On the existence of logarithmic and orbifold jet differentials

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0. Introduction and main definitions

The present research is concerned with the existence of logarithmic and orbifold jet differentials on projective varieties. For the sake of generality, and in view of potential applications to the case of foliations, we work throughout this paper in the category of directed varieties, and generalize them by introducing the concept of directed orbifold.

0.1. Definition. *Let X be a complex manifold or variety. A directed structure (X,V) on X is defined to be a subsheaf $V\subset \mathcal{O}(T_X)$ such that $\mathcal{O}(T_X)/V$ is torsion free. A morphism of directed varieties $\Psi:(X,V)\to(Y,W)$ is a holomorphic map $\Psi:X\to Y$ such that $d\Psi(V)\subset\Psi^*W$. We say that (X,V) is non singular if X is non singular and V is locally free, i.e., is a holomorphic subbundle of T_X.*

We refer to the *absolute case* as being the situation when $V=T_X$, the *relative case* when $V=T_{X/S}$ for some fibration $X\to S$, and the *foliated case* when V is integrable, i.e. $[V,V]\subset V$, that is, V is the tangent sheaf to a holomorphic foliation. We now combine these concepts with orbifold structures in the sense of Campana [Cam04].

0.2. Definition. *A directed orbifold is a triple (X,V,Δ) where (X,V) is a directed variety and $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$ an effective real divisor, where Δ_j is an irreducible hypersurface and $\rho_j\in\,]1,\infty]$ an associated "ramification number". We denote by $\lceil\Delta\rceil=\sum\Delta_j$ the corresponding reduced divisor, and by $|\Delta|=\bigcup\Delta_j$ its support.*

(a) *We will say that (X,V,Δ) is non singular if (X,V) is non singular and Δ is a simple normal crossing divisor such that Δ is transverse to V. If $r=\mathrm{rank}(V)$, we mean by this that there are at most r components Δ_j meeting at any point $x\in X$, and that for any p-tuple $(j_1,\ldots,j_p)$ of indices, $1\leqslant p\leqslant r$, we have $\dim V_x\cap\bigcap_{j=1}^p T_{\Delta_{j_\ell},x}=r-p$ at any point $x\in\bigcap_{j=1}^p\Delta_{j_\ell}$.*

(b) *If (X,V,Δ) is non singular, the canonical divisor of (X,V,Δ) is defined to be*

$$K_{V,\Delta}=K_V+\Delta$$

(*in additive notation*), *where $K_V=\det V^*$.*

(c) *The so called logarithmic case corresponds to all multiplicities $\rho_j=\infty$ being taken infinite, so that $\Delta=\sum\Delta_j=\lceil\Delta\rceil$.*

In case $V=T_X$, we recover the concept of orbifold introduced in [Cam04], except possibly for the fact that we allow here $\rho_j\in\mathbb{R}$, $\rho_j>1$ (even though the case $\rho_j\in\mathbb{N}^*$ is of greater interest). It would certainly be interesting to investigate the case when (X,V,Δ)

is singular, by allowing singularities in V and tangencies between V and Δ, and to study whether the results discussed in this paper can be extended in some way, e.g. by introducing suitable multiplier ideal sheaves taking care of singularities, as was done in [Dem15] for the study of directed varieties (X, V). For the sake of technical simplicity, we will refrain to do so here, and will therefore leave for future work the study of singular directed orbifolds.

0.3. Definition. *Let (X, V, Δ) be a singular directed orbifold. We say that $f : \mathbb{C} \to X$ is an orbifold entire curve if f is a non constant holomorphic map such that:*

(a) *f is tangent to V (i.e. $f'(t) \in V_{f(t)}$ at every point, or equivalently $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ is a morphism of directed varieties;*

(b) *$f(\mathbb{C})$ is not identically contained in $|\Delta|$;*

(c) *at every point $t_0 \in \mathbb{C}$ such that $f(t_0) \in \Delta_j$, f meets Δ_j with ramification number $\geqslant \rho_j$, i.e., if $\Delta_j = \{z_j = 0\}$ near $f(t_0)$, then $z_j \circ f(t)$ vanishes with multiplicity $\geqslant \rho_j$ at t_0.*

In the case of a logarithmic component Δ_j ($\rho_j = \infty$), condition (c) *is to be replaced by the assumption*

(c$'$) *$f(\mathbb{C})$ does not meet Δ_j.*

One can now consider a category of directed orbifolds as follows.

0.4. Definition. *Consider directed orbifolds (X, V, Δ), (Y, W, Λ) with*

$$\Delta = \sum \Big(1 - \frac{1}{\rho_{\Delta,i}}\Big)\Delta_i, \qquad \Lambda = \sum \Big(1 - \frac{1}{\rho_{\Lambda,j}}\Big)\Lambda_j.$$

A morphism $\Psi : (X, V, \Delta) \to (Y, W, \Lambda)$ is a morphism $\Psi : (X, V) \to (Y, W)$ of directed varieties satisfying the additional following properties (a,b,c)*.*

(a) *for every component Λ_j, $\Psi^{-1}(\Lambda_j)$ consists of a union of components Δ_i, $i \in I(j)$, eventually after adding a number of extra components Δ_i with $\rho_{\Delta,i} = 1$;*

(b) *in case $\rho_{\Lambda,j} < \infty$, for every $i \in I(j)$ and $z \in \Delta_i$, the derivatives $D^\alpha \Psi(z)$ of Ψ at z, computed in suitable local coordinates on X and Y, vanish for all multi-indices $\alpha \in \mathbb{N}^n$ with $0 < |\alpha| < \rho_{\Lambda,j}/\rho_{\Delta,i}$;*

(c) *if Λ_j is a logarithmic component ($\rho_{\Lambda,j} = \infty$), then $\Phi^{-1}(\Lambda_j) = \bigcup_{i \in I(j)} \Delta_i$ where the $(\Delta_i)_{i \in I(j)}$ consist of logarithmic components ($\rho_{\Delta,i} = \infty$).*

It is easy to check that the composite of directed orbifold morphisms is actually a directed orbifold morphism, and that the composition of an orbifold entire curve $f : \mathbb{C} \to (X, V, \Delta)$ with a directed orbifold morphism $\Psi : (X, V, \Delta) \to (Y, W, \Lambda)$ produces an orbifold entire curve $\Psi \circ f : \mathbb{C} \to (Y, W, \Lambda)$. One of our main goals is to investigate the following generalized Green-Griffiths conjecture

0.5. Conjecture. *Let (X, V, Δ) be a non singular directed orbifold of generated type, in the sense that the canonical divisor $K_V + \Delta$ is big. Then then should exist an algebraic subvariety $Y \subsetneq X$ containing all orbifold entire curves $f : \mathbb{C} \to (X, V, \Delta)$.*

As in the absolute case ($V = T_X$, $\Delta = 0$), the idea is to show, at least as a first step towards the conjecture, that orbifold entire curves must satisfy suitable algebraic differential equations. In section 1, we introduce graded algebras

$$\bigoplus_{m \in \mathbb{N}} E_{k,m} V^* \langle \Delta \rangle \tag{0.6}$$

of sheaves of "orbifold jet differentials". These sheaves correspond to algebraic differential operators $P(f;f',f'',\ldots,f^{(k)})$ acting on germs of k-jets of curves that are tangent to V and satisfy the ramification conditions prescribed by Δ. The strategy relies on the following standard vanishing theorem.

0.7. Proposition. *Let (X,V,Δ) be a projective non singular directed orbifold, and A an ample divisor on X. Then, for every orbifold entire curve $f:\mathbb{C}\to(X,V,\Delta)$ and every global jet differential operator $P\in H^0(X,E_{k,m}V^*\langle\Delta\rangle\otimes\mathcal{O}_X(-A))$, we have $P(f;f',f'',\ldots,f^{(k)})=0$.*

The next step consists precisely of finding sufficient conditions that ensure the existence of many global sections $P\in H^0(X,E_{k,m}V^*\langle\Delta\rangle\otimes\mathcal{O}_X(-A))$. In this direction, among other more general results, we prove

0.8. Theorem. *Let (X,V,Δ) be a n-dimensional projective non singular directed orbifold. We assume that $\Delta=(1-\frac{1}{\rho_1})\Delta_1$ has ramification index $\rho_1\geqslant n+1$, with a single component $\Delta_1\in|d_1A|$ of degree d_1 with respect to a* very ample *divisor A on X. Then, for $\rho_1\geqslant n+1$, $\varepsilon\in\mathbb{Q}_{>0}$ small and*

$$n-1+d_1>n\,2^{2n-1}\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)^n\frac{\rho_1^n}{\binom{\rho_1-1}{n}},$$

there exist many (i.e. at least $c\,m^{n+n^2-1}$, $c>0$) orbifold jet differentials of order n in

$$H^0(X,E_{n,m}T_X^*\langle\Delta\rangle\otimes\mathcal{O}_X(-m\varepsilon A))$$

for $m\gg1$ sufficiently divisible.

1. Logarithmic and orbifold jet differentials

1.A. Directed varieties and associated jet differentials

Let (X,V) be a non singular directed variety. We set $n=\dim_{\mathbb{C}}X$, $r=\operatorname{rank}_{\mathbb{C}}V$, and following the exposition of [Dem97], we denote by $\pi_k:J^kV\to X$ the bundle of k-jets of holomorphic curves tangent to V at each point. The canonical bundle of V is defined to be

$$K_V=\det(V^*)=\Lambda^rV^*. \tag{1.1}$$

If $f:(\mathbb{C},0)\to X$, $t\mapsto f(t)$ is a germ of holomorphic curve tangent to V, we denote by $f_{[k]}(0)$ its k-jet at $t=0$. For $x_0\in X$ given, we take a coordinate system $(z_1,\ldots,z_n)$ centered at x_0 such that $V_{x_0}=\mathrm{Span}(\frac{\partial}{\partial z_\mu})_{1\leqslant\mu\leqslant r}$. Then there exists a neighborhood U of x_0 such that $V_{|U}$ admits a holomorphic frame $(e_\mu)_{1\leqslant\mu\leqslant r}$ of the form

$$e_\mu(z)=\frac{\partial}{\partial z_\mu}+\sum_{r+1\leqslant\lambda\leqslant n}a_{\lambda\mu}(z)\frac{\partial}{\partial z_\lambda},\quad 1\leqslant\mu\leqslant r, \tag{1.2}$$

with $a_{\lambda\mu}(0)=0$. Germs of curves $f:(\mathbb{C},0)\to X$ tangent to $V_{|U}$ are obtained by integrating the system of ordinary differential equations

$$f'_\lambda(t)=\sum_{1\leqslant\mu\leqslant r}a_{\lambda\mu}(f(t))\,f'_\mu(t),\quad r+1\leqslant\lambda\leqslant n, \tag{1.3}$$

when we write $f = (f_1, \ldots, f_n)$ in coordinates. Therefore any such germ of curve f is uniquely determined by its initial point $z = f(0)$ and its projection $\tilde f = (f_1, \ldots, f_r)$ on the first r coordinates. By definition, every k-jet $f_{[k]} \in J^kV_z = \pi_k^{-1}(z)$ is uniquely determined by its initial point $f(0) = z \simeq (z_1, \ldots, z_n)$ and the Taylor expansion of order k

$$\tilde f(t) - \tilde f(0) = t\xi_1 + \frac{1}{2!}t^2\xi_2 + \cdots + \frac{1}{k!}t^k\xi_k + O(t^{k+1}), \quad t \in D(0,\varepsilon),\ \xi_s \in \mathbb{C}^r,\ 1 \leqslant s \leqslant k. \tag{1.4}$$

Alternatively, we can pick an arbitrary local holomorphic connection ∇ on $V_{|U}$ and represent the k-jet $f_{[k]}(0)$ by $(\xi_1, \ldots, \xi_k)$, where $\xi_s = \nabla^s f(0) \in V_z$ is defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local biholomorphic trivialization of $J^kV_{|U}$ of the form

$$J_kV_{|U} \to V_{|U}^{\oplus k}, \qquad f_{[k]}(0) \mapsto (\xi_1, \ldots, \xi_k) = (\nabla f(0), \ldots, \nabla f^k(0))\,; \tag{1.5}$$

the particular choice of the "trivial connection" ∇_0 of $V_{|U}$ that turns $(e_\mu)_{1\leqslant\mu\leqslant r}$ into a parallel frame precisely yields the components $\xi_s \in V_{|U} \simeq \mathbb{C}^r$ appearing in (1.4). We could of course also use a C^∞ connection $\nabla = \nabla_0 + \Gamma$ where $\Gamma \in C^\infty(U, T_X^* \otimes \mathrm{Hom}(V,V))$, and in this case, the corresponding trivialization (1.5) is just a C^∞ diffeomorphism; the advantage, though, is that we can always produce such a global C^∞ connection ∇ by using a partition of unity on X, and then (1.5) becomes a global C^∞ diffeomorphism. Now, there is a global holomorphic $\mathbb{C}^*$ action on J^kV given at the level of germs by $f \mapsto \alpha\cdot f$ where $\alpha\cdot f(t) := f(\alpha t)$, $\alpha \in \mathbb{C}^*$. With respect to our trivializations (1.5), this is the weighted $\mathbb{C}^*$ action defined by

$$\alpha \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\alpha\xi_1, \alpha^2\xi_2, \ldots, \alpha^k\xi_k), \quad \xi_s \in V. \tag{1.6}$$

We see that $J^kV \to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and that the projectivized k-jet bundle

$$X_k(V) := (J^kV \smallsetminus \{0\})/\mathbb{C}^*, \qquad \pi_k : X_k(V) \to X \tag{1.7}$$

is a $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]})$ weighted projective bundle over X, of total dimension

$$\dim X_k(V) = n + kr - 1. \tag{1.8}$$

1.9. Definition. *We define $\mathcal{O}_X(E_{k,m}V^*)$ to be the sheaf over X of holomorphic functions $P(z\,; \xi_1, \ldots, \xi_k)$ on J^kV that are weighted polynomials of degree m in $(\xi_1, \ldots, \xi_m)$.*

In coordinates and in multi-index notation, we can write

$$P(z\,; \xi_1, \ldots, \xi_k) = \sum_{\substack{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^r\\ |\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}} a_{\alpha_1\ldots\alpha_k}(z)\,\xi_1^{\alpha_1}\ldots\xi_k^{\alpha_k}$$

where the $a_{\alpha_1\ldots\alpha_k}(z)$ are holomorphic functions in $z = (z_1, \ldots, z_n)$ and $\xi_s^{\alpha_s}$ actually means

$$\xi_s^{\alpha_s} = \xi_{s,1}^{\alpha_{s,1}} \ldots \xi_{s,r}^{\alpha_{s,r}} \quad \text{for}\ \ \xi_s = (\xi_{s,1}, \ldots, \xi_{s,r}) \in \mathbb{C}^r,\ \ \alpha_s = (\alpha_{s,1}, \ldots, \alpha_{s,r}) \in \mathbb{N}^r,$$

and $|\alpha_s| = \sum_{j=1}^r \alpha_{s,j}$. Such sections can be interpreted as algebraic differential operators acting on holomorphic curves $f : D(0,R) \to X$ tangent to V, by putting $P(f) := u$ where

$$u(t) = \sum_{\substack{\alpha_1,\ldots,\alpha_k \in \mathbb{N}^r\\ |\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m}} a_{\alpha_1\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} \ldots f^{(k)}(t)^{\alpha_k}. \tag{1.10}$$

Here $f^{(s)}(t)^{\alpha_s}$ is actually to be expanded as

$$f^{(s)}(t)^{\alpha_s} = f_1^{(s)}(t)^{\alpha_{s,1}} \ldots f_r^{(s)}(t)^{\alpha_{s,r}}$$

with respect to the components $f_j^{(s)}$ defined in (1.4). We also set $u = P(f\,;\, f', f'', \ldots, f^{(k)})$ when we want to make more explicit the dependence of the expression in terms of the derivatives of f. We thus get a sheaf of graded algebras

$$\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*). \tag{1.11}$$

Locally in coordinates, the algebra is isomorphic to the weighted polynomial ring

$$\mathcal{O}_X\big[f_j^{(s)}\big]_{1\leqslant j\leqslant r,\,1\leqslant s\leqslant k}, \quad \deg f_j^{(s)} = s \tag{1.12}$$

over $\mathcal{O}_X$. An immediate consequence of these definitions is :

1.13. proposition. *The projectivized bundle $\pi_k : X_k(V) \to X$ can be identified with*

$$\text{(a)} \qquad \mathrm{Proj}\left(\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*)\right) \to X,$$

and, if $\mathcal{O}_{X_k(V)}(m)$ denote the associated tautological sheaves, we have the direct image formula

$$\text{(b)} \qquad (\pi_k)_*\mathcal{O}_{X_k(V)}(m) = \mathcal{O}_X(E_{k,m}V^*).$$

1.14. Remark. These objects where denoted X_k^{GG} and $E_{k,m}^{\mathrm{GG}}V^*$ in our previous paper [Dem97], as a reference to the work of Green-Griffiths [GG79], but we will avoid here the superscript GG to simplify the notation.

Thanks to the Faà di Bruno formula, a change of coordinates $w = \psi(z)$ on X leads to a transformation rule

$$(\psi\circ f)^{(k)} = \psi'\circ f\cdot f^{(k)} + Q_\psi(f', \ldots, f^{(k-1)})$$

where Q_ψ is a polynomial of weighted degree k in the lower order derivatives. This shows that the transformation rule of the top derivative is linear and, as a consequence, the partial degree in $f^{(k)}$ of the polynomial $P(f\,;\, f', \ldots, f^{k)})$ is intrinsically defined. By taking the corresponding filtration and factorizing the monomials $(f^{(k)})^{\alpha_k}$ with polynomials in $f', f'', \ldots, f^{(k-1)}$, we get graded pieces

$$G^\bullet(E_{k,m}V^*) = \bigoplus_{\ell_k\in\mathbb{N}} E_{k-1,m-k\ell_k}V^*\otimes S^{\ell_k}V^*.$$

By considering successively the partial degrees with respect to $f^{(k)}$, $f^{(k-1)}$, $\ldots$, f'', f' and merging inductively the resulting filtrations, we get a multi-filtration such that

$$G^\bullet(E_{k,m}V^*) = \bigoplus_{\ell_1,\ldots,\ell_k\in\mathbb{N},\,\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}V^*\otimes S^{\ell_2}V^*\otimes\cdots\otimes S^{\ell_k}V^*. \tag{1.15}$$

1.B. Logarithmic directed varieties

We now turn ourselves to the logarithmic case. Let (X, V, Δ) be a non singular logarithmic variety, where $\Delta = \sum \Delta_j$ is a simple normal crossing divisor. Fix a point $x_0 \in X$. By the assumption that Δ is transverse to V, we can then select holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x_0 such that $V_{x_0} = \mathrm{Span}(\frac{\partial}{\partial z_j})_{1 \leqslant j \leqslant r}$ and $\Delta_j = \{z_j = 0\}$, $1 \leqslant j \leqslant p$, are the components of Δ that contain x_0 (here $p \leqslant r$ and we can have $p = 0$ if $x_0 \notin |\Delta|$). What we want is to introduce an algebra of differential operators, defined locally near x_0 as the weighted polynomial ring

$$\mathcal{O}_X\big[(\log f_j)^{(s)}_{1 \leqslant j \leqslant p}, (f_j^{(s)})_{p+1 \leqslant j \leqslant r}\big]_{1 \leqslant s \leqslant k}, \quad \deg f_j^{(s)} = \deg(\log f_j)^{(s)} = s, \tag{1.16}$$

or equivalently

$$\mathcal{O}_X\big[(f_j^{-1} f_j^{(s)})_{1 \leqslant j \leqslant p}, (f_j^{(s)})_{p+1 \leqslant j \leqslant r}\big]_{1 \leqslant s \leqslant k}, \quad \deg f_j^{(s)} = s, \ \deg f_j^{-1} = 0. \tag{1.16$'$}$$

For this we notice that

$$(\log f_1)'' = (f_1^{-1} f_1')' = f_1^{-1} f_1'' - (f_1^{-1} f_1')^2,$$

$$(\log f_1)''' = f_1^{-1} f_1''' - 3(f_1^{-1} f_1')(f_1^{-1} f_1'') + 2(f_1^{-1} f_1')^3, \ldots .$$

A similar argument easily shows that the above graded rings do not depend on the particular choice of coordinates made, as soon as they satisty $\Delta_j = \{z_j = 0\}$.

Now (as is well known in the absolute case $V = T_X$), we have a corresponding logarithmic directed structure $V\langle\Delta\rangle$ and its dual $V^*\langle\Delta\rangle$. If the coordinates $(z_1, \ldots, z_n)$ are chosen so that $V_{x_0} = \{dz_{r+1} = \ldots = dz_n = 0\}$, then the fiber $V\langle\Delta\rangle_{x_0}$ is spanned by the derivations

$$z_1 \frac{\partial}{\partial z_1}, \ldots, z_p \frac{\partial}{\partial z_p}, \ \frac{\partial}{\partial z_{p+1}}, \ldots, \frac{\partial}{\partial z_r}.$$

The dual sheaf $\mathcal{O}_X(V^*\langle\Delta\rangle)$ is the locally free sheaf generated by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, \ dz_{p+1}, \ldots, dz_r$$

[where the 1-forms are considered in restriction to $\mathcal{O}_X(V\langle\Delta\rangle) \subset \mathcal{O}_X(V)$]. It follows from this that $\mathcal{O}_X(V\langle\Delta\rangle)$ and $\mathcal{O}_X(V^*\langle\Delta\rangle)$ are locally free sheaves of rank r. By taking $\det(V^*\langle\Delta\rangle)$ and using the above generators, we find

$$\det(V^*\langle\Delta\rangle) = \det(V^*) \otimes \mathcal{O}_X(\Delta) = K_V + \Delta \tag{1.17}$$

in additive notation. Quite similarly to 1.13 and 1.15, we have :

1.18. Proposition. *Let $\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle\Delta\rangle)$ be the graded algebra defined in coordinates by* (1.16) *or* (1.16$'$). *We define the logarithmic k-jet bundle to be*

(a) $$X_k(V\langle\Delta\rangle) := \mathrm{Proj}\left(\bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(E_{k,m} V^*\langle\Delta\rangle)\right) \to X.$$

If $\mathcal{O}_{X_k(V\langle\Delta\rangle)}(m)$ denote the associated tautological sheaves, we get the direct image formula

(b) $$(\pi_k)_* \mathcal{O}_{X_k(V\langle\Delta\rangle)}(m) = \mathcal{O}_X(E_{k,m} V^*\langle\Delta\rangle).$$

Moreover, the mult-filtration by the partial degrees in the derivatives $f_j^{(s)}$ has graded pieces

(c) $$G^\bullet\big(E_{k,m} V^*\langle\Delta\rangle\big) = \bigoplus_{\ell_1, \ldots, \ell_k \in \mathbb{N}, \, \ell_1 + 2\ell_2 + \cdots + k\ell_k = m} S^{\ell_1} V^*\langle\Delta\rangle \otimes S^{\ell_2} V^*\langle\Delta\rangle \otimes \cdots \otimes S^{\ell_k} V^*\langle\Delta\rangle.$$

1.C. Orbifold directed varieties

We finally consider a non singular directed orbifold (X, V, Δ), where $\Delta = \sum(1 - \frac{1}{\rho_j})\Delta_j$ is a simple normal crossing divisor transverse to V. Let $\lceil\Delta\rceil = \sum \Delta_j$ be the corresponding reduced divisor. By § 1.B, we have associated logarithmic sheaves $\mathcal{O}_X(E_{k,m}V^*\langle\lceil\Delta\rceil\rangle)$. We want to introduce a graded subalgebra

$$\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle\Delta\rangle) \quad\subset\quad \bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle\lceil\Delta\rceil\rangle) \tag{1.19}$$

in such a way that for every germ $P \in \mathcal{O}_X(E_{k,m}V^*\langle\Delta\rangle)$ and every germ of orbifold curve $f : (\mathbb{C}, 0) \to (X, V, \Delta)$ the germ of meromorphic function $P(f)(t)$ is bounded at $t = 0$ (hence holomorphic). Assume that $\Delta_1 = \{z_1 = 0\}$ and that f has multiplicity $q \geqslant \rho_1 > 1$ along Δ_1 at $t = 0$. Then $f_1^{(s)}$ still vanishes at order $\geqslant (q - s)_+$, thus $(f_1)^{-\beta} f_1^{(s)}$ is bounded as soon as $\beta q \leqslant (q - s)_+$, i.e. $\beta \leqslant (1 - \frac{s}{q})_+$. Thus, it is sufficient to ask that $\beta \leqslant (1 - \frac{s}{\rho_1})_+$. At a point $x_0 \in |\Delta_1| \cap \ldots \cap |\Delta_p|$, a sufficient condition for a monomial of the form

$$f_1^{-\beta_1} \ldots f_p^{-\beta_p} \prod_{s=1}^{k}\prod_{j=1}^{r} (f_j^{(s)})^{\alpha_{s,j}}, \quad \alpha_s = (\alpha_{s,j}) \in \mathbb{N}^r,\ \beta_1, \ldots, \beta_p \in \mathbb{N} \tag{1.20}$$

to be bounded is to require that the multiplicities of poles satisfy

$$\beta_j \leqslant \sum_{s=1}^{k} \alpha_{s,j}\Big(1 - \frac{s}{\rho_j}\Big)_+, \quad 1 \leqslant j \leqslant p. \tag{1.20′}$$

1.21. Definition. *The subalgebra $\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(E_{k,m}V^*\langle\Delta\rangle)$ is taken to be the graded ring generated by monomials* (1.20) *of degree $\sum s|\alpha_s| = m$, satisfying the pole multiplicity conditions* (1.20′)*. These conditions do not depend on the choice of coordinates, hence we get a globally and intrinsically defined sheaf of algebras on X.*

Proof. We only have to prove the last assertion. Consider a change of variables $w = \psi(z)$ such that Δ_j can still be expressed as $\Delta_j = \{w_j = 0\}$. Then, for $j = 1, \ldots, p$, we can write $w_j = z_j u_j(z)$ with an invertible holomorphic factor u_j. We need to check that the monomials (1.20) computed with $g = \psi\circ f$ are holomorphic combinations of those associated with f. However, we have $g_j = f_j u_j(f)$, hence $g_j^{(s)} = \sum_{0\leqslant\ell\leqslant s} \binom{s}{\ell} f_j^{(\ell)}(u_j(f))^{(s-\ell)}$ by the Leibniz formula, and we see that

$$g_1^{-\beta_1} \ldots g_p^{-\beta_p} \prod_{s=1}^{k}\prod_{j=1}^{r} (g_j^{(s)})^{\alpha_{s,j}}$$

expands as a linear combination of monomials

$$f_1^{-\beta_1} \ldots f_p^{-\beta_p} \prod_{s=1}^{k}\prod_{j=1}^{r}\prod_{m=1}^{\alpha_{s,j}} f_j^{(\ell_{s,j,m})}, \quad \ell_{s,j,m} \leqslant s,$$

multiplied by holomorphic factors of the form

$$\prod_{j=1}^{p} u_j(f)^{-\beta_j} \times \prod_{s=1}^{k}\prod_{j=1}^{r}\prod_{m=1}^{\alpha_{s,j}} (u_j(f))^{(s-\ell_{j,s,m})}.$$

However, we have

$$\beta_j \leqslant \sum_{s=1}^{k} \alpha_{s,j}\Big(1-\frac{s}{\rho_j}\Big)_+ \leqslant \sum_{s=1}^{k}\sum_{m=1}^{\alpha_{s,j}} \Big(1-\frac{\ell_{s,j,m}}{\rho_j}\Big)_+,$$

so the f-monomials satisfy again the required multiplicity conditions for the poles $f_j^{-\beta_j}$. $\square$

The above conditions (1.20′) suggest to introduce a sequence of "differentiated" orbifold divisors

$$\Delta^{(s)} = \sum_j \Big(1-\frac{s}{\rho_j}\Big)_+ \Delta_j. \tag{1.22}$$

We say that $\Delta^{(s)}$ is the order s orbifold divisor associated to Δ; its ramification numbers are $\rho_j^{(s)} = \max(\rho_j/s, 1)$. By definition, the logarithmic components ($\rho_j = \infty$) of Δ remain logarithmic in $\Delta^{(s)}$, while all others eventually disappear when s is large.

Now, we introduce (in a purely formal way) a sheaf of rings $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z_j^{\bullet}]$ by adjoining all positive real powers of coordinates z_j such that $\Delta_j = \{z_j = 0\}$ is locally a component of Δ. Locally over X, this can be done by taking the universal cover Y of a punctured polydisk

$$D^*(0,r) := \prod_{1\leqslant j\leqslant p} D^*(0,r_j) \times \prod_{p+1\leqslant j\leqslant n} D(0,r_j) \quad\subset\quad D(0,r) := \prod_{1\leqslant j\leqslant n} D(0,r_j)$$

in the local coordinates z_j on X. If $\gamma : Y \to D^*(0,r) \hookrightarrow X$ is the covering map and $U \subset D(0,r)$ is an open subset, we can then consider the functions of $\widetilde{\mathcal{O}}_X(U)$ as being defined on $\gamma^{-1}(U \cap D^*(0,r))$. In case X is projective, one can even achieve such a construction globally by taking Y to be the universal cover of a complement $X \smallsetminus (|\Delta| \cup |\Delta'|)$, where $\Delta' = \sum \Delta'_\ell$ is a sum of very ample divisors such that $\Delta + \Delta'$ has simple normal crossings, and $\Delta_j \sim \Delta'_{\ell_1(j,m)} - \Delta'_{\ell_2(j,m)}$ with $\bigcup_m X \smallsetminus (\Delta'_{\ell_1(j,m)} \cup \Delta'_{\ell_2(j,m)}) = X$ for each j.

In this setting, the subalgebra $\bigoplus_m \mathcal{O}_X(E_{k,m}V^*\langle\Delta\rangle)$ still has a multi-filtration induced by the one on $\bigoplus_m \mathcal{O}_X(E_{k,m}V^*\langle\lceil\Delta\rceil\rangle)$, and by extending the structure sheaf $\mathcal{O}_X$ into $\widetilde{\mathcal{O}}_X$, we get an inclusion

$$\widetilde{\mathcal{O}}_X(G^{\bullet}E_{k,m}V^*\langle\Delta\rangle) \subset \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} \widetilde{\mathcal{O}}_X(S^{\ell_1}V^*\langle\Delta^{(1)}\rangle) \otimes\cdots\otimes \widetilde{\mathcal{O}}_X(S^{\ell_k}V^*\langle\Delta^{(k)}\rangle), \tag{1.23}$$

$\widetilde{\mathcal{O}}_X(V^*\langle\Delta^{(s)}\rangle)$ is the "s-th orbifold (dual) directed structure", generated by the order s differentials

$$z_j^{-(1-s/\rho_j)_+} d^{(s)} z_j,\quad 1\leqslant j\leqslant p,\qquad d^{(s)}z_j,\quad p+1\leqslant j\leqslant r. \tag{1.24}$$

By construction, we have

$$\det(\widetilde{\mathcal{O}}_X(V^*\langle\Delta^{(s)}\rangle)) = \widetilde{\mathcal{O}}_X(K_V + \Delta^{(s)}). \tag{1.25}$$

1.26. Remark. When $\rho_j = a_j/b_j \in \mathbb{Q}_+$, one can find a finite ramified Galois cover $g : Y \to X$ from a smooth projective variety Y onto X, such that the compositions $(z_j \circ g)^{1/a_j}$

become single-valued functions w_j on Y. In this way, the pull-back $\mathcal{O}_Y(g^*V^*\langle\Delta^{(s)}\rangle)$ is actually a locally free $\mathcal{O}_Y$-module. On can also introduce a sheaf of algebras which we will denote by $\bigoplus\mathcal{O}_Y(E_{k,m}\widetilde{V}^*\langle\Delta\rangle)$, generated, according to the notation of §1.B, by the elements $g^*(z_j^{(1-s/\rho_j)_+}d^{(s)}z_j)$, $1\leqslant j\leqslant p$, and $g^*(d^{(s)}z_j)$, $p+1\leqslant j\leqslant r$. Then there is indeed a multifiltration on $\mathcal{O}_Y(E_{k,m}\widetilde{V}^*\langle\Delta\rangle)$ whose graded pieces are

$$(1.27)\quad \mathcal{O}_Y(G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle)=\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m}\mathcal{O}_Y(S^{\ell_1}\widetilde{V}^*\langle\Delta^{(1)}\rangle)\otimes\cdots\otimes\mathcal{O}_Y(S^{\ell_k}\widetilde{V}^*\langle\Delta^{(k)}\rangle).$$

However, we will adopt here an alternative viewpoint that avoids the introduction of finite or infinite covers, and suits better our approach. Using the general philosophy of [Laz??], the idea is to consider a "jet orbifold directed structure" $X_k(V\langle\Delta\rangle)$ as the underlying "jet logarithmic directed structure" $X_k(V\langle\lceil\Delta\rceil\rangle)$, equipped additionally with a submultiplicative sequence of ideal sheaves $\mathcal{J}_m\langle\Delta\rangle\subset\mathcal{O}_{X_k(V\langle\lceil\Delta\rceil\rangle)}$. These are precisely defined as the base loci ideals of the local sections defined by (1.20) and (1.20′), when these are seen as sections of the logarithmic tautological sheaves $\mathcal{O}_{X_k(V\langle\lceil\Delta\rceil\rangle)}(m)$. The corresponding analytic viewpoint is to consider ad hoc singular hermitian metrics on $\mathcal{O}_{X_k(V\langle\lceil\Delta\rceil\rangle)}(1)$ whose singularities are asymptotically described by the limit of the formal m-th root of $\mathcal{J}_m\langle\Delta\rangle$, see §3.B. It then becomes possible to deal without trouble with real coefficients $\rho_j\in\,]1,\infty]$, and since we no longer have to worry about the existence of Galois covers, the projectivity assumption on X can be dropped as well.

2. Preliminaries on holomorphic Morse inequalities

2.A. Basic results

We first recall the basic results concerning holomorphic Morse inequalities for smooth hermitian line bundles, first proved in [Dem85].

2.1. Theorem. *Let X be a compact complex manifolds, $E\to X$ a holomorphic vector bundle of rank r, and (L,h) a hermitian line bundle. We denote by $\Theta_{L,h}=\frac{i}{2\pi}D_h^2=-\frac{i}{2\pi}\partial\overline{\partial}\log h$ the curvature form of (L,h) and introduce the open subsets of X*

$$(*)\qquad\begin{cases}X(L,h,q)=\big\{x\in X\,;\ \Theta_{L,h}(x)\ \textit{has signature}\ (n-q,q)\big\},\\ X(L,h,S)=\displaystyle\bigcup_{q\in S}X(L,h,q),\quad \forall S\subset\{0,1,\ldots,n\}.\end{cases}$$

Then, for all $q=0,1,\ldots,n$, the dimensions $h^q(X,E\otimes L^m)$ of cohomology groups of the tensor powers $E\otimes L^m$ satisfy the following "Strong Morse inequalities" as $m\to+\infty$:

$$\mathrm{SM}(q):\qquad \sum_{0\leqslant j\leqslant q}(-1)^{q-j}h^j(X,E\otimes L^m)\leqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant q)}(-1)^q\Theta_{L,h}^n+o(m^n),$$

with equality $\chi(X,E\otimes L^m)=r\frac{m^n}{n!}\int_X\Theta_{L,h}^n+o(m^n)$ for the Euler characteristic $(q=n)$.

As a consequence, one gets upper and lower bounds for all cohomology groups, and especially a very useful criterion for the existence of sections of large multiples of L.

2.2. Corollary. *Under the above hypotheses, we have*

(a) *Upper bound for h^q (Weak Morse inequalities)* :

$$h^q(X, E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(L,h,q)} (-1)^q\Theta_{L,h}^n + o(m^n)\ .$$

(b) *Lower bound for h^0* :

$$h^0(X, E\otimes L^m) \geqslant h^0 - h^1 \geqslant r\frac{m^n}{n!}\int_{X(L,h,\leqslant 1)} \Theta_{L,h}^n - o(m^n)\ .$$

Especially L is big as soon as $\int_{X(L,h,\leqslant 1)}\Theta_{L,h}^n > 0$ for some hermitian metric h on L.

(c) *Lower bound for h^q* :

$$h^q(X, E\otimes L^m) \geqslant h^q - h^{q-1} - h^{q+1} \geqslant r\frac{m^n}{n!}\int_{X(L,h,\{q,q\pm 1\})} (-1)^q\Theta_{L,h}^n + o(m^n)\ .$$

Proof. (a) is obtained by taking $\mathrm{SM}(q) + \mathrm{SM}(q-1)$, (b) is equivalent to $-\mathrm{SM}(1)$ and (c) is equivalent to $-(\mathrm{SM}(q+1) + \mathrm{SM}(q-2))$. □

The following simple lemma is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

2.3. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\mathbb{1}_{\eta,\leqslant q}\ \eta^n \leqslant \sum_{0\leqslant j\leqslant q} (-1)^{q-j}\binom{n}{j}\alpha^{n-j}\wedge\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\ \eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha > 0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta = \alpha - \beta$ are then given by

$$1-\lambda_1 \leqslant \ldots \leqslant 1-\lambda_q \leqslant 1-\lambda_{q+1} \leqslant \ldots \leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1} < 1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n = 0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q} (-1)^{q-j}\sigma_n^j(\lambda) - \mathbb{1}_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j) \geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda) = \sigma_{n-1}^j(\lambda) + \sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). $\square$

2.4. Corollary. *Assume that $\eta = \Theta_{L,h}$ can be expressed as a difference $\eta = \alpha - \beta$ of smooth $(1,1)$-forms $\alpha, \beta \geqslant 0$. Then we have*

$$\mathrm{SM}(q): \quad \sum_{0\leqslant j\leqslant q} (-1)^{q-j} h^j(X, E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_X \sum_{0\leqslant j\leqslant q} (-1)^{q-j}\binom{n}{j}\alpha^{n-j}\wedge\beta^j + o(m^n),$$

and in particular, for $q = 1$,

$$h^0(X, E\otimes L^m) \geqslant h^0 - h^1 \geqslant r\frac{m^n}{n!}\int_X \alpha^n - n\alpha^{n-1}\wedge\beta + o(m^n).$$

2.5. Remark. These estimates are consequences of Theorem 2.1 and Lemma 2.3, by taking the integral over X. The estimate for h^0 was stated and studied by Trapani [Tra93]. In the special case $\alpha = \Theta_{A,h_A} > 0$, $\beta = \Theta_{B,h_B} > 0$ where A, B are ample line bundles, a direct proof can be obtained by purely algebraic means, via the Riemann-Roch formula. However, we will later have to use Corollary 2.4 in case α and β are not closed, a situation in which no algebraic proof seems to exist.

2.B. Singular holomorphic Morse inequalities

The case of singular hermitian metrics has been considered in Bonavero's PhD thesis [Bon93] and will be important for us. We assume that L is equipped with a singular hermitian metric $h = h_\infty e^{-\varphi}$ with analytic singularities, i.e., h_∞ is a smooth metric, and on an neighborhood $V \ni x_0$ of an arbitrary point $x_0 \in X$, the weight φ is of the form

$$\varphi(z) = c\log\sum_{1\leqslant j\leqslant N} |g_j|^2 + u(z) \tag{2.6}$$

where $g_j \in \mathcal{O}_X(V)$ and $u \in C^\infty(V)$. We then have $\Theta_{L,h} = \alpha + \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\varphi$ where $\alpha = \Theta_{L,h_\infty}$ is a smooth closed $(1,1)$-form on X. In this situation, the multiplier ideal sheaves

$$\mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \Big\{f \in \mathcal{O}_{X,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)}d\lambda(z) < +\infty\Big\} \tag{2.7}$$

play an important role. We define the singularity set of h by $\mathrm{Sing}(h) = \mathrm{Sing}(\varphi) = \varphi^{-1}(-\infty)$ which, by definition, is an analytic subset of X. The associated q-index sets are

$$X(L,h,q) = \big\{x \in X \smallsetminus \mathrm{Sing}(h)\,;\ \Theta_{L,h}(x) \text{ has signature } (n-q,q)\big\}. \tag{2.8}$$

We can then state:

2.9. Theorem ([Bon93]). *Morse inequalities still hold in the context of singular hermitian metric with analytic singularities, provided the cohomology groups under consideration are twisted by the appropriate multiplier ideal sheaves, i.e. replaced by $H^q(X, E\otimes L^m\otimes\mathcal{I}(h^m))$.*

2.10. Remark. The assumption (2.6) guarantees that the measure $\mathbb{1}_{X\smallsetminus\mathrm{Sing}(h)}(\Theta_{L,h})^n$ is locally integrable on X, as is easily seen by using the Hironaka desingularization theorem

and by taking a log resolution $\mu : \widetilde{X} \to X$ such that $\mu^*(g_j) = (\gamma) \subset \mathcal{O}_{\widetilde{X}}$ becomes a principal ideal associated with a simple normal crossing divisor $E = \mathrm{div}(\gamma)$. Then $\mu^*\Theta_{L,h} = c[E] + \beta$ where β is a smooth closed $(1,1)$-form on $\widetilde{X}$, hence

$$\mu^*(\mathbb{1}_{X\smallsetminus \mathrm{Sing}(h)}\Theta^n_{L,h}) = \beta^n \quad\Rightarrow\quad \int_{X\smallsetminus \mathrm{Sing}(h)} \Theta^n_{L,h} = \int_{\widetilde{X}} \beta^n.$$

It should be observed that the multiplier ideal sheaves $\mathcal{I}(h^m)$ and the integral $\int_{X\smallsetminus \mathrm{Sing}(h)} \Theta^n_{L,h}$ only depend on the equivalence class of singularities of h: if we have two metrics with analytic singularities $h_j = h_\infty e^{-\varphi_j}$, $j = 1,2$, such that $\psi = \varphi_2 - \varphi_1$ is bounded, then, with the above notation, we have $\mu^*\Theta_{L,h_j} = c[E] + \beta_j$ and $\beta_2 = \beta_1 + \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\psi$, therefore $\int_{\widetilde{X}} \beta_2^n = \int_{\widetilde{X}} \beta_1^n$ by Stokes theorem. By using Monge-Ampère operators in the sense of Bedford-Taylor [BT76], it is in fact enough to assume $u \in L^\infty_{\mathrm{loc}}(X)$ in (2.6), and $\psi \in L^\infty(X)$ here. In general, however, the Morse integrals $\int_{X(L,h_j,q)}(-1)^q\Theta^n_{L,h_j}$, $j = 1,2$, will differ.

2.C. Morse inequalities and semi-continuity

Let $\mathcal{X} \to S$ be a proper and flat morphism of reduced complex spaces, and let $(X_t)_{t\in S}$ be the fibers. Given a sheaf $\mathcal{E}$ over $\mathcal{X}$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules of rank r, inducing on the fibres a family of sheaves $(E_t \to X_t)_{t\in S}$, the following semicontinuity property holds ([CRAS]):

2.11. Proposition. *For every $q \geqslant 0$, the alternate sum*

$$t \mapsto h^q(X_t, E_t) - h^{q-1}(X_t, E_t) + ... + (-1)^q h^0(X_t, E_t)$$

is upper semicontinuous with respect to the (analytic) Zariski topology on S.

Now, if $\mathcal{L} \to \mathcal{X}$ is an invertible sheaf equipped with a smooth hermitian metric h, and if (h_t) are the fiberwise metrics on the family $(L_t \to X_t)_{t\in S}$, we get

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X_t, E_t\otimes L_t^{\otimes m}) \leqslant r\frac{m^n}{n!}\int_{X(L_0,h_0,\leqslant q)}(-1)^q\Theta^n_{L_0,h_0} + \delta(t)m^n, \tag{2.12}$$

where $\delta(t) \to 0$ as $t \to 0$. In fact, the proof of holomorphic Morse inequalities shows that the inequality holds uniformly on every relatively compact $S' \Subset S$, with

$$I(t) = \int_{X(L_t,h_t,\leqslant q)}(-1)^q\Theta^n_{L_t,h_t} = \int_X (-1)^q\mathbb{1}_{X(L_t,h_t,\leqslant q)}\Theta^n_{L_t,h_t}$$

in the right hand side, and $t \mapsto I(t)$ is clearly continuous with respect to the ordinary topology. In other words, the Morse integral computed on the central fibers provides uniform upper bounds for cohomology groups of $E_t \otimes L_t^{\otimes m}$ when t is close to 0 in ordinary topology (and also, as a consequence, for t in a complement $S \smallsetminus \bigcup S_m$ of at most countably many analytic strata $S_m \subsetneq S$).

2.13. Remark. Similar results would hold when h is a singular hermitian metric with analytic singularities on $\mathcal{L} \to \mathcal{X}$, under the restriction that the families of multiplier ideal sheaves $(\mathcal{I}(h_t^m))_{t\in S}$ "never jump".

2.D. Case of filtered bundles

Let $E \to X$ be a vector bundle over a variety, equipped with a filtration (or multi-filtration) $F^p(E)$, and let $G = \bigoplus G^p(E) \to X$ be the graded bundle associated to this filtration.

2.14. Lemma. *In the above setting, one has for every $q \geqslant 0$*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X,E) \leqslant \sum_{j=0}^{q}(-1)^{q-j}h^j(X,G).$$

Proof. One possible argument is to use the well known fact that there is a family of filtered bundles $(E_t \to X)_{t\in\mathbb{C}}$ (with the same graded pieces $G^p(E_t) = G^p(E)$), such that $E_t \simeq E$ for all $t \neq 0$ and $E_0 \simeq G$. The result is then an immediate consequence of the semi-continuity result 2.11. A more direct very elementary argument can be given as follows: by transitivity of inequalities, it is sufficient to prove the result for simple filtrations; then, by induction on the length of filtrations, it is sufficient to prove the result for exact sequences $0 \to S \to E \to Q \to 0$ of vector bundles on X. Consider the associated (truncated) long exact sequence in cohomology:

$$0 \to H^0(S) \to H^0(E) \to H^0(Q) \overset{\delta_1}{\to} \cdots$$
$$\overset{\delta_{q-1}}{\to} H^q(S) \to H^q(E) \to H^q(Q) \overset{\delta_q}{\to} \mathrm{Im}(\delta_q) \to 0.$$

By the rank theorem of linear algebra,

$$0 \leqslant \mathrm{rank}(\delta_q) = (-1)^q\sum_{j=0}^{q}(-1)^j(h^j(X,Q) - h^j(X,E) + h^j(X,S)).$$

The result follows, since here $h^j(X,G) = h^j(X,Q) + h^j(X,S)$. □

2.E. Rees deformation construction (after Cadorel)

In this short paragraph, we outline a nice algebraic interpretation by Benoît Cadorel of certain semi-continuity arguments for cohomology group dimensions that underline the analytic approach of [Dem11, Lemma 2.12 and Prop. 2.13] and [Dem12, Prop. 9.28] (we will anyway explain again its essential points in §3, since we have to deal here with a more general situation). Recall after [Cad17, Prop. 4.2, Prop. 4.5], that the Rees deformation construction allows one to construct natural deformations of Green-Griffiths jets spaces to weighted projectivized bundles.

Let (X,V,Δ) be a non singular directed orbifold, and let $g : Y \to (X,\Delta)$ be an adapted Galois cover, as briefly described in remark 1.26, see also [CDR18, §2.1] for more details. We then get a Green-Griffiths jet bundle of graded algebras $E_{k,\bullet}\widetilde{V}^\star\langle\Delta\rangle \to Y$ which admits a multifiltration of associated graded algebra

$$G^\bullet E_{k,\bullet}\widetilde{V}^*\langle\Delta\rangle = \bigoplus_{m\in\mathbb{N}}\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m} S^{\ell_1}\widetilde{V}^*\langle\Delta^{(1)}\rangle\otimes\cdots\otimes S^{\ell_k}\widetilde{V}^*\langle\Delta^{(k)}\rangle.$$

where the tilde means taking pull-backs by g^*. Applying the Proj functor, one gets a weighted projective bundle:

$$\mathbb{P}_{(1,\cdots,k)}\left(\widetilde{V}^*\langle\Delta^{(1)}\rangle\rangle\oplus\cdots\oplus\widetilde{V}^*\langle\Delta^{(k)}\rangle\right) = \mathrm{Proj}\left(G^\bullet E_{k,\bullet}\widetilde{V}^*\langle\Delta\rangle\right) \overset{\rho_k}{\to} Y,$$

Then, following mutadis mutandus the arguments of Cadorel, one constructs a family $Y \overset{p_k}{\leftarrow} \mathcal{Y}_k \to \mathbb{C}$ parametrized by $\mathbb{C}$, with a canonical line bundle $\mathcal{O}_{\mathcal{Y}_k}(1)$ such that:

- the central fiber $\mathcal{Y}_{k,0}$ is $\mathbb{P}_{(1,\cdots,k)}\left(\widetilde{V}^*\langle\Delta^{(1)}\rangle\oplus\cdots\oplus\widetilde{V}^*\langle\Delta^{(k)}\rangle\right)$ and the restriction of $\mathcal{O}_{\mathcal{Y}_k}(1)$ coincide with the canonical line bundle of this weighted projective bundle. Hence $(\pi_k)_*\mathcal{O}_{\mathcal{Y}_{k,0}}(m) = G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle$.
- the other fibers $\mathcal{Y}_{k,t}$ are isomorphic to the singular quotient $J^k(Y,\widetilde{V},\Delta)/\mathbb{C}^*$ for the natural $\mathbb{C}^*$-action by homotheties, where $J^k(Y,\widetilde{V},\Delta)$ is the affine algebraic bundle associated with the sheaf of algebras, and $(\pi_k)_*\mathcal{O}_{\mathcal{Y}_{k,t}}(m)\simeq E_{k,m}\widetilde{V}^*\langle\Delta\rangle$.

Applying the semicontinuity result of [Dem95], and working with holomorphic inequalities, we obtain a control about dimensions of cohomology spaces of $E_{k,m}\widetilde{V}^*\langle\Delta\rangle$ in terms of dimensions of cohomology spaces of the a priori simpler graded pieces $G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle$. This reduces the study of higher order jet differentials to sections of the tautological sheaves on the weighted projective space associated with a direct sum combination of symmetric differentials. In particular, we have

2.15. Lemma. *For every $q\in\mathbb{N}$*

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(Y,E_{k,m}\widetilde{V}^*\langle\Delta\rangle)\geqslant\sum_{j=0}^{q}(-1)^{q-j}h^j(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle).$$

Especially, for $q=1$, we get

$$\begin{aligned}h^0(Y,E_{k,m}\widetilde{V}^*\langle\Delta\rangle)&\geqslant h^0(Y,E_{k,m}\widetilde{V}^*\langle\Delta\rangle)-h^1(Y,E_{k,m}\widetilde{V}^*\langle\Delta\rangle)\\&\geqslant h^0(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle)-h^1(Y,G^\bullet E_{k,m}\widetilde{V}^*\langle\Delta\rangle).\end{aligned}$$

3. Construction of jet metrics and orbifold jet metrics

3.A. Jet metrics and curvature tensor of jet bundles

Let (X,V) be a non singular directed variety and h a hermitian metric on V. We assume that h is smooth at this point (but will later relax a little bit this assumption and allow certain singularities). Near any given point $z_0\in X$, we can choose local coordinates $z=(z_1,\ldots,z_n)$ centered at z_0 and a local holomorphic coordinate frame $(e_\lambda(z))_{1\leqslant\lambda\leqslant r}$ of V on an open set $U\ni z_0$, such that

$$\langle e_\lambda(z),e_\mu(z)\rangle_{h(z)}=\delta_{\lambda\mu}+\sum_{1\leqslant i,j\leqslant n,\,1\leqslant\lambda,\mu\leqslant r}c_{ij\lambda\mu}z_i\overline{z}_j+O(|z|^3)\tag{3.1}$$

for suitable complex coefficients $(c_{ij\lambda\mu})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{\mathrm{i}}{2\pi}\nabla^2_{V,h}$ of (V,h) at z_0 is given by

$$\Theta_{V,h}(z_0)=-\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,dz_i\wedge d\overline{z}_j\otimes e^*_\lambda\otimes e_\mu.\tag{3.2}$$

Therefore, $(c_{ij\lambda\mu})$ are the components of $-\Theta_{V,h}$. Up to taking the transposed tensor with respect to λ,μ, these coefficients are also the components of the curvature tensor

$\Theta_{V^*,h^*} = -{}^t\Theta_{V,h}$ of the dual bundle (V^*,h^*). By (1.5), the connection $\nabla = \nabla_h$ yields a C^∞ isomorphism $J_kV \to V^{\oplus k}$. Let us fix an integer $b \in \mathbb{N}^*$ that is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$, and positive numbers $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k > 0$. Following [Dem11], we define a global weighted Finsler metric on J^kV by putting for any k-jet $f \in J^kV_z$

$$\Psi_{h,b,\varepsilon}(f) := \Bigg(\sum_{1\leqslant s\leqslant k} \varepsilon_s^{2b}\|\nabla^s f(0)\|_{h(z)}^{2b/s}\Bigg)^{1/b}, \tag{3.3}$$

where $\|\ \ \|_{h(z)}$ is the hermitian metric h of V evaluated on the fiber V_z, $z = f(0)$. The function $\Psi_{h,b,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,b,\varepsilon}(\alpha\cdot f) = |\alpha|^2\,\Psi_{h,b,\varepsilon}(f) \tag{3.4}$$

with respect to the $\mathbb{C}^*$ action on J^kV, in other words, it induces a hermitian metric on the dual L_k^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k(V)}(1)$ over $X_k(V)$. The curvature of L_k is given by

$$\pi_k^*\Theta_{L_k,\Psi^*_{h,b,\varepsilon}} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log\Psi_{h,b,\varepsilon} \tag{3.5}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k(V)$ with the above metric. This might look a priori like an untractable problem, since the definition of $\Psi_{h,b,\varepsilon}$ is a rather complicated one, involving the hermitian metric in an intricate manner. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,b,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1} \to 0$ is in some sense uniquely defined, and "splits" according to the natural multifiltration on jet differentials (as already hinted in §2.E). This leads to a computable asymptotic formula, which is moreover simple enough to produce useful results.

3.6. Lemma. *Let us consider the global C^∞ bundle isomorphism $J^kV \to V^{\oplus k}$ associated with an arbitrary global C^∞ connection ∇ on $V \to X$, and let us introduce the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1,\xi_2,\ldots,\xi_k) = (\varepsilon_1^1\xi_1,\varepsilon_2^2\xi_2,\ldots,\varepsilon_k^k\xi_k) \quad \textit{on fibers } J^kV_z,\ z\in X.$$

Such a rescaling commutes with the $\mathbb{C}^*$-action. Moreover, if p is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$ *and the ratios $\varepsilon_s/\varepsilon_{s-1}$ tend to 0 for all $s = 2,\ldots,k$, the rescaled Finsler metric $\Psi_{h,b,\varepsilon}\circ\rho^{-1}_{\nabla,\varepsilon}(\xi_1,\ldots,\xi_k)$ converges towards the limit*

$$\Bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_h^{2b/s}\Bigg)^{1/b}$$

on every compact subset of $V^{\oplus k}\smallsetminus\{0\}$, uniformly in C^∞ topology, and the limit is independent of the connection ∇. The error is measured by a multiplicative factor $1 \pm O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$.

Proof. Let us pick another C^∞ connection $\widetilde{\nabla} = \nabla + \Gamma$ where $\Gamma \in C^\infty(U, T_X^*\otimes\mathrm{Hom}(V,V))$. Then $\widetilde{\nabla}^2 f = \nabla^2 f + \Gamma(f)(f')\cdot f'$, and inductively we get

$$\widetilde{\nabla}^s f = \nabla^s f + P_s(f\,;\,\nabla^1 f,\ldots,\nabla^{s-1}f)$$

where $P(z\,;\xi_1,\ldots,\xi_{s-1})$ is a polynomial with C^∞ coefficients in $z\in U$, which is of weighted homogeneous degree s in $(\xi_1,\ldots,\xi_{s-1})$. In other words, the corresponding isomorphisms $J^kV\simeq V^{\oplus k}$ correspond to each other by a $\mathbb{C}^*$-homogeneous transformation $(\xi_1,\ldots,\xi_k)\mapsto(\widetilde\xi_1,\ldots,\widetilde\xi_k)$ such that

$$\widetilde\xi_s=\xi_s+P_s(z\,;\xi_1,\ldots,\xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon},\ldots,\xi_{k,\varepsilon})=(\varepsilon_1^1\xi_1,\ldots,\varepsilon_k^k\xi_k),\qquad(\widetilde\xi_{1,\varepsilon},\ldots,\widetilde\xi_{k,\varepsilon})=(\varepsilon_1^1\widetilde\xi_1,\ldots,\varepsilon_k^k\widetilde\xi_k).$$

Then

$$\begin{aligned}\widetilde\xi_{s,\varepsilon}&=\xi_{s,\varepsilon}+\varepsilon_s^s\,P_s(x\,;\varepsilon_1^{-1}\xi_{1,\varepsilon},\ldots,\varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon})\\&=\xi_{s,\varepsilon}+O(\varepsilon_s/\varepsilon_{s-1})^s\,O(\|\xi_{1,\varepsilon}\|+\cdots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and it is easily seen, as a simple consequence of the mean value inequality $|\|x\|^\gamma-\|y\|^\gamma|\leqslant\gamma\sup_{z\in[x,y]}\|z\|^{\gamma-1}\|x-y\|$, that the "error term" in the difference $\|\widetilde\xi_{s,\varepsilon}\|^{2b/s}-\|\xi_{s,\varepsilon}\|^{2b/s}$ is bounded by

$$(\varepsilon_s/\varepsilon_{s-1})^s\left(\|\xi_{1,\varepsilon}\|+\cdots+\|\xi_{s-1,\varepsilon}\|^{1/(s-1)}+\|\xi_{s,\varepsilon}\|^{1/s}\right)^{2b}.$$

When b/s is an integer, similar bounds hold for all derivatives $D^\beta_{z,\xi}(\|\widetilde\xi_{s,\varepsilon}\|^{2b/s}-\|\xi_{s,\varepsilon}\|^{2b/s})$ and the lemma follows. $\square$

Now, we fix a point $z_0\in X$, a local holomorphic frame $(e_\lambda(z))_{1\leqslant\lambda\leqslant r}$ satisfying (3.1) on a neighborhood U of z_0, and the *holomorphic* connection ∇ on $V_{|U}$ such that $\nabla e_\lambda=0$. Since the uniform estimates of Lemma 3.6 also apply locally (provided they are applied on a relatively compact open subset $U'\Subset U$), we can use the corresponding holomorphic trivialization $J^kV_{|U}\simeq V_{|U}^{\oplus k}\simeq U\times(\mathbb{C}^r)^{\oplus k}$ to make our calculations. We do this in terms of the rescaled components $\xi_s=\varepsilon_s^s\nabla^sf(0)$. Then, uniformly on compact subsets of $J^kV_{|U}\smallsetminus\{0\}$, we have

$$\Psi_{h,b,\varepsilon}\circ\rho_{\nabla,\varepsilon}^{-1}(z\,;\xi_1,\ldots,\xi_k)=\Bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_{h(z)}^{2b/s}\Bigg)^{1/b}+O(\max((\varepsilon_s/\varepsilon_{s-1})^{1/b}),$$

and the error term remains of the same magnitude when we take any derivative $D^\beta_{z,\xi}$. By (3.1) we find

$$\|\xi_s\|_{h(z)}^2=\sum_\lambda|\xi_{s,\lambda}|^2+\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,z_i\overline z_j\,\xi_{s,\lambda}\overline\xi_{s,\mu}+O(|z|^3|\xi|^2).$$

The question is thus reduced to evaluating the curvature of the weighted Finsler metric on $V^{\oplus k}$ defined by

$$\begin{aligned}\Psi(z\,;\xi_1,\ldots,\xi_k)&=\Bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|_{h(z)}^{2b/s}\Bigg)^{1/b}\\&=\Bigg(\sum_{1\leqslant s\leqslant k}\Bigg(\sum_\lambda|\xi_{s,\lambda}|^2+\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\,z_i\overline z_j\,\xi_{s,\lambda}\overline\xi_{s,\mu}\Bigg)^{b/s}\Bigg)^{1/b}+O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2=\sum_\lambda|\xi_{s,\lambda}|^2$. A straightforward calculation yields the Taylor expansion

$$\begin{aligned}&\log\Psi(z\,;\,\xi_1,\ldots,\xi_k)\\&\quad=\frac{1}{b}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s}+\sum_{1\leqslant s\leqslant k}\frac{1}{s}\,\frac{|\xi_s|^{2b/s}}{\sum_t|\xi_t|^{2b/t}}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}z_i\overline{z}_j\frac{\xi_{s,\lambda}\overline{\xi}_{s,\mu}}{|\xi_s|^2}+O(|z|^3).\end{aligned}$$

By (3.5), the curvature form of $L_k=\mathcal{O}_{X_k(V)}(1)$ is given at the central point z_0 by the formula

$$\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z_0,[\xi])\simeq\omega_{r,k,b}(\xi)+\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\,\frac{|\xi_s|^{2b/s}}{\sum_t|\xi_t|^{2b/t}}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}\frac{\xi_{s,\lambda}\overline{\xi}_{s,\mu}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{3.7}$$

where $[\xi]=[\xi_1,\ldots,\xi_k]\in\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ and $\omega_{r,k,b}(\xi)=\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}(\frac{1}{b}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s})$. The fibers $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ of $X_k(V)\to X$ can be represented as a quotient of the "weighted ellipsoid" $\sum_{s=1}^k|\xi_s|^{2b/s}=1$ by the $\mathbb{S}^1$-action induced by the weighted $\mathbb{C}^*$-action. This suggests to make use of polar coordinates and to set

$$x_s=|\xi_s|^{2b/s},\quad x=(x_1,\ldots,x_k)\in\mathbb{R}^k, \tag{3.8}$$

$$u_s=\frac{\xi_s}{|\xi_s|}\in\mathbb{S}^{2r-1}\subset\mathbb{C}^r,\quad u=(u_1,\ldots,u_k)\in(\mathbb{S}^{2r-1})^k, \tag{3.8'}$$

so that

$$\sum_{s=1}^k x_s=1\quad\text{and}\quad\xi_s=x_s^{s/2b}u_s. \tag{3.8''}$$

The Morse integrals will then have to be computed for $(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, where $\Delta^{k-1}\subset\mathbb{R}^k$ is the $(k-1)$-dimensional simplex.

3.9. Proposition. *With respect to the rescaled components $\xi_s=\varepsilon_s^s\nabla^sf(0)$ at $z=f(0)\in X$ and the above choice of coordinates (3.8*), we have an approximate expression*

(a) $\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z,[\xi])=\omega_{r,k,b}(\xi)+g_{V,k}(z,x,u)+(\text{error terms}),$

where $(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k$, $\xi_s=x_s^{s/2b}u_s\in\mathbb{C}^r$,

(b) $\omega_{r,k,b}(\xi)=\dfrac{\mathrm{i}}{2\pi}\partial\overline{\partial}\Big(\dfrac{1}{b}\sum_{1\leqslant s\leqslant k}|\xi_s|^{2b/s}\Big)$

is a (slightly degenerate) Fubini-Study Kähler type metric on $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$, associated with the canonical $\mathbb{C}^*$ action on J^kV of weight $a=(1^{[r]},2^{[r]},\ldots,k^{[r]})$, and

(c) $g_{V,k}(z,x,u)=\dfrac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\dfrac{x_s}{s}\sum_{i,j,\lambda,\mu}c_{ij\lambda\mu}(z)\,u_{s,\lambda}\overline{u}_{s,\mu}\,dz_i\wedge d\overline{z}_j.$

Here $(c_{ij\lambda\mu})$ are the coefficients of $-\Theta_{V,h}$, and the error terms admit an upper bound

(d) $(\text{error terms})\leqslant O\Big(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s\Big)$ *uniformly on the compact variety $X_k(V)$.*

Proof. The error terms on Θ_{L_k} come from the differentiation of the error terms on the Finsler metric, found in Lemma 3.6. They can indeed be differentiated if b is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$, since $2b/s$ is then an even integer. □

For the calculation of Morse integrals, it is useful to find the expression of the volume form $\omega_{r,k,b}^{kr-1}$ on $\mathbb{P}(1^{[r]}, 2^{[r]}, \ldots, k^{[r]}) = (\mathbb{\Delta}^{k-1} \times (\mathbb{S}^{2r-1})^k)/\mathbb{S}^1$ in terms of the coordinates (x, u). We refer to [Dem11, Prop. 1.13] for the proof.

3.10. Proposition.

(a) *The volume form $\omega_{r,k,b}^{kr-1}$ is the quotient of the measure $\frac{1}{(k!)^r}\nu_{k,r} \otimes \mu$ on $\mathbb{\Delta}^{k-1} \times (\mathbb{S}^{2r-1})^k$, where*

$$d\nu_{k,r}(x) = (kr-1)!\frac{(x_1 \ldots x_k)^{r-1}}{(r-1)!^{\,k}} dx_1 \wedge \ldots \wedge dx_{k-1}, \quad d\mu(u) = d\mu_1(u_1) \ldots d\mu_k(u_k)$$

are probability measures on $\mathbb{\Delta}^{k-1}$ and $(\mathbb{S}^{2r-1})^k$ respectively (μ *being the rotation invariant one*).

(b) *We have the equality* $\displaystyle\int_{\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})} \omega_{r,k,b}^{kr-1} = \frac{1}{(k!)^r}$ (*independent of* b).

§3.B. Logarithmic and orbifold jet metrics

Consider now an arbifold directed structure (X, V, Δ), where $V \subset T_X$ is a subbundle, $r = \mathrm{rank}(V)$, and $\Delta = \sum(1 - \frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor that is assumed to intersect V transversally everywhere. One then performs very similar calculations to what we did in §3.A, but with adapted Finsler metrics. Fix a point z_0 at which p components Δ_j meet, and use coordinates $(z_1, \ldots, z_n)$ such that V_{z_0} is spanned by $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_r})$ and Δ_j is defined by $z_j = 0$, $1 \leqslant j \leqslant p \leqslant r$. In the logarithmic case $\rho_j = \infty$, the logarithmic dual bundle $\mathcal{O}(V^*\langle\Delta\rangle)$ is spanned by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p},\ dz_{p+1}, \ldots, dz_n.$$

The logarithmic jet differentials are just polynomials in

$$\frac{d^s z_1}{z_1}, \ldots, \frac{d^s z_p}{z_p},\ d^s z_{p+1}, \ldots, d^s z_n, \quad 1 \leqslant s \leqslant k,$$

and the corresponding $(\varepsilon_1, \ldots, \varepsilon_k)$-rescaled Finsler metric is

$$\left(\sum_{s=1}^{k} \varepsilon_s^{2b}\left(\sum_{j=1}^{p} |f_j|^{-2}|f_j^{(s)}|^2 + \sum_{j=p+1}^{r} |f_j^{(s)}|^2\right)^{2b/s}\right)^{1/b}. \tag{3.11}$$

Alternatively, we could replace $|f_j|^{-2}|f_j^{(s)}|^2$ by $|(\log f_j)^{(s)}|^2$ which has the same leading term and differs by a weighted degree s polynomial in the $f_j^{-1}f_j^{(\ell)}$, $\ell < s$; an argument very similar to the one used in the proof of Lemma 3.6 then shows that the difference is negligible when $\varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k$. However (3.11) is just the case of the model metric, in fact we get r-tuples $\xi_s = (\xi_{s,j})_{1 \leqslant j \leqslant r}$ of components produced by the trivialization of the logarithmic bundle $\mathcal{O}(V\langle\Delta\rangle)$, such that

$$\xi_{s,j} = f_j^{-1}f_j^{(s)} \quad \text{for } 1 \leqslant s \leqslant p \text{ and } \quad \xi_{s,j} = f_j^{(s)} \quad \text{for } p+1 \leqslant s \leqslant r. \tag{3.12}$$

In general, we are led to consider Finsler metrics of the form

$$\left(\sum_{s=1}^{k}\varepsilon_s^{2b}\|\xi_s\|_{h(z)}^{2b/s}\right)^{1/b},\quad \xi_s=(\xi_{s,j})_{1\leqslant j\leqslant r},\tag{3.13}$$

where $h(z)$ is a variable hermitian metric on the logarithmic bundle $V\langle\Delta\rangle$. In the orbifold case, the appropriate "model" Finsler metric is

$$\left(\sum_{s=1}^{k}\varepsilon_s^{2b}\left(\sum_{j=1}^{p}|f_j|^{-2(1-s/\rho_j)_+}|f_j^{(s)}|^2+\sum_{j=p+1}^{r}|f_j^{(s)}|^2\right)^{2b/s}\right)^{1/b}.\tag{3.14}$$

As a consequence of Remark 2.10, we would get a metric with equivalent singularities on the dual L_k^* of the tautological sheaf $L_k=\mathcal{O}_{X_k(V\langle\Delta\rangle)}(1)$ by replacing $\sum_{j=p+1}^{r}|f_j^{(s)}|^2$ with $\sum_{j=1}^{r}|f_j^{(s)}|^2$ (or by any smooth hermitian norm h on V), since the extra terms $\sum_{j=1}^{p}|f_j^{(s)}|^2$ are anyway controlled by the "orbifold part" of the summation. Of course, we need to find a suitable Finsler metric that is globally defined on X. This can be done by taking smooth metrics $h_{V,s}$ on V and h_j on $\mathcal{O}_X(\Delta_j)$ respectively, as well as smooth connections ∇ and ∇_j. One can then consider the globally defined metric

$$\left(\sum_{s=1}^{k}\varepsilon_s^{2b}\left(\sum_{j}\|\sigma_j(f)\|_{h_j}^{-2(1-s/\rho_j)_+}\|\nabla_j^{(s)}(\sigma_j\circ f)\|_{h_j}^2+\|\nabla^{(s)}f\|_{h_{V,s}}^2\right)^{2b/s}\right)^{1/b}\tag{3.15}$$

where $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$ and $\sigma_j\in H^0(X,\mathcal{O}_X(\Delta_j))$ are the tautological sections; here, we want the flexibility of not necessarily taking the same hermitian metrics on V to evaluate the various norms $\|\nabla^{(s)}f\|_{h_{V,s}}$. We obtain Finsler metrics with equivalent singularities by just changing the $h_{V,s}$ and h_j (and keeping ∇, ∇_j unchanged). If we also change the connections, then an argument very similar to the one used in the proof of Lemma 3.6 shows that the ratio of the corresponding metrics is $1\pm O(\max(\varepsilon_s/\varepsilon_{s-1}))$, and therefore arbitrary close to 1 whenever $\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k$; in any case, we get metrics with equivalent singularities. Fix $z_0\in X$ and use coordinates $(z_1,\ldots,z_n)$ as described at the beginning of §3.B, so that $\sigma_j(z)=z_j$, $1\leqslant j\leqslant p$, in a suitable trivialization of $\mathcal{O}_X(\Delta_j)$. Let f be a k-jet of curve such that $f(0)=z\in X\smallsetminus|\Delta|$ is in a sufficiently small neighborhood of z_0. By employing the trivial connections associated with the above coordinates, the derivative $f^{(s)}$ is described by components

$$\xi_{s,j}=f_j^{(s)},\quad 1\leqslant j\leqslant r,\quad \xi_{s,j}^{\log}=f_j^{-1}f_j^{(s)},\quad \xi_{s,j}^{\mathrm{orb}}=f_j^{-(1-s/\rho_j)_+}f_j^{(s)},\quad 1\leqslant j\leqslant p,$$

and $\xi_{s,j}^{\mathrm{orb}}=\xi_{s,j}^{\log}=\xi_{s,j}$ for $p+1\leqslant j\leqslant r$. Here $\xi_{s,j}^{\mathrm{orb}}$ are to be thought of as the components of $f^{(s)}$ in the "virtual" vector bundle $V\langle\Delta^{(s)}\rangle$, and the fact that the argument of these complex numbers is not uniquely defined is irrelevant, because the only thing we need to compute the norms is $|\xi_{s,j}^{\mathrm{orb}}|$. Accordingly, for $v\in V_z$, $v\simeq(v_j)_{1\leqslant j\leqslant r}\in\mathbb{C}^r$, we put

$$v_j^{\log}=z_j^{-1}v_j=\sigma_j(z)^{-1}d\sigma_j(v)\quad\text{and}\quad v_j^{\mathrm{orb}}=z_j^{-(1-s/\rho_j)_+}v_j,\quad 1\leqslant j\leqslant p,$$

and define the orbifold hermitian norm on $V\langle\Delta^{(s)}\rangle$ associated with h_j and $h_{V,s}$ by

$$\|v^{\mathrm{orb}}\|_{\widetilde{h}_s}^2=\sum_{j=1}^{p}\|\sigma_j(z)\|_{h_j}^{-2(1-s/\rho_j)_+)}\|d\sigma_j(v)\|_{h_j}^2+\|v\|_{h_{V,s}}^2\tag{3.16}$$

$$(3.16') \qquad = \sum_{j=1}^{p} \|\sigma_j(z)\|_{h_j}^{2(1-(1-s/\rho_j)_+)} |v_j^{\log}|^2 + \|v\|_{h_{V,s}}^2$$

$$(3.16'') \qquad = \sum_{j=1}^{p} \|v_j^{\mathrm{orb}}\|_{h_j^{1-(1-s/\rho_j)_+}}^2 + \|v\|_{h_{V,s}}^2.$$

With this notation, the orbifold Finsler metric (3.15) on k-jets is reduced to an expression

$$(3.17) \quad \|\xi^{\mathrm{orb}}\|_{\Psi_{h,b,\varepsilon}}^2 = \left(\sum_{s=1}^{k} \varepsilon_s^{2b} \|\xi_s^{\mathrm{orb}}\|_{\widetilde{h}_s}^{2b/s}\right)^{1/b}, \quad \xi_s^{\mathrm{orb}} = (\xi_{s,j}^{\mathrm{orb}})_{1\leqslant j\leqslant r}, \quad \xi^{\mathrm{orb}} = (\xi_s^{\mathrm{orb}})_{1\leqslant s\leqslant k},$$

formally identical to what we had in the compact or logarithmic cases. If v is a local holomorphic section of $\mathcal{O}_X(V)$, formula (3.16) shows that the norm $\|v^{\mathrm{orb}}\|_{\widetilde{h}_s}$ can take infinite values when $z \in |\Delta|$, while, by (3.16′), the norm is always bounded (but slightly degenerate along $|\Delta|$) if v is a section of the logarithmic sheaf $\mathcal{O}_X(V\langle\lceil\Delta\rceil\rangle)$; we think intuitively of the orbifold total space $V\langle\Delta^{(s)}\rangle$ as the subspace of V in which the tubular neighborhoods of the zero section are defined by $\|v^{\mathrm{orb}}\|_{\widetilde{h}_s} < \varepsilon$ for $\varepsilon > 0$.

3.18. Remark. When $\rho_j \in \mathbb{Q}$, we can take an adapted Galois cover $g : Y \to X$ such that $(z_j \circ g)^{1-(1-s/\rho_j)_+}$ is univalent on Y for all components Δ_j involved, and we then get a well defined locally free sheaf $\mathcal{O}_Y\big(g^* V\langle\Delta^{(s)}\rangle$ such that

$$g^*\big(\mathcal{O}_X(V\langle\lceil\Delta\rceil\rangle)\big) \subset \mathcal{O}_Y\big(g^* V\langle\Delta^{(s)}\rangle\big) \subset g^*\big(\mathcal{O}_X(V)\big).$$

However, as already stressed in Remark 1.26, this viewpoint is not needed in our analytic approach.

3.C. Orbifold tautological sheaves and their curvature

In this context, we define the orbifold tautological sheaves

$$(3.19) \qquad \mathcal{O}_{X_k(V\langle\Delta\rangle)}(m) := \mathcal{O}_{X_k(V\langle\lceil\Delta\rceil\rangle)}(m) \otimes \mathcal{I}((\Psi_{k,b,\varepsilon}^*)^m)$$

to be the logarithmic tautological sheaves $\mathcal{O}_{X_k(V\langle\lceil\Delta\rceil\rangle)}(m)$ twisted by the multiplier ideal sheaves associated with the dual metric $\Psi_{k,b,\varepsilon}^*$ (cf. (3.17)), when these are viewed as singular hermitian metrics over the logarithmic k-jet bundle $X_k(V\langle\lceil\Delta\rceil\rangle)$. In accordance with this viewpoint, we simply define the orbifold k-jet bundle to be $X_k(V\langle\Delta\rangle) = X_k(V\langle\lceil\Delta\rceil\rangle)$. The calculation of the curvature tensor is formally the same as in the case $\Delta = 0$, and we obtain :

3.20. Proposition. *With respect to the (rescaled) orbifold k-jet components*

$$\xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(1-(1-\rho_\lambda/s)_+)} f_\lambda^{(s)}(0), \quad 1 \leqslant \lambda \leqslant p, \quad \text{and} \quad \xi_{s,\lambda} = \varepsilon_s^s f_\lambda^{(s)}(0), \quad p+1 \leqslant \lambda \leqslant r,$$

and of the dual metric $\Psi_{h,b,\varepsilon}^*$, the curvature form of the tautological sheaf $L_k = \mathcal{O}_{X_k(V\langle\Delta\rangle)}(1)$ admits at any point $(z,[\xi]) \in X_k(V\langle\Delta\rangle)$ an approximate expression

(a) $\Theta_{L_k,\Psi_{h,b,\varepsilon}^*}(z,[\xi]) \simeq \omega_{r,k,b}(\xi) + g_{V,\Delta,k}(z,x,u)$,

where $x_s = |\xi_s|^{2b/s}$, $u_s = \frac{\xi_s}{|\xi_s|} \in \mathbb{S}^{2r-1}$ are polar coordinates associated with $\xi_s = (\xi_{s,\lambda})_{1\leqslant\lambda\leqslant k}$ in $\mathbb{C}^r$, $x = (x_1,\ldots,x_k) \in \Delta^{k-1}$, $[\xi] = [\xi_1,\ldots,\xi_k] \in \mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ and

(b) $g_{V,\Delta,k}(z,x,u) = \dfrac{\mathrm{i}}{2\pi} \displaystyle\sum_{1\leqslant s\leqslant k} \frac{x_s}{s} \sum_{i,j,\lambda,\mu} c_{ij\lambda\mu}^{(s)}(z)\, u_{s,\lambda}\overline{u}_{s,\mu}\, dz_i \wedge d\overline{z}_j.$

Here $(c^{(s)}_{ij\lambda\mu})$ *are the coefficients of the curvature tensor* $-\Theta_{V\langle\Delta^{(s)}\rangle,\widetilde{h}_s}$*, and the error terms are* $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$*, uniformly on the projectivized orbifold variety* $X_k(V\langle\Delta\rangle)$*.*

Notice, as is clear from the expressions (3.16″), (3.17) and the fact that $v_j = z_j v_j^{\rm orb}$, that our orbifold Finsler metrics always have fiberwise positive curvature, equal to $\omega_{k,r,b}(\xi)$, along the fibers of $X_k(V\langle\Delta\rangle) \to X$ (even after taking into account the so-called error terms, because fiberwise, the functions under consideration are just sums of even powers $|\widetilde{\xi}_s^{\rm orb}|^{2b/s}$ in suitable k-jet components, and are therefore plurisubharmonic.)

4. Existence theorems for jet differentials

4.A. Expression of the Morse integral

Thanks to the uniform approximation provided by proposition 3.20, we can (and will) neglect the $O(\varepsilon_s/\varepsilon_{s-1})$ error terms in our calculations. Since $\omega_{r,k,b}$ is positive definite on the fibers of $X_k(V\langle\Delta\rangle) \to X$ (at least outside of the axes $\xi_s = 0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,b,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form $g_{V,\Delta,k}(z,x,u)$. By the binomial formula, the q-index integral of $(L_k,\Psi^*_{h,b,\varepsilon})$ on $X_k(V\langle\Delta\rangle)$ is therefore equal to

$$\int_{X_k(V\langle\Delta\rangle)(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}}$$

$$= \frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in\mathbb{P}(1^{[r]},\ldots,k^{[r]})} \omega^{kr-1}_{r,k,b}(\xi)\wedge \mathbb{1}_{g_{V,\Delta,k},q}(z,x,u)\, g_{V,\Delta,k}(z,x,u)^n \tag{4.1}$$

where $\mathbb{1}_{g_{V,\Delta,k},q}(z,x,u)$ is the characteristic function of the open set of points where $g_{V,\Delta,k}(z,x,u)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $g_{V,\Delta,k}(z,x,u)^n$ is a determinant, the product $\mathbb{1}_{g_{V,\Delta,k},q}(z,x,u)\, g_{V,\Delta,k}(z,x,u)^n$ gives rise to a continuous function on $X_k(V\langle\Delta\rangle)$. By Formula 3.10 (a), we get

$$\int_{X_k(V\langle\Delta\rangle)(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}} = \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!} \times$$

$$\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \mathbb{1}_{g_{V,\Delta,k},q}(z,x,u)\, g_{V,\Delta,k}(z,x,u)^n\, d\nu_{k,r}(x)\, d\mu(u). \tag{4.2}$$

4.B. Probabilistic estimate of cohomology groups

We assume here that we are either in the "compact" case ($\Delta = 0$), or in the logarithmic case ($\rho_j = \infty$). Then the curvature coefficients $c^{(s)}_{ij\lambda\mu} = c_{ij\lambda\mu}$ do not depend on s and are those of the dual bundle V^* (resp. $V^*\langle\Delta\rangle$). In this situation, formula 3.20 (b) for $g_{V,\Delta,k}(z,x,u)$ can be thought of as a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s \in \mathbb{S}^{2r-1}$ with certain positive weights x_s/s; we then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ (resp. logarithmic derivatives) are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_{V,\Delta,k}(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{\mathbb{S}^{2r-1}} u_{s,\lambda}\overline{u}_{s,\mu}d\mu(u_s) = \frac{1}{r}\delta_{\lambda\mu}$ and $\int_{\Delta^{k-1}} x_s\, d\nu_{k,r}(x) = \frac{1}{k}$, we find

$$\mathbf{E}(g_{V,\Delta,k}(z,\bullet,\bullet)) = \frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda} c_{ij\lambda\lambda}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_{V,\Delta,k}(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\Theta_{\det(V^*\langle\Delta\rangle),\det h^*}, \tag{4.3}$$

where $\Theta_{\det(V^*\langle\Delta\rangle),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*\langle\Delta\rangle)$ with the metric induced by h. It is natural to guess that $g_{V,\Delta,k}(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_{V,\Delta,k}(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally $g_{V,\Delta,k}$ by its expected value in (4.2), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\,\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta:=\Theta_{\det(V^*\langle\Delta\rangle),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] in the compact case; the more general logarithmic case can be treated without any change, so we state the result in this situation by just transposing the results of [Dem11].

4.4. Probabilistic estimate. *Let (X,V,Δ) be a non singular logarithmic directed variety. Fix smooth hermitian metrics ω on T_X, h on $V\langle\Delta\rangle$, and write $\omega=\frac{\mathrm{i}}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V\langle\Delta\rangle,h}=-\frac{\mathrm{i}}{2\pi}\sum c_{ij\lambda\mu}dz_i\wedge d\overline{z}_j\otimes e_\lambda^*\otimes e_\mu$ the curvature tensor of $V\langle\Delta\rangle$ with respect to an h-orthonormal frame (e_λ), and put*

$$\eta(z):=\Theta_{\det(V^*\langle\Delta\rangle),\det h^*}=\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij}:=\sum_{1\leqslant\lambda\leqslant r}c_{ij\lambda\lambda}.$$

Finally consider the k-jet line bundle $L_k=\mathcal{O}_{X_k(V\langle\Delta\rangle)}(1)\to X_k(V\langle\Delta\rangle)$ equipped with the induced metric $\Psi^*_{h,b,\varepsilon}$ (as defined above, with $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k(V\langle\Delta\rangle)(L_k,q)$ is given by

$$\int_{X_k(V\langle\Delta\rangle)(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,b,\varepsilon}}=\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X\mathbb{1}_{\eta,q}\eta^n+O((\log k)^{-1})\bigg)$$

for all $q=0,1,\ldots,n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of $\Theta_{V\langle\Delta\rangle}$, η and ω. Moreover, the left hand side is identically zero for $q>n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k(V\langle\Delta\rangle)\to X$, by the plurisubharmonicity of the weight (this is true even when the error terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q>n$. It will be useful to extend the above estimates to the case of sections of

$$L_{F,k}=\mathcal{O}_{X_k(V\langle\Delta\rangle)}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\tag{4.5}$$

where $F\in\mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k:X_k(V\langle\Delta\rangle)\to X$ is the natural projection. We assume here that F is also equipped with a smooth hermitian metric h_F. In formulas (4.2–4.4), the curvature $\Theta_{L_{F,k}}$ of $L_{F,k}$ takes the form $\Theta_{L_{F,k}}=\omega_{r,k,b}(\xi)+g_{V,\Delta,F,k}(z,x,u)$ where

$$g_{V,\Delta,F,k}(z,x,u)=g_{V,\Delta,k}(z,x,u)+\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\Theta_{F,h_F}(z),\tag{4.6}$$

and by the same calculations its normalized expected value is

$$(4.7)\quad \eta_F(z) := \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\cdots+\frac{1}{k})}\mathbf{E}(g_{V,\Delta,F,k}(z,\bullet,\bullet)) = \Theta_{\det V^*\langle\Delta\rangle,\det h^*}(z) + \Theta_{F,h_F}(z).$$

Then the variance estimate for $g_{V,\Delta,F,k}$ is the same as the variance estimate for $g_{V,\Delta,k}$, and the recentered L^p bounds are still valid, since our forms are just shifted by adding the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 4.4 is therefore still true in exactly the same form for $L_{F,k}$, provided we use $g_{V,\Delta,F,k}$ and η_F instead of $g_{V,\Delta,k}$ and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q\Big(X, E_{k,m}V^*\langle\Delta\rangle \otimes \mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big) \\ = h^q(X_k(V\langle\Delta\rangle), \mathcal{O}_{X_k(V\langle\Delta\rangle)}(m) \otimes \pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big),\end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

4.8. Theorem. *Let $(X, V\langle\Delta\rangle)$ be a directed manifold, $F\to X$ a $\mathbb{Q}$-line bundle, $(V\langle\Delta\rangle, h)$ and (F, h_F) smooth hermitian structure on $V\langle\Delta\rangle$ and F respectively. We define*

$$\begin{aligned} L_{F,k} &= \mathcal{O}_{X_k(V\langle\Delta\rangle)}(1) \otimes \pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big),\\ \eta_F &= \Theta_{\det V^*\langle\Delta\rangle,\det h^*} + \Theta_{F,h_F} = \Theta_{\det V^*\langle\Delta\rangle\otimes F,\det h^*}.\end{aligned}$$

Then for all $q\geqslant 0$ and all $m\gg k\gg 1$ such that m is sufficiently divisible, we have

$$\text{(a)}\quad h^q(X_k(V\langle\Delta\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta_F,q)}(-1)^q\eta_F^n + O((\log k)^{-1})\bigg),$$

$$\text{(b)}\quad h^0(X_k(V\langle\Delta\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta_F,\leqslant 1)}\eta_F^n - O((\log k)^{-1})\bigg),$$

$$\text{(c)}\quad \chi(X_k(V\langle\Delta\rangle), \mathcal{O}(L_{F,k}^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\langle\Delta\rangle\otimes F)^n + O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr80] already checked the Riemann-Roch calculation (4.8 c) in the special case $\Delta = 0$, $V = T_X^*$ and $F = \mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\bigg(X, E_{k,m}T_X^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big)\bigg) = 0$$

as soon as $K_X\otimes F$ is big and $m\gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities (Theorem 2.9, cf. [Bon93]), everything works almost unchanged in the case where the metric h on V is taken to a product $h = h_\infty e^\varphi$ of a smooth metric h_∞ by the exponential of a quasi-plurisubarmonic

weight φ with analytic singularities (so that $\det(h^*) = \det(h^*_\infty)e^{-r\varphi}$). Then η is a $(1,1)$-current with logarithmic poles, and we just have to twist our cohomology groups by the appropriate multiplier ideal sheaves $\mathcal{I}_{k,m}$ associated with the weight $\frac{1}{k}(1+\frac{1}{2}+\cdots+\frac{1}{k})m\,\varphi$, since this is the multiple of $\det V^*$ that occurs in the calculation, up to the factor $\frac{1}{r}\times r\varphi$. The corresponding Morse integrals need only be evaluated in the complement of the poles, i.e., on $X(\eta,q)\smallsetminus S$ where $S=\mathrm{Sing}(\varphi)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L_{F,k}^{\otimes m})\otimes\mathcal{I}_{k,m}\big)\subset E_{k,m}V^*\otimes\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L_k^{\otimes m})$ on $X_k(V)$). If we assume that $K_V\otimes F$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of X.

4.9. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned}h^0\bigg(X_k(V),\mathcal{O}_{X_k(V)}(m)\otimes\pi_k^*\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)F\Big)\bigg)\\ \geqslant\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V\otimes F)-O((\log k)^{-1})\Big)-o(m^{n+kr-1}),\end{aligned}$$

when $m\gg k\gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V\otimes F$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu:\widetilde{X}\to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V\otimes F$ is not big, so we can assume $\mathrm{Vol}(K_V\otimes F)>0$. Let us fix smooth hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1}\otimes h_F)$ on $\mu^*(K_V\otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita [Fuj94] on approximate Zariski decomposition, for every $\delta>0$, one can find a modification $\mu_\delta:\widetilde{X}_\delta\to X$ dominating μ such that

$$\mu_\delta^*(K_V\otimes F)=\mathcal{O}_{\widetilde{X}_\delta}(A+E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A)=A^n\geqslant\mathrm{Vol}(K_V\otimes F)-\delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular hermitian metric h_Ah_E on $\mu_\delta^*(K_V\otimes F)$ with poles along E, i.e. the quotient $h_Ah_E/\mu^*(\det h_0^{-1}\otimes h_F)$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h=h_0e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta_F=\Theta_{K_V,\det h^*}+\Theta_{F,h_F}$ is such that

$$\mu_\delta^*\eta_F=\Theta_{A,h_A}+[E],\qquad [E]=\text{current of integration on }E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S}\eta_F^n=\int_{\widetilde X_\delta}\Theta^n_{A,h_A}=A^n\geqslant \mathrm{Vol}(K_V\otimes F)-\delta$$

and Corollary 4.9 follows from the fact that δ can be taken arbitrary small. $\square$

4.10. Remark. Since the probability estimate requires k to be very large, and since all non logarithmic components disappear from $\Delta^{(s)}$ when s is large, the above lower bound does not work in the general orbifold case. In that case, one can only hope to get an interesting result when k is fixed and not too large. This is what we aim at in the next section.

5. Non probabilistic estimate of the Morse integrals

5.A. Case of general directed orbifolds

The non probabilistic estimate uses more explicit curvature inequalities and has the advantage of producing results also in the general orbifold case. Let us fix an ample line bundle A on X equipped with a smooth hermitian metric h_A such that $\omega_A:=\Theta_{A,h_A}>0$. We assume here that the s-th directed (dual) orbifold bundle $V^*\langle\Delta^{(s)}\rangle$ (cf. § 1.B) possesses a hermitian metric $\widetilde h^*_s$ such that its curvature tensor satisfies an inequality

$$\Theta_{V^*\langle\Delta^{(s)}\rangle,\widetilde h^*_s}+\gamma_s\,\omega_A\otimes \mathrm{Id}_{V^*\langle\Delta^{(s)}\rangle}\geqslant 0 \tag{5.1}$$

in the sense of Griffiths, for some number $\gamma_s\geqslant 0$. Now, instead of exploiting a Monte Carlo convergence process for the curvature tensor, we replace $\Theta_{V^*\langle\Delta^{(s)}\rangle}$ with

$$\Theta^A_{V^*\langle\Delta^{(s)}\rangle}:=\Theta_{V^*\langle\Delta^{(s)}\rangle}+\gamma_s\,\omega_A\otimes\mathrm{Id}\geqslant 0,$$

and in this way get new curvature coefficients $c^{(s,A)}_{ij\lambda\mu}=c^{(s)}_{ij\lambda\mu}+\gamma_s\,\omega_{A,ij}\,\delta_{\lambda\mu}$. This has the effect of replacing $\Theta_{\det V^*\langle\Delta^{(s)}\rangle}=\mathrm{Tr}\,\Theta_{V^*\langle\Delta^{(s)}\rangle}$ by $\Theta_{\det V^*\langle\Delta^{(s)}\rangle}+r\gamma_s\,\omega_A$. Also, we take

$$L_{\varepsilon,k}:=\mathcal{O}_{X_k(V\langle\Delta\rangle)}(1)\otimes\pi_k^*\mathcal{O}_X(-\varepsilon A). \tag{5.2}$$

Then our earlier formulas 3.20 (a,b) become

$$\Theta_{L_{\varepsilon,k}}=\omega_{r,k,b}(\xi)+g_{\varepsilon,k}(z,x,u)\quad\text{where} \tag{5.3}$$

$$g_{\varepsilon,k}(z,x,u)=\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\sum_{i,j,\lambda,\mu}c^{(s)}_{ij\lambda\mu}(z)\,u_{s,\lambda}\overline u_{s,\mu}\,dz_i\wedge d\overline z_j-\varepsilon\,\omega_A. \tag{5.3′}$$

We want to express $g_{\varepsilon,k}(z,x,u)$ as a difference of two non negative terms. For this, we write

$$g_{\varepsilon,k}(z,x,u)=g^A_{V,\Delta,k}(z,x,u)-\bigg(\varepsilon+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\bigg)\omega_A\quad\text{where} \tag{5.4}$$

$$g^A_{V,\Delta,k}(z,x,u)=\frac{\mathrm{i}}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\sum_{i,j,\lambda,\mu}c^{(s,A)}_{ij\lambda\mu}(z)\,u_{s,\lambda}\overline u_{s,\mu}\,dz_i\wedge d\overline z_j\geqslant 0. \tag{5.4′}$$

Let us apply Corollary 2.4 with α,β replaced by

$$\alpha_k=g^A_{V,\Delta,k}(z,x,u),\qquad \beta_k=\bigg(\varepsilon+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\bigg)\omega_A,$$

both forms being semipositive by our assumptions. Then (4.2) leads to

$$\begin{aligned}
&\int_{X_k(V)(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_{\varepsilon,k},\Psi^*_{h,b,\varepsilon}}\\
&\quad= \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \mathbb{1}_{\alpha_k-\beta_k,\leqslant 1}\,(\alpha_k-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\quad\geqslant \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k} \big(\alpha_k^n-n\alpha_k^{n-1}\wedge\beta_k\big)\,d\nu_{k,r}(x)\,d\mu(u).
\end{aligned}\tag{5.6}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). It is then just a matter of routine to find a sufficient condition for the positivity of the integral. One can readily obtain an upper bound of α_k by taking the trace of $(c^{(s,A)}_{ij\lambda\mu})$. In this way we get

$$0\leqslant\alpha_k\leqslant\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\big(\Theta_s+r\gamma_s\,\omega_A\big)\quad\text{where}\quad\Theta_s=\Theta_{\det V^*\langle\Delta^{(s)}\rangle}\tag{5.7}$$

and where the right hand side no longer depends on $u\in(\mathbb{S}^{2r-1})^k$. Also, $\alpha_k=g^A_{V,\Delta,k}$ can be written as a sum of semipositive $(1,1)$-forms

$$g^A_{V,\Delta,k}=\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\theta^{s,A}(u_s),\qquad \theta^{s,A}(u)=\frac{\mathrm{i}}{2\pi}\sum_{i,j,\lambda,\mu}c^{(s,A)}_{ij\lambda\mu}u_\lambda\overline{u}_\mu\,dz_i\wedge d\overline{z}_j,$$

hence for $k\geqslant n$ we have

$$\alpha_k^n=(g^A_{V,\Delta,k})^n\geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^{s_1,A}(u_{s_1})\wedge\theta^{s_2,A}(u_{s_2})\wedge\ldots\wedge\theta^{s_n,A}(u_{s_n}).$$

Since $\int_{\mathbb{S}^{2r-1}}\theta^{s,A}(u)\,d\mu(u)=\frac{1}{r}\operatorname{Tr}(\Theta_{V^*\langle\Delta^{(s)}\rangle}+\gamma_s\,\omega_A\otimes\mathrm{Id})=\frac{1}{r}(\Theta_s+r\gamma_s\,\omega_A)$, we infer from this

$$\begin{aligned}
&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k}\alpha_k^n\,d\nu_{k,r}(x)\,d\mu(u)\\
&\qquad\geqslant\frac{n!}{r^n}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\int_{\Delta^{k-1}}x_1\ldots x_n\,d\nu_{k,r}(x)\bigwedge_{\ell=1}^n\big(\Theta_{s_\ell}+r\gamma_s\,\omega_A\big).
\end{aligned}\tag{5.8}$$

By formula 3.10 (a) and an elementary calculation (cf. [Dem11, Prop. 1.13]), one gets

$$\int_{\Delta^{k-1}}x_1\ldots x_n\,d\nu_{k,r}(x)=\frac{(kr-1)!}{(r-1)!^k}\,\frac{r!^n(r-1)!^{k-n}}{(n+rk-1)!}=\frac{(kr-1)!\,r^n}{(n+rk-1)!}.\tag{5.9}$$

Now, the upper bound (5.7) for α_k and the definition of β_k imply

$$n\alpha_k^{n-1}\wedge\beta_k\leqslant n\Big(\varepsilon+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1}\wedge\omega_A,\tag{5.10}$$

and we need an estimate of the integral, starting with $\int_{\Delta^{k-1}}(...)$. For every multi-index $a=(a_1,\ldots,a_k)\in\mathbb{N}^k$ with $\sum a_s = n$, we find

$$\int_{\Delta^{k-1}} x_1^{a_1}\ldots x_k^{a_k}\,d\nu_{k,r}(x)=\frac{(kr-1)!}{(r-1)!^k}\frac{\prod_{s=1}^k(r+a_s-1)!}{(n+kr-1)!}\quad\begin{cases}\leqslant \dfrac{(kr-1)!(n+r-1)!}{(r-1)!(n+kr-1)!},\\[2mm] \geqslant \dfrac{(kr-1)!\,r^n}{(n+kr-1)!},\end{cases}$$

because the maximum is attained for the length n multi-index $a=(n,0,\ldots,0)$, and the minimum for $a=(1,\ldots,1,0,\ldots,0)$ (or any permutation). An expansion of

$$\Big(\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1}$$

in terms of its monomials x^a then gives

$$\begin{aligned}&\int_{\Delta^{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{\gamma_s x_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}(\Theta_s+r\gamma)\Big)^{n-1}d\nu_{k,r}(x)\\ &\qquad\begin{cases}\leqslant \dfrac{(kr-1)!(n+r-1)!}{(r-1)!(n+kr-1)!}\Big(\displaystyle\sum_{1\leqslant s\leqslant k}\frac{\gamma_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1},\\[2mm] \geqslant \dfrac{(kr-1)!\,r^n}{(n+kr-1)!}\Big(\displaystyle\sum_{1\leqslant s\leqslant k}\frac{\gamma_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1}.\end{cases}\end{aligned}\tag{5.11}$$

The inequalities are to be understood as inequalities between $(n-1,n-1)$-forms, and they hold because our assumption (5.1) implies $\Theta_s+r\gamma_s\,\omega_1\geqslant 0$. Also observe that the ratio between the upper bound and the lower bound is $\frac{(n+r-1)!}{r^n(r-1)!}$ which, for $r=n$ is $\sim 2^{-1/2}(4/e)^n$ by Stirling's formula; thus, when taking the upper bound, the "inaccuracy" factor is at most exponential in n with a small constant $4/e<1.5$. By using integrals of monomials x^a of degree $|a|=n-1$, we would obtain in a similar way

$$\begin{aligned}&\int_{\Delta^{k-1}}\varepsilon\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}(\Theta_s+r\gamma)\Big)^{n-1}d\nu_{k,r}(x)\\ &\qquad\qquad\leqslant\varepsilon\,\frac{(kr-1)!(n+r-2)!}{(r-1)!(n+kr-2)!}\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1}.\end{aligned}\tag{5.12}$$

By putting (5.8 – 5.12) together we obtain

$$\begin{aligned}&\int_{(x,u)\in\Delta^{k-1}\times(\mathbb{S}^{2r-1})^k}\big(\alpha_k^n-n\alpha_k^{n-1}\wedge\beta_k\big)\,d\nu_{k,r}(x)\,d\mu(u)\\ &\geqslant\frac{n!}{r^n}\,\frac{(kr-1)!\,r^n}{(n+rk-1)!}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\bigwedge_{\ell=1}^n(\Theta_{s_\ell}+r\gamma_{s_\ell}\,\omega_A)\\ &-n\,\frac{(kr-1)!(n+r-1)!}{(r-1)!(n+kr-1)!}\Big(\frac{(n+kr-1)\varepsilon}{n+r-1}+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s}{s}\Big)\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\Big)^{n-1}\wedge\omega_A.\end{aligned}$$

The Morse integral lower bound (5.6) finally implies

5.13. Theorem. *Assume that the curvature of the orbifold bundles satisfy the lower bounds $\Theta_{V^*\langle\Delta^{(s)}\rangle} \geqslant -\gamma_s\,\omega_1 \otimes \mathrm{Id}_{V^*}$ (in the sense of Griffiths), for some number $\gamma_s \in \mathbb{R}_+$. Then the orbifold line bundle*

$$L_{\varepsilon,k} = \mathcal{O}_{X_k(V\langle\Delta\rangle)}(1) \otimes \pi_k^*\mathcal{O}(-\varepsilon A)$$

admits for all $k \geqslant n$ and $\varepsilon \in \mathbb{Q}_+$ a number of sections $h^0(X_k(V\langle\Delta\rangle), L_{\varepsilon,k}^{\otimes m})$ that is bounded below asymptotically, modulo an error term $o(m^{n+kr-1})$, by

$$\begin{aligned}\frac{m^{n+kr-1}}{(n+kr-1)!}\int_{X_k(V\langle\Delta\rangle)(L_{\varepsilon,k},\leqslant 1)} \Theta^{n+kr-1}_{L_{\varepsilon,k},\Psi^*_{h,b,\varepsilon}} \geqslant \frac{m^{n+kr-1}}{(k!)^r(n+kr-1)!}\times\\ \int_X \sum_{1\leqslant s_1<\ldots<s_n\leqslant k} \frac{1}{s_1\ldots\, s_n}\bigwedge_{\ell=1}^n(\Theta_{s_\ell}+r\gamma_{s_\ell}\,\omega_A)\\ -\frac{(n+r-1)!}{(n-1)!(r-1)!}\bigg(\frac{(n+kr-1)\varepsilon}{n+r-1}+\sum_{1\leqslant s\leqslant k}\frac{\gamma_s}{s}\bigg)\bigg(\sum_{1\leqslant s\leqslant k}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\bigg)^{n-1}\wedge\omega_A.\end{aligned}$$

where $\Theta_s = \Theta_{\det V^*\langle\Delta^{(s)}\rangle}$. Especially, for $m \gg 1$, we have a lot of sections in

$$H^0(X_k(V\langle\Delta\rangle), L_{\varepsilon,k}^{\otimes m}) = H^0(X, E_{k,m}V^*\langle\Delta\rangle \otimes \mathcal{O}_X(-m\varepsilon A)),$$

whenever the integral in the right hand side of the lower bound is positive.

The statement is also true for $k < n$, but then the first sum is equal to 0 and the lower bound cannot be positive (by Corollary 1.11, it still provides a non trivial lower bound for $h^0(X_k(V\langle\Delta\rangle), L_{\varepsilon,k}^{\otimes m}) - h^1(X_k(V\langle\Delta\rangle), L_{\varepsilon,k}^{\otimes m})$, though). For $k = n$, there is a single term $s_1 = 1, s_2 = 2, \ldots, s_n = n$, so that $\frac{1}{s_1\ldots\, s_n} = \frac{1}{n!}$, and we get the simpler estimate

$$\begin{aligned}\frac{m^{n+nr-1}}{(n+nr-1)!}\int_{X_n(V\langle\Delta\rangle)(L_{n,\varepsilon},\leqslant 1)} \Theta^{n+nr-1}_{L_{\varepsilon,k},\Psi^*_{h,b,\varepsilon}} \geqslant \frac{m^{n+nr-1}}{(n!)^{r+1}(n+nr-1)!}\int_X\bigwedge_{\ell=1}^n(\Theta_\ell+r\gamma_\ell\,\omega_A)\\ -\frac{n\,(n+r-1)!}{(r-1)!}\bigg(\frac{(n+nr-1)\varepsilon}{n+r-1}+\sum_{1\leqslant s\leqslant n}\frac{\gamma_s}{s}\bigg)\bigg(\sum_{1\leqslant s\leqslant n}\frac{1}{s}(\Theta_s+r\gamma_s\,\omega_A)\bigg)^{n-1}\wedge\omega_A.\end{aligned}\tag{5.14}$$

5.B. Case of orbifold structures on projective *n*-space

The following elementary result shows that the hypothesis made in Theorem 5.13 on the lower bound of the curvature is very natural and always holds true.

5.15. Proposition. *Let $X \subset \mathbb{P}^N$ be a projective variety and (X, V, Δ) an orbifold directed structure where $\Delta = \sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor transverse to V in X. Let a_j be the infimum of numbers $\lambda \in \mathbb{R}_+$ such that $\lambda\,\mathcal{O}_X(1) - \Delta_j$ is nef. Then for every $a > \max(a_j/\rho_j, 2)$, the orbifold vector bundle $V\langle\Delta\rangle$ possesses a hermitian metric h_a such that*

(a) *h_a is smooth on $X \smallsetminus |\Delta|$,*

(b) *h_a has the appropriate orbifold singularities along Δ,*

(c) *the curvature tensor of $V^*\langle\Delta\rangle \otimes \mathcal{O}_X(a)$ is Griffiths positive.*

Proof. Let $\Theta_{\mathcal{O}_{\mathbb{P}^N}(1),\mathrm{FS}} = \omega_{\mathrm{FS}}(\zeta) = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log|\zeta|^2$ be the Fubini-Study metric. Consider the tautological sections $\sigma_j \in H^0(X, \mathcal{O}_X(\Delta_j))$ such that $\Delta_j = \sigma_j^{-1}(0)$, and let h_j be a smooth hermitian metric on $\mathcal{O}_X(\Delta_j)$ for which

$$\frac{1}{\rho_j}\Theta_{\mathcal{O}_X(\Delta_j),h_j} < a\,\Theta_{\mathcal{O}_X(1),\mathrm{FS}} = a\,\omega_{\mathrm{FS}|X}, \tag{5.16}$$

as is possible by our choice of constants a_j and a. Finally, denote by ∇_j the associated Chern connection on $\mathcal{O}_X(\Delta_j)$. If we write $h_j = e^{-\varphi_j}$ in some local trivialization, then $\nabla_j\sigma_j = \nabla_j^{1,0}\sigma_j = \partial\sigma_j - \sigma_j\partial\varphi_j$. We are going to estimate the curvature of the orbifold metric on $V\langle\Delta\rangle$ defined by

$$\|v\|_h^2 = \sum_j |\sigma_j|_{h_j}^{-2(1-1/\rho_j)}\,|\nabla_j\sigma_j(v)|_{h_j}^2 + \big(K + K'|\sigma_j|_{h_j}^{2/\rho_j}\big)\,|v|_{\mathrm{FS}}^2, \quad K \gg K' \gg 1, \tag{5.17}$$

where the metric $|\bullet|_{\mathrm{FS}}^2$ on $V \subset T_X \subset T_{\mathbb{P}^N}$ is the restriction of the Fubini-Study metric ω_{FS}. What we need to prove is that over $X \smallsetminus |\Delta|$ we have

$$\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log\|v\|_h^2 + a\,p^*\omega_{\mathrm{FS}} \geqslant 0$$

on the total space of V, where $p : V \to X$ is the natural projection. We make this calculation at an arbitrary point $z_0 \in X$. By the homogeneity of $\mathbb{P}^N$, we can always find orthonormal coordinates such that $z_0 = 0 \in \mathbb{C}^N \subset \mathbb{P}^N$ in the affine chart $z \mapsto [1 : z]$, and $T_X = \mathrm{Span}(\frac{\partial}{\partial z_\ell})_{1\leqslant \ell\leqslant n}$. We take $(z_1, \ldots, z_n)$ as local coordinates on X and $v = \sum_{\ell=1}^n v_\ell\frac{\partial}{\partial z_\ell}$ in $V \subset T_X \simeq \mathbb{C}^n$. In terms of the standard hermitian metric on $\mathbb{C}^n$, we then find

$$\begin{aligned}
\|v\|_h^2 &= \sum_j \big(|\sigma_j|^2 e^{-\varphi_j}\big)^{-1+1/\rho_j}\,\big|\partial\sigma_j(v) - \sigma_j\partial\varphi_j(v)\big|^2 e^{-\varphi_j}\\
&\qquad\qquad + \big(K + K'e^{-\varphi_j/\rho_j}|\sigma_j|^{2/\rho_j}\big)\bigg(\frac{|v|^2}{1+|z|^2} - \frac{|\langle v, z\rangle|^2}{(1+|z|^2)^2}\bigg)\\
&= \sum_j e^{-\varphi_j/\rho_j}\,|\sigma_j|^{-2+2/\rho_j}\Big(|\partial\sigma_j(v)|^2 + |\sigma_j|^2|\partial\varphi_j(v)|^2 - 2\,\mathrm{Re}\,\big(\overline{\sigma}_j\,\partial\sigma_j(v)\,\overline{\partial}\varphi_j(v)\big)\Big)\\
&\qquad\qquad + \big(K + K'e^{-\varphi_j/\rho_j}|\sigma_j|^{2/\rho_j}\big)\,\frac{|v|^2 + |v\wedge z|^2}{(1+|z|^2)^2}.
\end{aligned}$$

By the homogeneity in v, it is enough to show that

$$\beta := i\partial\overline{\partial}\big(\|v\|_h^2(1+|z|^2)^a\big)_{(z_0,v_0)} \geqslant 0 \quad \text{on} \quad T(T_X)_{(z_0,v_0)} = \mathrm{Span}\Big(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial v_j}\Big) \simeq \mathbb{C}^N \times \mathbb{C}^N,$$

for every $v_0 \in \mathbb{C}^N$. In order to simplify our calculations, we take holomorphic trivializations of the line bundles $\mathcal{O}_X(\Delta_j)$ so that $\varphi_j(z) = \sum_{\ell,m}\alpha_{j,\ell,m}\,z_\ell\overline{z}_m + O(|z|^3)$ near $z_0 = 0$. Then

$$\partial\varphi_j = \sum \alpha_{j,\ell,m}\,\overline{z}_m dz_\ell + O(|z|^2), \quad \overline{\partial}\varphi_j = \sum \alpha_{j,\ell,m}\,z_\ell\,d\overline{z}_m + O(|z|^2). \tag{5.18}$$

It turns out that most of the terms occurring in $\beta_{(z_0,v_0)}$ are non negative, since we have many squares of holomorphic functions in (z, v) (such as $|\partial\sigma_j(v)|^2$), and likewise $i\partial\overline{\partial}|\partial\varphi_j(v)|^2 = |\partial\overline{\partial}\varphi_j(v, dz)|^2 \geqslant 0$ at $z = 0$, thanks to the Taylor expansion (5.18). In

what follows, we only keep track of the potentially negative terms and of a few positive ones that can be used to control them. Especially, one of the problematic terms is $2\,\mathrm{Re}(\ldots)$, which we rewrite $2\,\mathrm{Re}\big(e^{O(|z|^2)}\,\sigma_j^{-1+1/\rho_j}\,\overline{\sigma}_j^{1/\rho_j}\,\partial\sigma_j(v)\,\overline{\partial}\varphi_j(v)\big)$. This gives, modulo the usual identification of $(1,1)$-forms and hermitian forms:

$$\beta_{(z_0,v_0)} \geqslant \sum_j K\Big(|\partial v|^2+(a-2)|v|^2|\partial z|^2\Big)+K'\bigg(\mathrm{i}\partial\overline{\partial}(|\sigma_j|^{2/\rho_j}|v|^2)-\frac{1}{\rho_j}(\mathrm{i}\partial\overline{\partial}\varphi_j)\,|\sigma_j|^{2/\rho_j}|v|^2\bigg)$$

$$(5.19_1)\qquad +\Big(a|\partial z|^2-\frac{1}{\rho_j}\mathrm{i}\partial\overline{\partial}\varphi_j\Big)\,|\sigma_j|^{-2+2/\rho_j}\,|\partial\sigma_j(v)|^2$$

$$(5.19_2)\qquad -2\,\mathrm{Re}\bigg(\frac{1}{\rho_j}\,|\sigma_j|^{-2+2/\rho_j}\,\partial\sigma_j(v)\,\partial\overline{\partial}\varphi_j(dz,v)\,\overline{\partial\sigma_j}\bigg).$$

Since $K\gg K'$, the term $K'\cdot\frac{1}{\rho_j}(\mathrm{i}\partial\overline{\partial}\varphi_j)\,|\sigma_j|^{2/\rho_j}|v|^2$ is controlled by $K\cdot(a-2)|v|^2|\partial z|^2$, thanks to a compactness argument. The term (5.19_1) is positive by our curvature condition (5.16). Moreover, by the Cauchy-Schwarz inequality, there exists a constant $C>0$ (independent of z_0, K, K') such that

$$(5.20)\qquad \big|(5.19_2)\big|\leqslant\varepsilon\,|\sigma_j|^{-2+2/\rho_j}\,|\partial\sigma_j(v)|^2\,|\partial z|^2+\frac{C}{\varepsilon}\,|\sigma_j|^{-2+2/\rho_j}\,|v|^2\,|\partial\sigma_j|^2$$

for every $\varepsilon>0$. The ε-term can be absorbed in (5.19_1) after replacing a by $a-\varepsilon$. Finally

$$\begin{aligned}K'\cdot\mathrm{i}\partial\overline{\partial}(|\sigma_j|^{2/\rho_j}|v|^2)&=K'\bigg|\frac{1}{\rho_j}\sigma_j^{-1+1/\rho_j}\,v\,\partial\sigma_j+\sigma_j^{1/\rho_j}\,\partial v\bigg|^2\\&\geqslant\frac{K'}{2\rho_j^2}\,|\sigma_j|^{-2+2/\rho_j}|v|^2\,|\partial\sigma_j|^2-K'|\sigma_j|^{2/\rho_j}\,|\partial v|^2,\end{aligned}$$

where the term $\frac{K'}{2\rho_j^2}\ldots$ can be used to control the term $\frac{C}{\varepsilon}\ldots$ in (5.20), and the term $-K'|\sigma_j|^{2/\rho_j}\,|\partial v|^2$ is bounded by $K\,|\partial v|^2$. The proof is complete. □

5.21. Remark. The conclusion of Proposition 5.15 stills holds if we take a_j to be the infimum of numbers $\lambda\in\mathbb{R}_+$ such that $(\lambda\,\mathcal{O}_X(1)-\Delta_j)_{|\Delta_j}$ is nef on Δ_j. In fact, the curvature estimate (5.17) is merely needed in a neighborhood of $|\Delta_j|$, since all terms occurring in $\beta_{(z_0,v_0)}$ are controlled by the main term $K\big(|\partial v|^2+(a-2)|v|^2|\partial z|^2\big)$ in the complement of such a neighborhood.

An interesting special orbifold example is the case when $X=\mathbb{P}^n$, $V=T_X$, $A=\mathcal{O}(1)$ and $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor, with components Δ_j of degree d_j. Then $\Delta^{(s)}=\sum\big(1-\frac{s}{\rho_j}\big)_+\Delta_j$, hence

$$\det V^*\langle\Delta^{(s)}\rangle=\mathcal{O}_{\mathbb{P}^n}\big(-n-1+\sum_j d_j(1-s/\rho_j)_+\big)$$

and the associated curvature form is

$$\Theta_s=\Big(-n-1+\sum_j d_j(1-s/\rho_j)_+\Big)\omega_A.$$

Moreover, by Proposition 5.15, we have

$$\Theta_{V^*\langle\Delta^{(s)}\rangle} + \gamma_s\,\omega_{\mathrm{FS}} \otimes \mathrm{Id} > 0$$

as soon as $\gamma_s > 2$ and $\gamma_s > d_j/\max(\rho_j/s,1)$ for all components Δ_j in $\Delta^{(s)}$. We can take for instance $\gamma_s > \max(sd_j/\rho_j,2)$. Then, for $k=n$ and $\varepsilon\in\mathbb{Q}_+$ small, the estimate (5.14) guarantees the existence of jet differentials under the following complicated condition.

5.22. Proposition. *Let $\Delta = \sum_j(1-\frac{1}{\rho_j})\Delta_j$ a simple normal crossing orbifold divisor on $\mathbb{P}^n$. Then there exist jet differentials of order n and large degree m on $\mathbb{P}^n\langle\Delta\rangle$, with a small negative twist $\mathcal{O}_{\mathbb{P}^b}(-m\varepsilon)$, as soon as*

$$\prod_{s=1}^{n}\Big(n\max(sd_j/\rho_j,2)-(n-1)+\sum_j d_j(1-s/\rho_j)_+\Big) > \frac{n\,(2n-1)!}{(n-1)!}\times$$
$$\Big(\sum_{1\leqslant s\leqslant n}\frac{1}{s}\max(sd_j/\rho_j,2)\Big)\Big(\sum_{1\leqslant s\leqslant n}\frac{1}{s}\Big(n\max(sd_j/\rho_j,2)-(n-1)+\sum_j d_j(1-s/\rho_j)_+\Big)\Big)^{n-1}.$$

We are going to find a simpler sufficient condition. If we set $t_j := d_j/\rho_j$, $t=\max(t_j,2)$ and assume $\rho_j\geqslant\rho>n$, we get the condition

$$\prod_{s=1}^{n}\Big(nst-(n-1)+\sum_j(d_j-st_j)\Big) >$$
$$\frac{n\,(2n-1)!}{(n-1)!}\times nt\Big(\sum_{1\leqslant s\leqslant n}\frac{1}{s}\Big(nst-(n-1)+\sum_j(d_j-st_j)\Big)\Big)^{n-1},$$

which is implied by

$$\prod_{s=1}^{n}\Big(\Big(1-\frac{s}{\rho}\Big)\sum_j d_j\Big) > \frac{n^2\,(2n-1)!}{(n-1)!}\times t\Big(n^2t+\Big(1+\frac12+\cdots+\frac1n\Big)\sum_j d_j\Big)^{n-1}.$$

The latter condition is satisfied if $\sum_j d_j\geqslant c_n t\prod_{s=1}^n\big(1-\frac{s}{\rho}\big)^{-1}$ with

$$c_n = \frac{n^2\,(2n-1)!}{(n-1)!}\Big(1+\frac12+\cdots+\frac{1}{n-1}+\frac2n\Big)^{n-1},$$

since $c_n\geq n^3$, and so $n^2t\leqslant\frac1n\sum d_j$. The Stirling formula gives

$$c_n\leqslant 2^{-1/2}\,(4/e)^n\,n^{n+2}\,(1+\log n)^{n-1} = O((2n\log n)^n)$$

for n large. In this way we get

5.23. Corollary. *A sufficient condition for the existence of negatively twisted orbifold n-jet differentials on $\mathbb{P}^n\langle\Delta\rangle$ is*

$$\rho_j\geqslant\rho>n,\quad \sum d_j\geqslant c_n\,\max\Big(\frac{d_j}{\rho_j},2\Big)\prod_{s=1}^{n}\Big(1-\frac{s}{\rho}\Big)^{-1}.$$

For instance, one can take all components Δ_j possessing the same degree d and ramification number $\rho > n$, and a number of components

$$N \geqslant c_n \max\left(\frac{1}{\rho}, \frac{2}{d}\right) \prod_{s=1}^{n} \left(1 - \frac{s}{\rho}\right)^{-1},$$

or a single component $(1 - \frac{1}{\rho_1})\Delta_1$ with $\rho_1 \geqslant 2c_n$ and $d_1 \geqslant 4c_n$ (notice that $\prod(1 - \frac{s}{2c_n})^{-1} < 2$). Since we have neglected many terms in the above calculations, the "technological constant" c_n appearing in these estimates is probably much larger than needed.

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