On the existence of logarithmic and orbifold jet differentials

Frédéric Campana, Lionel Darondeau,
Jean-Pierre Demailly, Erwan Rousseau

0. Introduction and main definitions

The present research is concerned with the existence of logarithmic and orbifold jet differentials on projective varieties. For the sake of generality, and in view of potential applications to the case of foliations, we work throughout this paper in the category of directed varieties, and generalize them by introducing the concept of directed orbifold.

0.1. Definition. *Let X be a complex manifold or variety. A directed structure (X, V) on X is defined to be a subsheaf $V \subset \mathcal{O}(T_X)$ such that $\mathcal{O}(T_X)/V$ is torsion free. A morphism of directed varieties $\Psi : (X, V) \to (Y, W)$ is a holomorphic map $\Psi : X \to Y$ such that $d\Psi(V) \subset \Psi^* W$. We say that (X, V) is non singular if X is non singular and V is locally free, i.e., is a holomorphic subbundle of T_X.*

We refer to the *absolute case* as being the situation when $V = T_X$, the *relative case* when $V = T_{X/S}$ for some fibration $X \to S$, and the *foliated case* when V is integrable, i.e. $[V, V] \subset V$, that is, V is the tangent sheaf to a holomorphic foliation. We now combine these concepts with orbifold structures in the sense of Campana [Cam04].

0.2. Definition. *A directed orbifold is a triple (X, V, D) where (X, V) is a directed variety and $D = \sum(1 - \frac{1}{\rho_j})\Delta_j$ an effective real divisor, where Δ_j is an irreducible hypersurface and $\rho_j \in\ ]1, \infty]$ an associated "ramification number". We denote by $\lceil D\rceil = \sum \Delta_j$ the corresponding reduced divisor, and by $|D| = \bigcup \Delta_j$ its support.*

(a) *We will say that (X, V, D) is non singular if (X, V) is non singular and D is a simple normal crossing divisor such that D is transverse to V. If $r = \operatorname{rank}(V)$, we mean by this that there are at most r components Δ_j meeting at any point $x \in X$, and that for any p-tuple $(j_1, \ldots, j_p)$ of indices, $1 \leqslant p \leqslant r$, we have $\dim V_x \cap \bigcap_{j=1}^p T_{\Delta_{j_\ell},x} = r - p$ at any point $x \in \bigcap_{j=1}^p \Delta_{j_\ell}$.*

(b) *If (X, V, D) is non singular, the canonical divisor of (X, V, D) is defined to be*

$$K_{V,D} = K_V + D$$

(in additive notation), where $K_V = \det V^*$.

(c) *The so called logarithmic case corresponds to all multiplicities $\rho_j = \infty$ being taken infinite, so that $D = \sum \Delta_j = \lceil D\rceil$.*

In case $V = T_X$, we recover the concept of orbifold introduced in [Cam04], except possibly for the fact that we allow here $\rho_j \in \mathbb{R}$, $\rho_j > 1$ (even though the case $\rho_j \in \mathbb{N}^*$ is of greater interest). It would certainly be interesting to investigate the case when (X, V, D)

is singular, by allowing singularities in V and tangencies between V and D, and to study whether the results discussed in this paper can be extended in some way, e.g. by introducing suitable multiplier ideal sheaves taking care of singularities, as was done in [Dem15] for the study of directed varieties (X,V). For the sake of technical simplicity, we will refrain to do so here, and will therefore leave for future work the study of singular directed orbifolds.

0.3. Definition. *Let (X,V,D) be a singular directed orbifold. We say that $f:\mathbb{C}\to X$ is an orbifold entire curve if f is a non constant holomorphic map such that:*

(a) *f is tangent to V (i.e. $f'(t)\in V_{f(t)}$ at every point, or equivalently $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ is a morphism of directed varieties;*

(b) *$f(\mathbb{C})$ is not identically contained in $|D|$;*

(c) *at every point $t_0\in\mathbb{C}$ such that $f(t_0)\in\Delta_j$, f meets Δ_j with ramification number $\geqslant\rho_j$, i.e., if $\Delta_j=\{z_j=0\}$ near $f(t_0)$, then $z_j\circ f(t)$ vanishes with multiplicity $\geqslant\rho_j$ at t_0.*

In the case of a logarithmic component Δ_j ($\rho_j=\infty$), condition (c) *is to be replaced by the assumption*

(c′) *$f(\mathbb{C})$ does not meet Δ_j.*

One can now consider a category of directed orbifolds as follows.

0.4. Definition. *Consider directed orbifolds (X,V,D), (Y,W,Λ) with*

$$D=\sum\Big(1-\frac{1}{\rho_i}\Big)\Delta_i,\qquad D'=\sum\Big(1-\frac{1}{\rho'_j}\Big)\Delta'_j.$$

A morphism $\Psi:(X,V,D)\to(Y,W,D')$ is a morphism $\Psi:(X,V)\to(Y,W)$ of directed varieties satisfying the additional following properties (a,b,c)*.*

(a) *for every component Δ'_j, $\Psi^{-1}(\Delta'_j)$ consists of a union of components Δ_i, $i\in I(j)$, eventually after adding a number of extra components Δ_i with $\rho_i=1$;*

(b) *in case $\rho'_j<\infty$, for every $i\in I(j)$ and $z\in\Delta_i$, the derivatives $d^\alpha\Psi(z)$ of Ψ at z, computed in suitable local coordinates on X and Y, vanish for all multi-indices $\alpha\in\mathbb{N}^n$ with $0<|\alpha|<\rho'_j/\rho_i$;*

(c) *if Δ'_j is a logarithmic component ($\rho'_j=\infty$), then $\Phi^{-1}(\Delta'_j)=\bigcup_{i\in I(j)}\Delta_i$ where the $(\Delta_i)_{i\in I(j)}$ consist of logarithmic components ($\rho_i=\infty$).*

It is easy to check that the composite of directed orbifold morphisms is actually a directed orbifold morphism, and that the composition of an orbifold entire curve $f:\mathbb{C}\to(X,V,D)$ with a directed orbifold morphism $\Psi:(X,V,D)\to(Y,W,D')$ produces an orbifold entire curve $\Psi\circ f:\mathbb{C}\to(Y,W,D')$. One of our main goals is to investigate the following orbifold generalization of the Green-Griffiths conjecture.

0.5. Conjecture. *Let (X,V,D) be a non singular directed orbifold of generated type, in the sense that the canonical divisor K_V+D is big. Then then exists an algebraic subvariety $Y\subsetneq X$ containing all orbifold entire curves $f:\mathbb{C}\to(X,V,D)$.*

As in the absolute case ($V=T_X$, $D=0$), the idea is to show, at least as a first step towards the conjecture, that orbifold entire curves must satisfy suitable algebraic differential equations. In section 1, we introduce graded algebras

$$\bigoplus_{m\in\mathbb{N}}E_{k,m}V^*\langle D\rangle \tag{0.6}$$

of sheaves of "orbifold jet differentials". These sheaves correspond to algebraic differential operators $P(f;f',f'',\ldots,f^{(k)})$ acting on germs of k-jets of curves that are tangent to V and satisfy the ramification conditions prescribed by D. The strategy relies on the following standard vanishing theorem.

0.7. Proposition. *Let (X,V,D) be a projective non singular directed orbifold, and A an ample divisor on X. Then, for every orbifold entire curve $f:\mathbb{C}\to(X,V,D)$ and every global jet differential operator $P\in H^0(X,E_{k,m}V^*\langle D\rangle\otimes\mathcal{O}_X(-A))$, we have $P(f;f',f'',\ldots,f^{(k)})=0$.*

The next step consists precisely of finding sufficient conditions that ensure the existence of many global sections $P\in H^0(X,E_{k,m}V^*\langle D\rangle\otimes\mathcal{O}_X(-A))$. In this direction, among other more general results, we prove

0.8. Theorem. *Let $D=\sum_j(1-\frac{1}{\rho_j})\Delta_j$ a simple normal crossing orbifold divisor on $\mathbb{P}^n$ with $\deg\Delta_j=d_j$. Then there exist jet differentials of order n and large degree m on $\mathbb{P}^n\langle D\rangle$, with a small negative twist $\mathcal{O}_{\mathbb{P}^b}(-m\varepsilon)$, under any of the following two assumptions, where $c_n=O((2n\,\log n)^n)$:*

(a) *all components Δ_j possess the same degree d and ramification number $\rho>n$, and the number of components satisfies*

$$N\geqslant c_n\,\max\left(\frac{1}{\rho},\frac{2}{d}\right)\prod_{s=1}^{n}\left(1-\frac{s}{\rho}\right)^{-1},$$

(b) *D admits a component $(1-\frac{1}{\rho_1})\Delta_1$ with $\rho_1\geqslant 2c_n$ and $d_1\geqslant 4c_n$.*

The proof of Theorem 0.8 rests upon an application of holomorphic Morse inequalities. It is remarkable that a large part of the calculations use Chern forms and are non cohomological, although the final bounds are purely cohomological. At this point, we do not have a convincing or complete explanation of this "transcendental" phenomenon.

1. Logarithmic and orbifold jet differentials

1.A. Directed varieties and associated jet differentials

Let (X,V) be a non singular directed variety. We set $n=\dim_{\mathbb{C}}X$, $r=\operatorname{rank}_{\mathbb{C}}V$, and following the exposition of [Dem97], we denote by $\pi_k:J^kV\to X$ the bundle of k-jets of holomorphic curves tangent to V at each point. The canonical bundle of V is defined to be

$$K_V=\det(V^*)=\Lambda^rV^*. \tag{1.1}$$

If $f:(\mathbb{C},0)\to X$, $t\mapsto f(t)$ is a germ of holomorphic curve tangent to V, we denote by $f_{[k]}(0)$ its k-jet at $t=0$. For $x_0\in X$ given, we take a coordinate system $(z_1,\ldots,z_n)$ centered at x_0 such that $V_{x_0}=\mathrm{Span}(\frac{\partial}{\partial z_\mu})_{1\leqslant\mu\leqslant r}$. Then there exists a neighborhood U of x_0 such that $V_{|U}$ admits a holomorphic frame $(e_\mu)_{1\leqslant\mu\leqslant r}$ of the form

$$e_\mu(z)=\frac{\partial}{\partial z_\mu}+\sum_{r+1\leqslant\lambda\leqslant n}a_{\lambda\mu}(z)\frac{\partial}{\partial z_\lambda},\quad 1\leqslant\mu\leqslant r, \tag{1.2}$$

with $a_{\lambda\mu}(0)=0$. Germs of curves $f:(\mathbb{C},0)\to X$ tangent to $V_{|U}$ are obtained by integrating the system of ordinary differential equations

$$f'_\lambda(t)=\sum_{1\leqslant\mu\leqslant r}a_{\lambda\mu}(f(t))\,f'_\mu(t),\quad r+1\leqslant\lambda\leqslant n, \tag{1.3}$$

when we write $f = (f_1, \ldots, f_n)$ in coordinates. Therefore any such germ of curve f is uniquely determined by its initial point $z = f(0)$ and its projection $\tilde f = (f_1, \ldots, f_r)$ on the first r coordinates. By definition, every k-jet $f_{[k]} \in J^k V_z = \pi_k^{-1}(z)$ is uniquely determined by its initial point $f(0) = z \simeq (z_1, \ldots, z_n)$ and the Taylor expansion of order k

$$(1.4)\quad \tilde f(t) - \tilde f(0) = t\xi_1 + \frac{1}{2!}t^2\xi_2 + \cdots + \frac{1}{k!}t^k\xi_k + O(t^{k+1}), \quad t \in \mathbb{D}(0,\varepsilon),\ \xi_s \in \mathbb{C}^r,\ 1 \leqslant s \leqslant k.$$

Alternatively, we can pick an arbitrary local holomorphic connection ∇ on $V_{|U}$ and represent the k-jet $f_{[k]}(0)$ by $(\xi_1, \ldots, \xi_k)$, where $\xi_s = \nabla^s f(0) \in V_z$ is defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local biholomorphic trivialization of $J^k V_{|U}$ of the form

$$(1.5)\qquad J_k V_{|U} \to V_{|U}^{\oplus k}, \qquad f_{[k]}(0) \mapsto (\xi_1, \ldots, \xi_k) = (\nabla f(0), \ldots, \nabla f^k(0))\,;$$

the particular choice of the "trivial connection" ∇_0 of $V_{|U}$ that turns $(e_\mu)_{1\leqslant\mu\leqslant r}$ into a parallel frame precisely yields the components $\xi_s \in V_{|U} \simeq \mathbb{C}^r$ appearing in (1.4). We could of course also use a C^∞ connection $\nabla = \nabla_0 + \Gamma$ where $\Gamma \in C^\infty(U, T_X^* \otimes \mathrm{Hom}(V,V))$, and in this case, the corresponding trivialization (1.5) is just a C^∞ diffeomorphism; the advantage, though, is that we can always produce such a global C^∞ connection ∇ by using a partition of unity on X, and then (1.5) becomes a global C^∞ diffeomorphism. Now, there is a global holomorphic $\mathbb{C}^*$ action on $J^k V$ given at the level of germs by $f \mapsto \alpha\cdot f$ where $\alpha\cdot f(t) := f(\alpha t)$, $\alpha \in \mathbb{C}^*$. With respect to our trivializations (1.5), this is the weighted $\mathbb{C}^*$ action defined by

$$(1.6)\qquad \alpha\cdot(\xi_1, \xi_2, \ldots, \xi_k) = (\alpha\xi_1, \alpha^2\xi_2, \ldots, \alpha^k\xi_k), \quad \xi_s \in V.$$

We see that $J^k V \to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and that the projectivized k-jet bundle

$$(1.7)\qquad X_k(V) := (J^k V \smallsetminus \{0\})/\mathbb{C}^*, \qquad \pi_k : X_k(V) \to X$$

is a $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ weighted projective bundle over X, of total dimension

$$(1.8)\qquad \dim X_k(V) = n + kr - 1.$$

1.9. Definition. *We define $\mathcal{O}_X(E_{k,m}V^*)$ to be the sheaf over X of holomorphic functions $P(z\,;\xi_1, \ldots, \xi_k)$ on $J^k V$ that are weighted polynomials of degree m in $(\xi_1, \ldots, \xi_m)$.*

In coordinates and in multi-index notation, we can write

$$P(z\,;\xi_1, \ldots, \xi_k) = \sum_{\substack{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^r \\ |\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}} a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}$$

where the $a_{\alpha_1\ldots\alpha_k}(z)$ are holomorphic functions in $z = (z_1, \ldots, z_n)$ and $\xi_s^{\alpha_s}$ actually means

$$\xi_s^{\alpha_s} = \xi_{s,1}^{\alpha_{s,1}}\ldots\xi_{s,r}^{\alpha_{s,r}} \quad \text{for}\ \ \xi_s = (\xi_{s,1}, \ldots, \xi_{s,r}) \in \mathbb{C}^r, \ \ \alpha_s = (\alpha_{s,1}, \ldots, \alpha_{s,r}) \in \mathbb{N}^r,$$

and $|\alpha_s| = \sum_{j=1}^r \alpha_{s,j}$. Such sections can be interpreted as algebraic differential operators acting on holomorphic curves $f : \mathbb{D}(0,R) \to X$ tangent to V, by putting $P(f) := u$ where

$$(1.10)\qquad u(t) = \sum_{\substack{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^r \\ |\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}} a_{\alpha_1\ldots\alpha_k}(f(t))\, f'(t)^{\alpha_1}\ldots f^{(k)}(t)^{\alpha_k}.$$

Here $f^{(s)}(t)^{\alpha_s}$ is actually to be expanded as

$$f^{(s)}(t)^{\alpha_s} = f_1^{(s)}(t)^{\alpha_{s,1}} \ldots f_r^{(s)}(t)^{\alpha_{s,r}}$$

with respect to the components $f_j^{(s)}$ defined in (1.4). We also set $u = P(f\,;\,f', f'', \ldots, f^{(k)})$ when we want to make more explicit the dependence of the expression in terms of the derivatives of f. We thus get a sheaf of graded algebras

$$\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*). \tag{1.11}$$

Locally in coordinates, the algebra is isomorphic to the weighted polynomial ring

$$\mathcal{O}_X\big[f_j^{(s)}\big]_{1\leqslant j\leqslant r,\,1\leqslant s\leqslant k}, \quad \deg f_j^{(s)} = s \tag{1.12}$$

over $\mathcal{O}_X$. An immediate consequence of these definitions is :

1.13. proposition. *The projectivized bundle $\pi_k : X_k(V) \to X$ can be identified with*

$$\mathrm{Proj}\left(\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*)\right) \to X, \tag{a}$$

and, if $\mathcal{O}_{X_k(V)}(m)$ denote the associated tautological sheaves, we have the direct image formula

$$(\pi_k)_*\mathcal{O}_{X_k(V)}(m) = \mathcal{O}_X(E_{k,m}V^*). \tag{b}$$

1.14. Remark. These objects where denoted X_k^{GG} and $E_{k,m}^{\mathrm{GG}}V^*$ in our previous paper [Dem97], as a reference to the work of Green-Griffiths [GG79], but we will avoid here the superscript GG to simplify the notation.

Thanks to the Faà di Bruno formula, a change of coordinates $w = \psi(z)$ on X leads to a transformation rule

$$(\psi\circ f)^{(k)} = \psi'\circ f\cdot f^{(k)} + Q_\psi(f', \ldots, f^{(k-1)})$$

where Q_ψ is a polynomial of weighted degree k in the lower order derivatives. This shows that the transformation rule of the top derivative is linear and, as a consequence, the partial degree in $f^{(k)}$ of the polynomial $P(f\,;\,f', \ldots, f^{k)})$ is intrinsically defined. By taking the corresponding filtration and factorizing the monomials $(f^{(k)})^{\alpha_k}$ with polynomials in $f', f'', \ldots, f^{(k-1)}$, we get graded pieces

$$G^\bullet(E_{k,m}V^*) = \bigoplus_{\ell_k\in\mathbb{N}} E_{k-1,m-k\ell_k}V^*\otimes S^{\ell_k}V^*.$$

By considering successively the partial degrees with respect to $f^{(k)}$, $f^{(k-1)}$, $\ldots$, f'', f' and merging inductively the resulting filtrations, we get a multi-filtration such that

$$G^\bullet(E_{k,m}V^*) = \bigoplus_{\ell_1,\ldots,\ell_k\in\mathbb{N},\,\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*. \tag{1.15}$$

1.B. Logarithmic directed varieties

We now turn ourselves to the logarithmic case. Let (X, V, D) be a non singular logarithmic variety, where $D = \sum \Delta_j$ is a simple normal crossing divisor. Fix a point $x_0 \in X$. By the assumption that D is transverse to V, we can then select holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x_0 such that $V_{x_0} = \mathrm{Span}(\frac{\partial}{\partial z_j})_{1 \leqslant j \leqslant r}$ and $\Delta_j = \{z_j = 0\}$, $1 \leqslant j \leqslant p$, are the components of D that contain x_0 (here $p \leqslant r$ and we can have $p = 0$ if $x_0 \notin |D|$). What we want is to introduce an algebra of differential operators, defined locally near x_0 as the weighted polynomial ring

$$\mathcal{O}_X\big[(\log f_j)^{(s)}_{1 \leqslant j \leqslant p}, (f_j^{(s)})_{p+1 \leqslant j \leqslant r}\big]_{1 \leqslant s \leqslant k}, \quad \deg f_j^{(s)} = \deg(\log f_j)^{(s)} = s, \tag{1.16}$$

or equivalently

$$\mathcal{O}_X\big[(f_j^{-1} f_j^{(s)})_{1 \leqslant j \leqslant p}, (f_j^{(s)})_{p+1 \leqslant j \leqslant r}\big]_{1 \leqslant s \leqslant k}, \quad \deg f_j^{(s)} = s, \ \deg f_j^{-1} = 0. \tag{1.16$'$}$$

For this we notice that

$$(\log f_1)'' = (f_1^{-1} f_1')' = f_1^{-1} f_1'' - (f_1^{-1} f_1')^2,$$

$$(\log f_1)''' = f_1^{-1} f_1''' - 3(f_1^{-1} f_1')(f_1^{-1} f_1'') + 2(f_1^{-1} f_1')^3, \ldots .$$

A similar argument easily shows that the above graded rings do not depend on the particular choice of coordinates made, as soon as they satisty $\Delta_j = \{z_j = 0\}$.

Now (as is well known in the absolute case $V = T_X$), we have a corresponding logarithmic directed structure $V\langle D\rangle$ and its dual $V^*\langle D\rangle$. If the coordinates $(z_1, \ldots, z_n)$ are chosen so that $V_{x_0} = \{dz_{r+1} = \ldots = dz_n = 0\}$, then the fiber $V\langle D\rangle_{x_0}$ is spanned by the derivations

$$z_1 \frac{\partial}{\partial z_1}, \ldots, z_p \frac{\partial}{\partial z_p}, \ \frac{\partial}{\partial z_{p+1}}, \ldots, \frac{\partial}{\partial z_r}.$$

The dual sheaf $\mathcal{O}_X(V^*\langle D\rangle)$ is the locally free sheaf generated by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, \ dz_{p+1}, \ldots, dz_r$$

[where the 1-forms are considered in restriction to $\mathcal{O}_X(V\langle D\rangle) \subset \mathcal{O}_X(V)$]. It follows from this that $\mathcal{O}_X(V\langle D\rangle)$ and $\mathcal{O}_X(V^*\langle D\rangle)$ are locally free sheaves of rank r. By taking $\det(V^*\langle D\rangle)$ and using the above generators, we find

$$\det(V^*\langle D\rangle) = \det(V^*) \otimes \mathcal{O}_X(D) = K_V + D \tag{1.17}$$

in additive notation. Quite similarly to 1.13 and 1.15, we have :

1.18. Proposition. *Let $\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle D\rangle)$ be the graded algebra defined in coordinates by* (1.16) *or* (1.16$'$)*. We define the logarithmic k-jet bundle to be*

$$X_k(V\langle D\rangle) := \mathrm{Proj}\left(\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle D\rangle)\right) \to X. \tag{a}$$

If $\mathcal{O}_{X_k(V\langle D\rangle)}(m)$ denote the associated tautological sheaves, we get the direct image formula

$$(\pi_k)_* \mathcal{O}_{X_k(V\langle D\rangle)}(m) = \mathcal{O}_X(E_{k,m} V^*\langle D\rangle). \tag{b}$$

Moreover, the mult-filtration by the partial degrees in the derivatives $f_j^{(s)}$ has graded pieces

$$\text{(c)} \quad G^\bullet\big(E_{k,m} V^*\langle D\rangle\big) = \bigoplus_{\ell_1, \ldots, \ell_k \in \mathbb{N}, \, \ell_1 + 2\ell_2 + \cdots + k\ell_k = m} S^{\ell_1} V^*\langle D\rangle \otimes S^{\ell_2} V^*\langle D\rangle \otimes \cdots \otimes S^{\ell_k} V^*\langle D\rangle.$$

1.C. Orbifold directed varieties

We finally consider a non singular directed orbifold (X,V,D), where $D=\sum(1-\frac{1}{\rho_j})\Delta_j$ is a simple normal crossing divisor transverse to V. Let $\lceil D\rceil=\sum\Delta_j$ be the corresponding reduced divisor. By § 1.B, we have associated logarithmic sheaves $\mathcal{O}_X(E_{k,m}V^*\langle\lceil D\rceil\rangle)$. We want to introduce a graded subalgebra

$$\bigoplus_{m\in\mathbb{N}}\mathcal{O}_X(E_{k,m}V^*\langle D\rangle)\quad\subset\quad\bigoplus_{m\in\mathbb{N}}\mathcal{O}_X(E_{k,m}V^*\langle\lceil D\rceil\rangle)\tag{1.19}$$

in such a way that for every germ $P\in\mathcal{O}_X(E_{k,m}V^*\langle D\rangle)$ and every germ of orbifold curve $f:(\mathbb{C},0)\to(X,V,D)$ the germ of meromorphic function $P(f)(t)$ is bounded at $t=0$ (hence holomorphic). Assume that $\Delta_1=\{z_1=0\}$ and that f has multiplicity $q\geqslant\rho_1>1$ along Δ_1 at $t=0$. Then $f_1^{(s)}$ still vanishes at order $\geqslant(q-s)_+$, thus $(f_1)^{-\beta}f_1^{(s)}$ is bounded as soon as $\beta q\leqslant(q-s)_+$, i.e. $\beta\leqslant(1-\frac{s}{q})_+$. Thus, it is sufficient to ask that $\beta\leqslant(1-\frac{s}{\rho_1})_+$. At a point $x_0\in|\Delta_1|\cap\ldots\cap|\Delta_p|$, a sufficient condition for a monomial of the form

$$f_1^{-\beta_1}\ldots f_p^{-\beta_p}\prod_{s=1}^{k}\prod_{j=1}^{r}(f_j^{(s)})^{\alpha_{s,j}},\quad \alpha_s=(\alpha_{s,j})\in\mathbb{N}^r,\ \beta_1,\ldots,\beta_p\in\mathbb{N}\tag{1.20}$$

to be bounded is to require that the multiplicities of poles satisfy

$$\beta_j\leqslant\sum_{s=1}^{k}\alpha_{s,j}\Big(1-\frac{s}{\rho_j}\Big)_+,\quad 1\leqslant j\leqslant p.\tag{1.20'}$$

1.21. Definition. *The subalgebra $\bigoplus_{m\in\mathbb{N}}\mathcal{O}_X(E_{k,m}V^*\langle D\rangle)$ is taken to be the graded ring generated by monomials* (1.20) *of degree $\sum s|\alpha_s|=m$, satisfying the pole multiplicity conditions* (1.20′)*. These conditions do not depend on the choice of coordinates, hence we get a globally and intrinsically defined sheaf of algebras on X.*

Proof. We only have to prove the last assertion. Consider a change of variables $w=\psi(z)$ such that Δ_j can still be expressed as $\Delta_j=\{w_j=0\}$. Then, for $j=1,\ldots,p$, we can write $w_j=z_ju_j(z)$ with an invertible holomorphic factor u_j. We need to check that the monomials (1.20) computed with $g=\psi\circ f$ are holomorphic combinations of those associated with f. However, we have $g_j=f_ju_j(f)$, hence $g_j^{(s)}=\sum_{0\leqslant\ell\leqslant s}\binom{s}{\ell}f_j^{(\ell)}(u_j(f))^{(s-\ell)}$ by the Leibniz formula, and we see that

$$g_1^{-\beta_1}\ldots g_p^{-\beta_p}\prod_{s=1}^{k}\prod_{j=1}^{r}(g_j^{(s)})^{\alpha_{s,j}}$$

expands as a linear combination of monomials

$$f_1^{-\beta_1}\ldots f_p^{-\beta_p}\prod_{s=1}^{k}\prod_{j=1}^{r}\prod_{m=1}^{\alpha_{s,j}}f_j^{(\ell_{s,j,m})},\quad \ell_{s,j,m}\leqslant s,$$

multiplied by holomorphic factors of the form

$$\prod_{j=1}^{p}u_j(f)^{-\beta_j}\times\prod_{s=1}^{k}\prod_{j=1}^{r}\prod_{m=1}^{\alpha_{s,j}}(u_j(f))^{(s-\ell_{j,s,m})}.$$

However, we have

$$\beta_j \leqslant \sum_{s=1}^k \alpha_{s,j}\Big(1-\frac{s}{\rho_j}\Big)_+ \leqslant \sum_{s=1}^k \sum_{m=1}^{\alpha_{s,j}} \Big(1-\frac{\ell_{s,j,m}}{\rho_j}\Big)_+,$$

so the f-monomials satisfy again the required multiplicity conditions for the poles $f_j^{-\beta_j}$. $\square$

The above conditions (1.20′) suggest to introduce a sequence of "differentiated" orbifold divisors

$$D^{(s)} = \sum_j \Big(1-\frac{s}{\rho_j}\Big)_+ \Delta_j. \tag{1.22}$$

We say that $D^{(s)}$ is the order s orbifold divisor associated to D; its ramification numbers are $\rho_j^{(s)} = \max(\rho_j/s, 1)$. By definition, the logarithmic components ($\rho_j = \infty$) of D remain logarithmic in $D^{(s)}$, while all others eventually disappear when s is large.

Now, we introduce (in a purely formal way) a sheaf of rings $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z_j^\bullet]$ by adjoining all positive real powers of coordinates z_j such that $\Delta_j = \{z_j = 0\}$ is locally a component of D. Locally over X, this can be done by taking the universal cover Y of a punctured polydisk

$$\mathbb{D}^*(0,r) := \prod_{1\leqslant j\leqslant p} \mathbb{D}^*(0,r_j) \times \prod_{p+1\leqslant j\leqslant n} \mathbb{D}(0,r_j) \quad\subset\quad \mathbb{D}(0,r) := \prod_{1\leqslant j\leqslant n} \mathbb{D}(0,r_j)$$

in the local coordinates z_j on X. If $\gamma : Y \to \mathbb{D}^*(0,r) \hookrightarrow X$ is the covering map and $U \subset \mathbb{D}(0,r)$ is an open subset, we can then consider the functions of $\widetilde{\mathcal{O}}_X(U)$ as being defined on $\gamma^{-1}(U \cap \mathbb{D}^*(0,r))$. In case X is projective, one can even achieve such a construction "globally", by taking Y to be the universal cover of a complement $X \smallsetminus (|D| \cup |A|)$, where $A = \sum A_j$ is a very ample normal crossing divisor transverse to D, such that $\mathcal{O}_X(\Delta_j)_{|X\smallsetminus|A|}$ is trivial for every j.

In this setting, the subalgebra $\bigoplus_m \mathcal{O}_X(E_{k,m}V^*\langle D\rangle)$ still has a multi-filtration induced by the one on $\bigoplus_m \mathcal{O}_X(E_{k,m}V^*\langle\lceil D\rceil\rangle)$, and by extending the structure sheaf $\mathcal{O}_X$ into $\widetilde{\mathcal{O}}_X$, we get an inclusion

$$\widetilde{\mathcal{O}}_X(G^\bullet E_{k,m}V^*\langle D\rangle) \subset \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} \widetilde{\mathcal{O}}_X(S^{\ell_1}V^*\langle D^{(1)}\rangle) \otimes\cdots\otimes \widetilde{\mathcal{O}}_X(S^{\ell_k}V^*\langle D^{(k)}\rangle), \tag{1.23}$$

$\widetilde{\mathcal{O}}_X(V^*\langle D^{(s)}\rangle)$ is the "s-th orbifold (dual) directed structure", generated by the order s differentials

$$z_j^{-(1-s/\rho_j)_+} d^{(s)}z_j,\quad 1\leqslant j\leqslant p,\qquad d^{(s)}z_j,\quad p+1\leqslant j\leqslant r. \tag{1.24}$$

By construction, we have

$$\det(\widetilde{\mathcal{O}}_X(V^*\langle D^{(s)}\rangle)) = \widetilde{\mathcal{O}}_X(K_V + D^{(s)}). \tag{1.25}$$

1.26. Remark. When $\rho_j = a_j/b_j \in \mathbb{Q}_+$, one can find a finite ramified Galois cover $g : Y \to X$ from a smooth projective variety Y onto X, such that the compositions $(z_j\circ g)^{1/a_j}$

become single-valued functions w_j on Y. In this way, the pull-back $\mathcal{O}_Y(g^*V^*\langle D^{(s)}\rangle)$ is actually a locally free $\mathcal{O}_Y$-module. On can also introduce a sheaf of algebras which we will denote by $\bigoplus \mathcal{O}_Y(E_{k,m}\widetilde{V}^*\langle D\rangle)$, generated, according to the notation of §1.B, by the elements $g^*(z_j^{(1-s/\rho_j)_+}d^{(s)}z_j)$, $1\leqslant j\leqslant p$, and $g^*(d^{(s)}z_j)$, $p+1\leqslant j\leqslant r$. Then there is indeed a multifiltration on $\mathcal{O}_Y(E_{k,m}\widetilde{V}^*\langle D\rangle)$ whose graded pieces are

$$(1.27)\quad \mathcal{O}_Y(G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle)=\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m}\mathcal{O}_Y(S^{\ell_1}\widetilde{V}^*\langle D^{(1)}\rangle)\otimes\cdots\otimes\mathcal{O}_Y(S^{\ell_k}\widetilde{V}^*\langle D^{(k)}\rangle).$$

However, we will adopt here an alternative viewpoint that avoids the introduction of finite or infinite covers, and suits better our approach. Using the general philosophy of [Laz??], the idea is to consider a "jet orbifold directed structure" $X_k(V\langle D\rangle)$ as the underlying "jet logarithmic directed structure" $X_k(V\langle\lceil D\rceil\rangle)$, equipped additionally with a submultiplicative sequence of ideal sheaves $\mathcal{J}_m\langle D\rangle\subset\mathcal{O}_{X_k(V\langle\lceil D\rceil\rangle)}$. These are precisely defined as the base loci ideals of the local sections defined by (1.20) and (1.20′), when these are seen as sections of the logarithmic tautological sheaves $\mathcal{O}_{X_k(V\langle\lceil D\rceil\rangle)}(m)$. The corresponding analytic viewpoint is to consider ad hoc singular hermitian metrics on $\mathcal{O}_{X_k(V\langle\lceil D\rceil\rangle)}(1)$ whose singularities are asymptotically described by the limit of the formal m-th root of $\mathcal{J}_m\langle D\rangle$, see §3.B. It then becomes possible to deal without trouble with real coefficients $\rho_j\in\,]1,\infty]$, and since we no longer have to worry about the existence of Galois covers, the projectivity assumption on X can be dropped as well.

2. Preliminaries on holomorphic Morse inequalities

2.A. Basic results

We first recall the basic results concerning holomorphic Morse inequalities for smooth hermitian line bundles, first proved in [Dem85].

2.1. Theorem. *Let X be a compact complex manifolds, $E\to X$ a holomorphic vector bundle of rank r, and (L,h) a hermitian line bundle. We denote by $\Theta_{L,h}=\frac{\mathrm{i}}{2\pi}\nabla_h^2=-\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log h$ the curvature form of (L,h) and introduce the open subsets of X*

$$(*)\qquad \begin{cases} X(L,h,q)=\big\{x\in X\,;\ \Theta_{L,h}(x)\ \textit{has signature}\ (n-q,q)\big\},\\ X(L,h,S)=\displaystyle\bigcup_{q\in S}X(L,h,q),\quad \forall S\subset\{0,1,\ldots,n\}.\end{cases}$$

Then, for all $q=0,1,\ldots,n$, the dimensions $h^q(X,E\otimes L^m)$ of cohomology groups of the tensor powers $E\otimes L^m$ satisfy the following "Strong Morse inequalities" as $m\to+\infty$:

$$\mathrm{SM}(q):\qquad \sum_{0\leqslant j\leqslant q}(-1)^{q-j}h^j(X,E\otimes L^m)\leqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant q)}(-1)^q\Theta_{L,h}^n+o(m^n),$$

with equality $\chi(X,E\otimes L^m)=r\frac{m^n}{n!}\int_X\Theta_{L,h}^n+o(m^n)$ for the Euler characteristic $(q=n)$.

As a consequence, one gets upper and lower bounds for all cohomology groups, and especially a very useful criterion for the existence of sections of large multiples of L.

2.2. Corollary. *Under the above hypotheses, we have*

(a) *Upper bound for h^q (Weak Morse inequalities)* :

$$h^q(X, E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,q)} (-1)^q\Theta^n_{L,h} + o(m^n)\ .$$

(b) *Lower bound for h^0* :

$$h^0(X, E\otimes L^m) \geqslant h^0 - h^1 \geqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant 1)} \Theta^n_{L,h} - o(m^n)\ .$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)} \Theta^n_{L,h} > 0$ for some hermitian metric h on L.

(c) *Lower bound for h^q* :

$$h^q(X, E\otimes L^m) \geqslant h^q - h^{q-1} - h^{q+1} \geqslant r\frac{m^n}{n!}\int_{X(L,h,\{q,q\pm 1\})} (-1)^q\Theta^n_{L,h} + o(m^n)\ .$$

Proof. (a) is obtained by taking $\mathrm{SM}(q)+\mathrm{SM}(q-1)$, (b) is equivalent to $-\mathrm{SM}(1)$ and (c) is equivalent to $-(\mathrm{SM}(q+1)+\mathrm{SM}(q-2))$. □

The following simple lemma is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

2.3. Lemma. *Let $\eta = \alpha-\beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\,\mathbb{1}_{\eta,\leqslant q}\ \eta^n \leqslant \sum_{0\leqslant j\leqslant q} (-1)^{q-j}\binom{n}{j}\alpha^{n-j}\wedge\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\ \eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha>0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta=\alpha-\beta$ are then given by

$$1-\lambda_1 \leqslant \ldots \leqslant 1-\lambda_q \leqslant 1-\lambda_{q+1} \leqslant \ldots \leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1}<1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n=0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma^j_n(\lambda)\,\alpha^n,$$

where $\sigma^j_n(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q} (-1)^{q-j}\sigma^j_n(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j) \geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). $\square$

2.4. Corollary. *Assume that $\eta = \Theta_{L,h}$ can be expressed as a difference $\eta = \alpha - \beta$ of smooth $(1,1)$-forms $\alpha, \beta \geqslant 0$. Then we have*

$$\mathrm{SM}(q): \quad \sum_{0\leqslant j\leqslant q} (-1)^{q-j} h^j(X, E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_X \sum_{0\leqslant j\leqslant q} (-1)^{q-j}\binom{n}{j}\alpha^{n-j}\wedge\beta^j + o(m^n),$$

and in particular, for $q = 1$,

$$h^0(X, E\otimes L^m) \geqslant h^0 - h^1 \geqslant r\frac{m^n}{n!}\int_X \alpha^n - n\alpha^{n-1}\wedge\beta + o(m^n).$$

2.5. Remark. These estimates are consequences of Theorem 2.1 and Lemma 2.3, by taking the integral over X. The estimate for h^0 was stated and studied by Trapani [Tra93]. In the special case $\alpha = \Theta_{A,h_A} > 0$, $\beta = \Theta_{B,h_B} > 0$ where A, B are ample line bundles, a direct proof can be obtained by purely algebraic means, via the Riemann-Roch formula. However, we will later have to use Corollary 2.4 in case α and β are not closed, a situation in which no algebraic proof seems to exist.

2.B. Singular holomorphic Morse inequalities

The case of singular hermitian metrics has been considered in Bonavero's PhD thesis [Bon93] and will be important for us. We assume that L is equipped with a singular hermitian metric $h = h_\infty e^{-\varphi}$ with analytic singularities, i.e., h_∞ is a smooth metric, and on an neighborhood $V \ni x_0$ of an arbitrary point $x_0 \in X$, the weight φ is of the form

$$\varphi(z) = c\log\sum_{1\leqslant j\leqslant N}|g_j|^2 + u(z) \tag{2.6}$$

where $g_j \in \mathcal{O}_X(V)$ and $u \in C^\infty(V)$. We then have $\Theta_{L,h} = \alpha + \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\varphi$ where $\alpha = \Theta_{L,h_\infty}$ is a smooth closed $(1,1)$-form on X. In this situation, the multiplier ideal sheaves

$$\mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \Big\{f \in \mathcal{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)}d\lambda(z) < +\infty\Big\} \tag{2.7}$$

play an important role. We define the singularity set of h by $\mathrm{Sing}(h) = \mathrm{Sing}(\varphi) = \varphi^{-1}(-\infty)$ which, by definition, is an analytic subset of X. The associated q-index sets are

$$X(L,h,q) = \big\{x \in X \smallsetminus \mathrm{Sing}(h)\,;\ \Theta_{L,h}(x) \text{ has signature } (n-q,q)\big\}. \tag{2.8}$$

We can then state:

2.9. Theorem ([Bon93])**.** *Morse inequalities still hold in the context of singular hermitian metric with analytic singularities, provided the cohomology groups under consideration are twisted by the appropriate multiplier ideal sheaves, i.e. replaced by $H^q(X, E\otimes L^m\otimes\mathcal{I}(h^m))$.*

2.10. Remark. The assumption (2.6) guarantees that the measure $\mathbb{1}_{X\smallsetminus\mathrm{Sing}(h)}(\Theta_{L,h})^n$ is locally integrable on X, as is easily seen by using the Hironaka desingularization theorem

and by taking a log resolution $\mu : \widetilde{X} \to X$ such that $\mu^*(g_j) = (\gamma) \subset \mathcal{O}_{\widetilde{X}}$ becomes a principal ideal associated with a simple normal crossing divisor $E = \mathrm{div}(\gamma)$. Then $\mu^*\Theta_{L,h} = c[E] + \beta$ where β is a smooth closed $(1,1)$-form on $\widetilde{X}$, hence

$$\mu^*(\mathbb{1}_{X\smallsetminus \mathrm{Sing}(h)}\Theta^n_{L,h}) = \beta^n \quad \Rightarrow \quad \int_{X\smallsetminus \mathrm{Sing}(h)} \Theta^n_{L,h} = \int_{\widetilde{X}} \beta^n.$$

It should be observed that the multiplier ideal sheaves $\mathcal{I}(h^m)$ and the integral $\int_{X\smallsetminus \mathrm{Sing}(h)} \Theta^n_{L,h}$ only depend on the equivalence class of singularities of h: if we have two metrics with analytic singularities $h_j = h_\infty e^{-\varphi_j}$, $j = 1, 2$, such that $\psi = \varphi_2 - \varphi_1$ is bounded, then, with the above notation, we have $\mu^*\Theta_{L,h_j} = c[E] + \beta_j$ and $\beta_2 = \beta_1 + \frac{i}{2\pi}\partial\overline{\partial}\psi$, therefore $\int_{\widetilde{X}} \beta_2^n = \int_{\widetilde{X}} \beta_1^n$ by Stokes theorem. By using Monge-Ampère operators in the sense of Bedford-Taylor [BT76], it is in fact enough to assume $u \in L^\infty_{\mathrm{loc}}(X)$ in (2.6), and $\psi \in L^\infty(X)$ here. In general, however, the Morse integrals $\int_{X(L,h_j,q)} (-1)^q \Theta^n_{L,h_j}$, $j = 1, 2$, will differ.

2.C. Morse inequalities and semi-continuity

Let $\mathcal{X} \to S$ be a proper and flat morphism of reduced complex spaces, and let $(X_t)_{t\in S}$ be the fibers. Given a sheaf $\mathcal{E}$ over $\mathcal{X}$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules of rank r, inducing on the fibres a family of sheaves $(E_t \to X_t)_{t\in S}$, the following semicontinuity property holds ([CRAS]):

2.11. Proposition. *For every $q \geqslant 0$, the alternate sum*

$$t \mapsto h^q(X_t, E_t) - h^{q-1}(X_t, E_t) + ... + (-1)^q h^0(X_t, E_t)$$

is upper semicontinuous with respect to the (analytic) Zariski topology on S.

Now, if $\mathcal{L} \to \mathcal{X}$ is an invertible sheaf equipped with a smooth hermitian metric h, and if (h_t) are the fiberwise metrics on the family $(L_t \to X_t)_{t\in S}$, we get

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X_t, E_t \otimes L_t^{\otimes m}) \leqslant r\frac{m^n}{n!} \int_{X(L_0,h_0,\leqslant q)} (-1)^q \Theta^n_{L_0,h_0} + \delta(t)m^n, \tag{2.12}$$

where $\delta(t) \to 0$ as $t \to 0$. In fact, the proof of holomorphic Morse inequalities shows that the inequality holds uniformly on every relatively compact $S' \Subset S$, with

$$I(t) = \int_{X(L_t,h_t,\leqslant q)} (-1)^q \Theta^n_{L_t,h_t} = \int_X (-1)^q \mathbb{1}_{X(L_t,h_t,\leqslant q)} \Theta^n_{L_t,h_t}$$

in the right hand side, and $t \mapsto I(t)$ is clearly continuous with respect to the ordinary topology. In other words, the Morse integral computed on the central fibers provides uniform upper bounds for cohomology groups of $E_t \otimes L_t^{\otimes m}$ when t is close to 0 in ordinary topology (and also, as a consequence, for t in a complement $S \smallsetminus \bigcup S_m$ of at most countably many analytic strata $S_m \subsetneq S$).

2.13. Remark. Similar results would hold when h is a singular hermitian metric with analytic singularities on $\mathcal{L} \to \mathcal{X}$, under the restriction that the families of multiplier ideal sheaves $(\mathcal{I}(h_t^m))_{t\in S}$ "never jump".

2.D. Case of filtered bundles

Let $E \to X$ be a vector bundle over a variety, equipped with a filtration (or multi-filtration) $F^p(E)$, and let $G = \bigoplus G^p(E) \to X$ be the graded bundle associated to this filtration.

2.14. Lemma. *In the above setting, one has for every $q \geqslant 0$*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X,E) \leqslant \sum_{j=0}^{q}(-1)^{q-j}h^j(X,G).$$

Proof. One possible argument is to use the well known fact that there is a family of filtered bundles $(E_t \to X)_{t\in\mathbb{C}}$ (with the same graded pieces $G^p(E_t) = G^p(E)$), such that $E_t \simeq E$ for all $t \neq 0$ and $E_0 \simeq G$. The result is then an immediate consequence of the semi-continuity result 2.11. A more direct very elementary argument can be given as follows: by transitivity of inequalities, it is sufficient to prove the result for simple filtrations; then, by induction on the length of filtrations, it is sufficient to prove the result for exact sequences $0 \to S \to E \to Q \to 0$ of vector bundles on X. Consider the associated (truncated) long exact sequence in cohomology:

$$0 \to H^0(S) \to H^0(E) \to H^0(Q) \xrightarrow{\delta_1} \cdots$$

$$\xrightarrow{\delta_{q-1}} H^q(S) \to H^q(E) \to H^q(Q) \xrightarrow{\delta_q} \mathrm{Im}(\delta_q) \to 0.$$

By the rank theorem of linear algebra,

$$0 \leqslant \mathrm{rank}(\delta_q) = (-1)^q \sum_{j=0}^{q}(-1)^j(h^j(X,Q) - h^j(X,E) + h^j(X,S)).$$

The result follows, since here $h^j(X,G) = h^j(X,Q) + h^j(X,S)$. □

2.E. Rees deformation construction (after Cadorel)

In this short paragraph, we outline a nice algebraic interpretation by Benoît Cadorel of certain semi-continuity arguments for cohomology group dimensions that underline the analytic approach of [Dem11, Lemma 2.12 and Prop. 2.13] and [Dem12, Prop. 9.28] (we will anyway explain again its essential points in §3, since we have to deal here with a more general situation). Recall after [Cad17, Prop. 4.2, Prop. 4.5], that the Rees deformation construction allows one to construct natural deformations of Green-Griffiths jets spaces to weighted projectivized bundles.

Let (X, V, D) be a non singular directed orbifold, and let $g : Y \to (X, D)$ be an adapted Galois cover, as briefly described in remark 1.26, see also [CDR18, §2.1] for more details. We then get a Green-Griffiths jet bundle of graded algebras $E_{k,\bullet}\widetilde{V}^\star\langle D\rangle \to Y$ which admits a multifiltration of associated graded algebra

$$G^\bullet E_{k,\bullet}\widetilde{V}^*\langle D\rangle = \bigoplus_{m\in\mathbb{N}} \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}\widetilde{V}^*\langle D^{(1)}\rangle \otimes \cdots \otimes S^{\ell_k}\widetilde{V}^*\langle D^{(k)}\rangle.$$

where the tilde means taking pull-backs by g^*. Applying the Proj functor, one gets a weighted projective bundle:

$$\mathbb{P}_{(1,\cdots,k)}\left(\widetilde{V}^*\langle D^{(1)}\rangle \oplus \cdots \oplus \widetilde{V}^*\langle D^{(k)}\rangle\right) = \mathrm{Proj}\left(G^\bullet E_{k,\bullet}\widetilde{V}^*\langle D\rangle\right) \xrightarrow{\rho_k} Y,$$

Then, following mutadis mutandus the arguments of Cadorel, one constructs a family $Y \overset{p_k}{\leftarrow} \mathcal{Y}_k \to \mathbb{C}$ parametrized by $\mathbb{C}$, with a canonical line bundle $\mathcal{O}_{\mathcal{Y}_k}(1)$ such that:

- the central fiber $\mathcal{Y}_{k,0}$ is $\mathbb{P}_{(1,\cdots,k)}\left(\widetilde{V}^*\langle D^{(1)}\rangle \oplus \cdots \oplus \widetilde{V}^*\langle D^{(k)}\rangle\right)$ and the restriction of $\mathcal{O}_{\mathcal{Y}_k}(1)$ coincide with the canonical line bundle of this weighted projective bundle. Hence $(\pi_k)_*\mathcal{O}_{\mathcal{Y}_{k,0}}(m) = G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle$.
- the other fibers $\mathcal{Y}_{k,t}$ are isomorphic to the singular quotient $J^k(Y,\widetilde{V},D)/\mathbb{C}^*$ for the natural $\mathbb{C}^*$-action by homotheties, where $J^k(Y,\widetilde{V},D)$ is the affine algebraic bundle associated with the sheaf of algebras, and $(\pi_k)_*\mathcal{O}_{\mathcal{Y}_{k,t}}(m) \simeq E_{k,m}\widetilde{V}^*\langle D\rangle$.

Applying the semicontinuity result of [Dem95], and working with holomorphic inequalities, we obtain a control about dimensions of cohomology spaces of $E_{k,m}\widetilde{V}^*\langle D\rangle$ in terms of dimensions of cohomology spaces of the a priori simpler graded pieces $G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle$. This reduces the study of higher order jet differentials to sections of the tautological sheaves on the weighted projective space associated with a direct sum combination of symmetric differentials. In particular, we have

2.15. Lemma. *For every $q \in \mathbb{N}$*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(Y,E_{k,m}\widetilde{V}^*\langle D\rangle) \geqslant \sum_{j=0}^{q}(-1)^{q-j}h^j(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle).$$

Especially, for $q=1$, we get

$$\begin{aligned} h^0(Y,E_{k,m}\widetilde{V}^*\langle D\rangle) &\geqslant h^0(Y,E_{k,m}\widetilde{V}^*\langle D\rangle) - h^1(Y,E_{k,m}\widetilde{V}^*\langle D\rangle)\\ &\geqslant h^0(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle) - h^1(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle D\rangle). \end{aligned}$$

3. Construction of jet metrics and orbifold jet metrics

3.A. Jet metrics and curvature tensor of jet bundles

Let (X,V) be a non singular directed variety and h a hermitian metric on V. We assume that h is smooth at this point (but will later relax a little bit this assumption and allow certain singularities). Near any given point $z_0 \in X$, we can choose local coordinates $z=(z_1,\ldots,z_n)$ centered at z_0 and a local holomorphic coordinate frame $(e_\lambda(z))_{1\leqslant\lambda\leqslant r}$ of V on an open set $U \ni z_0$, such that

$$\langle e_\lambda(z), e_\mu(z)\rangle_{h(z)} = \delta_{\lambda\mu} + \sum_{1\leqslant i,j\leqslant n,\ 1\leqslant\lambda,\mu\leqslant r} c_{ij\lambda\mu} z_i\overline{z}_j + O(|z|^3) \tag{3.1}$$

for suitable complex coefficients $(c_{ij\lambda\mu})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{\mathrm{i}}{2\pi}\nabla^2_{V,h}$ of (V,h) at z_0 is given by

$$\Theta_{V,h}(z_0) = -\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}\, dz_i \wedge d\overline{z}_j \otimes e_\lambda^* \otimes e_\mu. \tag{3.2}$$

Therefore, $(c_{ij\lambda\mu})$ are the coefficients of $-\Theta_{V,h}$. Up to taking the transposed tensor with respect to λ,μ, these coefficients are also the components of the curvature tensor

$\Theta_{V^*,h^*} = -{}^t\Theta_{V,h}$ of the dual bundle (V^*, h^*). By (1.5), the connection $\nabla = \nabla_h$ yields a C^∞ isomorphism $J_k V \to V^{\oplus k}$. Let us fix an integer $b \in \mathbb{N}^*$ that is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$, and positive numbers $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k > 0$. Following [Dem11], we define a global weighted Finsler metric on $J^k V$ by putting for any k-jet $f \in J^k V_z$

$$\Psi_{h,b,\varepsilon}(f) := \Bigg(\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2b} \|\nabla^s f(0)\|_{h(z)}^{2b/s}\Bigg)^{1/b}, \tag{3.3}$$

where $\|\ \ \|_{h(z)}$ is the hermitian metric h of V evaluated on the fiber V_z, $z = f(0)$. The function $\Psi_{h,b,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,b,\varepsilon}(\alpha \cdot f) = |\alpha|^2\, \Psi_{h,b,\varepsilon}(f) \tag{3.4}$$

with respect to the $\mathbb{C}^*$ action on $J^k V$, in other words, it induces a hermitian metric on the dual L_k^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k(V)}(1)$ over $X_k(V)$. The curvature of L_k is given by

$$\pi_k^* \Theta_{L_k, \Psi^*_{h,b,\varepsilon}} = \frac{\mathrm{i}}{2\pi} \partial\overline{\partial} \log \Psi_{h,b,\varepsilon} \tag{3.5}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k(V)$ with the above metric. This might look a priori like an untractable problem, since the definition of $\Psi_{h,b,\varepsilon}$ is a rather complicated one, involving the hermitian metric in an intricate manner. However, the “miracle” is that the asymptotic behavior of $\Psi_{h,b,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1} \to 0$ is in some sense uniquely defined, and “splits” according to the natural multifiltration on jet differentials (as already hinted in §2.E). This leads to a computable asymptotic formula, which is moreover simple enough to produce useful results.

3.6. Lemma. *Let us consider the global C^∞ bundle isomorphism $J^k V \to V^{\oplus k}$ associated with an arbitrary global C^∞ connection ∇ on $V \to X$, and let us introduce the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1, \xi_2, \ldots, \xi_k) = (\varepsilon_1^1 \xi_1, \varepsilon_2^2 \xi_2, \ldots, \varepsilon_k^k \xi_k) \quad \text{on fibers } J^k V_z,\ z \in X.$$

Such a rescaling commutes with the $\mathbb{C}^*$-action. Moreover, if p is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$ *and the ratios $\varepsilon_s/\varepsilon_{s-1}$ tend to 0 for all $s = 2, \ldots, k$, the rescaled Finsler metric $\Psi_{h,b,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1, \ldots, \xi_k)$ converges towards the limit*

$$\Bigg(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{2b/s}\Bigg)^{1/b}$$

on every compact subset of $V^{\oplus k} \smallsetminus \{0\}$, uniformly in C^∞ topology, and the limit is independent of the connection ∇. The error is measured by a multiplicative factor $1 \pm O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$.

Proof. Let us pick another C^∞ connection $\widetilde{\nabla} = \nabla + \Gamma$ where $\Gamma \in C^\infty(U, T_X^* \otimes \mathrm{Hom}(V, V))$. Then $\widetilde{\nabla}^2 f = \nabla^2 f + \Gamma(f)(f') \cdot f'$, and inductively we get

$$\widetilde{\nabla}^s f = \nabla^s f + P_s(f\,;\, \nabla^1 f, \ldots, \nabla^{s-1} f)$$

where $P(z\,;\xi_1,\ldots,\xi_{s-1})$ is a polynomial with C^∞ coefficients in $z\in U$, which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding isomorphisms $J^kV\simeq V^{\oplus k}$ correspond to each other by a $\mathbb{C}^*$-homogeneous transformation $(\xi_1,\ldots,\xi_k)\mapsto(\widetilde\xi_1,\ldots,\widetilde\xi_k)$ such that

$$\widetilde\xi_s=\xi_s+P_s(z\,;\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon})=(\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k),\qquad(\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon})=(\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon}&=\xi_{s,\varepsilon}+\varepsilon_s^s\,P_s(x\,;\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\&=\xi_{s,\varepsilon}+O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\|+\cdots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and it is easily seen, as a simple consequence of the mean value inequality $|\|x\|^\gamma-\|y\|^\gamma|\leqslant\gamma\sup_{z\in[x,y]}\|z\|^{\gamma-1}\|x-y\|$, that the "error term" in the difference $\|\widetilde\xi_{s,\varepsilon}\|^{2b/s}-\|\xi_{s,\varepsilon}\|^{2b/s}$ is bounded by

$$(\varepsilon_s/\varepsilon_{s-1})^s\left(\|\xi_{1,\varepsilon}\|+\cdots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)}+\|\xi_{s,\varepsilon}\|^{1/s}\right)^{2b}.$$

When b/s is an integer, similar bounds hold for all derivatives $D^\beta_{z,\xi}(\|\widetilde\xi_{s,\varepsilon}\|^{2b/s}-\|\xi_{s,\varepsilon}\|^{2b/s})$ and the lemma follows. □

Now, we fix a point $z_0\in X$, a local holomorphic frame $(e_\lambda(z))_{1\leqslant\lambda\leqslant r}$ satisfying (3.1) on a neighborhood U of z_0, and the *holomorphic* connection ∇ on $V_{|U}$ such that $\nabla e_\lambda=0$. Since the uniform estimates of Lemma 3.6 also apply locally (provided they are applied on a relatively compact open subset $U'\Subset U$), we can use the corresponding holomorphic trivialization $J^kV_{|U}\simeq V_{|U}^{\oplus k}\simeq U\times(\mathbb{C}^r)^{\oplus k}$ to make our calculations. We do this in terms of the rescaled components $\xi_s=\varepsilon_s^s\nabla^sf(0)$. Then, uniformly on compact subsets of $J^kV_{|U}\smallsetminus\{0\}$, we have

$$\Psi_{h,b,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(z\,;\xi_1,\ldots,\xi_k)=\bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|^{2b/s}_{h(z)}\bigg)^{1/b}+O(\max((\varepsilon_s/\varepsilon_{s-1})^{1/b}),$$

and the error term remains of the same magnitude when we take any derivative $D^\beta_{z,\xi}$. By (3.1) we find

$$\|\xi_s\|^2_{h(z)}=\sum_\lambda|\xi_{s,\lambda}|^2+\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,z_i\overline z_j\,\xi_{s,\lambda}\overline\xi_{s,\mu}+O(|z|^3|\xi|^2).$$

The question is thus reduced to evaluating the curvature of the weighted Finsler metric on $V^{\oplus k}$ defined by

$$\begin{aligned}\Psi(z\,;\xi_1,\ldots,\xi_k)&=\bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|^{2b/s}_{h(z)}\bigg)^{1/b}\\&=\bigg(\sum_{1\leqslant s\leqslant k}\Big(\sum_\lambda|\xi_{s,\lambda}|^2+\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,z_i\overline z_j\,\xi_{s,\lambda}\overline\xi_{s,\mu}\Big)^{b/s}\bigg)^{1/b}+O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2=\sum_\lambda|\xi_{s,\lambda}|^2$. A straightforward calculation yields the Taylor expansion

$$\begin{aligned}&\log\Psi(z\,;\,\xi_1,\ldots,\xi_k)\\&\quad=\frac{1}{b}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s}+\sum_{1\leqslant s\leqslant k}\frac{1}{s}\,\frac{|\xi_s|^{2b/s}}{\sum_t|\xi_t|^{2b/t}}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}z_i\overline{z}_j\,\frac{\xi_{s,\lambda}\overline{\xi}_{s,\mu}}{|\xi_s|^2}+O(|z|^3).\end{aligned}$$

By (3.5), the curvature form of $L_k=\mathcal{O}_{X_k(V)}(1)$ is given at the central point z_0 by the formula

$$\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z_0,[\xi])\simeq\omega_{r,k,b}(\xi)+\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\,\frac{|\xi_s|^{2b/s}}{\sum_t|\xi_t|^{2b/t}}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,\frac{\xi_{s,\lambda}\overline{\xi}_{s,\mu}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j\tag{3.7}$$

where $[\xi]=[\xi_1,\ldots,\xi_k]\in\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ and $\omega_{r,k,b}(\xi)=\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}(\frac{1}{b}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s})$. The fibers $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ of $X_k(V)\to X$ can be represented as a quotient of the “weighted ellipsoid” $\sum_{s=1}^k|\xi_s|^{2b/s}=1$ by the $\mathbb{S}^1$-action induced by the weighted $\mathbb{C}^*$-action. This suggests to make use of polar coordinates and to set

$$x_s=|\xi_s|^{2b/s},\quad x=(x_1,\ldots,x_k)\in\mathbb{R}^k,\tag{3.8}$$

$$u_s=\frac{\xi_s}{|\xi_s|}\in\mathbb{S}^{2r-1}\subset\mathbb{C}^r,\quad u=(u_1,\ldots,u_k)\in(\mathbb{S}^{2r-1})^k,\tag{3.8′}$$

so that

$$\sum_{s=1}^k x_s=1\quad\text{and}\quad\xi_s=x_s^{s/2b}u_s.\tag{3.8″}$$

The Morse integrals will then have to be computed for $(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, where $\Delta^{k-1}\subset\mathbb{R}^k$ is the $(k-1)$-dimensional simplex.

3.9. Proposition. *With respect to the rescaled components $\xi_s=\varepsilon_s^s\nabla^sf(0)$ at $z=f(0)\in X$ and the above choice of coordinates* (3.8*)*, we have an approximate expression*

(a) $\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z,[\xi])=\omega_{r,k,b}(\xi)+g_{V,k}(z,x,u)+(\text{error terms})$,

where $(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, $\xi_s=x_s^{s/2b}u_s\in\mathbb{C}^r$,

(b) $\omega_{r,k,b}(\xi)=\dfrac{\mathrm{i}}{2\pi}\partial\overline{\partial}\Big(\dfrac{1}{b}\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s}\Big)$

is a (slightly degenerate) Fubini-Study Kähler type metric on $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$, associated with the canonical $\mathbb{C}^*$ action on J^kV of weight $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$, and

(c) $g_{V,k}(z,x,u)=\dfrac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\dfrac{x_s}{s}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}(z)\,u_{s,\lambda}\overline{u}_{s,\mu}\,dz_i\wedge d\overline{z}_j.$

Here $(c_{ij\lambda\mu})$ are the coefficients of $-\Theta_{V,h}$, and the error terms admit an upper bound

(d) $(\text{error terms})\leqslant O\Big(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s\Big)$ *uniformly on the compact variety $X_k(V)$.*

Proof. The error terms on Θ_{L_k} come from the differentiation of the error terms on the Finsler metric, found in Lemma 3.6. They can indeed be differentiated if b is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$, since $2b/s$ is then an even integer. □

For the calculation of Morse integrals, it is useful to find the expression of the volume form $\omega_{r,k,b}^{kr-1}$ on $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]}) = (\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k)/\mathbb{S}^1$ in terms of the coordinates (x,u). We refer to [Dem11, Prop. 1.13] for the proof.

3.10. Proposition.

(a) *The volume form $\omega_{r,k,b}^{kr-1}$ is the quotient of the measure $\frac{1}{k!^r}\nu_{k,r}\otimes\mu$ on $\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, where*

$$d\nu_{k,r}(x) = (kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^{\,k}}\,dx_1\wedge\ldots\wedge dx_{k-1},\quad d\mu(u) = d\mu_1(u_1)\ldots d\mu_k(u_k)$$

are probability measures on Δ^{k-1} and $(\mathbb{S}^{2r-1})^k$ respectively (μ being the rotation invariant one).

(b) *We have the equality* $\displaystyle\int_{\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})}\omega_{r,k,b}^{kr-1} = \frac{1}{k!^r}$ *(independent of b).*

§3.B. Logarithmic and orbifold jet metrics

Consider now an arbifold directed structure (X,V,D), where $V\subset T_X$ is a subbundle, $r=\mathrm{rank}(V)$, and $D=\sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor that is assumed to intersect V transversally everywhere. One then performs very similar calculations to what we did in §3.A, but with adapted Finsler metrics. Fix a point z_0 at which p components Δ_j meet, and use coordinates $(z_1,\ldots,z_n)$ such that V_{z_0} is spanned by $(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_r})$ and Δ_j is defined by $z_j=0$, $1\leqslant j\leqslant p\leqslant r$. In the logarithmic case $\rho_j=\infty$, the logarithmic dual bundle $\mathcal{O}(V^*\langle D\rangle)$ is spanned by

$$\frac{dz_1}{z_1},\ldots,\frac{dz_p}{z_p},\ dz_{p+1},\ldots,dz_n.$$

The logarithmic jet differentials are just polynomials in

$$\frac{d^sz_1}{z_1},\ldots,\frac{d^sz_p}{z_p},\ d^sz_{p+1},\ldots,d^sz_n,\quad 1\leqslant s\leqslant k,$$

and the corresponding $(\varepsilon_1,\ldots,\varepsilon_k)$-rescaled Finsler metric is

$$\left(\sum_{s=1}^k\varepsilon_s^{2b}\left(\sum_{j=1}^p|f_j|^{-2}|f_j^{(s)}|^2+\sum_{j=p+1}^r|f_j^{(s)}|^2\right)^{2b/s}\right)^{1/b}. \tag{3.11}$$

Alternatively, we could replace $|f_j|^{-2}|f_j^{(s)}|^2$ by $|(\log f_j)^{(s)}|^2$ which has the same leading term and differs by a weighted degree s polynomial in the $f_j^{-1}f_j^{(\ell)}$, $\ell<s$; an argument very similar to the one used in the proof of Lemma 3.6 then shows that the difference is negligible when $\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k$. However (3.11) is just the case of the model metric, in fact we get r-tuples $\xi_s=(\xi_{s,j})_{1\leqslant j\leqslant r}$ of components produced by the trivialization of the logarithmic bundle $\mathcal{O}(V\langle D\rangle)$, such that

$$\xi_{s,j}=f_j^{-1}f_j^{(s)}\quad\text{for } 1\leqslant s\leqslant p \text{ and }\quad \xi_{s,j}=f_j^{(s)}\quad\text{for } p+1\leqslant s\leqslant r. \tag{3.12}$$

In general, we are led to consider Finsler metrics of the form

$$(3.13)\qquad \bigg(\sum_{s=1}^{k}\varepsilon_s^{2b}\|\xi_s\|_{h(z)}^{2b/s}\bigg)^{1/b},\quad \xi_s=(\xi_{s,j})_{1\leqslant j\leqslant r},$$

where $h(z)$ is a variable hermitian metric on the logarithmic bundle $V\langle D\rangle$. In the orbifold case, the appropriate "model" Finsler metric is

$$(3.14)\qquad \bigg(\sum_{s=1}^{k}\varepsilon_s^{2b}\bigg(\sum_{j=1}^{p}|f_j|^{-2(1-s/\rho_j)_+}|f_j^{(s)}|^2+\sum_{j=p+1}^{r}|f_j^{(s)}|^2\bigg)^{2b/s}\bigg)^{1/b}.$$

As a consequence of Remark 2.10, we would get a metric with equivalent singularities on the dual L_k^* of the tautological sheaf $L_k=\mathcal{O}_{X_k(V\langle D\rangle)}(1)$ by replacing $\sum_{j=p+1}^{r}|f_j^{(s)}|^2$ with $\sum_{j=1}^{r}|f_j^{(s)}|^2$ (or by any smooth hermitian norm h on V), since the extra terms $\sum_{j=1}^{p}|f_j^{(s)}|^2$ are anyway controlled by the "orbifold part" of the summation. Of course, we need to find a suitable Finsler metric that is globally defined on X. This can be done by taking smooth metrics $h_{V,s}$ on V and h_j on $\mathcal{O}_X(\Delta_j)$ respectively, as well as smooth connections ∇ and ∇_j. One can then consider the globally defined metric

$$(3.15)\qquad \bigg(\sum_{s=1}^{k}\varepsilon_s^{2b}\bigg(\|\nabla^{(s)}f\|_{h_{V,s}}^2+\sum_j\|\sigma_j(f)\|_{h_j}^{-2(1-s/\rho_j)_+}\|\nabla_j^{(s)}(\sigma_j\circ f)\|_{h_j}^2\bigg)^{2b/s}\bigg)^{1/b}$$

where $D=\sum(1-\frac{1}{\rho_j})\Delta_j$ and $\sigma_j\in H^0(X,\mathcal{O}_X(\Delta_j))$ are the tautological sections; here, we want the flexibility of not necessarily taking the same hermitian metrics on V to evaluate the various norms $\|\nabla^{(s)}f\|_{h_{V,s}}$. We obtain Finsler metrics with equivalent singularities by just changing the $h_{V,s}$ and h_j (and keeping ∇, ∇_j unchanged). If we also change the connections, then an argument very similar to the one used in the proof of Lemma 3.6 shows that the ratio of the corresponding metrics is $1\pm O(\max(\varepsilon_s/\varepsilon_{s-1}))$, and therefore arbitrary close to 1 whenever $\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k$; in any case, we get metrics with equivalent singularities. Fix $z_0\in X$ and use coordinates $(z_1,\ldots,z_n)$ as described at the beginning of §3.B, so that $\sigma_j(z)=z_j$, $1\leqslant j\leqslant p$, in a suitable trivialization of $\mathcal{O}_X(\Delta_j)$. Let f be a k-jet of curve such that $f(0)=z\in X\smallsetminus|D|$ is in a sufficiently small neighborhood of z_0. By employing the trivial connections associated with the above coordinates, the derivative $f^{(s)}$ is described by components

$$\xi_{s,j}=f_j^{(s)},\quad 1\leqslant j\leqslant r,\quad \xi_{s,j}^{\log}=f_j^{-1}f_j^{(s)},\quad \xi_{s,j}^{\rm orb}=f_j^{-(1-s/\rho_j)_+}f_j^{(s)},\quad 1\leqslant j\leqslant p,$$

and $\xi_{s,j}^{\rm orb}=\xi_{s,j}^{\log}=\xi_{s,j}$ for $p+1\leqslant j\leqslant r$. Here $\xi_{s,j}^{\rm orb}$ are to be thought of as the components of $f^{(s)}$ in the "virtual" vector bundle $V\langle D^{(s)}\rangle$, and the fact that the argument of these complex numbers is not uniquely defined is irrelevant, because the only thing we need to compute the norms is $|\xi_{s,j}^{\rm orb}|$. Accordingly, for $v\in V_z$, $v\simeq(v_j)_{1\leqslant j\leqslant r}\in\mathbb{C}^r$, we put

$$v_j^{\log}=z_j^{-1}v_j=\sigma_j(z)^{-1}\nabla_j\sigma_j(v)\quad\text{and}\quad v_j^{\rm orb}=z_j^{-(1-s/\rho_j)_+}v_j,\quad 1\leqslant j\leqslant p,$$

and define the orbifold hermitian norm on $V\langle D^{(s)}\rangle$ associated with $h_{V,s}$ and h_j by

$$(3.16)\qquad \|v^{\rm orb}\|_{\widetilde{h}_s}^2=\|v\|_{h_{V,s}}^2+\sum_{j=1}^{p}\|\sigma_j(z)\|_{h_j}^{-2(1-s/\rho_j)_+)}\|\nabla_j\sigma_j(v)\|_{h_j}^2$$

$$(3.16') \qquad = \|v\|^2_{h_{V,s}} + \sum_{j=1}^{p} \|\sigma_j(z)\|_{h_j}^{2(1-(1-s/\rho_j)_+)} |v_j^{\log}|^2$$

$$(3.16'') \qquad = \|v\|^2_{h_{V,s}} + \sum_{j=1}^{p} \|v_j^{\mathrm{orb}}\|^2_{h_j^{1-(1-s/\rho_j)_+}}.$$

With this notation, the orbifold Finsler metric (3.15) on k-jets is reduced to an expression

$$(3.17) \quad \|\xi^{\mathrm{orb}}\|^2_{\Psi_{h,b,\varepsilon}} = \left(\sum_{s=1}^{k} \varepsilon_s^{2b} \|\xi_s^{\mathrm{orb}}\|^{2b/s}_{\widetilde{h}_s}\right)^{1/b}, \quad \xi_s^{\mathrm{orb}} = (\xi_{s,j}^{\mathrm{orb}})_{1\leqslant j\leqslant r}, \quad \xi^{\mathrm{orb}} = (\xi_s^{\mathrm{orb}})_{1\leqslant s\leqslant k},$$

formally identical to what we had in the compact or logarithmic cases. If v is a local holomorphic section of $\mathcal{O}_X(V)$, formula (3.16) shows that the norm $\|v^{\mathrm{orb}}\|_{\widetilde{h}_s}$ can take infinite values when $z \in |D|$, while, by (3.16′), the norm is always bounded (but slightly degenerate along $|D|$) if v is a section of the logarithmic sheaf $\mathcal{O}_X(V\langle\lceil D\rceil\rangle)$; we think intuitively of the orbifold total space $V\langle D^{(s)}\rangle$ as the subspace of V in which the tubular neighborhoods of the zero section are defined by $\|v^{\mathrm{orb}}\|_{\widetilde{h}_s} < \varepsilon$ for $\varepsilon > 0$.

3.18. Remark. When $\rho_j \in \mathbb{Q}$, we can take an adapted Galois cover $g : Y \to X$ such that $(z_j \circ g)^{1-(1-s/\rho_j)_+}$ is univalent on Y for all components Δ_j involved, and we then get a well defined locally free sheaf $\mathcal{O}_Y\big(g^*V\langle D^{(s)}\rangle$ such that

$$g^*\big(\mathcal{O}_X(V\langle\lceil D\rceil\rangle)\big) \subset \mathcal{O}_Y\big(g^*V\langle D^{(s)}\rangle\big) \subset g^*\big(\mathcal{O}_X(V)\big).$$

However, as already stressed in Remark 1.26, this viewpoint is not needed in our analytic approach.

3.C. Orbifold tautological sheaves and their curvature

In this context, we define the orbifold tautological sheaves

$$(3.19) \qquad \mathcal{O}_{X_k(V\langle D\rangle)}(m) := \mathcal{O}_{X_k(V\langle\lceil D\rceil\rangle)}(m) \otimes \mathcal{I}((\Psi^*_{k,b,\varepsilon})^m)$$

to be the logarithmic tautological sheaves $\mathcal{O}_{X_k(V\langle\lceil D\rceil\rangle)}(m)$ twisted by the multiplier ideal sheaves associated with the dual metric $\Psi^*_{k,b,\varepsilon}$ (cf. (3.17)), when these are viewed as singular hermitian metrics over the logarithmic k-jet bundle $X_k(V\langle\lceil D\rceil\rangle)$. In accordance with this viewpoint, we simply define the orbifold k-jet bundle to be $X_k(V\langle D\rangle) = X_k(V\langle\lceil D\rceil\rangle)$. The calculation of the curvature tensor is formally the same as in the case $D = 0$, and we obtain :

3.20. Proposition. *With respect to the (rescaled) orbifold k-jet components*

$$\xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(1-(1-\rho_\lambda/s)_+)} f_\lambda^{(s)}(0), \quad 1 \leqslant \lambda \leqslant p, \quad \textit{and} \quad \xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(s)}(0), \quad p+1 \leqslant \lambda \leqslant r,$$

and of the dual metric $\Psi^*_{h,b,\varepsilon}$, the curvature form of the tautological sheaf $L_k = \mathcal{O}_{X_k(V\langle D\rangle)}(1)$ admits at any point $(z,[\xi]) \in X_k(V\langle D\rangle)$ an approximate expression

(a) $\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z,[\xi]) \simeq \omega_{r,k,b}(\xi) + g_{V,D,k}(z,x,u)$,

where $x_s = |\xi_s|^{2b/s}$, $u_s = \frac{\xi_s}{|\xi_s|} \in \mathbb{S}^{2r-1}$ are polar coordinates associated with $\xi_s = (\xi_{s,\lambda})_{1\leqslant\lambda\leqslant k}$ in $\mathbb{C}^r$, $x = (x_1,\ldots,x_k) \in \Delta^{k-1}$, $[\xi] = [\xi_1,\ldots,\xi_k] \in \mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ and

(b) $g_{V,D,k}(z,x,u) = \dfrac{\mathrm{i}}{2\pi} \displaystyle\sum_{1\leqslant s\leqslant k} \frac{x_s}{s} \sum_{i,j,\lambda,\mu} c^{(s)}_{ij\lambda\mu}(z)\, u_{s,\lambda}\overline{u}_{s,\mu}\, dz_i \wedge d\overline{z}_j.$

Here $(c^{(s)}_{ij\lambda\mu})$ *are the coefficients of the curvature tensor* $-\Theta_{V\langle D^{(s)}\rangle,\widetilde{h}_s}$, *and the error terms are* $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$, *uniformly on the projectivized orbifold variety* $X_k(V\langle D\rangle)$.

Notice, as is clear from the expressions (3.16″), (3.17) and the fact that $v_j = z_j v_j^{\rm orb}$, that our orbifold Finsler metrics always have fiberwise positive curvature, equal to $\omega_{k,r,b}(\xi)$, along the fibers of $X_k(V\langle D\rangle) \to X$ (even after taking into account the so-called error terms, because fiberwise, the functions under consideration are just sums of even powers $|\widetilde{\xi}_s^{\rm orb}|^{2b/s}$ in suitable k-jet components, and are therefore plurisubharmonic.)

4. Existence theorems for jet differentials

4.A. Expression of the Morse integral

Thanks to the uniform approximation provided by proposition 3.20, we can (and will) neglect the $O(\varepsilon_s/\varepsilon_{s-1})$ error terms in our calculations. Since $\omega_{r,k,b}$ is positive definite on the fibers of $X_k(V\langle D\rangle) \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form $g_{V,D,k}(z,x,u)$. By the binomial formula, the q-index integral of $(L_k,\Psi^*_{h,b,\varepsilon})$ on $X_k(V\langle D\rangle)$ is therefore equal to

$$\int_{X_k(V\langle D\rangle)(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}}$$

$$= \frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in\mathbb{P}(1^{[r]},\ldots,k^{[r]})} \omega^{kr-1}_{r,k,b}(\xi)\wedge \mathbb{1}_{g_{V,D,k},q}(z,x,u)\, g_{V,D,k}(z,x,u)^n \tag{4.1}$$

where $\mathbb{1}_{g_{V,D,k},q}(z,x,u)$ is the characteristic function of the open set of points where $g_{V,D,k}(z,x,u)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $g_{V,D,k}(z,x,u)^n$ is a determinant, the product $\mathbb{1}_{g_{V,D,k},q}(z,x,u)\, g_{V,D,k}(z,x,u)^n$ gives rise to a continuous function on $X_k(V\langle D\rangle)$. By Formula 3.10 (a), we get

$$\int_{X_k(V\langle D\rangle)(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}} = \frac{(n+kr-1)!}{n!\,k!^r(kr-1)!}\ \times$$

$$\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \mathbb{1}_{g_{V,D,k},q}(z,x,u)\, g_{V,D,k}(z,x,u)^n\, d\nu_{k,r}(x)\, d\mu(u). \tag{4.2}$$

4.B. Probabilistic estimate of cohomology groups

We assume here that we are either in the "compact" case ($D=0$), or in the logarithmic case ($\rho_j=\infty$). Then the curvature coefficients $c^{(s)}_{ij\lambda\mu} = c_{ij\lambda\mu}$ do not depend on s and are those of the dual bundle V^* (resp. $V^*\langle D\rangle$). In this situation, formula 3.20 (b) for $g_{V,D,k}(z,x,u)$ can be thought of as a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in\mathbb{S}^{2r-1}$ with certain positive weights x_s/s; we then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ (resp. logarithmic derivatives) are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_{V,D,k}(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{\mathbb{S}^{2r-1}} u_{s,\lambda}\overline{u}_{s,\mu} d\mu(u_s) = \frac{1}{r}\delta_{\lambda\mu}$ and $\int_{\Delta^{k-1}} x_s\, d\nu_{k,r}(x) = \frac{1}{k}$, we find

$$\mathbf{E}(g_{V,D,k}(z,\bullet,\bullet)) = \frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda} c_{ij\lambda\lambda}(z)\, dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_{V,D,k}(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\Theta_{\det(V^*\langle D\rangle),\det h^*}, \tag{4.3}$$

where $\Theta_{\det(V^*\langle D\rangle),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*\langle D\rangle)$ with the metric induced by h. It is natural to guess that $g_{V,D,k}(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_{V,D,k}(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally $g_{V,D,k}$ by its expected value in (4.2), we get the integral

$$\frac{(n+kr-1)!}{n!\,k!^r(kr-1)!}\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta := \Theta_{\det(V^*\langle D\rangle),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!\,k!^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] in the compact case; the more general logarithmic case can be treated without any change, so we state the result in this situation by just transposing the results of [Dem11].

4.4. Probabilistic estimate. *Let (X,V,D) be a non singular logarithmic directed variety. Fix smooth hermitian metrics ω on T_X, h on $V\langle D\rangle$, and write $\omega=\frac{\mathrm{i}}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V\langle D\rangle,h}=-\frac{\mathrm{i}}{2\pi}\sum c_{ij\lambda\mu}dz_i\wedge d\overline{z}_j\otimes e_\lambda^*\otimes e_\mu$ the curvature tensor of $V\langle D\rangle$ with respect to an h-orthonormal frame (e_λ), and put*

$$\eta(z):=\Theta_{\det(V^*\langle D\rangle),\det h^*}=\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij}:=\sum_{1\leqslant\lambda\leqslant r}c_{ij\lambda\lambda}.$$

Finally consider the k-jet line bundle $L_k=\mathcal{O}_{X_k(V\langle D\rangle)}(1)\to X_k(V\langle D\rangle)$ equipped with the induced metric $\Psi^*_{h,b,\varepsilon}$ (as defined above, with $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k(V\langle D\rangle)(L_k,q)$ is given by

$$\int_{X_k(V\langle D\rangle)(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}}=\frac{(\log k)^n}{n!\,k!^r}\bigg(\int_X\mathbb{1}_{\eta,q}\eta^n+O((\log k)^{-1})\bigg)$$

for all $q=0,1,\ldots,n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of $\Theta_{V\langle D\rangle}$, η and ω. Moreover, the left hand side is identically zero for $q>n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k(V\langle D\rangle)\to X$, by the plurisubharmonicity of the weight (this is true even when the error terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q>n$. It will be useful to extend the above estimates to the case of sections of

$$L_{F,k}=\mathcal{O}_{X_k(V\langle D\rangle)}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\tag{4.5}$$

where $F\in\mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k: X_k(V\langle D\rangle)\to X$ is the natural projection. We assume here that F is also equipped with a smooth hermitian metric h_F. In formulas (4.2–4.4), the curvature $\Theta_{L_{F,k}}$ of $L_{F,k}$ takes the form $\Theta_{L_{F,k}}=\omega_{r,k,b}(\xi)+g_{V,D,F,k}(z,x,u)$ where

$$g_{V,D,F,k}(z,x,u)=g_{V,D,k}(z,x,u)+\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\Theta_{F,h_F}(z),\tag{4.6}$$

and by the same calculations its normalized expected value is

$$(4.7)\qquad \eta_F(z) := \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\cdots+\frac{1}{k})}\mathbf{E}(g_{V,D,F,k}(z,\bullet,\bullet)) = \Theta_{\det V^*\langle D\rangle,\det h^*}(z) + \Theta_{F,h_F}(z).$$

Then the variance estimate for $g_{V,D,F,k}$ is the same as the variance estimate for $g_{V,D,k}$, and the recentered L^p bounds are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 4.4 is therefore still true in exactly the same form for $L_{F,k}$, provided we use $g_{V,D,F,k}$ and η_F instead of $g_{V,D,k}$ and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} &h^q\Big(X, E_{k,m}V^*\langle D\rangle \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big) \\ &\qquad\qquad = h^q(X_k(V\langle D\rangle), \mathcal{O}_{X_k(V\langle D\rangle)}(m)\otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big), \end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

4.8. Theorem. *Let $(X, V\langle D\rangle)$ be a non singular logarithmic directed variety, $F\to X$ a $\mathbb{Q}$-line bundle, $(V\langle D\rangle, h)$ and (F, h_F) smooth hermitian structure on $V\langle D\rangle$ and F respectively. We define*

$$\begin{aligned} L_{F,k} &= \mathcal{O}_{X_k(V\langle D\rangle)}(1)\otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big),\\ \eta_F &= \Theta_{\det V^*\langle D\rangle,\det h^*} + \Theta_{F,h_F} = \Theta_{\det V^*\langle D\rangle\otimes F,\det h^*}. \end{aligned}$$

Then for all $q\geqslant 0$ and all $m\gg k\gg 1$ such that m is sufficiently divisible, we have

$$\text{(a)}\qquad h^q(X_k(V\langle D\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,k!^r}\bigg(\int_{X(\eta_F,q)}(-1)^q\eta_F^n + O((\log k)^{-1})\bigg),$$

$$\text{(b)}\qquad h^0(X_k(V\langle D\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,k!^r}\bigg(\int_{X(\eta_F,\leqslant 1)}\eta_F^n - O((\log k)^{-1})\bigg),$$

$$\text{(c)}\qquad \chi(X_k(V\langle D\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,k!^r}\big(c_1(V^*\langle D\rangle\otimes F)^n + O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr80] already checked the Riemann-Roch calculation (4.8 c) in the special case $D=0$, $V=T_X^*$ and $F=\mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0\geqslant\chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\bigg(X, E_{k,m}T_X^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\bigg) = 0$$

as soon as $K_X\otimes F$ is big and $m\gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities (Theorem 2.9, cf. [Bon93]), everything works almost unchanged in the case where the metric h on V is taken to

a product $h = h_\infty e^\varphi$ of a smooth metric h_∞ by the exponential of a quasi-plurisubarmonic weight φ with analytic singularities (so that $\det(h^*) = \det(h^*_\infty)e^{-r\varphi}$). Then η is a $(1,1)$-current with logarithmic poles, and we just have to twist our cohomology groups by the appropriate multiplier ideal sheaves $\mathcal{I}_{k,m}$ associated with the weight $\frac{1}{k}(1+\frac{1}{2}+\cdots+\frac{1}{k})m\,\varphi$, since this is the multiple of $\det V^*$ that occurs in the calculation, up to the factor $\frac{1}{r}\times r\varphi$. The corresponding Morse integrals need only be evaluated in the complement of the poles, i.e., on $X(\eta,q)\smallsetminus S$ where $S = \mathrm{Sing}(\varphi)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_{F,k}^{\otimes m})\otimes\mathcal{I}_{k,m}\big)\subset E_{k,m}V^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on $X_k(V)$). If we assume that $K_V\otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of X.

4.9. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned}&h^0\Big(X_k(V),\mathcal{O}_{X_k(V)}(m)\otimes\pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big)\\ &\qquad\geqslant\frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,k!^r}\Big(\mathrm{Vol}(K_V\otimes F)-O((\log k)^{-1})\Big)-o(m^{n+kr-1}),\end{aligned}$$

when $m\gg k\gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V\otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu:\widetilde{X}\to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V\otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V\otimes F)>0$. Let us fix smooth hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1}\otimes h_F)$ on $\mu^*(K_V\otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta>0$, one can find a modification $\mu_\delta:\widetilde{X}_\delta\to X$ dominating μ such that

$$\mu_\delta^*(K_V\otimes F)=\mathcal{O}_{\widetilde{X}_\delta}(A+E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A)=A^n\geqslant\mathrm{Vol}(K_V\otimes F)-\delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular hermitian metric h_Ah_E on $\mu_\delta^*(K_V\otimes F)$ with poles along E, i.e. the quotient $h_Ah_E/\mu^*(\det h_0^{-1}\otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h=h_0e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta_F=\Theta_{K_V,\det h^*}+\Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta_F=\Theta_{A,h_A}+[E],\qquad [E]=\text{current of integration on }E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S}\eta_F^n=\int_{\widetilde{X}_\delta}\Theta^n_{A,h_A}=A^n\geqslant \mathrm{Vol}(K_V\otimes F)-\delta$$

and Corollary 4.9 follows from the fact that δ can be taken arbitrary small. □

4.10. Remark. Since the probability estimate requires k to be very large, and since all non logarithmic components disappear from $D^{(s)}$ when s is large, the above lower bound does not work in the general orbifold case. In that case, one can only hope to get an interesting result when k is fixed and not too large. This is what we aim at in the next sections.

5. Curvature of orbifold tangent bundles

5.A. Positivity concepts for vector bundles

Let $E\to X$ be a holomorphic vector bundle equipped with a hermitian metric. Then E possesses a uniquely defined Chern connection ∇_h compatible with h and such that $\nabla_h^{0,1}=\overline\partial$. The curvature tensor of (E,h) is defined to be

$$\Theta_{E,h}:=\frac{\mathrm{i}}{2\pi}\,\mathrm{i}\,\partial\overline\partial\nabla_h^2\in C^\infty(X,\Lambda^{1,1}T_X^*\otimes\mathrm{Hom}(E,E)). \tag{5.1}$$

One can then associate bijectively to $\Theta_{E,h}$ a hermitian form $\widetilde\Theta_{E,h}$ on $TX\otimes E$, such that

$$\widetilde\Theta_{E,h}(\xi\otimes v,\xi\otimes v)=\langle\Theta_{E,h}(\xi,\xi)\cdot v,v\langle_h. \tag{5.2}$$

and can be written

$$\Theta_{E,h}=\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,dz_i\wedge d\overline z_j\otimes e_\lambda^*\otimes e_\mu$$

Let $(z_1,\ldots,z_n)$ be a holomorphic coordinate system and $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ a smooth frame of E. If (e_λ) is chosen to be orthonormal, then we can write

$$\Theta_{E,h}=\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,dz_i\wedge d\overline z_j\otimes e_\lambda^*\otimes e_\mu, \tag{5.3}$$

$$\widetilde\Theta_{E,h}(\xi\otimes v,\xi\otimes v)=\frac{1}{2\pi}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,\xi_i\overline\xi_j\,v_\lambda\overline v_\mu, \tag{5.3'}$$

and more generally $\widetilde\Theta_{E,h}(\tau,\tau)=\frac{1}{2\pi}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,\tau_{i\lambda}\overline\tau_{j\mu}$ for every tensor $\tau\in T_X\otimes E$. We now consider three concepts of (semi-)positivity, the first two being very classical.

5.4. Definition. *Let θ be a hermitian form on a tensor product $T\otimes E$ of complex vector spaces. We say that*

(a) *θ is Griffiths semi-positive if $\theta(\xi\otimes v,\xi\otimes v)\geq 0$ for every $\xi\in T$ and every $v\in E$;*

(b) *θ is Nakano semi-positive if $\theta(\tau,\tau)\geq 0$ for every $\tau\in T\otimes E$;*

(c) *θ is strongly semi-positive if there exist a finite collection of linear forms $\alpha_j\in T^*$, $w_j\in E^*$ such that $\theta=\sum_j|\alpha_j\otimes w_j|^2$, i.e.*

$$\theta(\tau,\tau)=\sum_j|(\alpha_j\otimes w_j)\cdot\tau|^2,\quad\forall\tau\in T\otimes E.$$

Semi-negativity concepts are introduced in a similar way.

(d) *We say that the hermitian bundle* (E, h) *is Griffiths semi-positive, resp. Nakano semi-positive, resp. strongly semi-positive, if* $\widetilde{\Theta}_{E,h}(x) \in \mathrm{Herm}(T_{X,x} \otimes E_x)$ *satisfies the corresponding property for every point* $x \in X$.

(e) (*Strict*) *Griffiths positivity means that* $\widetilde{\Theta}_{E,h}(\xi \otimes v, \xi \otimes v) > 0$ *for every non zero vectors* $\xi \in T_{X,x}$, $v \in E_x$.

(f) (*Strict*) *strong positivity means that at every point* $x \in X$ *we can decompose* $\widetilde{\Theta}_{E,h}$ *as* $\widetilde{\Theta}_{E,h} = \sum_j |\alpha_j \otimes w_j|^2$ *where* $\mathrm{Span}(\alpha_j \otimes w_j) = T^*_{X,x} \otimes E^*_x$.

We will denote respectively by $\geqslant_G$, $\geqslant_N$, $\geqslant_S$ (and $>_G$, $>_N$, $>_S$) the Griffiths, Nakano, strong (semi-)positivity relations. It is obvious that

$$\theta \geqslant_S 0 \quad \Rightarrow \quad \theta \geqslant_N 0 \quad \Rightarrow \quad \theta \geqslant_G 0,$$

and one can show that the reverse implications do not hold when $\dim T > 1$ and $\dim E > 1$. The following result from [Dem80] will be useful.

5.5. Proposition. *Let* $\theta \in \mathrm{Herm}(T \otimes E)$, *where* (E, h) *is a hermitian vector space. We define* $\mathrm{Tr}_E(\theta) \in \mathrm{Herm}(T)$ *to be the hermitian form such that*

$$\mathrm{Tr}_E(\theta)(\xi, \xi) = \sum_{1 \leqslant \lambda \leqslant r} \theta(\xi \otimes e_\lambda, \xi \otimes e_\lambda)$$

where $(e_\lambda)_{1 \leqslant \lambda \leqslant r}$ *is an arbitrary orthonormal basis of* E. *Then*

$$\theta \geqslant_G 0 \quad \Longrightarrow \quad \theta + \mathrm{Tr}_E(\theta) \otimes h \geqslant_S 0.$$

As a consequence, if (E, h) *is a Griffiths* (*semi-*)*positive vector bundle, then the tensor product* $(E \otimes \det E, h \otimes \det(h))$ *is strongly* (*semi-*)*positive.*

Proof. Since [Dem80] is written in French and perhaps not so easy to find, we repeat here briefly the arguments. They are based on a Fourier inversion formula for discrete Fourier transforms.

5.6. Lemma. *Let* q *be an integer* $\geqslant 3$, *and* x_λ, y_μ, $1 \leqslant \lambda, \mu \leqslant r$, *be complex numbers. Let* χ *describe the set* U_q^r *of* r*-tuples of* q*-th roots of unity and put*

$$x'_\chi = \sum_{1 \leqslant \lambda \leqslant r} x_\lambda \overline{\chi}_\lambda, \qquad y'_\chi = \sum_{1 \leqslant \mu \leqslant r} y_\mu \overline{\chi}_\mu, \qquad \chi \in U_q^r.$$

Then for every pair (α, β), $1 \leqslant \alpha, \beta \leqslant r$, *the following identity holds:*

$$q^{-r} \sum_{\chi \in U_q^r} x'_\chi \overline{y}'_\chi \chi_\alpha \overline{\chi}_\beta = \begin{cases} x_\alpha \overline{y}_\beta & \text{if } \alpha \neq \beta, \\ \sum_{1 \leqslant \mu \leqslant r} x_\mu \overline{y}_\mu & \text{if } \alpha = \beta. \end{cases}$$

In fact, the coefficient of $x_\lambda \overline{y}_\mu$ in the summation $q^{-r} \sum_{\chi \in U_q^r} x'_\chi \overline{y}'_\chi \chi_\alpha \overline{\chi}_\beta$ is given by

$$q^{-r} \sum_{\chi \in U_q^r} \chi_\alpha \overline{\chi}_\beta \overline{\chi}_\lambda \chi_\mu,$$

so it is equal to 1 when the pairs $\{\alpha, \mu\}$ and $\{\beta, \lambda\}$ coincide, and is equal to 0 otherwise. The identity stated in Lemma 5.6 follows immediately. $\square$

Now, let $(t_j)_{1\leqslant j\leqslant n}$ be a basis of T, $(e_\lambda)_{1\leqslant\lambda\leqslant r}$ an orthonormal basis of E and $\xi = \sum_j \xi_j t_j \in T$, $u = \sum_{j,\lambda} u_{j\lambda}\, t_j \otimes e_\lambda \in T \otimes E$. The coefficients $c_{jk\lambda\mu}$ of θ with respect to the basis $t_j \otimes e_\lambda$ satisfy the symmetry relation $\overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}$, and we have the formulas

$$\theta(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu}, \quad \mathrm{Tr}_E\, \theta(\xi,\xi) = \sum_{j,k,\lambda} c_{jk\lambda\lambda} \xi_j \overline{\xi}_k,$$

$$(\theta + \mathrm{Tr}_E\, \theta \otimes h)(u,u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu} + c_{jk\lambda\lambda} u_{j\mu} \overline{u}_{k\mu}.$$

For every $\chi \in U_q^r$, let us put

$$u'_{j\chi} = \sum_{1\leqslant\lambda\leqslant r} u_{j\lambda} \overline{\chi}_\lambda \in \mathbb{C}, \quad \widehat{u}_\chi = \sum_j u'_{j\chi} t_j \in T\,, \quad \widehat{e}_\chi = \sum_\lambda \chi_\lambda e_\lambda \in E.$$

Lemma 5.6 implies

$$\begin{aligned} q^{-r} \sum_{\chi\in U_q^r} \theta(\widehat{u}_\chi \otimes \widehat{e}_\chi, \widehat{u}_\chi \otimes \widehat{e}_\chi) &= q^{-r} \sum_{\chi\in U_q^r} c_{jk\lambda\mu} u'_{j\chi} \overline{u}'_{k\chi} \chi_\lambda \overline{\chi}_\mu \\ &= \sum_{j,k,\lambda\neq\mu} c_{jk\lambda\mu} u_{j\lambda} \overline{u}_{k\mu} + \sum_{j,k,\lambda,\mu} c_{jk\lambda\lambda} u_{j\mu} \overline{u}_{k\mu}. \end{aligned}$$

The Griffiths positivity assumption $\theta_G \geqslant 0$ shows that $\xi \mapsto q^{-r}\theta(\xi \otimes \widehat{e}_\chi, \xi \otimes \widehat{e}_\chi)$ is a semi-positive hermitian form on T, hence there are linear forms $\ell_{\chi,j} \in T^*$ such that $\theta(\xi \otimes \widehat{e}_\chi, \xi \otimes \widehat{e}_\chi) = \sum_j |\ell_{\chi,j}(\xi)|^2$ for all $\xi \in T$. Our final identity can ve rewritten

$$(\theta + \mathrm{Tr}_E\, \theta \otimes h)(u,u) = \sum_{\chi\in U_q^r} \sum_j |\ell_{\chi,j}(\widehat{u}_\chi)|^2 = \sum_{\chi\in U_q^r} \sum_j |\ell_{\chi,j} \otimes \chi^*(u)|^2$$

where $\chi^* = \langle \bullet, \chi\rangle \in E^*$, thus $\theta + \mathrm{Tr}_E\, \theta \otimes h \geqslant_S 0$. $\square$

5.7. Corollary. *Let $r = \dim E$ and $\Theta \in \mathrm{Herm}(T \otimes E)$.*

(a) *If $\theta \geqslant_G 0$, then* $\quad -\mathrm{Tr}_E\, \theta \otimes h \;\leqslant_S\; \theta \;\leqslant_S\; r\ \mathrm{Tr}_E\, \theta \otimes h.$

(b) *If $\theta \leqslant_G 0$, then* $\; -r\ \mathrm{Tr}_E(-\theta) \otimes h \;\leqslant_S\; \theta \;\leqslant_S\; \mathrm{Tr}_E(-\theta) \otimes h.$

(c) *If $\pm\theta \leqslant_G \tau \otimes h$ where $\tau \in \mathrm{Herm}(T)$ is semi-positive, then*

$$-(2r+1)\,\tau \otimes h \;\leqslant_S\; \theta \;\leqslant_S\; (2r+1)\,\tau \otimes h.$$

Proof. (a) It is easy to chech that $\theta' = \mathrm{Tr}_E\, \theta \otimes h - \theta$ satisfies $\theta' \geqslant_G 0$ and that we have $\mathrm{Tr}_E\, \theta' = (r-1)\, \mathrm{Tr}_E\, \theta$. Lemma 5.6 implies

$$\theta' + \mathrm{Tr}_E\, \theta' \otimes h = r\ \mathrm{Tr}_E\, \theta \otimes h - \theta \geqslant_S 0.$$

(b) follows from (a), after replacing θ with $-\theta$.

(c) also follows from Lemma 5.6 by taking $\theta' = \tau \otimes h + \theta$ (resp. $\theta' = \tau \otimes h - \theta$), since $\mathrm{Tr}_E\, \theta \leqslant r\,\tau$ and we have e.g.

$$0 \leqslant_S \theta' + \mathrm{Tr}_E\, \theta' \otimes h = \theta + \mathrm{Tr}_E\, \theta \otimes h + (r+1)\tau \otimes h \leqslant_S \theta + (2r+1)\tau \otimes h. \qquad \square$$

5.B. Estimate of the curvature tensor in the orbifold setting

The main qualitative result is summarized in the following statement.

5.8. Proposition. *Let X be a projective variety, A an ample line bundle, and (X,V,D) an orbifold directed structure where $D=\sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor transverse to V in X. Let d_j be the infimum of numbers $\lambda\in\mathbb{R}_+$ such that $\lambda A-\Delta_j$ is nef, and γ_V be the infimum of numbers $\gamma\geqslant 0$ such that $\gamma\,\Theta_{A,h_A}\otimes\mathrm{Id}_V-\Theta_{V,h_V}\geqslant_G 0$ for suitable hermitian metrics h_V on V. Then for every $\gamma>\gamma_{V,D}:=\max(\max_j(d_j/\rho_j),\gamma_V)$, the orbifold vector bundle $V\langle D\rangle$ possesses a hermitian metric $h_{V\langle D\rangle,\gamma}$ such that*

(a) *$h_{V\langle D\rangle,\gamma}$ is smooth on $X\smallsetminus|D|$,*

(b) *$h_{V\langle D\rangle,\gamma}$ has the appropriate orbifold singularities along D,*

(c) *we have $\gamma\,\Theta_{A,h_A}\otimes\mathrm{Id}-\Theta_{V\langle D\rangle,h_{V\langle D\rangle,\gamma}}\geqslant_G 0$ on $X\smallsetminus|D|$.*

Proof. Let h_A be a metric on A such that $\Theta_{A,h_A}>0$, written locally as $h_A=e^{-\psi}$, and take $\gamma>\max(\max_j(d_j/\rho_j),\gamma_V)$. Consider the tautological sections $\sigma_j\in H^0(X,\mathcal{O}_X(\Delta_j))$ defining $\Delta_j=\sigma_j^{-1}(0)$, and let h_j be a smooth hermitian metric on $\mathcal{O}_X(\Delta_j)$ for which

$$\gamma\,\Theta_A-\frac{1}{\rho_j}\Theta_{\mathcal{O}_X(\Delta_j),h_j}>0,\tag{5.9}$$

as is possible by our choice of constants d_j and γ. Finally, denote by ∇_j the associated Chern connection on $\mathcal{O}_X(\Delta_j)$. If we write $h_j=e^{-\varphi_j}$ in some local trivialization, then $\nabla_j\sigma_j=\nabla_j^{1,0}\sigma_j=\partial\sigma_j-\sigma_j\partial\varphi_j$. We are going to estimate the curvature of the orbifold metric $h_{V\langle D\rangle,\varepsilon}$ on $V\langle D\rangle$ defined by

$$\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}=|v|^2_{h_V}+\sum_j\varepsilon_j\,|\sigma_j|^{-2(1-1/\rho_j)}_{h_j}\,|\nabla_j\sigma_j(v)|^2_{h_j},\quad \varepsilon_j\ll 1,\tag{5.10}$$

where the metric h_V is chosen so that $\gamma\,\Theta_{A,h_A}\otimes\mathrm{Id}_V-\Theta_{V,h_V}\geqslant_G 0$ (resp. $\geqslant_S 0$). We will later to slightly perturb the metric as $\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}e^{\chi_\varepsilon}$ with an extra weight χ_ε, but we ignore this minor twist for the time being. Since

$$\mathrm{i}\,\partial\overline\partial\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}=\mathrm{i}\langle\nabla v,\nabla v\rangle_{h_{V\langle D\rangle,\varepsilon}}-2\pi\,\langle\Theta_{V\langle D\rangle,h_{V\langle D\rangle,\varepsilon}}(v),v\rangle_{h_{V\langle D\rangle,\varepsilon}}$$

where $\nabla v=dv+\Gamma(dz)\cdot v$ is the Chern connection of $(V\langle D\rangle,h_{V\langle D\rangle,\varepsilon})$, what we need to prove is that on the total space of V over $X\smallsetminus|D|$, the $(1,1)$-form

$$Q(z,v):=\mathrm{i}\,\partial\overline\partial\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}+2\pi\,\gamma\,\Theta_{A,h_A}\,\|v\|^2_{h_{V\langle D\rangle,\varepsilon}},\tag{5.11}$$

is non negative. For this, we calculate the associated hermitian quadratic form on T_V

$$\widetilde Q(z,v)(\xi,\eta)^2,\quad (\xi,\eta)\in T_{V,(z,v)},\quad \xi=\sum_{j=1}^n\xi\frac{\partial}{\partial z},\quad \eta=\sum_{\lambda=1}^r\eta_\lambda\frac{\partial}{\partial v_\lambda},$$

and observe that the curvature tensor is obtained by taking the restriction to the "parallel" directions $\nabla v=0$, that is, by substituting $dv=-\Gamma(dz)\cdot v$, i.e. $\eta=-\Gamma(\xi)\cdot v$. Let us fix an arbitrary point $z_0\in X\smallsetminus|D|$. We take local holomorphic coordinates $(z_1,\ldots,z_n)$ centered at z_0, and let $(e_1,\ldots,e_r)$ be a local holomorphic frame of V such that

$$\langle e_\lambda,e_\mu\rangle_{h_V}=\delta_{\lambda\mu}+\sum_{\ell,m,\lambda,\mu}c_{\ell m\lambda\mu}\,z_\ell\overline z_m+O(|z|^3),$$

where the $c_{\ell m\lambda\mu}$ are the coefficients of $-2\pi\,\Theta_{V,h_V}$. Let us write $v=\sum_{\lambda=1}^r v_\lambda e_\lambda$ and denote by $\langle v,w\rangle=\sum_{1\leqslant\lambda\leqslant r} v_\lambda\overline{w}_\lambda$ the standard hermitian form, $|v|$ the associated norm. We find

$$\begin{aligned}\|v\|^2_{h_{V\langle D\rangle,\varepsilon}} &= |v|^2+\sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\, z_\ell\overline{z}_m v_\lambda\overline{v}_\mu + O(|z|^3)\\ &\qquad+\sum_j \varepsilon_j\,\big(|\sigma_j|^2e^{-\varphi_j}\big)^{-1+1/\rho_j}\,\big|\partial\sigma_j(v)-\sigma_j\partial\varphi_j(v)\big|^2e^{-\varphi_j},\end{aligned}\tag{5.12}$$

since $\overline{\partial}\sigma_j=0$. In order to simplify the calculation, we set formally

$$\begin{aligned}&\tilde\sigma_j=\sigma_j^{1/\rho_j},\quad \tilde\varepsilon_j=\rho_j^2\varepsilon_j,\quad \tilde\varphi_j=\rho_j^{-1}\varphi_j,\qquad \text{if } \rho_j<\infty,\\ &\tilde\sigma_j=\log\sigma_j,\quad \tilde\varepsilon_j=\varepsilon_j,\quad\ \ \tilde\varphi_j=\varphi_j,\qquad\quad\ \ \text{if } \rho_j=\infty.\end{aligned}\tag{5.13}$$

Respectively to the non logarithmic and logarithmic situations, we then get the more tractable expression

$$\begin{aligned}\|v\|^2_{h_{V\langle D\rangle,\varepsilon}} &= |v|^2+\sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\, z_\ell\overline{z}_m v_\lambda\overline{v}_\mu + O(|z|^3)+\sum_j\tilde\varepsilon_j\,\big|\partial\tilde\sigma_j(v)-\sigma_j\partial\tilde\varphi_j(v)\big|^2e^{-\tilde\varphi_j},\\ \|v\|^2_{h_{V\langle D\rangle,\varepsilon}} &= |v|^2+\sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\, z_\ell\overline{z}_m v_\lambda\overline{v}_\mu + O(|z|^3)+\sum_j\tilde\varepsilon_j\,\big|\partial\tilde\sigma_j(v)-\partial\tilde\varphi_j(v)\big|^2.\end{aligned}\tag{5.14}$$

In what follows, for the sake of simplicity, we remove the tildes in the notation, and conduct the calculation only in the non logarithmic situation ($\rho_j<\infty$), since the logarithmic case can be recovered by taking ρ_j very large; this actually amounts to using a ramified change of variable $\tilde z'_\ell=z_\ell^{1/\rho_\ell}$ in suitable coordinates, allowing us in this way to take $\rho_j=1$ in (5.12). We then obtain

$$\begin{aligned}\|v\|^2_{h_{V\langle D\rangle,\varepsilon}} &= |v|^2+\sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\, z_\ell\overline{z}_m v_\lambda\overline{v}_\mu + O(|z|^3)\\ &\qquad+\sum_j\Big(|\partial\sigma_j(v)|^2-2\,\mathrm{Re}\,\big(\partial\sigma_j(v)\,\overline{\sigma}_j\,\overline{\partial}\varphi_j(v)\big)+|\sigma_j|^2|\partial\varphi_j(v)|^2\Big)\,e^{-\varphi_j}.\end{aligned}\tag{5.15}$$

We also take holomorphic trivializations of the line bundles $\mathcal{O}_X(\Delta_j)$ so that the associated weight φ_j satisfies $\varphi_j(z)=\sum_{\ell,m}\alpha_{j\ell m}\,z_\ell\overline{z}_m+O(|z|^3)$ near $z_0=0$. Then

$$\partial\varphi_j=\sum\alpha_{j\ell m}\,\overline{z}_m dz_\ell+O(|z|^2),\quad \overline{\partial}\varphi_j=\sum\alpha_{j\ell m}\,z_\ell\,d\overline{z}_m+O(|z|^2).\tag{5.16}$$

At the point $z=z_0$, we have $\partial\varphi(z_0)=\partial\varphi_j(z_0)=0$, $\nabla_j\sigma_j=\partial\sigma_j$, and our norm admits the expression

$$\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}=|v|^2+\sum_j\varepsilon_j\,|\partial\sigma_j(v)|^2.\tag{5.15$_0$}$$

Let v,w be arbitrary local holomorphic sections of V, and denote by ∇_ξ the Chern covariant differentiation of $(V\langle D\rangle,h_{V\langle D\rangle,\varepsilon})$ in the direction $\xi\in T_X$. By polarizing the quadratic form $\|v\|^2_{h_{V\langle D\rangle,\varepsilon}}$ into a hermitian inner product $\partial_\xi\langle\!\langle v,w\rangle\!\rangle_{h_{V\langle D\rangle,\varepsilon}}$ and setting $\nabla_\xi v=\nabla^{1,0}_\xi v=\partial_\xi v+\Gamma(\xi)\cdot v$, a differentiation of (5.15) at $z=z_0$ yields

$$\begin{aligned}\partial_\xi\langle\!\langle v,w\rangle\!\rangle_{h_{V\langle D\rangle,\varepsilon}} &= \langle\nabla_\xi v,w\rangle+\sum_j\varepsilon_j\,\partial\sigma_j\,(\nabla_\xi v)\,\overline{\partial\sigma_j(w)}\\ &= \langle\partial_\xi v\,,\,w\rangle+\sum_j\varepsilon_j\,\partial\sigma_j\,(\partial_\xi v)\,\overline{\partial\sigma_j(w)}\\ &\qquad+\varepsilon_j\,\partial^2\sigma_j(\xi,v)\,\overline{\partial\sigma_j(w)}-\varepsilon_j\,\partial\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline{\partial}\varphi_j(\xi,w),\end{aligned}$$

where $\partial^2\sigma_j(\xi,v) := \sum_\lambda \partial_\xi\big(\partial\sigma_j(e_\lambda)\big)\,v_\lambda$ is viewed as an element of $(T^*_X\otimes V^*)_{z_0}$ and $\partial\overline\partial\varphi_j$ as a hermitian form on T_X, operating on $T_X\otimes\overline{V}\subset T_X\otimes\overline{T}_X$. In fact, $v\mapsto\partial\sigma_j(v)$ and $(\xi,v)\mapsto\partial^2\sigma_j(\xi,v)$ can be intrinsically defined as $\nabla^{1,0}_j\sigma_{j|V}$ and $\nabla^{1,0}_{V^*\otimes A_j}(\nabla^{1,0}_j\sigma_{j|V})$ at z_0, and we will denote them by $\nabla_j\sigma_j$ and $\nabla^2_j\sigma_j$. In this setting, the $(1,0)$-form Γ of the connection of $(V\langle D\rangle, h_{V\langle D\rangle})$ is given at z_0 by the formula

$$\begin{aligned}&\langle\Gamma(\xi)\cdot v,w\rangle+\sum_j\varepsilon_j\,\nabla_j\sigma_j(\Gamma(\xi)\cdot v)\,\overline{\nabla_j\sigma_j(w)}\\ &\qquad=\sum_j\varepsilon_j\,\nabla^2_j\sigma_j(\xi,v)\,\overline{\nabla_j\sigma_j(w)}-\varepsilon_j\,\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline\partial\varphi_j(\xi,w).\end{aligned}\tag{5.17}$$

This equality if valid pointwise for any $v,w\in V_{z_0}$. As a consequence

$$\begin{aligned}&\Gamma(\xi)\cdot v+\sum_j\varepsilon_j\,\nabla_j\sigma_j(\Gamma(\xi)\cdot v)\,(\nabla\sigma_j)^*\\ &\qquad=\sum_j\varepsilon_j\,\nabla^2_j\sigma_j(\xi,v)\,(\nabla\sigma_j)^*-\varepsilon_j\,\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,(\partial\overline\partial\varphi_j(\bullet,\xi))^*\end{aligned}\tag{5.18}$$

where $\alpha^*\in V$ is the dual vector to a 1-form $\alpha\in V^*$, such that $\langle\alpha^*,\bullet\rangle_{h_V}=\overline{\alpha}$. The special choice $w=\Gamma(\xi)\cdot v$ yields a (non negative) real value in the left hand side of (5.17), and by taking the real part of the right hand side, we obtain

$$\begin{aligned}&|\Gamma(\xi)\cdot v|^2+\sum_j\varepsilon_j\,\big|\nabla_j\sigma_j(\Gamma(\xi)\cdot v)\big|^2\\ &=\sum_j\varepsilon_j\ \mathrm{Re}\big(\nabla^2_j\sigma_j(\xi,v)\,\overline{\nabla_j\sigma_j(\Gamma(\xi)\cdot v)}\,\big)-\varepsilon_j\ \mathrm{Re}\big(\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline\partial\varphi_j(\xi,\Gamma(\xi)\cdot v)\big).\end{aligned}\tag{5.19_0}$$

Also, by applying $\nabla_j\sigma_j$ to (5.18), we obtain

$$\begin{aligned}&\nabla\sigma_j(\Gamma(\xi)\cdot v)+\sum_\ell\varepsilon_\ell\,\nabla_\ell\sigma_\ell(\Gamma(\xi)\cdot v)\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\\ &\qquad=\sum_\ell\varepsilon_\ell\,\nabla^2_\ell\sigma_\ell(\xi,v)\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle-\varepsilon_\ell\,\nabla_\ell\sigma_\ell(v)\,\overline{\sigma}_\ell\,\langle\nabla_j\sigma_j,\partial\overline\partial\varphi_\ell(\bullet,\xi)\rangle,\end{aligned}\tag{5.19_1}$$

in other terms,

$$(I+M_\varepsilon)p_\varepsilon=M_\varepsilon q_\varepsilon-r_\varepsilon\quad\text{where}\tag{5.19_2}$$

$$p_\varepsilon=\Big(\varepsilon_j^{1/2}\nabla_j\sigma_j(\Gamma(\xi)\cdot v)\Big)_j,\ \ q_\varepsilon=\Big(\varepsilon_j^{1/2}\nabla^2_j\sigma_j(\xi,v)\Big)_j,\tag{5.19_3}$$

$$r_\varepsilon=\Big(\sum_\ell\varepsilon_j^{1/2}\varepsilon_\ell\nabla_\ell\sigma_\ell(v)\,\overline{\sigma}_\ell\,\langle\nabla_j\sigma_j,\partial\overline\partial\varphi_\ell(\bullet,\xi)\rangle\Big)_j,\ \ M_\varepsilon=\Big(\varepsilon_j^{1/2}\varepsilon_\ell^{1/2}\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\Big)_{j,\ell}.\tag{5.19_4}$$

It will be useful to observe that M_ε is a semi-positive hermitian matrix. As $2\pi\,\Theta_{A,h_A}=\mathrm{i}\,\partial\overline\partial\psi$, we infer by a brute force calculation from (5.15) that

$$\widetilde{Q}(z,v)(\xi,\eta)^2=\partial\overline\partial\|v\|^2_{h_{V\langle D\rangle,\varepsilon_j}}\cdot(\xi,\eta)^2+\gamma\,\partial\overline\partial\psi(\xi,\xi)\,\|v\|^2_{h_{V\langle D\rangle,\varepsilon_j}}$$

$$(5.20_1) \qquad = \gamma\, \partial\overline{\partial}\psi(\xi,\xi)\,|v|^2 + \sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\, \xi_\ell\overline{\xi}_m\, v_\lambda\overline{v}_\mu$$

$$(5.20_2) \qquad + \sum_j \varepsilon_j\, \big(\gamma\, \partial\overline{\partial}\psi(\xi,\xi) - \partial\overline{\partial}\varphi_j(\xi,\xi)\big)\, |\nabla_j\sigma_j(v)|^2$$

$$(5.20_3) \qquad + |\eta|^2 + \sum_j \varepsilon_j \left|\nabla_j\sigma_j(\eta) + \nabla_j^2\sigma_j(\xi,v)\right|^2$$

$$(5.20_4) \qquad - 2\varepsilon_j\ \mathrm{Re}\big(\nabla_j\sigma_j(v)\, \overline{\sigma}_j\, \partial\overline{\partial}\varphi_j(\xi,\eta)\big)$$

$$(5.20_5) \qquad - 2\varepsilon_j\ \mathrm{Re}\big(\nabla_j\sigma_j(v)\, \partial\overline{\partial}\varphi_j(\xi,v)\, \overline{\nabla_j\sigma_j(\xi)}\,\big)$$

$$(5.20_6) \qquad - 2\varepsilon_j\ \mathrm{Re}\big(\nabla_j\sigma_j(v)\, \overline{\sigma}_j\, \partial\overline{\partial}^2\varphi_j(\xi,\xi,v)\big)$$

$$(5.20_7) \qquad + \varepsilon_j\, |\sigma_j|^2\, |\partial\overline{\partial}\varphi_j(v,\xi)|^2,$$

where we identify a $(1,1)$-form such as $\partial\overline{\partial}\varphi_j$ with a hermitian form, and take $\eta = -\Gamma(\xi)\cdot v$. The second term in (5.20_2) is obtained by differentiating $\varepsilon_j\, |\nabla_j\sigma_j(v)|^2$, while (5.20_3), (5.20_4) and (5.20_5) actually come from the differentiation of the term $\ldots\mathrm{Re}(\ldots)$ in (5.15). By our assumptions, the first two terms (5.20_1), (5.20_2) are positive (in the sense of Griffiths, at least), and such that

$$(5.20_1) \geqslant c\,|\xi|^2\,|v|^2, \quad (5.20_2) \geqslant c\,|\xi|^2 \sum_j \varepsilon_j\, |\nabla_j\sigma(v)|^2, \quad c > 0.$$

In order to improve the positivity of $\widetilde{Q}$, we actually replace the metric $|v|^2_{h_{V\langle D\rangle,\varepsilon}}$ by $|v|^2_{h_{V\langle D\rangle,\varepsilon}}\, e^{\chi_\varepsilon(z)}$ with

$$\chi_\varepsilon(z) = K \sum_j \varepsilon_j |\sigma_j(z)|^2_{h_j} = K \sum_j \varepsilon_j |\sigma_j(z)|^2\, e^{-\varphi_j(z)}, \quad K \gg 1.$$

At $z = z_0$, we then get in $\widetilde{Q}$ an additional term

$$\begin{aligned} &\partial\overline{\partial}\chi_\varepsilon(\xi,\xi)\, |v|^2_{h_{V\langle D\rangle,\varepsilon}} \\ (5.20_0) \qquad &= K\left(\sum_j \varepsilon_j |\nabla\sigma_j(\xi)|^2 - \sum_j \varepsilon_j |\sigma_j|^2\, \partial\overline{\partial}\varphi_j(\xi,\xi)\right)\left(|v|^2 + \sum_j \varepsilon_j |\nabla\sigma_j(v)|^2\right), \\ &\geqslant K \sum_j \varepsilon_j |\nabla\sigma_j(\xi)|^2\, |v|^2 - CK\, \max\varepsilon_j\, |\xi|^2 \left(|v|^2 + \sum_j \varepsilon_j |\nabla\sigma_j(v)|^2\right), \end{aligned}$$

and, as a consequence, for $\varepsilon_j \leqslant c/2CK$, we have

$$\sum_{j=0,1,2} (5.20_j) \geqslant \frac{c}{2}\left(|\xi|^2\,|v|^2 + \sum_j \varepsilon_j\, |\xi|^2\, |\nabla\sigma_j(v)|^2\right) + K \sum_j \varepsilon_j\, |\nabla\sigma_j(\xi)|^2\, |v|^2.$$

(Here the last two summations are significant, because we will later replace σ_j by σ_j^{1/ρ_j} in the orbifold case, and then $\nabla_j\sigma_j^{1/\rho_j}$ is unbounded). The third term (5.20_3) is semi-positive. We claim that the terms $(5.20_4 \ldots 5.20_7)$ are negligible for $\varepsilon_j \ll 1$, in the sense that $\widetilde{Q}(z,v)(\xi,\eta)^2$ is comprised between $(1\pm\delta)\sum_{j=0,1,2,3}(5.20_j)$, with $\delta > 0$ as small as we want when $\varepsilon_j \leqslant \varepsilon_0(\delta)$. In fact, as $\partial\overline{\partial}\varphi_j$ is smooth, there exists $C > 0$ such that

$$\begin{aligned} \left|(5.20_4)\right| &\leqslant C\,\varepsilon_j |\sigma_j|\, |\nabla_j\sigma_j(v)|\, |\xi|\, |\eta| \\ &\leqslant \varepsilon_j^{3/2} |\xi|^2\, |\nabla_j\sigma_j(v)|^2 + C^2\, \varepsilon_j^{1/2}\, |\sigma_j|^2\, |\eta|^2 \ll (5.20_2) + (5.20_3). \end{aligned}$$

Similarly

$$\begin{aligned}\big|(5.20_5)\big| &\leqslant C\,\varepsilon_j\,|\xi|\,|v|\,|\nabla_j\sigma_j(\xi)|\,|\nabla_j\sigma_j(v)|\\ &\leqslant K^{-1/2}\,\varepsilon_j\,|\xi|^2\,|\nabla_j\sigma_j(v)|^2+C^2\,K^{1/2}\,\varepsilon_j\,|\nabla_j\sigma_j(\xi)|^2\,|v|^2\ll\sum_{j=0,1,2}(5.20_j)\end{aligned}$$

for $K\gg 1$. The last two terms $(5.20_{6,7})$ are even easier, since

$$\begin{aligned}\big|(5.20_6)\big| &\leqslant C\,\varepsilon_j\,|\sigma_j|\,|\xi|^2\,|v|\,|\nabla_j\sigma_j(v)|\leqslant\varepsilon_j^{1/2}|\xi|^2\,|v|^2+C^2\,\varepsilon_j^{3/2}\,|\sigma_j|^2\,|\xi|^2\,|\nabla_j\sigma_j(v)|^2\\ &\ll(5.20_1)+(5.20_2),\\ \big|(5.20_7)\big| &\leqslant C\,\varepsilon_j\,|\xi|^2|v|^2\ll(5.20_1).\end{aligned}$$

Finally, by replacing η with $-\Gamma(\xi)\cdot v$ and using (5.19_0), we find

$$\begin{aligned}(5.20_3)+(5.20_4)=&\ \big|\Gamma(\xi)\cdot v\big|^2\\ &+\sum_j\varepsilon_j\,\big|\nabla_j\sigma_j(\Gamma(\xi)\cdot v)-\nabla_j^2\sigma_j(\xi,v)\big|^2+2\varepsilon_j\ \mathrm{Re}\,\big(\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline{\partial}\varphi_j(\xi,\Gamma(\xi)\cdot v)\big)\\ =&\ (5.19_0)+\sum_j\varepsilon_j\,\big|\nabla_j^2\sigma_j(\xi,v)\big|^2-2\varepsilon_j\ \mathrm{Re}\,\big(\nabla_j^2\sigma_j(\xi,v)\,\overline{\nabla_j\sigma_j(\Gamma(\xi)\cdot v)}\,\big)\\ &+2\varepsilon_j\ \mathrm{Re}\,\big(\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline{\partial}\varphi_j(\xi,\Gamma(\xi)\cdot v)\big)\\ =&\ \sum_j\varepsilon_j\,\big|\nabla_j^2\sigma_j(\xi,v)\big|^2-\varepsilon_j\ \mathrm{Re}\,\big(\nabla_j^2\sigma_j(\xi,v)\,\overline{\nabla_j\sigma_j(\Gamma(\xi)\cdot v)}\,\big)\\ &+\varepsilon_j\ \mathrm{Re}\,\big(\nabla_j\sigma_j(v)\,\overline{\sigma}_j\,\partial\overline{\partial}\varphi_j(\xi,\Gamma(\xi)\cdot v)\big).\end{aligned}$$

The last term equals $\frac{1}{2}(5.20_4)$, thus it is negligible, and by $(5.19_{2,3,4})$ we get

$$\begin{aligned}(5.20_3)+(5.20_4)&\simeq|q_\varepsilon|^2-\mathrm{Re}\langle p_\varepsilon,q_\varepsilon\rangle\\ &=|q_\varepsilon|^2-\mathrm{Re}\langle(I+M_\varepsilon)^{-1}(M_\varepsilon q_\varepsilon-r_\varepsilon),q_\varepsilon\rangle=\mathrm{Re}\langle(I+M_\varepsilon)^{-1}q_\varepsilon,q_\varepsilon+r_\varepsilon\rangle.\\ &=\big|(I+M_\varepsilon)^{-1/2}q_\varepsilon\big|^2+\mathrm{Re}\langle(I+M_\varepsilon)^{-1}q_\varepsilon,r_\varepsilon\rangle.\end{aligned}$$

However, with $\varepsilon=\max\varepsilon_j$, we have

$$\begin{aligned}\big|\,\mathrm{Re}\langle(I+M_\varepsilon)^{-1}q_\varepsilon,r_\varepsilon\rangle\big|&\leqslant\varepsilon^{1/2}\,\big|(I+M_\varepsilon)^{-1/2}q_\varepsilon\big|^2+\varepsilon^{-1/2}\,\big|(I+M_\varepsilon)^{-1/2}r_\varepsilon\big|^2\\ &\leqslant\varepsilon^{1/2}\,\big|(I+M_\varepsilon)^{-1/2}q_\varepsilon\big|^2+\varepsilon^{-1/2}\,\big|M_\varepsilon^{-1/2}r_\varepsilon\big|^2,\end{aligned}$$

and, for any $t=(t_j)$,

$$\begin{aligned}\big|\langle r_\varepsilon,t\rangle\big|&=\Big|\sum_\ell\varepsilon_\ell\nabla_\ell\sigma_\ell(v)\,\overline{\sigma}_\ell\,\Big\langle\sum_j\bar{t}_j\,\varepsilon_j^{1/2}\nabla_j\sigma_j,\partial\overline{\partial}\varphi_\ell(\bullet,\xi)\Big\rangle\Big|\\ &\leqslant\Big|\sum_j\bar{t}_j\,\varepsilon_j^{1/2}\nabla_j\sigma_j\Big|\ \sum_\ell\varepsilon_\ell\,|\nabla_\ell\sigma_\ell(v)|\,|\sigma_\ell|\,\big|\partial\overline{\partial}\varphi_\ell(\bullet,\xi)\big|\\ &\leqslant C\,\varepsilon^{1/2}\,\langle M_\varepsilon t,t\rangle^{1/2}\,\Big(\sum_\ell\varepsilon_\ell\,|\xi|^2\,|\nabla_\ell\sigma_\ell(v)|^2\Big)^{1/2},\end{aligned}$$

whence

$$\varepsilon^{-1/2}\,|M_\varepsilon^{-1/2}r_\varepsilon|^2 \leqslant C^2\,\varepsilon^{1/2}\sum_\ell \varepsilon_\ell\,|\xi|^2\,|\nabla_\ell\sigma_\ell(v)|^2 \ll (5.20_1),$$

and

$$(5.20_3)+(5.20_4) \simeq \big|(I+M_\varepsilon)^{-1/2}q_\varepsilon\big|^2,\quad q_\varepsilon = \Big(\varepsilon_j^{1/2}\nabla_j^2\sigma_j(\xi,v)\Big)_j.$$

At this point, we come back to the orbifold situation, and thus replace σ_j by σ_j^{1/ρ_j}, φ_j by $\rho_j^{-1}\varphi_j$ and ε_j by $\rho_j^2\,\varepsilon_j$. The vector q_ε becomes

$$q_\varepsilon = \Big(\varepsilon_j^{1/2}\,\sigma_j^{-1+1/\rho_j}\,\nabla_j^2\sigma_j(\xi,v) - \varepsilon_j^{1/2}\,(1-1/\rho_j)\,\sigma_j^{-2+1/\rho_j}\,\nabla_j\sigma_j(\xi)\,\nabla_j\sigma_j(v)\Big)_j.$$

By collecting all non negligible terms (5.20_i), $i=0,1,2,3$, we obtain

5.21. Corollary. *With a choice of $\gamma > \gamma_{V,D} := \max(\max_j(\delta_j/\rho_j),\gamma_V)\geqslant 0$ determined by the curvature assumptions of Proposition* 5.8 *and the hermitian metric on (V,D) defined as above, i.e.*

(a) $$|v|_{h_{V\langle D\rangle,\varepsilon}} := e^{\chi_\varepsilon}\bigg(|v|_{h_V}^2 + \sum_j \varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|_{h_j}^2\bigg),\quad \chi_\varepsilon = K\sum_j \varepsilon_j\,\rho_j^2\,|\sigma_j|^{2/\rho_j},$$

with $K\gg 1$ and $\varepsilon_j \ll K^{-1}$, the hermitian quadratic form associated with the curvature tensor $\gamma\,\Theta_{A,h_A}\otimes\mathrm{Id}-\Theta_{V\langle D\rangle,h_{V\langle D\rangle,\gamma}}$ satisfies $\widetilde{Q}(z)(\xi\otimes v)^2 \simeq Q_{\varepsilon,K}(z)(\xi\otimes v)^2$ where

(b) $$\begin{aligned} Q_{\varepsilon,K}(z)(\xi\otimes v)^2 &= \gamma\,\partial\overline\partial\psi(\xi,\xi)\,|v|^2 + \sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\,\xi_\ell\overline\xi_m\,v_\lambda\overline v_\mu\\ &\quad+\sum_j \varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,\big(\gamma\,\partial\overline\partial\psi(\xi,\xi)-\rho_j^{-1}\,\partial\overline\partial\varphi_j(\xi,\xi)\big)\,|\nabla_j\sigma_j(v)|^2\\ &\quad+K\bigg(\sum_j\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla\sigma_j(\xi)|^2\bigg)\bigg(|v|^2+\sum_j\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla\sigma_j(v)|^2\bigg)\\ &\quad+\Big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,\nabla^2_{\mathrm{orb}}\,\sigma(\xi,v)-\varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\,\nabla\sigma(\xi)\,\nabla_{\mathrm{orb}}\sigma(v)\big)\Big|^2,\end{aligned}$$

and

$$\nabla^2_{A,h_A}=\partial\overline\partial\psi,\quad \nabla^2_{\Delta_j,h_j}=\partial\overline\partial\varphi_j,\quad (c_{\ell m\lambda\mu}) = \textit{coefficients of } -2\pi\,\Theta_{V,h_V},$$

$$M_{\rho,\sigma,\varepsilon} = \Big(\varepsilon_j^{1/2}\varepsilon_\ell^{1/2}\,\sigma_j^{-1+1/\rho_j}\,\overline\sigma_\ell^{-1+1/\rho_\ell}\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\Big)_{j,\ell},$$

$$\varepsilon^{1/2}\,\nabla^2_{\mathrm{orb}}\sigma(\xi,v) = \Big(\varepsilon_j^{1/2}\,\sigma_j^{-1+1/\rho_j}\,\nabla_j^2\sigma_j(\xi,v)\Big)_j,$$

$$\varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\,\nabla\sigma(\xi)\,\nabla_{\mathrm{orb}}(v) = \Big(\varepsilon_j^{1/2}\,(1-1/\rho_j)\,\sigma_j^{-2+1/\rho_j}\,\nabla_j\sigma_j(\xi)\,\nabla_j(v)\Big)_j.$$

Here, the symbol $\simeq$ means that the ratio $\widetilde{Q}/Q_{\varepsilon,K}$ is in $[1-\delta,1+\delta]$ as soon as $K\geqslant K_0(\gamma,\delta)$ and $\varepsilon_j\leqslant\varepsilon_0(\gamma,\delta,K)$.

5.22. Remark. If c is a lower bound for the curvature coefficients of $\gamma\,\Theta_{A,h_A}-\frac{1}{\rho_j}\Theta_{\Delta_j,h_j}$ and $\gamma\,\Theta_{A,h_A}\otimes\mathrm{Id}-\Theta_{V,h_V}$ with respect to $\omega=\Theta_{A,h_A}$, an examination of the estimates shows

that $\widetilde{Q}/Q_{\varepsilon,K} \in [1-\delta, 1+\delta]$ as soon as $K \geqslant C_0\, c^{-2}\delta^{-2}$ and $\varepsilon_j \leqslant c\,(C_0 K\,\rho_j^2)^{-1}$ with $C_0 > 1$ large enough.

5.23. Proposition. *The term* $\big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,(\ldots)\big|^2$ *can be estimated as follows.*

(a) *In case there is only one component* Δ_j*, its expression becomes*

$$\frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2}\,\Big|\nabla_j^2\sigma_j(\xi,v) - (1-1/\rho_j)\,\sigma_j^{-1}\,\nabla_j\sigma_j(\xi)\,\nabla_j\sigma_j(v)\Big|^2.$$

(b) *In general, we have a uniform upper bound*

$$\Big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,\nabla^2_{\rm orb}\,\sigma(\xi,v)\big)\Big|^2 \leqslant C\sum_j \frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla\sigma_j|^2}\,|\xi|^2\,|v|^2,$$

where the sum is bounded and converges pointwise to 0 *on* $X \smallsetminus \mathrm{Supp}(D)$*.*

(c) *With the same constant* C*, we have for every* $\delta > 0$

$$\Big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,\nabla^2_{\rm orb}\sigma(\xi,v) - \varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\nabla\sigma(\xi)\,\nabla_{\rm orb}\sigma(v)\big)\Big|^2$$

$$\begin{cases} \leqslant (1+\delta)\big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\nabla\sigma(\xi)\,\nabla_{\rm orb}\sigma(v)\big)\big|^2 \\ \qquad\qquad + C\,(1+\delta^{-1})\displaystyle\sum_j \frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla\sigma_j|^2}\,|\xi|^2\,|v|^2 \\ \geqslant (1-\delta)\big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\nabla\sigma(\xi)\,\nabla_{\rm orb}\sigma(v)\big)\big|^2 \\ \qquad\qquad - C\,\delta^{-1}\displaystyle\sum_j \frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla\sigma_j|^2}\,|\xi|^2\,|v|^2. \end{cases}$$

Proof. Formula (a) is immediate from 5.21 (b). For estimate (b), let us fix $\eta > 0$ such that $(\nabla_j\sigma_j)_{j\in J}$ are transverse whenever we are at a point $z \in X$ such that $|\sigma_j(z)| < \eta$, $\forall j \in J \subset \{1,2,\ldots,N\}$. Then $\sum_{j\in J}\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\,t_j\bar t_\ell \geqslant c\sum_{j\in J}|t_j|^2$ for some $c > 0$ (uniformly with respect to $z \in X$). Then, taking $J = \{j\,;\ |\sigma_j(z)| < \eta\}$, we obtain the existence of constants C, C' such that

$$\begin{aligned}&\sum_j \varepsilon_j^{1/2}\varepsilon_\ell^{1/2}\,\sigma_j^{-1+1/\rho_j}\overline{\sigma}_\ell^{-1+1/\rho_\ell}\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\,t_j\bar t_\ell \\ &\qquad\geqslant \frac{c}{2}\sum_{j\in J}\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2\,|t_j|^2 - C\sum_{\ell\notin J}\varepsilon_\ell\,|\sigma_\ell|^{-2+2/\rho_\ell}\,|\nabla_\ell\sigma_\ell|^2\,|t_\ell|^2 \\ &\qquad\geqslant \frac{c}{2}\sum_{j\in J}\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2\,|t_j|^2 - C'\sum_{j\notin J}|t_j|^2,\end{aligned}$$

where

$$C' = C\sum_{j\notin J}\varepsilon_j\eta^{-2+2/\rho_j}\,\sup|\nabla_j\sigma_j|^2.$$

For $\delta=\min(1,\frac{2}{c},\frac{1}{2C'})$, this implies

$$
\begin{aligned}
|t|^2+\sum_j &\varepsilon_j^{1/2}\varepsilon_\ell^{1/2}\,\sigma_j^{-1+1/\rho_j}\overline{\sigma}_\ell^{-1+1/\rho_\ell}\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\,t_j\bar t_\ell\\
&\geqslant |t|^2+\delta\sum_j\varepsilon_j^{1/2}\varepsilon_\ell^{1/2}\,\sigma_j^{-1+1/\rho_j}\overline{\sigma}_\ell^{-1+1/\rho_\ell}\,\langle\nabla_j\sigma_j,\nabla_\ell\sigma_\ell\rangle\,t_j\bar t_\ell\\
&\geqslant \frac{\delta c}{2}\sum_{j\in J}\big(1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2\big)\,|t_j|^2+(1-\delta\,C')\sum_{j\notin J}|t_j|^2,
\end{aligned}
$$

and by inverting the matrices of these quadratic forms we get

$$
\begin{aligned}
\langle (I+M_{\rho,\sigma,\varepsilon})^{-1}t,t\rangle &\leqslant \frac{2}{\delta c}\sum_{j\in J}\big(1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2\big)^{-1}\,|t_j|^2+2\sum_{j\notin J}|t_j|^2\\
&\leqslant C''\sum_{j\in J\cup\complement J}\big(1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2\big)^{-1}\,|t_j|^2.
\end{aligned}
$$

Estimate (b) follows by taking $t=\varepsilon^{1/2}\,\nabla^2_{\rm orb}\sigma(\xi,v)=(\varepsilon_j^{1/2}\,\sigma_j^{-1+1/\rho_j}\,\nabla_j^2\sigma(\xi,v))$, since we have $|\nabla_j^2\sigma(\xi,v)|\leqslant C_3|\xi|\,|v|$, and (c) is an immediate consequence. □

In view of the estimates developed in section 6, we will have to evaluate integrals involving powers of curvature tensors, and the following basic inequalities will be useful.

5.24. Lemma. *Let $\ell_j\in(\mathbb{C}^r)^*$, $1\leqslant j\leqslant p$, be non zero complex linear forms on $\mathbb{C}^r$, where $(\mathbb{C}^r)^*\simeq\mathbb{C}^r$ is equipped with its standard hermitian form, and let μ the rotation invariant probability measure on $\mathbb{S}^{2r-1}\subset\mathbb{C}^r$. Then*

$$
I(\ell_1,\ldots,\ell_p)=\int_{\mathbb{S}^{2r-1}}|\ell_1(u)|^2\ldots|\ell_p(u)|^2\,d\mu(u)
$$

satisfies the following inequalities :

(a) $I(\ell_1,\ldots,\ell_p)\leqslant\dfrac{p!\,(r-1)!}{(p+r-1)!}\displaystyle\prod_{j=1}^p|\ell_j|^2,$

and the equality occurs if and only if the ℓ_j are proportional;

(b) $I(\ell_1,\ldots,\ell_p)\geqslant\dfrac{(r-1)!}{(p+r-1)!}\displaystyle\prod_{j=1}^p|\ell_j|^2,$

and the equality occurs if and only if $p\leqslant r$ and the ℓ_j are pairwise orthogonal.

Proof. Denote by $d\lambda$ the Lebesgue measure on Euclidean space and by $d\sigma$ the area measure of the sphere. One can easily check that the projection

$$
\mathbb{S}^{2r-1}\to\mathbb{B}^{2r-2},\quad u=(u_1,\ldots,u_r)\mapsto v=(u_1,\ldots,u_{r-1}),
$$

yields $d\sigma(u)=d\theta\wedge d\lambda(v)$ where $u_r=|u_r|\,e^{i\theta}$ [just check that the wedge products of both sides with $\frac12 d|u|^2$ are equal to $d\lambda(u)$, and use the fact that $d\theta=\frac{1}{2i}(du_r/u_r-d\overline{u}_r/\overline{u}_r)$], thus, in terms of polar coordinates $v=t\,u'$, $u'\in\mathbb{S}^{2r-1}$, we have $d\sigma(u)=d\theta\wedge t^{2r-3}\,dt\wedge d\sigma'(u')$,

and going back to the invariant probability measures μ on $\mathbb{S}^{2r-1}$ and μ' on $\mathbb{S}^{2r-3}$, we get $|u_r|^2 = 1 - |v|^2 = 1 - t^2$ and an equality

$$d\mu(u) = \frac{2r-2}{2\pi}\, d\theta \wedge t^{2r-3}\, dt \wedge d\mu'(u'). \tag{5.25}$$

If $\ell_1, \ldots, \ell_p$ are independent of u_r, (6.8) and the Fubini theorem imply by homogeneity

$$\int_{\mathbb{S}^{2r-1}} |\ell_1(u')|^2 \ldots |\ell_p(u')|^2\, d\mu(u) = \frac{r-1}{p+r-1} \int_{\mathbb{S}^{2r-3}} |\ell_1(u')|^2 \ldots |\ell_p(u')|^2\, d\mu'(u'), \tag{5.26}$$

$$\int_{\mathbb{S}^{2r-1}} |\ell_1(u')|^2 \ldots |\ell_{p-1}(u')|^2\, |u_r|^2\, d\mu(u) =$$

$$\frac{r-1}{(p+r-2)(p+r-1)} \int_{\mathbb{S}^{2r-3}} |\ell_1(u')|^2 \ldots |\ell_{p-1}(u')|^2\, d\mu'(u') \tag{5.26$'$}$$

(for instance, in case (5.26′), we have to integrate $t^{2p-2}(1-t^2) \times t^{2r-3}\, dt$). The formulas

$$\int_{\mathbb{S}^{2r-1}} |u_1|^{2p}\, d\mu(u) = \frac{p!\,(r-1)!}{(p+r-1)!}, \quad \int_{\mathbb{S}^{2r-1}} |u_1|^2 \ldots |u_p|^2\, d\mu(u) = \frac{(r-1)!}{(p+r-1)!} \quad (p \le r),$$

are then obtained by induction on r and p.

(a) For any $\ell \in (\mathbb{C}^r)^*$, we can find orthonormal coordinates on $\mathbb{C}^r$ such that $\ell(u) = |\ell|\, u_1$ in the new coordinates. Hence

$$\int_{\mathbb{S}^{2r-1}} |\ell(u)|^{2p}\, d\mu(u) = m_{r,p}\, |\ell|^{2p} \quad \text{where} \quad m_{r,p} = \int_{\mathbb{S}^{2r-1}} |u_1|^{2p}\, d\mu(u) = \frac{p!\,(r-1)!}{(p+r-1)!}.$$

It follows from Hölder's inequality that

$$I(\ell_1, \ldots, \ell_p) \leqslant \prod_{j=1}^{p} \bigg(\int_{\mathbb{S}^{2r-1}} |\ell_j|^{2p}\, d\mu(u) \bigg)^{1/p} = m_{r,p} \prod_{j=1}^{p} |\ell_j|^2,$$

and that the equality occurs if and only if all ℓ_j are proportional.

(b) We prove the inequality

$$I(\ell_1, \ldots, \ell_p) \geqslant \frac{(r-1)!}{(p+r-1)!} \prod_{j=1}^{p} |\ell_j|^2$$

by induction on p, the result being clear for $p = 0$ or $p = 1$. If we choose an orthonormal basis $(e_1, \ldots, e_r) \in \mathbb{C}^r$ such that $\ell_j(e_r) \neq 0$ for all j and replace ℓ_j by $(\ell_j(e_r))^{-1}\ell_j$, we can assume $\ell_j(e_r) = 1$. We then write $u = u' + u_r e_r$ with $u' \in e_r^{\perp} \simeq \mathbb{C}^{r-1}$ and

$$\ell_j(u) = \ell'_j(u') + u_r, \quad 1 \leqslant j \leqslant p, \quad \ell'_j := \ell_{j|e_r^{\perp}}.$$

Let $s_k(\ell'_\bullet(u'))$ be the elementary symmetric functions in $\ell'_j(u')$, $1 \leqslant j \leqslant p$, with $s_0 := 1$. We have

$$I(\ell_1, \ldots, \ell_p) = \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{p} |\ell'_j(u') + u_r|^2\, d\mu(u) = \int_{\mathbb{S}^{2r-1}} \bigg| \sum_{k=0}^{p} s_k(\ell'_\bullet(u'))\, u_r^{p-k} \bigg|^2 d\mu(u).$$

We make a change of variable $u_r \mapsto u_r\, e^{i\theta}$ and take the average over $\theta \in [0, 2\pi]$. Parseval's formula gives

$$I(\ell_1, \ldots, \ell_p) = \int_{\mathbb{S}^{2r-1}} \sum_{k=0}^{p} \big|s_k(\ell'_\bullet(u'))\big|^2\, |u_r|^{2(p-k)} d\mu(u),$$

and since

$$(2r-2)\int_0^1 t^{2k}(1-t^2)^{p-k}\, t^{2r-3} dt = \frac{(r-1)\,(k+r-2)!\,(p-k)!}{(p+r-1)!},$$

formula (5.25) implies

$$I(\ell_1, \ldots, \ell_p) = \int_{\mathbb{S}^{2r-3}} \sum_{k=0}^{p} \frac{(r-1)\,(k+r-2)!\,(p-k)!}{(p+r-1)!}\, \big|s_k(\ell'_\bullet(u'))\big|^2\, d\mu'(u').$$

As $|\ell_j|^2 = 1 + |\ell'_j|^2$, our inequality (5.24 (b)) is equivalent to

$$\int_{\mathbb{S}^{2r-3}} \sum_{k=0}^{p} \frac{(k+r-2)!\,(p-k)!}{(r-2)!}\, \big|s_k(\ell'_\bullet(u'))\big|^2\, d\mu'(u') \geqslant \prod_{j=1}^{p}(1+|\ell'_j|^2) \tag{5.27}$$

for all linear forms $\ell'_j \in (\mathbb{C}^{r-1})^*$. We actually prove (5.27) by induction on p (observing that the inequality is a trivial equality for $p = 0, 1$). Assume that (5.27) (and hence (5.24 (b))) is known for any $(p-1)$-tuple of linear forms $(\ell'_1, \ldots, \ell'_{p-1})$. As (5.24 (b)) is invariant under the action of $U(r)$, it is sufficient to consider the case when $\ell_p(u) = u_r$, i.e. $\ell'_p = 0$. The induction hypothesis tells us that

$$\int_{\mathbb{S}^{2r-3}} \sum_{k=0}^{p-1} \frac{(k+r-2)!\,(p-1-k)!}{(r-2)!}\, \big|s_k(\ell'_\bullet(u'))\big|^2\, d\mu'(u') \geqslant \prod_{j=1}^{p-1}(1+|\ell'_j|^2).$$

However, when we add the factor ℓ_p, the elementary symmetric functions $s_k(\ell'_\bullet(u'))$ are left unchanged for $k \leqslant p-1$, while $s_p(\ell'_\bullet(u')) = 0$ and $1 + |\ell'_p|^2 = 1$. Therefore (5.27) holds true for p, since $(p-k)! \geqslant (p-1-k)!$ for all $k = 0, 1, \ldots, p-1$. We have proved the inequality at order p whenever $\ell_p = \alpha_p\langle \bullet, e_r\rangle$ and $\ell_j(e_r) \neq 0$ for $j \leqslant p-1$. Since those $(\ell_1, \ldots, \ell_p)$ are dense in the space $((\mathbb{C}^r)^*)^p$ of p-tuples of linear forms, the proof of the lower bound is complete.

(b, equality case) We argue by induction on r. For $r = 1$, we have in fact $\ell_j(u) = \alpha_j u_1$, $\alpha_j \in \mathbb{C}^*$, and $I(\ell_1, \ldots, \ell_r) = \prod |\ell_j|^2$, thus the coefficient $\frac{1}{(p+r-1)!} = \frac{1}{p!}$ is reached if and only if $p \leqslant 1$. Now, assume $r \geqslant 2$ and the equality case solved for dimension $r-1$. By rescaling and reordering the ℓ_j, we can always assume that $\ell_j(e_r) \neq 0$ (and hence $\ell_j(e_r) = 1$) for $q+1 \leqslant j \leqslant p$, while $\ell_j(e_r) = 0$ for $1 \leqslant j \leqslant q$ (we can possibly have $q = 0$ here). Then we write $\ell_j(u) = \ell'_j(u')$ for $1 \leqslant j \leqslant q$ and $\ell_j(u) = \ell'_j(u') + u_r$ for $q+1 \leqslant j \leqslant p$. Therefore, if

$s_k(\ell'(u'))$ denotes the k-th elementary symmetric function in $(\ell'_j(u')_{q+1\leqslant j\leqslant p}$, we find

$$\begin{aligned}
I(\ell_1,\ldots,\ell_p) &= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{q} |\ell'_j(u')|^2 \prod_{j=q+1}^{p} |\ell'_j(u')+u_r|^2\,d\mu(u)\\
&= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{q} |\ell'_j(u')|^2 \bigg|\sum_{k=0}^{p-q} s_k(\ell'(u'))\,u_r^{p-q-k}\bigg|^2\,d\mu(u)\\
&= \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{q} |\ell'_j(u')|^2 \sum_{k=0}^{p-q} \big|s_k(\ell'(u'))\big|^2\,|u_r|^{2(p-q-k)}\,d\mu(u)\\
&= \int_{\mathbb{S}^{2r-3}} \prod_{j=1}^{q} |\ell'_j(u')|^2 \sum_{k=0}^{p-q} \frac{(r-1)\,(k+r-2)!\,(p-q-k)!}{(p-q+r-1)!}\,\big|s_k(\ell'(u'))\big|^2\,d\mu'(u')\\
&\geqslant \frac{(r-1)!}{(p+r-1)!}\prod_{j=1}^{q} |\ell'_j|^2 \prod_{j=q+1}^{p}(1+|\ell'_j|^2)
\end{aligned}$$

by what we have just proved. In an equivalent way, we get

$$\begin{aligned}
&\int_{\mathbb{S}^{2r-3}} \prod_{j=1}^{q} |\ell'_j(u')|^2 \sum_{k=0}^{p-q} \frac{(k+r-2)!\,(p-q-k)!\,(p+r-1)!}{(r-2)!\,(p-q+r-1)!}\,\big|s_k(\ell'(u'))\big|^2\,d\mu'(u')\\
&\qquad\geqslant \prod_{j=1}^{q} |\ell'_j|^2 \prod_{j=q+1}^{p}(1+|\ell'_j|^2)
\end{aligned}$$

for all $0\leqslant q\leqslant p-1$ and all choices of the forms $\ell'_j\in(\mathbb{C}^{r-1})^*$. In general, we can rotate coordinates in such a way that $\ell_p(u)=u_r$ and $\ell'_p=0$, and we see that the above inequality holds when p is replaced by $p-1$, as soon as $q\leqslant p-2$. Then the corresponding coefficients $k=0$ for p, $p-1$ are

$$\frac{(p-q)!\,(p+r-1)!}{(p-q+r-1)!} > \frac{(p-1-q)!\,(p-1+r-1)!}{(p-1-q+r-1)!},$$

and since $s_0=1$, we infer that the inequality is strict. The only possibility for the equality case is $q=p-1$, but then

$$I(\ell_1,\ldots,\ell_p) = \int_{\mathbb{S}^{2r-1}} \prod_{j=1}^{p-1} |\ell'_j(u')|^2\,|u_r|^2\,d\mu(u) = \frac{r-1}{p+r-1}\int_{\mathbb{S}^{2r-3}} \prod_{j=1}^{p-1} |\ell'_j(u')|^2\,d\mu'(u'),$$

and we see that we must have equality in the case $(r-1,p-1)$. By induction, we conclude that $p-1\leqslant r-1$ and that the $\ell_j(u)=\ell'_j(u')$ are orthogonal for $j\leqslant p-1$, as desired. □

5.28. Remark. When $r=2$, our inequality (5.27) is equivalent to the "elementary" inequality

$$(*)\qquad \prod_{j=1}^{p}(1+|a_j|^2) \leqslant \sum_{k=0}^{p} k!\,(p-k)!\,|s_k|^2,$$

relating a polynomial $X^p - s_1 X^{p-1} + \cdots + (-1)^p s_p$ and its complex roots a_j (just consider $\ell'_j(u') = a_j u_1$ and $\ell_j(u) = a_j u_1 + u_2$ on $\mathbb{C}^2$ to get this). It should be observed that $(*)$ is not optimal symptotically when $p \to +\infty$; in fact, Landau's inequality [Land05] gives $\prod \max(1, |a_j|) \leqslant (\sum |s_k|^2)^{1/2}$, from which one can easily derive that $\prod(1 + |a_j|^2) \leqslant 2^p \sum |s_k|^2$, which improves $(*)$ as soon as $p \geqslant 7$ (observe that $2^7 = 128$ and $k!(7-k)! \geqslant 3!\,4! = 144$). Our discussion of the equality case shows that inequality (5.24 (b)) is never sharp when $p > r$. It would be interesting, but probably challenging, if not impossible, to compute the optimal constant for all pairs (r,p), $p > r$, since this is an optimization problem involving the distribution of a large number of points in projective space.

We finally state one of the main consequences of these estimates concerning the Chern curvature form of a hermitian holomorphic vector bundle.

5.29. Proposition. *Let T, E be complex vector spaces of respective dimensions $\dim T = n$, $\dim E = r$. Assume that E is equipped with a hermitian structure h and denote by μ the unitary invariant probability measure μ on the unit sphere bundle $S(E) = \{v \in E\,;|v|_h\} \subset E$.*

(a) *If $\theta_1, \ldots, \theta_p \geqslant_S 0$ are strongly semi-positive hermitian tensors in $\mathrm{Herm}(T \otimes E) \simeq \Lambda^{1,1}_{\mathbb{R}} T^* \otimes_{\mathbb{R}} \mathrm{Herm}(E,E)$ then*

$$\int_{v\in S(E)} \langle\theta_1(v), v\rangle_h \wedge \ldots \wedge \langle\theta_p(v), v\rangle_h \, d\mu(v) \begin{cases} \geqslant \dfrac{(r-1)!}{(p+r-1)!}\ \mathrm{Tr}_h\,\theta_1 \wedge \ldots \wedge \mathrm{Tr}_h\,\theta_p, \\[2ex] \leqslant \dfrac{p!\,(r-1)!}{(p+r-1)!}\ \mathrm{Tr}_h\,\theta_1 \wedge \ldots \wedge \mathrm{Tr}_h\,\theta_p, \end{cases}$$

as pointwise strong inequalities of (p,p)-forms.

(b) *If $\theta \geqslant_G 0$ in $\Lambda^{1,1}_{\mathbb{R}} T^* \otimes_{\mathbb{R}} \mathrm{Herm}(E,E)$ and $\ell_j \in E^*$, then*

$$\int_{v\in S(E)} |\ell_1(v)|^2 \ldots |\ell_k(v)|^2 \, \langle\theta(v), v\rangle_h^{p-k} \, d\mu(v) \leqslant \frac{p!\,(r-1)!}{(p+r-1)!}\bigg(\prod_{j=1}^k |\ell_j|^2\bigg)(\mathrm{Tr}_h\,\theta)^{p-k}$$

as a pointwise weak inequality of $(p-k, p-k)$-forms.

In particular, the above inequalities apply when (E,h) is a hermitian holomorphic vector bundle of rank r on a complex n-dimensional manifold X, and one takes $\theta_j = \Theta_{E,h}$ to be the curvature tensor of E, so that $\mathrm{Tr}_h\,\theta_j = c_1(E,h)$ is the first Chern form of (E,h).

Proof. (a) The assumption $\theta_q \geqslant_S 0$ means that at every point $x \in X$ we can write θ as

$$\theta_q = \sum_{1\leqslant j\leqslant N_q} |\beta_{qj} \otimes \ell_{qj}|^2 \simeq \sum_{1\leqslant j\leqslant N_q} \mathrm{i}\,\beta_{qj} \wedge \overline{\beta}_{qj} \otimes \ell_{qj} \otimes \ell^*_{qj}, \quad \beta_{qj} \in T^*, \quad \ell_{qj} \in E^*$$

as an element of $\Lambda^{1,1}_{\mathbb{R}} T^* \otimes_{\mathbb{R}} \mathrm{Herm}(E,E)$, hence

$$\langle\theta_q(v), v\rangle_h = \sum_{1\leqslant j\leqslant N_q} \mathrm{i}\,\beta_{qj} \wedge \overline{\beta}_{qj}\, |\ell_{qj}(v)|^2.$$

Without loss of generality, we can assume $|\ell_{qj}|_{h^*} = 1$. Then

$$\langle\theta_1(v), v\rangle_h \wedge \ldots \wedge \langle\theta_p(v), v\rangle_h = \sum_{j_1,\ldots,j_p} \mathrm{i}\,\beta_{1j_1} \wedge \overline{\beta}_{1j_1} \wedge \ldots \wedge \mathrm{i}\,\beta_{pj_p} \wedge \overline{\beta}_{pj_p}\, |\ell_{1j_1}(v)|^2 \ldots |\ell_{pj_p}(v)|^2,$$

and since $|\ell_{qj}|_{h^*} = 1$, Lemma 5.24 (b) implies

$$\begin{aligned}\int_{v\in S(E)} &\langle\theta_1(v), v\rangle_h \wedge \ldots \wedge \langle\theta_p(v), v\rangle_h \, d\mu(v)\\ &\geqslant \frac{(r-1)!}{(p+r-1)!} \sum_{j_1,\ldots,j_p} \mathrm{i}\,\beta_{1j_1} \wedge \overline{\beta}_{1j_1} \wedge \ldots \wedge \mathrm{i}\,\beta_{pj_p} \wedge \overline{\beta}_{pj_p}\\ &= \frac{(r-1)!}{(p+r-1)!} \,\mathrm{Tr}_h\,\theta_1 \wedge \ldots \wedge \mathrm{Tr}_h\,\theta_p,\end{aligned}$$

where $\geqslant$ is in the sense of the strong positivity of (p,p)-forms. The upper bound is obtained by the same argument, via 5.24 (a).

(b) By the definition of weak positivity of forms, it is enough to show the inequality in restriction to every $(p-k)$-dimensional subspace $T' \subset T$. Without loss of generality, we can assume that $\dim T = p - k$ (and then take $T' = T$), that $|\ell_j| = 1$, and also that $\theta >_G 0$ (otherwise take a positive definite form $\eta \in \Lambda^{1,1}_{\mathbb{R}} T^*$, replace θ with $\theta_\varepsilon = \theta + \varepsilon\,\eta \otimes h$, and let ε tend to 0). For any $v \in S(E)$, let

$$0 \leqslant \lambda_1(v) \leqslant \cdots \leqslant \lambda_{p-k}(v)$$

be the eigenvalues of the hermitian form $q_v(\bullet) = \langle\theta(v), v\rangle$ on T with respect to

$$\omega = \mathrm{Tr}_h\,\theta = \sum_{j=1}^{r} \langle\theta(e_j), e_j\rangle \in \mathrm{Herm}(T), \quad \omega > 0,$$

(e_j) being any orthonormal frame of E. We have to show that

$$\int_{v\in S(E)} |\ell_1(v)|^2 \ldots |\ell_k(v)|^2 \,\lambda_1(v) \cdots \lambda_{p-k}(v)\, d\mu(v) \leqslant \frac{p!\,(r-1)!}{(p+r-1)!}.$$

However, the inequality between geometric and arithmetic means implies

$$\lambda_1(v) \cdots \lambda_p(v) \leqslant \bigg(\frac{1}{p-k} \sum_{j=1}^{p-k} \lambda_j(v)\bigg)^p,$$

thus, putting $Q(v) = \frac{1}{p-k} \langle \mathrm{Tr}_\omega\,\theta(v), v\rangle$, $Q \in \mathrm{Herm}(E)$, it is enough to prove that

$$\int_{v\in S(E)} |\ell_1(v)|^2 \ldots |\ell_k(v)|^2 \,Q(v)^{p-k}\, d\mu(v) \leqslant \frac{p!\,(r-1)!}{(p+r-1)!}. \tag{5.30}$$

Our assumption $\theta >_G 0$ implies $Q(v) = \sum_{1\leqslant j\leqslant r} c_j |\ell'_{qj}(v)|^2$ for some $c_j > 0$ and some orthonormal basis $(\ell'_{qj})_{1\leqslant j\leqslant r}$ of E^*, and

$$\sum_{j=1}^{r} c_j = \mathrm{Tr}_h\,Q = \frac{1}{p-k}\,\mathrm{Tr}_h(\mathrm{Tr}_\omega\,\theta) = \frac{1}{p-k}\,\mathrm{Tr}_\omega(\mathrm{Tr}_h\,\theta) = \frac{1}{p-k}\,\mathrm{Tr}_\omega(\omega) = 1.$$

Inequality (5.30) is a consequence of Lemma 5.24 (a), by Newton's multinomial expansion. □

5.31. Remark. For $p = 1$, the inequalities of Proposition 5.29 are identities, and no semi-positivity assumption is needed in that case. However, when $p \geqslant 2$, inequality 5.29 (a) does not hold under the assumption that $E \geqslant_G 0$ (or even that E is dual Nakano semi-positive, i.e. E^* Nakano semi-negative). Let us take for instance $E = T_{\mathbb{P}^n} \otimes \mathcal{O}(1)$. It is well known that E is isomorphic to the tautological quotient vector bundle $\mathbb{C}^{n+1}/\mathcal{O}(-1)$ over $\mathbb{P}^n$, and that its curvature tensor form for the Fubini-Study metric is given by

$$\theta_E(\xi \otimes v, \xi \otimes v) = |\langle \xi, v\rangle|^2 \geqslant 0$$

(where v is identified which a tangent vector via the choice of a unit element $e \in \mathcal{O}(-1)$). Then $\det E = \mathcal{O}(1)$ and thus $c_1(E, h) = \omega_{\rm FS} > 0$, although $\langle \Theta_{E,h}(v), v\rangle_h^p = 0$ for all $p \geqslant 2$, as one can easily check.

Our aim is to apply Proposition 5.29 to the curvature tensor $\theta = \Theta_{V\langle D\rangle}$ of a directed orbifold (V, D). Under ad hoc hypotheses, Proposition 5.8 implies Griffiths positivity, but we want to invoke strong positivity to be able to apply the lower bound of 5.29 (a). The main observation is that one can somehow separate the contribution of V and the contribution of D in the calculation. For the sake of generality (and the needs of §6), we introduce the possibility of combining different orbifold divisors D_s with the same support.

5.32. Proposition. *Let X be a projective variety, A an ample line bundle, and let (X, V, D_s), $1 \leqslant s \leqslant k$, be orbifold directed structures where $D_s = \sum(1 - \frac{1}{\rho_{sj}})\Delta_j$ are normal crossing divisors on X transverse to V, sharing the same components Δ_j. Let d_j be the infimum of numbers $\lambda \in \mathbb{R}_+$ such that $\lambda\, A - \Delta_j$ is nef, and let γ_V (resp. $\widetilde{\gamma}_V$) be the infimum of numbers $\gamma \geqslant 0$ such that $\theta_{V,\gamma} := \gamma\, \Theta_{A,h_A} \otimes \mathrm{Id}_V - \Theta_{V,h_V} \geqslant_G 0$ (resp. $\geqslant_S 0$) for suitable hermitian metrics h_V on V. Take $p_1, \ldots, p_k \in \mathbb{N}$ such that $q = n - (p_1 + \ldots + p_k) \geqslant 0$ and a weakly positive smooth (q, q) form $\beta \geqslant_W 0$ on X. Then for every*

$$\gamma_s > \widetilde{\gamma}_{V,D_s} := \max(\max_j(d_j/\rho_{sj}), \widetilde{\gamma}_V)$$

there exist hermitian metrics $h_{V\langle D_s\rangle,\varepsilon_s}$ on the orbifold vector bundles $V\langle D_s\rangle$ such that

$$\theta_{D_s,\gamma_s,\varepsilon_s} := \gamma_s\, \Theta_{A,h_A} \otimes \mathrm{Id}_V - \Theta_{V\langle D_s\rangle, h_{V\langle D_s\rangle,\varepsilon_s}} >_G 0$$

in the sense of Griffiths, and

$$\begin{aligned}&\lim_{\varepsilon_{ij}\to 0} \int_X \int_{v_s \in S(V\langle D_s\rangle)} \langle \theta_{D_1,\gamma_1,\varepsilon_1}(v_1), v_1\rangle^{p_1} \wedge \ldots \wedge \langle \theta_{D_k,\gamma_k,\varepsilon_k}(v_k), v_k\rangle^{p_k} \wedge \beta\, d\mu(v_1) \ldots d\mu(v_k)\\ &\geqslant \sum_{J_1 \amalg \ldots \amalg J_k \subset \{1,\ldots,N\}} \ \sum_{\ell_j \geqslant 1,\, \Sigma_{j\in J_s} \ell_j \leqslant p_s} \ \prod_{1\leqslant s\leqslant k} \bigg(\frac{(r-1)!}{(p_s + r - 1)!} \prod_{j\in J_s} (1 - 1/\rho_{sj})\bigg) \int_{\Delta_{J_1 \amalg \ldots \amalg J_k}}\\ &\frac{\prod_{1\leqslant s\leqslant k} p_s!}{\prod \ell_j! \prod_s (p_s - \sum_{j\in J_s} \ell_j)!} \bigwedge_{1\leqslant s\leqslant k} \bigg((\mathrm{Tr}\, \theta_{V,\gamma})^{p_s - \Sigma_{j\in J_s}\ell_j} \wedge \bigwedge_{j\in J_s} \big(\gamma\omega_A - \rho_{sj}^{-1}\theta_{\Delta_j,h_j}\big)^{\ell_j - 1}\bigg) \wedge \beta,\end{aligned}$$

where the limit $\lim_{\varepsilon_{ij}\to 0}$ is an iterated limit $\lim_{\varepsilon_{11}\to 0} \ldots \lim_{\varepsilon_{kN}\to 0}$ with respect to the lexicographic order $(i, j) < (i', j')$ if $i < i'$ or $i = i'$ and $j < j'$. The summation is taken over all disjoint subsets $J_1, \ldots, J_k \subset \{1, 2, \ldots, N\}$, and we have set here $\Delta_J := \bigcap_{j\in J} \Delta_j$. For $\beta \geqslant_S 0$ and γ_s satisfying the Griffiths type condition

$$\gamma_s > \gamma_{V,D_s} := \max(\max_j(d_j/\rho_{sj}), \gamma_V),$$

a similar upper bound holds with constants $\frac{(r-1)!}{(p_s+r-1)!}$ replaced by $\frac{p_s!\,(r-1)!}{(p_s+r-1)!}$.

Proof. For the sake of simplicity, we first deal with the case where a single divisor $D = \sum(1-1/\rho_j)\Delta_j$ is involved. We apply Corollary 5.21 (b) and compute the limit as $\varepsilon \to 0$ when the curvature tensor is replaced by $Q_{\varepsilon,K}$, since the other terms are negligible. The formula shows that the metric $h_{V\langle D\rangle,\varepsilon}$ converges to h_V, and that its curvature tensor converges uniformly to $\theta_{V,\gamma}$ in the complement of any fixed neighborhood of $|D|$. This contribution corresponds to taking powers of the terms in the first line of 5.21 (b), and yields the term $J_1 = \ldots = J_k = \emptyset$ in the right hand side of the limit. In order to evaluate the other terms, which produce contributions supported in $|D|$, we introduce the "orbifold" coordinates

$$t_j = \varepsilon_j^{-1/2}\,\sigma_j(z)^{1-1/\rho_j}\,|\nabla_j\sigma_j(z)|^{-1}, \quad j = j_1,\ldots,j_s,$$

in a neigborhood of any point $z_0 \in \Delta_{j_1}\cap\ldots\cap\Delta_{j_s}$ (and complete those coordinates with $n-s$ variables z_ℓ that define coordinates on $\Delta_{j_1}\cap\ldots\cap\Delta_{j_s}$). These coordinates t_j are not single valued near z_0, but we can always (locally) make a "cut" in X along Δ_j to exclude the negligible set of points where $\sigma_j(z) \in \mathbb{R}_-$, and take the argument in $]-\pi,\pi[$, so that $\mathrm{Arg}(t_j) \in\,]-(1-1/\rho_j)\pi,(1-1/\rho_j)\pi[$. If we integrate over complex numbers t_j without such a restriction on the argument, the integral will have to be multiplied by the factor $(1-1/\rho_j)$ to get the correct value. Since $|\nabla_j\sigma_j(z)|$ does not vanish near Δ_j, the range of the absolute value $|t_j|$ is an interval $]0, C_j\,\varepsilon_j^{-1/2}[$, thus t_j will cover asymptotically an entire angular sector in $\mathbb{C}$. With the above coordinates coordinates, we can write

$$\sigma_j^{1/\rho_j} = (\varepsilon_j^{1/2}\,t_j)^{1/(\rho_j-1)}\,|\nabla_j\sigma_j(z)|^{1/(\rho_j-1)},$$

thus by differentiation along a given tangent vector $\xi \in T_{X,z}$

$$\varepsilon_j^{1/2}\,\sigma_j^{-1+1/\rho_j}\,\nabla\sigma_j(\xi) = O\Big(\varepsilon_j\,\big(\varepsilon_j^{1/2}|t_j|\big)^{-1+1/(\rho_j-1)}\,|dt_j(\xi)| + \varepsilon_j^{1/2}\,|\xi|\Big)$$

as $\varepsilon_j^{1/2}|t_j| \leqslant C_j$ and $d(|\nabla_j\sigma_j(z)|^{1/(\rho_j-1)})$ is bounded. By 5.21 (b), we find

$$|v|^2_{h_{V\langle D\rangle,\varepsilon}} = |v|^2 + \sum |t_j|^{-2}\,|e_j^*(v)|^2, \quad e_j^* = \frac{\nabla_j\sigma_j}{|\nabla_j\sigma_j|} \in S(V^*),$$

$$\begin{aligned}
Q_{\varepsilon,K}(z)(\xi\otimes v)^2 &= \gamma\,\partial\overline{\partial}\psi(\xi,\xi)\,|v|^2 + \sum_{\ell,m,\lambda,\mu} c_{\ell m\lambda\mu}\,\xi_\ell\overline{\xi}_m\,v_\lambda\overline{v}_\mu\\
&\quad + \sum_j |t_j|^{-2}\,\big(\gamma\,\partial\overline{\partial}\psi(\xi,\xi) - \rho_j^{-1}\,\partial\overline{\partial}\varphi_j(\xi,\xi)\big)\,|e_j^*(v)|^2\\
&\quad + O\bigg(\sum_j \varepsilon_j^2\,\big(\varepsilon_j^{1/2}\,|t_j|\big)^{-2+2/(\rho_j-1)}\,|dt_j(\xi)|^2 + \varepsilon_j\,|\xi|^2\bigg)\bigg(|v|^2 + \sum_j |t_j|^{-2}\,|e_j^*(v)|^2\bigg)\\
&\quad + \Big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,\big(\varepsilon^{1/2}\,\nabla^2_{\mathrm{orb}}\,\sigma(\xi,v) - \varepsilon^{1/2}\,(1-1/\rho)\,\sigma^{-1}\,\nabla\sigma(\xi)\,\nabla_{\mathrm{orb}}\sigma(v)\big)\Big|^2,
\end{aligned} \tag{5.33}$$

and with respect to the variables v_ℓ, we have to integrate on the sphere

$$S(V\langle D\rangle, h_{V\langle D\rangle,\varepsilon}) = \big\{v\,;\ |v|^2 + \sum |t_j|^{-2}\,|e_j^*(v)|^2 = 1\big\}.$$

We argue by induction on the number of components Δ_j. When there is only one component Δ_j, Proposition 5.23 (a) gives

$$\begin{aligned}
&\big|(I+M_{\rho,\sigma,\varepsilon})^{-1/2}\,(\ldots)\big|^2\\
&\quad = \frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2}\,\Big|\nabla_j^2\sigma_j(\xi,v) - (1-1/\rho_j)\,\sigma_j^{-1}\,\nabla_j\sigma_j(\xi)\,\nabla_j\sigma_j(v)\Big|^2\\
&\quad = \frac{|t_j|^{-2}}{1+|t_j|^{-2}}\,\bigg|\,\frac{dt_j}{t_j}\,e_j^*(v) + O\big(|\xi|\,|v|\big)\bigg|^2 = \frac{1}{1+|t_j|^2}\,\bigg|\,\frac{dt_j}{t_j}\,e_j^*(v) + O\big(|\xi|\,|v|\big)\bigg|^2.
\end{aligned} \tag{5.34}$$

At this point, we have to make several observations. The most important one is that (5.34) can be viewed as a rank one $(1,1)$ form $\mathrm{i}\,\psi_v\wedge\overline{\psi}_v\in\Lambda^{1,1}T^*_X$ (the associated hermitian form is the square of a linear form), thus if we expand $\langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p$ via its $Q_{K,\varepsilon}$ approximation, this term will only appear with an exponent equal to 0 or 1. The complicated term $O(\varepsilon_j^2\ldots)$ in (5.33) will have a zero contribution in the limit (the right hand side factor $|v|^2+\sum_j|t_j|^{-2}\,|e_j^*(v)|^2$ is equal to 1 and there is a sufficiently large exponent in ε_j in the left hand side factor, so that even after integrating on a large disc $|t_j|<C_j\varepsilon_j^{-1/2}$, one factor ε_j is left). Finally, the term $O(|\xi|\,|v|)^2$ in (5.34) does not contribute, because it will only appear in products of factors that either come from (5.33), or are forms with uniformly bounded coefficients on X multiplied by the bounded factor

$$\frac{|t_j|^{-2}}{1+|t_j|^{-2}}=\frac{\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2}{1+\varepsilon_j\,|\sigma_j|^{-2+2/\rho_j}\,|\nabla_j\sigma_j|^2}\leqslant 1,$$

which has the pleasant property of converging pointwise to 0 almost everywhere on X, namely on $X\smallsetminus\sigma_j^{-1}(0)$; the double product term $2\frac{dt_j}{t_j}e_j^*(v)\,O(|\xi|\,|v|)$ can be bounded by $\delta\,\big|\frac{dt_j}{t_j}e_j^*(v)\big|^2+\delta^{-1}O(|\xi|\,|v|)^2$, $\delta\ll 1$. Putting everything together, $Q_{K,\varepsilon}$ can be simplified and approximated as

$$\begin{aligned}Q_{\varepsilon,K}(z)(\xi\otimes v)^2&\simeq\gamma\,\partial\overline{\partial}\psi(\xi,\xi)\,|v|^2+\sum_{\ell,m,\lambda,\mu}c_{\ell m\lambda\mu}\,\xi_\ell\overline{\xi}_m\,v_\lambda\overline{v}_\mu\\&+|t_j|^{-2}\left(\gamma\,\partial\overline{\partial}\psi(\xi,\xi)-\rho_j^{-1}\,\partial\overline{\partial}\varphi_j(\xi,\xi)\right)|e_j^*(v)|^2+\frac{|t_j|^{-2}}{1+|t_j|^2}\,\big|dt_j(\xi)\big|^2\,|e_j^*(v)|^2.\end{aligned}\tag{5.35}$$

It is also important to notice that since $|v|^2\geqslant|e_j^*(v)|^2$, we have $|e_j^*(v)|^2\leqslant(1+|t_j|^{-2})^{-1}$ on the unit sphere $S(V\langle D\rangle)$, thus

$$|t_j|^{-2}\,|e_j^*(v)|^2\leqslant\frac{1}{1+|t_j|^2}\leqslant 1,\qquad\frac{|t_j|^{-2}}{1+|t_j|^2}\,\big|dt_j(\xi)\big|^2\,|e_j^*(v)|^2\leqslant\frac{1}{(1+|t_j|^2)^2}\,\big|dt_j(\xi)\big|^2,$$

and all terms yield convergent integrals in the limit, even after integrating over the whole complex line $t_j\in\mathbb{C}$. We use the approximation (5.35) to compute $\langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p$ and get

$$\begin{aligned}\langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p&=\Big(\langle\theta_{V,\gamma}(v),v\rangle+\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)|t_j|^{-2}\,|e_j^*(v)|^2\Big)^p\\&\quad+p\Big(\langle\theta_{V,\gamma}(v),v\rangle+\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)|t_j|^{-2}\,|e_j^*(v)|^2\Big)^{p-1}\wedge\frac{|t_j|^{-2}\,|e_j^*(v)|^2}{1+|t_j|^2}\,\frac{\mathrm{i}\,dt_j\wedge d\bar{t}_j}{2\pi}\\&=\sum_{\ell=0}^{p}\binom{p}{\ell}\langle\theta_{V,\gamma}(v),v\rangle^{p-\ell}\wedge\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)^\ell\big(|t_j|^{-2}\,|e_j^*(v)|^2\big)^\ell\\&\quad+p\sum_{\ell=0}^{p-1}\binom{p-1}{\ell}\langle\theta_{V,\gamma}(v),v\rangle^{p-1-\ell}\wedge\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)^\ell\wedge\frac{(|t_j|^{-2}\,|e_j^*(v)|^2)^{\ell+1}}{1+|t_j|^2}\,\frac{\mathrm{i}\,dt_j\wedge d\bar{t}_j}{2\pi}\\&=\langle\theta_{V,\gamma}(v),v\rangle^p+\sum_{\ell=0}^{p-1}\binom{p}{\ell}\langle\theta_{V,\gamma}(v),v\rangle^{p-\ell}\wedge\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)^\ell\big(|t_j|^{-2}\,|e_j^*(v)|^2\big)^\ell\\&\quad+\sum_{\ell=1}^{p}\binom{p}{\ell}\langle\theta_{V,\gamma}(v),v\rangle^{p-\ell}\wedge\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)^{\ell-1}\wedge\frac{\ell\,(|t_j|^{-2}\,|e_j^*(v)|^2)^\ell}{1+|t_j|^2}\,\frac{\mathrm{i}\,dt_j\wedge d\bar{t}_j}{2\pi}\end{aligned}$$

after a change of index $\ell+1 =: \ell'$ in the last summation. As

$$(|t_j|^{-2}\,|e_j^*(v)|^2)^\ell \leqslant (1+|t_j|^2)^{-\ell} = \frac{1}{(1+\varepsilon_j^{-1}\,|\sigma_j|^{2-2/\rho_j}|\nabla_j\sigma_j|^2)^\ell}$$

converges boundedly almost everywhere to 0 on X for $\ell \geqslant 1$, the corresponding term (the one that does not include $dt_j\wedge d\bar t_j$) has an integral over X converging to 0 for all v, and this is uniform in v since $0 \leqslant \langle\theta_{V,\gamma}(v),v\rangle \leqslant \operatorname{Tr}\theta_{V,\gamma}$. When D has N components Δ_j, we can argue by picking the components one by one, each of these giving rise to a factor $\ell_j\,(|t_j|^{-2}\,|e_j^*(v)|^2)^{\ell_j}$, $\ell_j \geqslant 1$, for $j\in J\subset\{1,2,\ldots,N\}$. For the other components $j\notin J$, we set $\ell_j = 0$. We get in that way

$$\begin{aligned}(5.36)\quad \langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p = \langle\theta_{V,\gamma}(v),v\rangle^p + \sum_{\emptyset\neq J\subset\{1,\ldots,N\}}\ \sum_{\ell_j\geqslant 1,\,\Sigma\ell_j\leqslant p} \frac{p!}{\prod\ell_j!\,(p-\sum\ell_j)!}\\ \langle\theta_{V,\gamma}(v),v\rangle^{p-\Sigma\ell_j}\wedge\bigwedge_{j\in J}\Big(\big(\gamma\omega_A-\rho_j^{-1}\theta_{\Delta_j,h_j}\big)^{\ell_j-1}\wedge\frac{\ell_j\,(|t_j|^{-2}\,|e_j^*(v)|^2)^{\ell_j}}{1+|t_j|^2}\,\frac{\mathrm{i}\,dt_j\wedge d\bar t_j}{2\pi}\Big)\\ +\ \text{asymptotically vanishing terms,}\end{aligned}$$

as we compute the iterated limit $\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\ldots\lim_{\varepsilon_N\to 0}$. Notice that since this is an iterated limit, all terms associated with indices $j'<j$ are uniformly bounded in orbifold coordinates when we let $\varepsilon_j\to 0$ and keep $\varepsilon_{j'}>0$ fixed. The term $\langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p$ in the right hand side of (5.36) corresponds to the choice $J=\emptyset$ in the summation. The square of the norm of the linear form $v\mapsto t_j^{-1}\,e_j^*(v)$ is $(1+|t_j|^2)^{-1}$, thus by Lemma 5.24, a partial integration $\int_{S(V\langle D\rangle}(\ldots)\,d\mu(v)$ produces a factor $\ell_j(1+|t_j|^2)^{-(\ell_j+1)}\,\frac{\mathrm{i}\,dt_j\wedge d\bar t_j}{2\pi}$. An elementary calculation gives

$$\int_{t\in\mathbb{C},\,\operatorname{Arg}t\in\,]-\eta,\eta[}\frac{\ell}{(1+|t|^2)^{\ell+1}}\,\frac{\mathrm{i}\,dt\wedge d\bar t}{2\pi} = \frac{\eta}{\pi},$$

hence the current

$$\int_{t_j\in\mathbb{C},\,\operatorname{Arg}t\in\,]-(1-1/\rho_j)\pi,(1-1/\rho_j)\pi[}\ \bigwedge_{j\in J}\frac{\ell_j}{(1+|t_j|^2)^{\ell_j+1}}\,\frac{\mathrm{i}\,dt_j\wedge d\bar t_j}{2\pi}$$

converges weakly to the effective cycle $\prod_{j\in J}(1-1/\rho_j)\,[\Delta_J]$, where $\Delta_J=\bigcap_{j\in J}\Delta_j$. Therefore Proposition 5.29 (a) implies

$$\begin{aligned}&\lim_{\varepsilon_j\to 0}\int_X\int_{v\in S(V\langle D\rangle)}\langle\theta_{D,\gamma,\varepsilon}(v),v\rangle^p\wedge\beta\,d\mu(v)\\ &\geqslant\sum_{J\subset\{1,\ldots,N\}}\ \sum_{\ell_j\geqslant 1,\,\Sigma\ell_j\leqslant p}\frac{(r-1)!}{(p+r-1)!}\,\frac{p!}{\prod\ell_j!\,(p-\sum_{j\in J}\ell_j)!}\\ &\qquad\prod_{j\in J}(1-1/\rho_j)\int_{\Delta_J}(\operatorname{Tr}\theta_{V,\gamma})^{p-\Sigma\ell_j}\wedge\bigwedge_{j\in J}\big(\gamma\omega_A-\rho_{sj}^{-1}\theta_{\Delta_j,h_j}\big)^{\ell_j-1}\wedge\beta,\end{aligned}$$

Now, we consider the case of a product of terms $\langle\theta_{D_s,\gamma_s,\varepsilon_s}(v_s),v_s\rangle^{p_s}$ associated with orbifold divisors $D_s=\sum_j(1-1/\rho_{sj})\Delta_j$, and metrics $h_{V\langle D_s\rangle,\varepsilon_s}$, $\varepsilon_s=(\varepsilon_{s1},\ldots,\varepsilon_{sN})\to 0$, $1\leqslant s\leqslant k$.

By (5.36), we obtain a similar more general expression, given by a summation on all disjoint subsets $J_1, \ldots, J_k \subset \{1, 2, \ldots, N\}$,

$$\begin{aligned}&\langle \theta_{D_1,\gamma_1,\varepsilon_1}(v_1), v_1\rangle^{p_1} \wedge \ldots \wedge \langle \theta_{D_k,\gamma_k,\varepsilon_k}(v_k), v_k\rangle^{p_k}\\ &= \sum_{J_1 \amalg \ldots \amalg J_k \subset \{1,\ldots,N\}} \; \sum_{\ell_j \geqslant 1,\, \Sigma_{j\in J_s} \ell_j \leqslant p_s} \frac{\prod_{1\leqslant i\leqslant k} p_s!}{\prod \ell_j! \prod_s (p_s - \sum_{j\in J_s} \ell_j)!} \bigwedge_{1\leqslant i\leqslant k} \bigg(\langle \theta_{V,\gamma}(v_s), v_s\rangle^{p_s - \Sigma_{j\in J_s}\ell_j}\\ &\qquad\qquad \wedge \bigwedge_{j\in J_s} \bigg(\big(\gamma\omega_A - \rho_j^{-1}\theta_{\Delta_j,h_j}\big)^{\ell_j - 1} \wedge \frac{\ell_j\, (|t_j|^{-2}\, |e_j^*(v_s)|^2)^{\ell_j}}{1 + |t_j|^2} \, \frac{\mathrm{i}\, dt_j \wedge d\bar t_j}{2\pi}\bigg)\bigg)\\ &\qquad\qquad + \text{asymptotically vanishing terms.}\end{aligned}$$

The arguments explained above for the case of a single term $\langle \theta_{D,\gamma,\varepsilon}(v), v\rangle^p$ can be generalized to such products, and easily imply the lower bound stated in Proposition 5.32. The upper bound is proved in a similar way, except that we use 5.29 (b) instead of 5.29 (a), and merely need a Griffiths semi-positivity condition for $\theta_{V,\gamma}$ instead of strong semi-positivity. One has to observe that this bound involves quantities of the form

$$\bigwedge_{s=1}^{k} \Big(\langle \theta_s(v_s), v_s\rangle^{p_s - m_s}\Big) \wedge \prod_{1\leqslant j\leqslant m_s} |\ell'_{sj}(v_s)|^2\, \beta', \quad \theta_s \geqslant_G 0, \quad \beta' \geqslant_S 0.$$

A priori, the inequality provided by 5.29 (b) for each integral $\int_{v_s \in S(V\langle D_s\rangle)}$ is merely a weak inequality, but we are anyway integrating products of strongly semi-positive forms, so the expected inequalities still hold, and we can apply the Fubini theorem without trouble. By 5.29, the constant $\frac{(r-1)!}{(p_s+r-1)!}$ has to be replaced by $\frac{p_s!\,(r-1)!}{(p_s+r-1)!}$ for the upper bound. □

5.37. Remark. One of the technical difficulties is that, strictly speaking, the fourth line term of $Q_{\varepsilon,K}$ in (5.33) is not strongly semi-positive, while the other terms are, if we assume $\gamma\Theta_{A,h_A} \otimes \mathrm{Id} - \Theta_{V,h_V} >_S 0$, as implied by the condition $\gamma > \widetilde\gamma_{V,D}$. However, near any point of $\Delta_j \smallsetminus \bigcup_{\ell\neq j} \Delta_\ell$, we have seen that the fourth line term has the same limit as the strongly semi-positive rank 1 tensor

$$\frac{\varepsilon_j\, |\sigma_j|^{-2+2/\rho_j}}{1 + \varepsilon_j\, |\sigma_j|^{-2+2/\rho_j}\, |\nabla_j \sigma_j|^2} \, \Big|(1 - 1/\rho_j)\, \sigma_j^{-1}\, \nabla_j\sigma_j(\xi)\, \nabla_j\sigma_j(v)\Big|^2.$$

The latter converges weakly in the sense of currents to the integral on Δ_j of a strongly semi-positive term; inductively, via the iterated limit process, the remaining terms combining several components Δ_j also produce strongly semi-positive terms with support in higher codimensional strata $\Delta_J = \bigcap_{j\in J} \Delta_j$.

6. Non probabilistic estimates of the Morse integrals

6.A. Case of general directed orbifolds

The non probabilistic estimate uses more explicit curvature inequalities and has the advantage of producing results also in the general orbifold case. Let us fix an ample line bundle A on X equipped with a smooth hermitian metric h_A such that $\omega_A := \Theta_{A,h_A} > 0$. We assume here that the s-th directed (dual) orbifold bundle $V^*\langle D^{(s)}\rangle$ (cf. § 1.B) possesses a hermitian metric $h^*_{V,s}$ such that its curvature tensor satisfies an inequality

$$\Theta_{V^*\langle D^{(s)}\rangle, h^*_{V,s}} + \gamma_s\, \omega_A \otimes \mathrm{Id}_{V^*\langle D^{(s)}\rangle} \geqslant_G 0 \tag{6.1}$$

in the sense of Griffiths, for some number $\gamma_s \geqslant 0$. Now, instead of exploiting a Monte Carlo convergence process for the curvature tensor, we replace $\Theta_{V^*\langle D^{(s)}\rangle}$ with

$$\Theta^A_{V^*\langle D^{(s)}\rangle} := \Theta_{V^*\langle D^{(s)}\rangle} + \gamma_s\,\omega_A \otimes \mathrm{Id} \geqslant_G 0,$$

and in this way get new curvature coefficients $c^{(s,A)}_{ij\lambda\mu} = c^{(s)}_{ij\lambda\mu} + \gamma_s\,\omega_{A,ij}\,\delta_{\lambda\mu}$. This has the effect of replacing $\Theta_{\det V^*\langle D^{(s)}\rangle} = \mathrm{Tr}\,\Theta_{V^*\langle D^{(s)}\rangle}$ by $\Theta_{\det V^*\langle D^{(s)}\rangle} + r\gamma_s\,\omega_A$. Also, we take

$$L_{\varepsilon,k} := \mathcal{O}_{X_k(V\langle D\rangle)}(1) \otimes \pi_k^*\mathcal{O}_X(-\varepsilon A). \tag{6.2}$$

Then our earlier formulas 3.20 (a,b) become

$$\Theta_{L_{\varepsilon,k}} = \omega_{r,k,b}(\xi) + g_{\varepsilon,k}(z,x,u) \quad \text{where} \tag{6.3}$$

$$g_{\varepsilon,k}(z,x,u) = \frac{\mathrm{i}}{2\pi}\sum_{s=1}^{k}\frac{x_s}{s}\sum_{i,j,\lambda,\mu} c^{(s)}_{ij\lambda\mu}(z)\,u_{s,\lambda}\overline{u}_{s,\mu}\,dz_i\wedge d\overline{z}_j - \varepsilon\,\omega_A. \tag{6.3$'$}$$

We want to express $g_{\varepsilon,k}(z,x,u)$ as a difference of two non negative terms. For this, we write

$$g_{\varepsilon,k}(z,x,u) = \sum_{s=1}^{k}\frac{x_s}{s}\,\theta_{s,A}(u) - \Big(\varepsilon + \sum_{s=1}^{k}\frac{\gamma_s x_s}{s}\Big)\omega_A \quad \text{where} \tag{6.4}$$

$$\theta_{s,A}(u_s) := \frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu} c^{(s,A)}_{ij\lambda\mu} u_{s,\lambda}\overline{u}_{s,\mu}\,dz_i\wedge d\overline{z}_j \geqslant 0, \tag{6.4$'$}$$

and we also consider

$$\mathrm{Tr}\,\theta_{s,A} = \frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda} c^{(s,A)}_{ij\lambda\mu}\,dz_i\wedge d\overline{z}_j = \Theta_{\det V^*\langle D^{(s)}\rangle} + r\gamma_s\,\omega_A.$$

We apply Corollary 2.4 with α,β replaced by

$$\alpha_k = \sum_{s=1}^{k}\frac{x_s}{s}\,\theta_{s,A}(u_s), \qquad \beta_k = \Big(\varepsilon + \sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\Big)\omega_A,$$

both forms being semipositive by our assumptions. Then (4.2) leads to

$$\begin{aligned}
&\int_{X_k(V)(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_{\varepsilon,k},\Psi^*_{h,b,\varepsilon}}\\
&\quad= \frac{(n+kr-1)!}{n!\,k!^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \mathbb{1}_{\alpha_k-\beta_k,\leqslant 1}\,(\alpha_k-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\quad\geqslant \frac{(n+kr-1)!}{n!\,k!^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \big(\alpha_k^n - n\alpha_k^{n-1}\wedge\beta_k\big)\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned} \tag{6.5}$$

The resulting integral now produces a "closed formula" which, as we will see, can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). At this stage, we invoke two types of expressions or lower

bounds for α_k^n. The first simply uses an expansion of α_k^n by Newton's multinomial formula and is valid for every $k \geqslant 1$. We get

$$\alpha_k^n = \sum_{p\in\mathbb{N}^k,\,|p|=n} \frac{n!}{p_1!\ldots p_k!}\prod_{s=1}^k \Big(\frac{x_s}{s}\,\theta_{s,A}(u_s)\Big)^{p_s}. \tag{6.6}$$

The second (and weaker) bound consists of keeping only the terms for which $p_s = 0$ or 1; the existence of such terms requires $k \geqslant n$. As $\theta_{s,A} \geqslant 0$, we find in that case

$$\alpha_k^n \geqslant \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} n!\,\frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta_{s_1,A}(u_{s_1})\wedge\theta_{s_2,A}(u_{s_2})\wedge\ldots\wedge\theta_{s_n,A}(u_{s_n}). \tag{6.6$'$}$$

In the case of expression (6.6), we need a strong semi-positivity assumption $\theta_{s,A} \geqslant_S 0$, and Proposition 5.29 (a) then yields

$$\int_{\mathbb{S}^{2r-1}} (\theta_{s,A}(u_s))^p\,d\mu_s(u_s) \geqslant \frac{(r-1)!}{(p+r-1)!}\,(\mathrm{Tr}\,\theta_{s,A})^p$$

When $p = 1$, we have $\frac{(r-1)!}{(p+r-1)!} = \frac{1}{r}$, and the assumption $\theta_{s,A} \geqslant_G 0$ is sufficient. We infer

$$\begin{aligned}&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \alpha_k^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\geqslant \sum_{p\in\mathbb{N}^k,\,|p|=n} n!\prod_{s=1}^k\frac{(r-1)!}{p_s!\,(p_s+r-1)!\,s^{p_s}}\int_{\Delta^{k-1}} x_1^{p_1}\ldots x_k^{p_k}\,d\nu_{k,r}(x)\bigwedge_{s=1}^k\big(\mathrm{Tr}\,\theta_{s,A}\big)^{p_s}, &(6.7)\\
&\geqslant \sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{n!}{s_1\ldots s_n\,r^n}\int_{\Delta^{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x)\bigwedge_{\ell=1}^n \mathrm{Tr}\,\theta_{s_\ell,A}, \quad\text{respectively.} &(6.7')\end{aligned}$$

By formula 3.10 (a) and an elementary calculation (cf. [Dem11, Prop. 1.13]), one gets

$$\int_{\Delta^{k-1}} x_1^{p_1}\ldots x_k^{p_k}\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\,\frac{\prod_{1\leqslant s\leqslant k}(p_s+r-1)!}{(n+kr-1)!}, \tag{6.8}$$

in particular

$$\int_{\Delta^{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\,\frac{r!^n(r-1)!^{k-n}}{(n+kr-1)!} = \frac{(kr-1)!\,r^n}{(n+kr-1)!}. \tag{6.8$'$}$$

The combination of (6.7^*) and (6.8^*) implies

$$\begin{aligned}&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \alpha_k^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\geqslant \frac{(kr-1)!}{(n+kr-1)!}\bigg(\sum_{s=1}^k\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A}\bigg)^n, \quad\text{resp.} &(6.9)\\
&\geqslant \frac{n!\,(kr-1)!}{(n+kr-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\bigwedge_{\ell=1}^n \mathrm{Tr}\,\theta_{s_\ell,A}. &(6.9')\end{aligned}$$

Now, we compute an upper bound for the integral of $n\alpha_k^{n-1}\wedge\beta_k$ over $\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, where

$$\begin{aligned}
n\alpha_k^{n-1}\wedge\beta_k &= n\Big(\varepsilon+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\,\theta_{s,A}(u_s)\Big)^{n-1}\wedge\omega_A\\
&= n\varepsilon\Big(\sum_{p\in\mathbb{N}^k,\,|p|=n-1}\frac{n!}{p_1!\ldots p_k!}\prod_{s=1}^k\Big(\frac{x_s}{s}\,\theta_{s,A}(u_s)\Big)^{p_s}\Big)\wedge\omega_A\\
&\quad+n\Big(\sum_{1\leqslant j\leqslant k,\,p\in\mathbb{N}^k,\,|p|=n-1}\frac{n!}{p_1!\ldots p_k!}\frac{\gamma_j x_j}{j}\prod_{s=1}^k\Big(\frac{x_s}{s}\,\theta_{s,A}(u_s)\Big)^{p_s}\Big)\wedge\omega_A.
\end{aligned}\tag{6.10}$$

We apply Proposition 5.29 (b) and (6.8) to infer

$$\begin{aligned}
&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} n\alpha_k^{n-1}\wedge\beta_k\,d\nu_{k,r}(x)\,d\mu(u)\\
&\leqslant n\int_{u\in(\mathbb{S}^{2r-1})^k}\Big(\varepsilon\sum_{|p|=n-1}\frac{n!}{p_1!\ldots p_k!}\frac{(kr-1)!}{(r-1)!^k}\prod_{s=1}^k\frac{(p_s+r-1)!}{(n+kr-1)!}\Big(\frac{1}{s}\,\theta_{s,A}(u_s)\Big)^{p_s}\\
&\quad+\sum_{j,\,|p|=n-1}\frac{n!}{p_1!\ldots p_k!}\frac{\gamma_j}{j}\frac{(kr-1)!}{(r-1)!^k}\prod_{s=1}^k\frac{(p_s+\delta_{js}+r-1)!}{(n+kr-1)!}\Big(\frac{1}{s}\,\theta_{s,A}(u_s)\Big)^{p_s}\Big)\wedge\omega_A\,d\mu(u)\\
&\leqslant n\Big(\varepsilon\sum_{|p|=n-1}\frac{n!\,(kr-1)!}{(n+kr-1)!}\prod_{s=1}^k\Big(\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A}\Big)^{p_s}\\
&\quad+\sum_{j,\,|p|=n-1}\frac{n!\,(kr-1)!}{(n+kr-1)!}\frac{\gamma_j(p_j+r)}{j}\prod_{s=1}^k\Big(\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A}\Big)^{p_s}\Big)\wedge\omega_A,
\end{aligned}\tag{6.11}$$

where the inequalities are to be understood as inequalities between (n,n)-forms. By putting $(6.7-6.11)$ together, we obtain

$$\begin{aligned}
&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k}\big(\alpha_k^n-n\alpha_k^{n-1}\wedge\beta_k\big)\,d\nu_{k,r}(x)\,d\mu(u)\\
&\geqslant\frac{n!\,(kr-1)!}{(n+kr-1)!}\Big(\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\bigwedge_{\ell=1}^n\mathrm{Tr}\,\theta_{s_\ell,A}\\
&\quad-\sum_{p\in\mathbb{N}^k,\,|p|=n-1}n\Big(\varepsilon\prod_{s=1}^k\Big(\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A}\Big)^{p_s}+\sum_{j=1}^k\frac{\gamma_j(p_j+r)}{j}\prod_{s=1}^k\Big(\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A}\Big)^{p_s}\Big)\wedge\omega_A\Big),
\end{aligned}$$

and the first summation $\sum_{s_1,\ldots,s_n}$ can be replaced by the larger term $\frac{1}{n!}\,(\sum_{s=1}^k\frac{1}{s}\,\mathrm{Tr}\,\theta_{s,A})^n$ in case we have strong semi-positivity. The Morse integral lower bound (6.5) finally implies

6.12. Theorem. *Assume that the curvature of the orbifold bundles satisfy the lower bounds $\Theta_{V^*\langle D^{(s)}\rangle}\geqslant_G-\gamma_s\,\omega_A\otimes\mathrm{Id}_{V^*}$ (in the sense of Griffiths), for some number $\gamma_s\in\mathbb{R}_+$. Then the orbifold line bundle*

$$L_{\varepsilon,k}=\mathcal{O}_{X_k(V\langle D\rangle)}(1)\otimes\pi_k^*\mathcal{O}(-\varepsilon A)$$

admits for all $k \geqslant n$ and $\varepsilon \in \mathbb{Q}_+$ a number of sections $h^0(X_k(V\langle D\rangle), L_{\varepsilon,k}^{\otimes m})$ that is bounded below asymptotically, modulo an error term $o(m^{n+kr-1})$, by

$$\frac{m^{n+kr-1}}{(n+kr-1)!}\int_{X_k(V\langle D\rangle)(L_{\varepsilon,k},\leqslant 1)} \Theta^{n+kr-1}_{L_{\varepsilon,k},\Psi^*_{h,b,\varepsilon}} \geqslant \frac{m^{n+kr-1}}{k!^r}\times$$

$$\int_X \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{1}{s_1\ldots\, s_n}\bigwedge_{\ell=1}^{n}\big(\Theta_{s_\ell}+r\gamma_{s_\ell}\,\omega_A\big)$$

$$-\sum_{p\in\mathbb{N}^k,\,|p|=n-1} n\bigg(\varepsilon\prod_{s=1}^{k}\frac{1}{s^{p_s}}\Big(\Theta_s+r\gamma_s\,\omega_A\Big)^{p_s}+\sum_{j=1}^{k}\frac{\gamma_j(p_j+r)}{j}\prod_{s=1}^{k}\frac{1}{s^{p_s}}\Big(\Theta_s+r\gamma_s\,\omega_A\Big)^{p_s}\bigg)\wedge\omega_A,$$

where $\Theta_s = \Theta_{\det V^*\langle D^{(s)}\rangle}$. The first summation $\sum_{s_1,\ldots,s_n}$ can be replaced by the larger term $\frac{1}{n!}\,(\sum_{s=1}^{k}\frac{1}{s}(\Theta_s+r\gamma_s\omega_A))^n$ in case we have strong semi-positivity instead of Griffiths semi-positivity.

Especially, for $m \gg 1$, we have a lot of sections in

$$H^0(X_k(V\langle D\rangle), L_{\varepsilon,k}^{\otimes m}) = H^0(X, E_{k,m}V^*\langle D\rangle\otimes \mathcal{O}_X(-m\varepsilon A)),$$

whenever the integral providing the lower bound is positive; by Corollary 1.11, when the integral is non positive, we still get a non trivial lower bound for the difference $h^0(X_k(V\langle D\rangle), L_{\varepsilon,k}^{\otimes m}) - h^1(X_k(V\langle D\rangle), L_{\varepsilon,k}^{\otimes m})$. In the compact or logarithmic situation, we can take $\theta_{s,A} = \theta_A$ independent of s, $\Theta = \Theta_{\det(V\langle D\rangle)}$ and then obtain the simple lower bound

$$\frac{m^{n+kr-1}}{k!^r}\int_X H_{n,k}\big(\Theta+r\gamma\,\omega_A\big)^n - n\,H'_{n,k,\varepsilon}\,\big(\Theta+r\gamma\,\omega_A\big)^{n-1}\wedge\omega_A \tag{6.13}$$

where

$$H_{n,k} = \sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots\, s_n},\quad H'_{n,k,\varepsilon} = \sum_{p\in\mathbb{N}^k,\,|p|=n-1}\bigg(\varepsilon\prod_{s=1}^{k}\frac{1}{s^{p_s}}+\gamma\sum_{j=1}^{k}\frac{p_j+r}{j}\prod_{s=1}^{k}\frac{1}{s^{p_s}}\bigg).$$

In the case of strong semi-positivity, $H_{n,k}$ can be replaced with the larger value $\frac{1}{n!}\,(H_k)^n$, where $H_k = 1+\frac{1}{2}+\cdots+\frac{1}{k}$ is the harmonic sequence. When $\varepsilon \ll 1$, we get sections as soon as

$$\Theta - \bigg(n\,\frac{H'_{n,k,\varepsilon}}{H_{n,k}} - r\gamma\bigg)\omega_A > 0,\quad \text{resp.}\quad \Theta - \bigg(n\,n!\,\frac{H'_{n,k,\varepsilon}}{(H_k)^n} - r\gamma\bigg)\omega_A > 0. \tag{6.14}$$

This last condition is substantially sharper than the one stated in [Dem12] (thanks to much improved estimates of the integrals involved in the calculation). We have

$$\sum_{j=1}^{k}\frac{p_j+r}{j}\leqslant\sum_{j=1}^{k}p_j+\frac{r}{j}\leqslant n-1+r\,H_k$$

and

$$\sum_{p\in\mathbb{N}^k,\,|p|=n-1}\prod_{s=1}^{k}\frac{1}{s^{p_s}}\leqslant\begin{cases}\displaystyle\sum_{0\leqslant p_2,\ldots,p_k<+\infty}\frac{1}{2^{p_2}}\,\frac{1}{3^{p_3}}\cdots\frac{1}{k^{p_k}}\leqslant\frac{1}{1-1/2}\,\frac{1}{1-1/3}\cdots\frac{1}{1-1/k}=k,\\[2ex] \displaystyle\bigg(\sum_{s=1}^{k}\frac{1}{s}\bigg)^{n-1}=(H_k)^{n-1}.\end{cases}$$

Therefore

$$H'_{n,k,\varepsilon} \leqslant \Big(\varepsilon + \gamma\big(n-1+r\,H_k\big)\Big)\min\big(k,(H_k)^{n-1}\big). \tag{6.15}$$

On the other hand, we have $H_{n,k} \geqslant \frac{1}{n!}$ for $k \geqslant n$, and asymptotically when $k \to +\infty$, if we let s_j vary in the range $\lfloor k^{\frac{j-1}{n}}\rfloor \leqslant s_j < \lfloor k^{\frac{j}{n}}\rfloor$, $1 \leqslant j \leqslant n$, we get

$$H_{n,k} \geqslant \prod_{j=1}^{n} \log\frac{\lfloor k^{\frac{j}{n}}\rfloor}{\lfloor k^{\frac{j-1}{n}}\rfloor} \sim \Big(\frac{1}{n}\log k\Big)^n \sim n^{-n}\,(H_k)^n. \tag{6.16}$$

This implies

$$\limsup_{k\to+\infty}\left(n\,\frac{H'_{n,k,\varepsilon}}{H_{n,k}} - r\gamma\right) \leqslant r\gamma(n^{n+1}-1), \quad \limsup_{k\to+\infty}\left(n\,\frac{H'_{n,k,\varepsilon}}{(H_k)^n} - r\gamma\right) \leqslant r\gamma(n\,n!-1).$$

Asymptotically as $k \to +\infty$, we get the sufficient condition $\Theta - r\gamma(n^{n+1}-1)\,\omega_A > 0$, resp. $\Theta - r\gamma(n\,n!-1)\,\omega_A > 0$, which are much more restrictive than the condition $\Theta > 0$ we would get by the probabilistic estimate. The case $k = n$ is especially interesting. We then find $H_{n,k} = H_{n,n} = \frac{1}{n!}$ and

$$n\,\frac{H'_{n,n,\varepsilon}}{H_{n,n}} - r\gamma \leqslant n^2\,n!\Big(\varepsilon + \gamma\big(n-1+r\,H_n\big)\Big) - r\gamma.$$

6.17. Application. In the case where X is a smooth hypersurface of $\mathbb{P}^{n+1}$ of degree d and $V = T_X$, thus $r = n$, we have $\Theta = c_1(\mathcal{O}(d-n-2), h_{\mathrm{FS}})$. Also, we can take $A = \mathcal{O}(1)$ and $\gamma = 2$, since the surjective morphisms

$$T_{\mathbb{P}^{n+1}|X} \to T_X \to V^*$$

imply $V^* \otimes \mathcal{O}(2) \geqslant_G 0$. Condition (6.13) is satisfied, and therefore we have many n-jet differentials with a negative twist $\mathcal{O}(-m\varepsilon)$, as soon as $d+n-2 \geqslant 2\,n^2\,n!\,(n-1+n\,H_n)$. In the logarithmic situation where $X = \mathbb{P}^n$, $V = T_{\mathbb{P}^n}$, and $D = \sum \Delta_j$ is a divisor of total degree d, we can still take $\gamma = 2$ by Proposition 5.8, and $\Theta = c_1(\mathcal{O}(d-n-1), h_{\mathrm{FS}})$; a similar degree bound $d+n-1 \geqslant 2\,n^2\,n!\,(n-1+n\,H_n)$ holds in that case.

6.B. Case of orbifold structures on projective n-space

An interesting orbifold example is the case when $X = \mathbb{P}^n$, $V = T_X$, $A = \mathcal{O}(1)$ and $D = \sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor, with components Δ_j of degree d_j. Since

$$D^{(s)} = \sum_j \Big(1 - \frac{s}{\rho_j}\Big)_+ \Delta_j,$$

we have

$$\det V^*\langle D^{(s)}\rangle = \mathcal{O}_{\mathbb{P}^n}\big(-n-1+\sum_j d_j(1-s/\rho_j)_+\big)$$

and the associated curvature form is

$$\Theta_s = \Big(-n-1+\sum_j d_j(1-s/\rho_j)_+\Big)\omega_A.$$

Moreover, by Proposition 5.8, we have

$$\Theta_{V^*\langle D^{(s)}\rangle} + \gamma_s\,\omega_{\mathrm{FS}} \otimes \mathrm{Id} >_G 0$$

as soon as $\gamma_s > 2$ and $\gamma_s > \max_j(d_j/\max(\rho_j/s, 1))$ for all components Δ_j in $D^{(s)}$. We can take for instance $\gamma_s > st$ where $t = \max(\max_j(d_j/\rho_j), 2)$. Then, for $k = n$ and $\varepsilon \in \mathbb{Q}_+$ small, the estimate (6.14) guarantees the existence of jet differentials under the complicated condition

$$\prod_{s=1}^{n}\Big(nst - n - 1 + \sum_j d_j(1 - s/\rho_j)_+\Big) > \frac{n\,(2n-1)!}{(n-1)!} \times$$

$$nt\Big(\sum_{1\leqslant s\leqslant n} \frac{1}{s}\Big(nst - n - 1 + \sum_j d_j(1 - s/\rho_j)_+\Big)\Big)^{n-1}. \tag{6.15}$$

If we take $\rho_j \geqslant \rho > n$, then $(1 - s/\rho_j)_+ \geq 1 - s/\rho$ for $s \leqslant n$, and as $nst - n - 1 \geqslant 0$ and $\sum_{1\leqslant s\leqslant n} \frac{1}{s}(nst - n - 1) \leqslant n^2 t$, we get a sufficient condition

$$\prod_{s=1}^{n}\Big(\Big(1 - \frac{s}{\rho}\Big)\sum_j d_j\Big) > \frac{n^2\,(2n-1)!}{(n-1)!} \times t\Big(n^2 t + \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n}\Big)\sum_j d_j\Big)^{n-1}.$$

The latter condition is satisfied if $\sum_j d_j \geqslant c_n t \prod_{s=1}^{n}\big(1 - \frac{s}{\rho}\big)^{-1}$ with

$$c_n = \frac{n^2\,(2n-1)!}{(n-1)!}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n^3}\Big)^{n-1},$$

since $c_n \geq n^5$ for all $n \in \mathbb{N}^*$, and so $n^2 t \leqslant \frac{1}{n^3}\sum d_j$. The Stirling formula gives

$$c_n \leqslant 2^{-1/2}\,(4/e)^n\,n^{n+2}\,(1 + \log n)^{n-1} = O((2n\log n)^n) \tag{6.16}$$

for n large. In this way we get

6.17. Proposition. *Let $D = \sum_j (1 - \frac{1}{\rho_j})\Delta_j$ a simple normal crossing orbifold divisor on $\mathbb{P}^n$ with $\deg \Delta_j = d_j$. Then there exist jet differentials of order n and large degree m on $\mathbb{P}^n\langle D\rangle$, with a small negative twist $\mathcal{O}_{\mathbb{P}^b}(-m\varepsilon)$, provided that*

$$\rho_j \geqslant \rho > n, \quad \sum d_j \geqslant c_n \max\Big(\max\Big(\frac{d_j}{\rho_j}\Big), 2\Big)\prod_{s=1}^{n}\Big(1 - \frac{s}{\rho}\Big)^{-1}.$$

For instance, one can take all components Δ_j possessing the same degree d and ramification number $\rho > n$, and a number of components

$$N \geqslant c_n \max\Big(\frac{1}{\rho}, \frac{2}{d}\Big)\prod_{s=1}^{n}\Big(1 - \frac{s}{\rho}\Big)^{-1},$$

or a single component $(1 - \frac{1}{\rho_1})\Delta_1$ with $\rho_1 \geqslant 2c_n$ and $d_1 \geqslant 4c_n$ (notice that $\prod(1 - \frac{s}{2c_n})^{-1} < 2$). Since we have neglected many terms in the above calculations, the "technological constant" c_n appearing in these estimates is probably much larger than needed.

References

[Cad17] Cadorel, B.: *Jet differentials on toroidal compactifications of ball quotients.* arXiv: math.AG/1707.07875.

[CDR18] Campana, F., Darondeau, L., Rousseau, E.: *Orbifold hyperbolicity.* arXiv: math.AG/1803.10716.

[Dem80] Demailly, J.-P.: *Relations entre les différentes notions de fibrés et de courants positifs.* Sém. P. Lelong-H. Skoda (Analyse) 1980/81, Lecture Notes in Math. n° 919, Springer-Verlag, 56–76.

[Dem95] Demailly, J.-P.: *Propriétés de semi-continuité de la cohomologie et de la dimension de Kodaira-Iitaka.* C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), 341–346.

[Dem97] Demailly, J.-P.: *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials.* AMS Summer School on Algebraic Geometry, Santa Cruz 1995, Proc. Symposia in Pure Math., ed. by J. Kollár and R. Lazarsfeld, Amer. Math. Soc., Providence, RI (1997), 285—360.

[Dem11] Demailly, J.-P.: *Holomorphic Morse Inequalities and the Green-Griffiths-Lang Conjecture.* Pure and Applied Math. Quarterly **7** (2011), 1165–1208.

[Dem12] Demailly, J.-P.: *Hyperbolic algebraic varieties and holomorphic differential equations.* expanded version of the lectures given at the annual meeting of VIASM, Acta Math. Vietnam. **37** (2012), 441-–512.

[GrGr80] Green, M., Griffiths, P.: *Two applications of algebraic geometry to entire holomorphic mappings.* The Chern Symposium 1979, Proc. Internal. Sympos. Berkeley, CA, 1979, Springer-Verlag, New York (1980), 41–74.

[Lan05] Landau, E.: *Sur quelques théorèmes de M. Petrovitch relatifs aux zéros des fonctions analytiques.* Bull. Soc. Math. France **33** (1905), 251–261.

(version of October 23, 2019, printed on June 19, 2021, 23:12)

Frédéric Campana
Institut de Mathématiques Élie Cartan, Université de Lorraine, B.P. 70239
54506 Vandœuvre-lès-Nancy, France
E-mail : frederic.campana@univ-lorraine.fr

Lionel Darondeau
Université Montpellier II, Institut Montpellierain Alexander Grothendieck,
Case courrier 051, Place Eugène Bataillon, 34090 Montpellier, France
E-mail : lionel.darondeau@normalesup.org

Jean-Pierre Demailly
Université Grenoble Alpes,
Institut Fourier, 100 rue des Maths, 38610 Gières, France
E-mail : jean-pierre.demailly@univ-grenoble-alpes.fr

Erwan Rousseau
Institut Universitaire de France,
CMI, Université d'Aix-Marseille, 39, rue Frédéric Joliot-Curie, 13453 Marseille, France
E-mail : erwan.rousseau@univ-amu.fr