to two types of terms, those containing  $(q\tau + \gamma) \frac{x_q}{q} \omega_A(z)$  and those containing  $\frac{x_q}{q} (\rho_\ell^{(q)})^{-1} \theta_{\Delta_j, h_j} |t_j|^{-2} |e_j^*(u_q)|^2$ . Finally, the factor  $(r - 1)! / (p_s + r - 1)!$  is replaced by  $p_s! (r - 1)! / (p_s + r - 1)!$ , the integral of  $x^p x_q$  produces an extra factor  $(p_q + r)$  by (6.7), and we get as well as a new factor  $(p_q + 1) / (p_q + r)$  in the application of 5.29 (b) for the integral  $\int(\dots) d\mu(u_q)$ , when  $u_q$  appears with an exponent  $p_q + 1$  instead of  $p_q$ . In this way, we find

$$\begin{aligned} I_1(\varepsilon) & \leq \frac{1}{n! k!^r (n + kr - 1)!} \sum_{p \in \mathbb{N}^k, |p|=n-1} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \frac{n! \prod_{s=1}^k p_s!}{\prod_{s=1}^k (p_s - |J_s|)! s^{p_s}} \times \\ & \quad \prod_{s=1}^k \prod_{j \in J_s} \left(1 - \frac{1}{\rho_j^{(s)}}\right) \int_X (r\gamma \omega_A - \text{Tr } \theta_V)^{n-1 - \sum |J_s|} \wedge \bigwedge_{j \in J_1 \amalg \dots \amalg J_k} \frac{i}{2\pi} \frac{dt_j \wedge \bar{d}t_j}{(1 + |t_j|^2)^2} \\ & \quad \wedge \sum_{q=1}^k \frac{1}{q} \left( (p_q + r)(q\tau + \gamma) \omega_A + (p_q + 1) \sum_{\ell=1}^N \frac{1}{\rho_\ell^{(q)}} \frac{\theta_{\Delta_\ell}}{1 + |t_\ell|^2} \right). \end{aligned}$$

When we take the limit as  $\varepsilon \rightarrow 0$ , all terms for which  $\ell \notin J_1 \amalg \dots \amalg J_k$  give a zero contribution by Lebesgue's bounded convergence theorem. Therefore, the limit  $I_1 = \lim_{\varepsilon \rightarrow 0} I_1(\varepsilon)$  admits an upper bound

$$(6.9) \quad I_1 \leq \frac{1}{k!^r (n + kr - 1)!} \sum_{p \in \mathbb{N}^k, |p|=n-1} \sum_{J_1 \amalg \dots \amalg J_k \subset \{1, \dots, N\}, |J_s| \leq p_s} \frac{\prod_{s=1}^k p_s!}{\prod_{s=1}^k (p_s - |J_s|)! s^{p_s}} \times \\ \prod_{s=1}^k \prod_{j \in J_s} \left(1 - \frac{1}{\rho_j^{(s)}}\right) \int_{\Delta_{J_1 \amalg \dots \amalg J_k}} (r\gamma c_1(A) + c_1(V^*))^{n-1-\sum |J_s|} \cdot \\ \sum_{s=1}^k \frac{1}{s} \left( (p_s + r)(s\tau + \gamma) c_1(A) + (p_s + 1) \sum_{j \in J_1 \amalg \dots \amalg J_k} \frac{c_1(\Delta_j)}{2\rho_j^{(s)}} \right).$$

The reason for the appearance of the factor 2 in  $2\rho_j^{(s)}$  is that, in the final summation of the previous formula, we have to take  $\ell = j$  for some  $j \in J_1 \amalg \dots \amalg J_k$ , and

$$\int_{t_j \in \mathbb{C}} \frac{i}{2\pi} \frac{dt_j \wedge d\bar{t}_j}{(1 + |t_j|^2)^3} = \frac{1}{2}.$$

We can summarize these results as follows.

**6.10. Theorem.** *Let  $(X, V, D)$  with  $D = \sum_{1 \leq j \leq N} (1 - 1/\rho_j) \Delta_j$  be a directed orbifold, and let  $A$  be an ample line bundle on  $X$ . We assume that  $\Delta_j$  is nef for every  $j$ , and let  $\gamma_V, \tilde{\gamma}_V$  be the infimum of real values  $\gamma \in \mathbb{R}_+$  such that*

$$\gamma \omega_A \otimes \text{Id} - \Theta_V >_G 0 \quad \text{for } \gamma > \gamma_V, \quad (\text{resp. } >_S 0 \text{ for } \gamma > \tilde{\gamma}_V),$$

with suitable smooth hermitian metrics on  $V$ . We set

$$D^{(s)} = \sum_{1 \leq j \leq N} \left(1 - \frac{1}{\rho_j^{(s)}}\right) \Delta_j \quad \text{with} \quad \rho_j^{(s)} = \max(1, \rho_j/s)$$

and take  $\gamma = \gamma_V$  (resp.  $\gamma = \tilde{\gamma}_V$ ). Then the  $m$ -th power of the orbifold line bundle

$$L_{\tau, k} = \mathcal{O}_{X_k(V\langle D \rangle)}(1) \otimes \pi_k^* \mathcal{O}(-\tau A)$$

admits for all  $k \geq n$  and  $\tau \in \mathbb{Q}_+$  a number of sections bounded below asymptotically by

$$\frac{m^{n+kr-1}}{(n + kr - 1)!} \int_{X_k(V\langle D \rangle)(L_{\tau, k}, \leq 1)} \Theta_{L_{\tau, k}, \varepsilon}^{n+kr-1} \geq m^{n+kr-1} (I_0 - I_1 - o(1)),$$

where  $I_0$  satisfies the estimates (6.8) (resp. (6.8')), and  $I_1$  satisfies (6.9).

Especially, for  $m \gg 1$ , we have a lot of sections in

$$H^0(X_k(V\langle D \rangle), L_{\tau, k}^{\otimes m}) = H^0(X, E_{k, m} V^* \langle D \rangle \otimes \mathcal{O}_X(-m\tau A))$$

whenever  $I_0 - I_1 > 0$ ; by Corollary 1.11, then  $I_0 - I_1$  is non positive, we still get a non trivial lower bound for the difference  $h^0(X_k(V\langle D \rangle), L_{\tau, k}^{\otimes m}) - h^1(X_k(V\langle D \rangle), L_{\tau, k}^{\otimes m})$ .

### 6.B. Compact case (no boundary divisor)

We address here the compact case, i.e.  $D = 0$ , and deal only with the Griffiths semi-positivity assumption. In that case, we take  $k \geq n$  and  $\gamma_s = \gamma_V$  for all  $s$ , and (6.8'), (6.9) imply the simple bounds

$$(6.11) \quad I_0 \geq \frac{H_{n,k}}{k!^r (n + kr - 1)!} (r\gamma_V c_1(A) + c_1(V^*))^n,$$

$$(6.12) \quad I_1 \leq \frac{H'_{n,k,\tau}}{k!^r (n + kr - 1)!} (r\gamma_V c_1(A) + c_1(V^*))^{n-1} \cdot c_1(A),$$

hence

$$I_0 - I_1 \geq \frac{H_{n,k}}{k!^r (n + kr - 1)!} (r\gamma_V c_1(A) + c_1(V^*))^{n-1} \cdot \left( c_1(V^*) - \left( \frac{H'_{n,k,\tau}}{H_{n,k}} - r\gamma_V \right) c_1(A) \right),$$

where

$$(6.13) \quad H_{n,k} = \sum_{1 \leq s_1 < \dots < s_n \leq k} \frac{1}{s_1 \dots s_n},$$

$$(6.13') \quad H'_{n,k,\tau} = \sum_{p \in \mathbb{N}^k, |p|=n-1} \left( \prod_{1 \leq s \leq k} \frac{1}{s^{p_s}} \right) \left( \sum_{1 \leq s \leq k} \frac{1}{s} (p_s + r)(s\tau + \gamma_V) \right).$$

In this way, we get a sufficient condition for the existence of jet differentials that is substantially sharper than the one stated in [Dem12] (thanks to much improved estimates of the integrals involved in the calculation).

**6.14. Corollary.** *The line bundle  $L_{\tau,k}$  on  $X_k(V)$  is big as soon as*

$$c_1(V^*) - \left( \frac{H'_{n,k,\tau}}{H_{n,k}} - r\gamma_V \right) c_1(A) > 0.$$

Let us give here some estimates about these constants. Let us put

$$H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}, \quad \hat{H}_{n-1,k} = \sum_{p \in \mathbb{N}^k, |p|=n-1} \prod_{s=1}^k \frac{1}{s^{p_s}}.$$

We have  $H_{n,k} \geq \frac{1}{n!}$  for  $k \geq n$ , and asymptotically when  $k \rightarrow +\infty$ ,  $H_k \sim \log k$ . By looking at terms with distinct factors in  $H_k^n$ , we find

$$(6.15) \quad \frac{1}{n!} H_k^n \geq H_{n,k} \geq \frac{1}{n!} \left( H_k^n - \binom{n}{2} \left( \sum_{s=1}^k \frac{1}{s^2} \right) H_k^{n-2} \right) \underset{k \rightarrow +\infty}{\sim} \frac{1}{n!} (H_k)^n,$$

hence  $H_{n,k} \sim \frac{1}{n!} (\log k)^n$  as  $k \rightarrow +\infty$ . Next, we have

$$\hat{H}_{n-1,k} \leq \sum_{0 \leq p_2, \dots, p_k < +\infty} \frac{1}{2^{p_2}} \frac{1}{3^{p_3}} \dots \frac{1}{k^{p_k}} \leq \frac{1}{1-1/2} \frac{1}{1-1/3} \dots \frac{1}{1-1/k} = k,$$

and also

$$H_{n-1,k} \leq \widehat{H}_{n-1,k} \leq H_k^{n-1} \leq (1 + \log k)^{n-1}.$$

In fact, by slitting  $p \in \mathbb{N}^k$  between those multi-indices for which  $p_s = 0$  or 1, and those for which there is at least one index  $p_s \geq 2$ , we get

$$\widehat{H}_{n-1,k} \leq \frac{1}{(n-1)!} H_k^{n-1} + \left( \sum_{s=1}^k \frac{1}{s^2} \right) \widehat{H}_{n-3,k}.$$

This implies

$$(6.16) \quad \widehat{H}_{n-1,k} \sim \frac{1}{(n-1)!} H_k^{n-1}$$

as  $k \rightarrow +\infty$ . Since

$$\sum_{1 \leq s \leq k} \frac{1}{s} (p_q + r)(s\tau + \gamma_V) \leq (n-1 + kr)\tau + (n-1 + r H_k)\gamma_V,$$

we obtain

$$H'_{n,k,\tau} \leq \widehat{H}_{n-1,k} \left( (n + kr - 1)\tau + (n - 1 + r H_k)\gamma_V \right),$$

therefore

$$\limsup_{k \rightarrow +\infty} \left( \frac{H'_{n,k,0}}{H_{n,k}} - r\gamma_V \right) \leq nr\gamma_V - r\gamma_V = (n-1)r\gamma_V.$$

For  $k \gg 1$  and  $\tau > 0$  very small, the above inequalities imply that  $L_{\tau,k}$  is big as soon as

$$(6.17) \quad c_1(V^*) - (n-1)r\gamma_V c_1(A) > 0.$$

This is more restrictive than the condition  $c_1(V^*) > 0$  we would get for  $k \gg 1$  very large by the probabilistic estimate: we have lost here a factor  $n$  in the process, probably as a consequence of the use of Corollary 2.4 (even though the latter is optimal). Still, the result is interesting in view of the fact that we get explicit bounds, even for low values of  $k$ . Since  $H_{n,k} \geq H_{n,n} = \frac{1}{n!}$ , we obtain for all  $k \geq n$

$$\frac{H'_{n,k,\tau}}{H_{n,k}} - r\gamma_V \leq kn! \left( (n + kr - 1)\tau + (n - 1 + r H_k)\gamma_V \right) - r\gamma_V.$$

**6.18. Example.** In the case where  $X$  is a smooth hypersurface of  $\mathbb{P}^{n+1}$  of degree  $d$  and  $V = T_X$ , we have  $r = n$  and  $V^* = \mathcal{O}(d - n - 2)$ . Also, we can take  $A = \mathcal{O}(1)$  and  $\gamma_V = 2$ , since the surjective morphisms

$$T_{\mathbb{P}^{n+1}|X} \rightarrow T_X \rightarrow V^*$$

imply  $V^* \otimes \mathcal{O}(2) \geq_G 0$  (see Lemma 6.20 below). Therefore, for  $k \geq n$ ,  $L_{\tau,k}$  is big as soon as

$$d + n - 2 \geq kn! \left( (n + kn - 1)\tau + 2(n - 1 + n H_k) \right).$$

This lower bound is unfortunately far from being optimal, especially when  $k$  is large (for  $k \gg 1$ , we know that  $d \geq n + 3$  is sufficient).

### 6.C. Logarithmic case

In the logarithmic situation, we have  $\rho_j^{(s)} = \infty$  for all  $j, s$ , and the estimates for  $I_0$  and  $I_1$  become

$$(6.19_0) \quad I_0 \geq \sum_{J \subset \{1, \dots, N\}, |J| \leq n} \frac{H_k^n}{k!^r (n + kr - 1)!} \frac{n!}{(n - |J|)!} (r\tilde{\gamma}_V c_1(A) + c_1(V^*))^{n - |J|} \cdot \Delta_J,$$

$$(6.19'_0) \quad I_0 \geq \sum_{J \subset \{1, \dots, N\}, |J| \leq n} \frac{H_{n,k}}{k!^r (n + kr - 1)!} \frac{n!}{(n - |J|)!} (r\gamma_V c_1(A) + c_1(V^*))^{n - |J|} \cdot \Delta_J,$$

$$(6.19_1) \quad I_1 \leq \sum_{J \subset \{1, \dots, N\}, |J| \leq n-1} \frac{H''_{n,|J|,\tau}}{k!^r (n + kr - 1)!} (r\gamma c_1(A) + c_1(V^*))^{n-1-|J|} \cdot c_1(A) \cdot \Delta_J,$$

where

$$H''_{n,|J|,\tau} = \sum_{p \in \mathbb{N}^k, |p|=n-1} \sum_{J_1 \sqcup \dots \sqcup J_k = J, |J_s| \leq p_s} \frac{\prod_{s=1}^k p_s!}{\prod_{s=1}^k (p_s - |J_s|)! s^{p_s}} \sum_{s=1}^k \frac{1}{s} (p_s + r)(s\tau + \gamma)$$

(Clearly  $H''_{n,|J|,\tau}$  depends only on the cardinal of  $J$ ).

**6.20. Lemma.** *When  $X = \mathbb{P}^n$ ,  $V = T_{\mathbb{P}^n}$ , one can take  $\gamma_V = 2$  and  $\tilde{\gamma}_V = n + 1$ , in fact*

$$T_{\mathbb{P}^n}^* \otimes \mathcal{O}(2) \geq_G 0 \quad \text{and} \quad T_{\mathbb{P}^n}^* \otimes \mathcal{O}(n+1) \geq_S 0.$$

*Proof.* The Euler exact sequence shows that  $T_{\mathbb{P}^n} = Q \otimes \mathcal{O}(1)$  where  $Q = \mathbb{C}^{n+1}/\mathcal{O}(-1)$  is the tautological quotient bundle on  $\mathbb{P}^n$ , which is such that  $\det(Q) = \mathcal{O}(1)$ . Then

$$T_{\mathbb{P}^n}^* = Q^* \otimes \mathcal{O}(-1) = \bigwedge^{n-1} Q \otimes \det Q^* \otimes \mathcal{O}(-1) = (\bigwedge^{n-1} Q) \otimes \mathcal{O}(-2).$$

In particular  $T_{\mathbb{P}^n}^* \otimes \mathcal{O}(2) \geq_G 0$ . Moreover, Proposition 5.5 shows that  $Q \otimes \mathcal{O}(1) \geq_S 0$ , hence

$$T_{\mathbb{P}^n}^* \otimes \mathcal{O}(n+1) = \bigwedge^{n-1} (Q \otimes \mathcal{O}(1)) \geq_S 0. \quad \square$$

In case we take  $X = \mathbb{P}^n$ ,  $V = T_{\mathbb{P}^n}$  and  $D$  a smooth irreducible divisor of degree  $d$ , it is more interesting to use the strong semi-positivity bound. For  $r = n$ ,  $\tilde{\gamma}_V = n + 1$  and  $c_1(T_{\mathbb{P}^n}^*) = n + 1$ , we find

$$\begin{aligned} I_0 &\geq \frac{H_k^n}{k!^r (n + kr - 1)!} (n + 1)^{2n-2} ((n + 1)^2 + nd) \\ I_1 &\leq \frac{\hat{H}_{n-1,k}}{k!^r (n + kr - 1)!} (n + 1)^{2n-4} ((n + 1)^2 + (n - 1)d) \times \\ &\quad ((n - 1 + kn)\tau + (n - 1 + n H_k)(n + 1)). \end{aligned}$$

we can still take  $\gamma = 2$  by Proposition 5.8. Then a similar degree bound  $d + n - 1 \geq 2n^2 n! (n - 1 + n H_n)$  holds in that case.

### 6.D. Case of orbifold structures on projective $n$ -space

An interesting orbifold example is the case when  $X = \mathbb{P}^n$ ,  $V = T_X$ ,  $A = \mathcal{O}(1)$  and  $D = \sum (1 - \frac{1}{\rho_j}) \Delta_j$  is a normal crossing divisor, with components  $\Delta_j$  of degree  $d_j$ . Since

$$D^{(s)} = \sum_j \left(1 - \frac{s}{\rho_j}\right)_+ \Delta_j,$$

we have

$$\det V^* \langle D^{(s)} \rangle = \mathcal{O}_{\mathbb{P}^n} \left( -n - 1 + \sum_j d_j (1 - s/\rho_j)_+ \right)$$

and the associated curvature form is

$$\Theta_s = \left( -n - 1 + \sum_j d_j (1 - s/\rho_j)_+ \right) \omega_A.$$

Moreover, by Proposition 5.8, we have

$$\Theta_{V^* \langle D^{(s)} \rangle} + \gamma_s \omega_{\text{FS}} \otimes \text{Id} >_G 0$$

as soon as  $\gamma_s > 2$  and  $\gamma_s > \max_j (d_j / \max(\rho_j/s, 1))$  for all components  $\Delta_j$  in  $D^{(s)}$ . We can take for instance  $\gamma_s > st$  where  $t = \max(\max_j (d_j/\rho_j), 2)$ . Then, for  $k = n$  and  $\varepsilon \in \mathbb{Q}_+$  small, the estimate (6.14) guarantees the existence of jet differentials under the complicated condition

$$(6.15) \quad \prod_{s=1}^n \left( nst - n - 1 + \sum_j d_j (1 - s/\rho_j)_+ \right) > \frac{n(2n-1)!}{(n-1)!} \times \\ nt \left( \sum_{1 \leq s \leq n} \frac{1}{s} \left( nst - n - 1 + \sum_j d_j (1 - s/\rho_j)_+ \right) \right)^{n-1}.$$

If we take  $\rho_j \geq \rho > n$ , then  $(1 - s/\rho_j)_+ \geq 1 - s/\rho$  for  $s \leq n$ , and as  $nst - n - 1 \geq 0$  and  $\sum_{1 \leq s \leq n} \frac{1}{s} (nst - n - 1) \leq n^2 t$ , we get a sufficient condition

$$\prod_{s=1}^n \left( \left(1 - \frac{s}{\rho}\right) \sum_j d_j \right) > \frac{n^2 (2n-1)!}{(n-1)!} \times t \left( n^2 t + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \sum_j d_j \right)^{n-1}.$$

The latter condition is satisfied if  $\sum_j d_j \geq c_n t \prod_{s=1}^n \left(1 - \frac{s}{\rho}\right)^{-1}$  with

$$c_n = \frac{n^2 (2n-1)!}{(n-1)!} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n^3} \right)^{n-1},$$

since  $c_n \geq n^5$  for all  $n \in \mathbb{N}^*$ , and so  $n^2 t \leq \frac{1}{n^3} \sum_j d_j$ . The Stirling formula gives

$$(6.16) \quad c_n \leq 2^{-1/2} (4/e)^n n^{n+2} (1 + \log n)^{n-1} = O((2n \log n)^n)$$

for  $n$  large. In this way we get

**6.17. Proposition.** *Let  $D = \sum_j (1 - \frac{1}{\rho_j}) \Delta_j$  a simple normal crossing orbifold divisor on  $\mathbb{P}^n$  with  $\deg \Delta_j = d_j$ . Then there exist jet differentials of order  $n$  and large degree  $m$  on  $\mathbb{P}^n \langle D \rangle$ , with a small negative twist  $\mathcal{O}_{\mathbb{P}^n}(-m\varepsilon)$ , provided that*

$$\rho_j \geq \rho > n, \quad \sum d_j \geq c_n \max \left( \max \left( \frac{d_j}{\rho_j} \right), 2 \right) \prod_{s=1}^n \left( 1 - \frac{s}{\rho} \right)^{-1}.$$

For instance, one can take all components  $\Delta_j$  possessing the same degree  $d$  and ramification number  $\rho > n$ , and a number of components

$$N \geq c_n \max \left( \frac{1}{\rho}, \frac{2}{d} \right) \prod_{s=1}^n \left( 1 - \frac{s}{\rho} \right)^{-1},$$

or a single component  $(1 - \frac{1}{\rho_1}) \Delta_1$  with  $\rho_1 \geq 2c_n$  and  $d_1 \geq 4c_n$  (notice that  $\prod (1 - \frac{s}{2c_n})^{-1} < 2$ ). Since we have neglected many terms in the above calculations, the “technological constant”  $c_n$  appearing in these estimates is probably much larger than needed.

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