Hyperbolic algebraic varieties and holomorphic differential equations

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By elementary integrations by parts and induction on k, $r_1, \ldots, r_k$, it can be checked that

$$\int_{x\in\Delta_{k-1}} \prod_{1\leqslant s\leqslant k} x_s^{r_s-1}dx_1\ldots dx_{k-1} = \frac{1}{(|r|-1)!}\prod_{1\leqslant s\leqslant k}(r_s-1)!\ . \tag{9.20}$$

This implies that $(|r|-1)!\prod_{1\leqslant s\leqslant k}\frac{x_s^{r_s-1}}{(r_s-1)!}\,dx$ is a probability measure on Δ_{k-1}.

§9.C. Probabilistic estimate of the curvature of k-jet bundles

Let (X,V) be a compact complex directed non singular variety. To avoid any technical difficulty at this point, we first assume that V is a holomorphic vector subbundle of T_X, equipped with a smooth Hermitian metric h.

According to the notation already specified in § 7, we denote by J^kV the bundle of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V at each point. Let us set $n=\dim_{\mathbb{C}} X$ and $r=\operatorname{rank}_{\mathbb{C}} V$. Then $J^kV\to X$ is an algebraic fiber bundle with typical fiber $\mathbb{C}^{rk}$, and we get a projectivized k-jet bundle

$$X_k^{\mathrm{GG}} := (J^kV\smallsetminus\{0\})/\mathbb{C}^*, \qquad \pi_k : X_k^{\mathrm{GG}}\to X \tag{9.21}$$

which is a $P(1^{[r]},2^{[r]},\ldots,k^{[r]})$ weighted projective bundle over X, and we have the direct image formula $(\pi_k)_*\mathcal{O}_{X_k^{\mathrm{GG}}}(m)=\mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)$ (cf. Proposition 7.9). In the sequel, we do not make a direct use of coordinates, because they need not be related in any way to the Hermitian metric h of V. Instead, we choose a local holomorphic coordinate frame $(e_\alpha(z))_{1\leqslant\alpha\leqslant r}$ of V on a neighborhood U of x_0, such that

$$\langle e_\alpha(z),e_\beta(z)\rangle = \delta_{\alpha\beta}+\sum_{1\leqslant i,j\leqslant n,\,1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta}z_i\overline{z}_j+O(|z|^3) \tag{9.22}$$

for suitable complex coefficients $(c_{ij\alpha\beta})$. It is a standard fact that such a normalized coordinate system always exists, and that the Chern curvature tensor $\frac{i}{2\pi}D^2_{V,h}$ of (V,h) at x_0 is then given by

$$\Theta_{V,h}(x_0) = -\frac{i}{2\pi}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\,dz_i\wedge d\overline{z}_j\otimes e_\alpha^*\otimes e_\beta. \tag{9.23}$$

Consider a local holomorphic connection ∇ on $V_{|U}$ (e.g. the one which turns (e_α) into a parallel frame), and take $\xi_k = \nabla^k f(0) \in V_x$ defined inductively by $\nabla^1 f = f'$ and $\nabla^s f = \nabla_{f'}(\nabla^{s-1} f)$. This gives a local identification

$$J_k V_{|U} \to V_{|U}^{\oplus k}, \qquad f \mapsto (\xi_1, \ldots, \xi_k) = (\nabla f(0), \ldots, \nabla f^k(0)),$$

and the weighted $\mathbb{C}^*$ action on $J_k V$ is expressed in this setting by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k).$$

Now, we fix a finite open covering $(U_\alpha)_{\alpha \in I}$ of X by open coordinate charts such that $V_{|U_\alpha}$ is trivial, along with holomorphic connections ∇_α on $V_{|U_\alpha}$. Let θ_α be a partition of unity of X subordinate to the covering (U_α). Let us fix $p > 0$ and small parameters $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$. Then we define a global weighted Finsler metric on $J^k V$ by putting for any k-jet $f \in J^k_x V$

$$\Psi_{h,p,\varepsilon}(f) := \Big(\sum_{\alpha \in I} \theta_\alpha(x) \sum_{1 \leqslant s \leqslant k} \varepsilon_s^{2p} \|\nabla_\alpha^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p} \tag{9.24}$$

where $\|\ \|_{h(x)}$ is the Hermitian metric h of V evaluated on the fiber V_x, $x = f(0)$. The function $\Psi_{h,p,\varepsilon}$ satisfies the fundamental homogeneity property

$$\Psi_{h,p,\varepsilon}(\lambda \cdot f) = \Psi_{h,p,\varepsilon}(f)\, |\lambda|^2 \tag{9.25}$$

with respect to the $\mathbb{C}^*$ action on $J^k V$, in other words, it induces a Hermitian metric on the dual L^* of the tautological $\mathbb{Q}$-line bundle $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ over X_k^{GG}. The curvature of L_k is given by

$$\pi_k^* \Theta_{L_k, \Psi^*_{h,p,\varepsilon}} = dd^c \log \Psi_{h,p,\varepsilon} \tag{9.26}$$

Our next goal is to compute precisely the curvature and to apply holomorphic Morse inequalities to $L \to X_k^{\mathrm{GG}}$ with the above metric. It might look a priori like an untractable problem, since the definition of $\Psi_{h,p,\varepsilon}$ is a rather unnatural one. However, the "miracle" is that the asymptotic behavior of $\Psi_{h,p,\varepsilon}$ as $\varepsilon_s/\varepsilon_{s-1} \to 0$ is in some sense uniquely defined and very natural. It will lead to a computable asymptotic formula, which is moreover simple enough to produce useful results.

9.27. Lemma. *On each coordinate chart U equipped with a holomorphic connection ∇ of $V_{|U}$, let us define the components of a k-jet $f \in J^k V$ by $\xi_s = \nabla^s f(0)$, and consider the rescaling transformation*

$$\rho_{\nabla,\varepsilon}(\xi_1, \xi_2, \ldots, \xi_k) = (\varepsilon_1^1 \xi_1, \varepsilon_2^2 \xi_2, \ldots, \varepsilon_k^k \xi_k) \quad \text{on } J^k_x V,\ x \in U$$

(it commutes with the $\mathbb{C}^*$-action but is otherwise unrelated and not canonically defined over X as it depends on the choice of ∇). Then, if p is a multiple of $\mathrm{lcm}(1, 2, \ldots, k)$ and $\varepsilon_s/\varepsilon_{s-1} \to 0$ for all $s = 2, \ldots, k$, the rescaled function $\Psi_{h,p,\varepsilon} \circ \rho_{\nabla,\varepsilon}^{-1}(\xi_1, \ldots, \xi_k)$ converges towards

$$\Big(\sum_{1 \leqslant s \leqslant k} \|\xi_s\|_h^{2p/s} \Big)^{1/p}$$

on every compact subset of $J^k V_{|U} \smallsetminus \{0\}$, uniformly in C^∞ topology.

Proof. Let $U \subset X$ be an open set on which $V_{|U}$ is trivial and equipped with some holomorphic connection ∇. Let us pick another holomorphic connection $\widetilde{\nabla} = \nabla + \Gamma$ where $\Gamma \in H^0(U, \Omega^1_X \otimes \mathrm{Hom}(V, V)$. Then $\widetilde{\nabla}^2 f = \nabla^2 f + \Gamma(f)(f') \cdot f'$, and inductively we get

$$\widetilde{\nabla}^s f = \nabla^s f + P_s(f\,;\,\nabla^1 f, \ldots, \nabla^{s-1} f)$$

where $P(x\,;\,\xi_1, \ldots, \xi_{s-1})$ is a polynomial with holomorphic coefficients in $x \in U$ which is of weighted homogeneous degree s in $(\xi_1, \ldots, \xi_{s-1})$. In other words, the corresponding change in the parametrization of $J^k V_{|U}$ is given by a $\mathbb{C}^*$-homogeneous transformation

$$\widetilde{\xi}_s = \xi_s + P_s(x\,;\,\xi_1, \ldots, \xi_{s-1}).$$

Let us introduce the corresponding rescaled components

$$(\xi_{1,\varepsilon}, \ldots, \xi_{k,\varepsilon}) = (\varepsilon_1^1 \xi_1, \ldots, \varepsilon_k^k \xi_k), \qquad (\widetilde{\xi}_{1,\varepsilon}, \ldots, \widetilde{\xi}_{k,\varepsilon}) = (\varepsilon_1^1 \widetilde{\xi}_1, \ldots, \varepsilon_k^k \widetilde{\xi}_k).$$

Then

$$\begin{aligned}\widetilde{\xi}_{s,\varepsilon} &= \xi_{s,\varepsilon} + \varepsilon_s^s\, P_s(x\,;\,\varepsilon_1^{-1}\xi_{1,\varepsilon}, \ldots, \varepsilon_{s-1}^{-(s-1)}\xi_{s-1,\varepsilon}) \\ &= \xi_{s,\varepsilon} + O(\varepsilon_s/\varepsilon_{s-1})^s\, O(\|\xi_{1,\varepsilon}\| + \ldots + \|\xi_{s-1,\varepsilon}\|^{1/(s-1)})^s\end{aligned}$$

and the error terms are thus polynomials of fixed degree with arbitrarily small coefficients as $\varepsilon_s/\varepsilon_{s-1} \to 0$. Now, the definition of $\Psi_{h,p,\varepsilon}$ consists of glueing the sums

$$\sum_{1 \leqslant s \leqslant k} \varepsilon_s^{2p} \|\xi_k\|_h^{2p/s} = \sum_{1 \leqslant s \leqslant k} \|\xi_{k,\varepsilon}\|_h^{2p/s}$$

corresponding to $\xi_k = \nabla_\alpha^s f(0)$ by means of the partition of unity $\sum \theta_\alpha(x) = 1$. We see that by using the rescaled variables $\xi_{s,\varepsilon}$ the changes occurring when replacing a connection ∇_α by an alternative one ∇_β are arbitrary small in C^∞ topology, with error terms uniformly controlled in terms of the ratios $\varepsilon_s/\varepsilon_{s-1}$ on all compact subsets of $V^k \smallsetminus \{0\}$. This shows that in C^∞ topology, $\Psi_{h,p,\varepsilon} \circ \rho^{-1}_{\nabla,\varepsilon}(\xi_1, \ldots, \xi_k)$ converges uniformly towards $(\sum_{1 \leqslant s \leqslant k} \|\xi_k\|_h^{2p/s})^{1/p}$, whatever the trivializing open set U and the holomorphic connection ∇ used to evaluate the components and perform the rescaling are. $\square$

Now, we fix a point $x_0 \in X$ and a local holomorphic frame $(e_\alpha(z))_{1 \leqslant \alpha \leqslant r}$ satisfying (9.22) on a neighborhood U of x_0. We introduce the rescaled components $\xi_s = \varepsilon_s^s \nabla^s f(0)$ on $J^k V_{|U}$ and compute the curvature of

$$\Psi_{h,p,\varepsilon} \circ \rho^{-1}_{\nabla,\varepsilon}(z\,;\,\xi_1, \ldots, \xi_k) \simeq \Bigg(\sum_{1 \leqslant s \leqslant k} \|\xi_s\|_h^{2p/s}\Bigg)^{1/p}$$

(by Lemma 9.27, the errors can be taken arbitrary small in C^∞ topology). We write $\xi_s = \sum_{1 \leqslant \alpha \leqslant r} \xi_{s\alpha} e_\alpha$. By (9.22) we have

$$\|\xi_s\|_h^2 = \sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha} \overline{\xi}_{s\beta} + O(|z|^3 |\xi|^2).$$

The question is to evaluate the curvature of the weighted metric defined by

$$\begin{aligned}\Psi(z\,;\,\xi_1, \ldots, \xi_k) &= \Bigg(\sum_{1 \leqslant s \leqslant k} \|\xi_s\|_h^{2p/s}\Bigg)^{1/p} \\ &= \Bigg(\sum_{1 \leqslant s \leqslant k} \Big(\sum_\alpha |\xi_{s\alpha}|^2 + \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} z_i \overline{z}_j \xi_{s\alpha} \overline{\xi}_{s\beta}\Big)^{p/s}\Bigg)^{1/p} + O(|z|^3).\end{aligned}$$

We set $|\xi_s|^2 = \sum_\alpha |\xi_{s\alpha}|^2$. A straightforward calculation yields

$$\begin{aligned}\log\Psi(z\,;\,\xi_1,\ldots,\xi_k) &= \\ &= \frac{1}{p}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s} + \sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}z_i\overline{z}_j\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}+O(|z|^3).\end{aligned}$$

By (9.26), the curvature form of $L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given at the central point x_0 by the following formula.

9.28. Proposition. *With the above choice of coordinates and with respect to the rescaled components $\xi_s = \varepsilon_s^s\nabla^s f(0)$ at $x_0\in X$, we have the approximate expression*

$$\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(x_0,[\xi]) \simeq \omega_{a,r,p}(\xi) + \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where the error terms are $O(\max_{2\leqslant s\leqslant k}(\varepsilon_s/\varepsilon_{s-1})^s)$ uniformly on the compact variety X_k^{GG}. Here $\omega_{a,r,p}$ is the (degenerate) Kähler metric associated with the weight $a = (1^{[r]},2^{[r]},\ldots,k^{[r]})$ of the canonical $\mathbb{C}^*$ action on J^kV.

Thanks to the uniform approximation, we can (and will) neglect the error terms in the calculations below. Since $\omega_{a,r,p}$ is positive definite on the fibers of $X_k^{\mathrm{GG}}\to X$ (at least outside of the axes $\xi_s=0$), the index of the $(1,1)$ curvature form $\Theta_{L_k,\Psi^*_{h,p,\varepsilon}}(z,[\xi])$ is equal to the index of the $(1,1)$-form

$$\gamma_k(z,\xi) := \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j \tag{9.29}$$

depending only on the differentials $(dz_j)_{1\leqslant j\leqslant n}$ on X. The q-index integral of $(L_k,\Psi^*_{h,p,\varepsilon})$ on X_k^{GG} is therefore equal to

$$\begin{aligned}\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} &= \\ &= \frac{(n+kr-1)!}{n!(kr-1)!}\int_{z\in X}\int_{\xi\in P(1^{[r]},\ldots,k^{[r]})}\omega^{kr-1}_{a,r,p}(\xi)\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n\end{aligned}$$

where $\mathbb{1}_{\gamma_k,q}(z,\xi)$ is the characteristic function of the open set of points where $\gamma_k(z,\xi)$ has signature $(n-q,q)$ in terms of the dz_j's. Notice that since $\gamma_k(z,\xi)^n$ is a determinant, the product $\mathbb{1}_{\gamma_k,q}(z,\xi)\gamma_k(z,\xi)^n$ gives rise to a continuous function on X_k^{GG}. Formula 9.20 with $r_1=\ldots=r_k=r$ and $a_s=s$ yields the slightly more explicit integral

$$\begin{aligned}\int_{X_k^{\mathrm{GG}}(L_k,q)}\Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(n+kr-1)!}{n!(k!)^r}\ \times \\ \int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx\,d\mu(u),\end{aligned}$$

where $g_k(z,x,u) = \gamma_k(z,x_1^{1/2p}u_1,\ldots,x_k^{k/2p}u_k)$ is given by

$$g_k(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \tag{9.30}$$

and $\mathbb{1}_{g_k,q}(z,x,u)$ is the characteristic function of its q-index set. Here

$$d\nu_{k,r}(x) = (kr-1)!\,\frac{(x_1\ldots x_k)^{r-1}}{(r-1)!^k}\,dx \tag{9.31}$$

is a probability measure on Δ_{k-1}, and we can rewrite

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\ \times$$

$$\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k} \mathbb{1}_{g_k,q}(z,x,u)g_k(z,x,u)^n\,d\nu_{k,r}(x)\,d\mu(u). \tag{9.32}$$

Now, formula (9.30) shows that $g_k(z,x,u)$ is a "Monte Carlo" evaluation of the curvature tensor, obtained by averaging the curvature at random points $u_s\in S^{2r-1}$ with certain positive weights x_s/s; we should then think of the k-jet f as some sort of random variable such that the derivatives $\nabla^k f(0)$ are uniformly distributed in all directions. Let us compute the expected value of $(x,u)\mapsto g_k(z,x,u)$ with respect to the probability measure $d\nu_{k,r}(x)\,d\mu(u)$. Since $\int_{S^{2r-1}} u_{s\alpha}\overline{u}_{s\beta}d\mu(u_s)=\frac{1}{r}\delta_{\alpha\beta}$ and $\int_{\Delta_{k-1}} x_s\,d\nu_{k,r}(x)=\frac{1}{k}$, we find

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\sum_{1\leqslant s\leqslant k}\frac{1}{s}\cdot\frac{i}{2\pi}\sum_{i,j,\alpha} c_{ij\alpha\alpha}(z)\,dz_i\wedge d\overline{z}_j.$$

In other words, we get the normalized trace of the curvature, i.e.

$$\mathbf{E}(g_k(z,\bullet,\bullet)) = \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{\det(V^*),\det h^*}, \tag{9.33}$$

where $\Theta_{\det(V^*),\det h^*}$ is the $(1,1)$-curvature form of $\det(V^*)$ with the metric induced by h. It is natural to guess that $g_k(z,x,u)$ behaves asymptotically as its expected value $\mathbf{E}(g_k(z,\bullet,\bullet))$ when k tends to infinity. If we replace brutally g_k by its expected value in (9.32), we get the integral

$$\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\,\frac{1}{(kr)^n}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\int_X \mathbb{1}_{\eta,q}\eta^n,$$

where $\eta:=\Theta_{\det(V^*),\det h^*}$ and $\mathbb{1}_{\eta,q}$ is the characteristic function of its q-index set in X. The leading constant is equivalent to $(\log k)^n/n!(k!)^r$ modulo a multiplicative factor $1+O(1/\log k)$. By working out a more precise analysis of the deviation, the following result has been proved in [Dem11] and [Dem12].

9.34. Probabilistic estimate. *Fix smooth Hermitian metrics h on V and $\omega=\frac{i}{2\pi}\sum\omega_{ij}dz_i\wedge d\overline{z}_j$ on X. Denote by $\Theta_{V,h}=-\frac{i}{2\pi}\sum c_{ij\alpha\beta}dz_i\wedge d\overline{z}_j\otimes e^*_\alpha\otimes e_\beta$ the curvature tensor of V with respect to an h-orthonormal frame (e_α), and put*

$$\eta(z)=\Theta_{\det(V^*),\det h^*}=\frac{i}{2\pi}\sum_{1\leqslant i,j\leqslant n}\eta_{ij}dz_i\wedge d\overline{z}_j,\qquad \eta_{ij}=\sum_{1\leqslant\alpha\leqslant r}c_{ij\alpha\alpha}.$$

Finally consider the k-jet line bundle $L_k=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)\to X_k^{\mathrm{GG}}$ equipped with the induced metric $\Psi^*_{h,p,\varepsilon}$ (as defined above, with $1=\varepsilon_1\gg\varepsilon_2\gg\ldots\gg\varepsilon_k>0$). When k tends to infinity, the integral of the top power of the curvature of L_k on its q-index set $X_k^{\mathrm{GG}}(L_k,q)$ is given by

$$\int_{X_k^{\mathrm{GG}}(L_k,q)} \Theta^{n+kr-1}_{L_k,\Psi^*_{h,p,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_X \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg)$$

for all $q = 0, 1, \ldots, n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of Θ_V, η and ω. Moreover, the left hand side is identically zero for $q > n$.

The final statement follows from the observation that the curvature of L_k is positive along the fibers of $X_k^{\mathrm{GG}} \to X$, by the plurisubharmonicity of the weight (this is true even when the partition of unity terms are taken into account, since they depend only on the base); therefore the q-index sets are empty for $q > n$. It will be useful to extend the above estimates to the case of sections of

$$L_{F,k} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big) \tag{9.35}$$

where $F \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is an arbitrary $\mathbb{Q}$-line bundle on X and $\pi_k : X_k^{\mathrm{GG}} \to X$ is the natural projection. We assume here that F is also equipped with a smooth Hermitian metric h_F. In formulas (9.32–9.34), the curvature $g_{F,k}(z,x,u)$ of $L_{F,k}$ takes the form

$$g_{F,k}(z,x,u) = g_k(z,x,u) - \frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\Theta_{F,h_F}(z), \tag{9.36}$$

and by the same calculations its normalized expected value is

$$\eta_F(z) := \frac{1}{\frac{1}{kr}(1+\frac{1}{2}+\ldots+\frac{1}{k})}\mathbf{E}(g_k^F(z,\bullet,\bullet)) = \Theta_{\det V^*,\det h^*}(z) - \Theta_{F,h_F}(z). \tag{9.37}$$

Then the variance estimate for $g_{F,k}$ is the same as the variance estimate for g_k, and the recentered L^p bounds are still valid, since our forms are just shifted by subtracting the constant smooth term $\Theta_{F,h_F}(z)$. The probabilistic estimate 9.34 is therefore still true in exactly the same form for $L_{F,k}$, provided we use $g_{F,k}$ and η_F instead of g_k and η. An application of holomorphic Morse inequalities gives the desired cohomology estimates for

$$\begin{aligned} h^q\Big(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big) \\ = h^q(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^*\mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big), \end{aligned}$$

provided m is sufficiently divisible to give a multiple of F which is a $\mathbb{Z}$-line bundle.

9.38. Theorem. *Let (X,V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) smooth Hermitian structure on V and F respectively. We define*

$$\begin{aligned} L_{F,k} &= \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big), \\ \eta_F &= \Theta_{\det V^*,\det h^*} - \Theta_{F,h_F} = \Theta_{\det V^*\otimes F^*,\det h^*}. \end{aligned}$$

Then for all $q \geqslant 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

(a) $$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_{F,k}^{\otimes m})) \leqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta_F,q)}(-1)^q\eta_F^n + O((\log k)^{-1})\bigg),$$

(b) $$h^0(X_k^{\mathrm{GG}}, \mathcal{O}(L_{F,k}^{\otimes m})) \geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta_F,\leqslant 1)}\eta_F^n - O((\log k)^{-1})\bigg),$$

(c) $$\chi(X_k^{\mathrm{GG}}, \mathcal{O}(L_{F,k}^{\otimes m})) = \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\big(c_1(V^*\otimes F^*)^n + O((\log k)^{-1})\big).$$

Green and Griffiths [GrGr79] already checked the Riemann-Roch calculation (9.38 c) in the special case $V = T^*_X$ and $F = \mathcal{O}_X$. Their proof is much simpler since it relies only on Chern class calculations, but it cannot provide any information on the individual cohomology groups, except in very special cases where vanishing theorems can be applied; in fact in dimension 2, the Euler characteristic satisfies $\chi = h^0 - h^1 + h^2 \leqslant h^0 + h^2$, hence it is enough to get the vanishing of the top cohomology group H^2 to infer $h^0 \geqslant \chi$; this works for surfaces by means of a well-known vanishing theorem of Bogomolov which implies in general

$$H^n\Big(X, E^{\mathrm{GG}}_{k,m}T^*_X \otimes \mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\Big) = 0$$

as soon as $K_X \otimes F^*$ is big and $m \gg 1$.

In fact, thanks to Bonavero's singular holomorphic Morse inequalities [Bon93], everything works almost unchanged in the case where $V \subset T_X$ has singularities and h is an admissible metric on V (see Definition 9.7). We only have to find a blow-up $\mu : \widetilde{X}_k \to X_k$ so that the resulting pull-backs μ^*L_k and μ^*V are locally free, and $\mu^*\det h^*$, $\mu^*\Psi_{h,p,\varepsilon}$ only have divisorial singularities. Then η is a $(1,1)$-current with logarithmic poles, and we have to deal with smooth metrics on $\mu^*L^{\otimes m}_{F,k} \otimes \mathcal{O}(-mE_k)$ where E_k is a certain effective divisor on X_k (which, by our assumption in 9.7, does not project onto X). The cohomology groups involved are then the twisted cohomology groups

$$H^q(X^{\mathrm{GG}}_k, \mathcal{O}(L^{\otimes m}_{F,k}) \otimes \mathcal{J}_{k,m})$$

where $\mathcal{J}_{k,m} = \mu_*(\mathcal{O}(-mE_k))$ is the corresponding multiplier ideal sheaf, and the Morse integrals need only be evaluated in the complement of the poles, that is on $X(\eta, q) \smallsetminus S$ where $S = \mathrm{Sing}(V) \cup \mathrm{Sing}(h)$. Since

$$(\pi_k)_*\big(\mathcal{O}(L^{\otimes m}_{F,k}) \otimes \mathcal{J}_{k,m}\big) \subset E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

we still get a lower bound for the H^0 of the latter sheaf (or for the H^0 of the un-twisted line bundle $\mathcal{O}(L^{\otimes m}_k)$ on X^{GG}_k). If we assume that $K_V \otimes F^*$ is big, these considerations also allow us to obtain a strong estimate in terms of the volume, by using an approximate Zariski decomposition on a suitable blow-up of (X, V). The following corollary implies in particular Theorem 9.3.

9.39. Corollary. *If F is an arbitrary $\mathbb{Q}$-line bundle over X, one has*

$$\begin{aligned} &h^0\Big(X^{\mathrm{GG}}_k, \mathcal{O}_{X^{\mathrm{GG}}_k}(m) \otimes \pi^*_k\mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)\\ &\qquad\geqslant \frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\Big(\mathrm{Vol}(K_V \otimes F^*) - O((\log k)^{-1})\Big) - o(m^{n+kr-1}),\end{aligned}$$

when $m \gg k \gg 1$, in particular there are many sections of the k-jet differentials of degree m twisted by the appropriate power of F if $K_V \otimes F^*$ is big.

Proof. The volume is computed here as usual, i.e. after performing a suitable modification $\mu : \widetilde{X} \to X$ which converts K_V into an invertible sheaf. There is of course nothing to prove if $K_V \otimes F^*$ is not big, so we can assume $\mathrm{Vol}(K_V \otimes F^*) > 0$. Let us fix smooth Hermitian metrics h_0 on T_X and h_F on F. They induce a metric $\mu^*(\det h_0^{-1} \otimes h_F^{-1})$ on $\mu^*(K_V \otimes F)$ which, by our definition of K_V, is a smooth metric. By the result of Fujita

[Fuj94] on approximate Zariski decomposition, for every $\delta > 0$, one can find a modification $\mu_\delta : \widetilde{X}_\delta \to X$ dominating μ such that

$$\mu_\delta^*(K_V \otimes F^*) = \mathcal{O}_{\widetilde{X}_\delta}(A + E)$$

where A and E are $\mathbb{Q}$-divisors, A ample and E effective, with

$$\mathrm{Vol}(A) = A^n \geqslant \mathrm{Vol}(K_V \otimes F^*) - \delta.$$

If we take a smooth metric h_A with positive definite curvature form Θ_{A,h_A}, then we get a singular Hermitian metric $h_A h_E$ on $\mu_\delta^*(K_V \otimes F^*)$ with poles along E, i.e. the quotient $h_A h_E / \mu^*(\det h_0^{-1} \otimes h_F^{-1})$ is of the form $e^{-\varphi}$ where φ is quasi-psh with log poles $\log|\sigma_E|^2$ (mod $C^\infty(\widetilde{X}_\delta)$) precisely given by the divisor E. We then only need to take the singular metric h on T_X defined by

$$h = h_0 e^{\frac{1}{r}(\mu_\delta)^*\varphi}$$

(the choice of the factor $\frac{1}{r}$ is there to correct adequately the metric on $\det V$). By construction h induces an admissible metric on V and the resulting curvature current $\eta_F = \Theta_{K_V,\det h^*} - \Theta_{F,h_F}$ is such that

$$\mu_\delta^* \eta_F = \Theta_{A,h_A} + [E], \qquad [E] = \text{current of integration on } E.$$

Then the 0-index Morse integral in the complement of the poles is given by

$$\int_{X(\eta,0)\smallsetminus S} \eta_F^n = \int_{\widetilde{X}_\delta} \Theta_{A,h_A}^n = A^n \geqslant \mathrm{Vol}(K_V \otimes F) - \delta$$

and (9.39) follows from the fact that δ can be taken arbitrary small. $\square$

9.40. Example. In some simple cases, the above estimates can lead to very explicit results. Take for instance X to be a smooth complete intersection of multidegree $(d_1, d_2, \ldots, d_s)$ in $\mathbb{P}^{n+s}_{\mathbb{C}}$ and consider the absolute case $V = T_X$. Then $K_X = \mathcal{O}_X(d_1 + \ldots + d_s - n - s - 1)$ and one can check via explicit bounds of the error terms (cf. [Dem11], [Dem12]) that a sufficient condition for the existence of sections is

$$k \geqslant \exp\Big(7.38\, n^{n+1/2}\Big(\frac{\sum d_j + 1}{\sum d_j - n - s - a - 1}\Big)^n\Big).$$

This is good in view of the fact that we can cover arbitrary smooth complete intersections of general type. On the other hand, even when the degrees d_j tend to $+\infty$, we still get a large lower bound $k \sim \exp(7.38\, n^{n+1/2})$ on the order of jets, and this is far from being optimal: Diverio [Div08, Div09] has shown e.g. that one can take $k = n$ for smooth hypersurfaces of high degree, using the algebraic Morse inequalities of Trapani [Tra95]. The next paragraph uses essentially the same idea, in our more analytic setting. $\square$

§9.D. Non probabilistic estimate of the Morse integrals

We assume here that the curvature tensors $(c^{(s)}_{ij\alpha\beta})$ satisfy a lower bound

$$\sum_{i,j,\alpha,\beta} c^{(s)}_{ij\alpha\beta} \xi_i \overline{\xi}_j u_\alpha \overline{u}_\beta \geqslant -\sum \gamma_{ij} \xi_i \overline{\xi}_j\, |u|^2, \qquad \forall \xi \in T_X,\ \forall u \in V\langle\Delta^{(s)}\rangle \tag{9.41}$$

for some semipositive $(1,1)$-form $\gamma = \frac{i}{2\pi}\sum\gamma_{ij}(z)\,dz_i\wedge d\overline{z}_j$ on X. This is the same as assuming that the curvature tensor of $V^*\langle\Delta^{(s)}\rangle, h^*)$ satisfies the semipositivity condition

$$(9.41') \qquad \Theta_{V\langle\Delta^{(s)}\rangle^*,h^*} + \gamma\otimes \mathrm{Id}_{V^*\langle\Delta^{(s)}\rangle} \geqslant 0$$

in the sense of Griffiths. Thanks to the compactness of X, such a form γ always exists if h is an admissible metric on V. Now, instead of replacing Θ_V with its trace free part $\widetilde{\Theta}_V$ and exploiting a Monte Carlo convergence process, we replace Θ_V with $\Theta_V^\gamma = \Theta_V - \gamma\otimes\mathrm{Id}_V \leqslant 0$, i.e. $c^{(s)}_{ij\alpha\beta}$ by $c^{(s,\gamma)}_{ij\alpha\beta} = c^{(s)}_{ij\alpha\beta} + \gamma_{ij}\delta_{\alpha\beta}$. Also, we take a line bundle $F = A$ with $\Theta_{A,h_A}\geqslant 0$, i.e. F semipositive. Then our earlier formulas (9.28), (9.35), (9.36) become instead

$$(9.42) \qquad g_k^\gamma(z,x,u) = \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{1}{s}x_s\sum_{i,j,\alpha,\beta} c^{(s,\gamma)}_{ij\alpha\beta}(z)\,u_{s\alpha}\overline{u}_{s\beta}\,dz_i\wedge d\overline{z}_j \geqslant 0,$$

$$(9.43) \qquad L_{A,k} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big),$$

$$(9.44) \qquad \Theta_{L_{A,k}} = \omega_{a,r,p}(\xi) + g_{A,k}(z,x,u),$$

$$g_{A,k}(z,x,u) = g_k^\gamma(z,x,u) - \frac{1}{kr}\Big(\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\big(\Theta_{A,h_A}(z)+r\gamma(z)\big).$$

In fact, replacing Θ_V by $\Theta_V - \gamma\otimes\mathrm{Id}_V$ has the effect of replacing $\Theta_{\det V^*} = \mathrm{Tr}\,\Theta_{V^*}$ by $\Theta_{\det V^*} + r\gamma$. The major gain that we have is that $\Theta_{L_{A,k}} = g_{A,k}$ is now expressed as a difference of semipositive $(1,1)$-forms, and we can exploit the following simple lemma, which is the key to derive algebraic Morse inequalities from their analytic form (cf. [Dem94], Theorem 12.3).

9.45. Lemma. *Let $\eta = \alpha - \beta$ be a difference of semipositive $(1,1)$-forms on an n-dimensional complex manifold X, and let $\mathbb{1}_{\eta,\leqslant q}$ be the characteristic function of the open set where η is non degenerate with a number of negative eigenvalues at most equal to q. Then*

$$(-1)^q\mathbb{1}_{\eta,\leqslant q}\,\eta^n \leqslant \sum_{0\leqslant j\leqslant q}(-1)^{q-j}\alpha^{n-j}\beta^j,$$

in particular

$$\mathbb{1}_{\eta,\leqslant 1}\,\eta^n \geqslant \alpha^n - n\alpha^{n-1}\wedge\beta \qquad \textit{for } q=1.$$

Proof. Without loss of generality, we can assume $\alpha > 0$ positive definite, so that α can be taken as the base hermitian metric on X. Let us denote by

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$$

the eigenvalues of β with respect to α. The eigenvalues of $\eta = \alpha - \beta$ are then given by

$$1-\lambda_1 \leqslant \ldots \leqslant 1-\lambda_q \leqslant 1-\lambda_{q+1} \leqslant \ldots \leqslant 1-\lambda_n,$$

hence the open set $\{\lambda_{q+1} < 1\}$ coincides with the support of $\mathbb{1}_{\eta,\leqslant q}$, except that it may also contain a part of the degeneration set $\eta^n = 0$. On the other hand we have

$$\binom{n}{j}\alpha^{n-j}\wedge\beta^j = \sigma_n^j(\lambda)\,\alpha^n,$$

where $\sigma_n^j(\lambda)$ is the j-th elementary symmetric function in the λ_j's. Thus, to prove the lemma, we only have to check that

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}\sigma_n^j(\lambda)-1\!\!1_{\{\lambda_{q+1}<1\}}(-1)^q\prod_{1\leqslant j\leqslant n}(1-\lambda_j)\geqslant 0.$$

This is easily done by induction on n (just split apart the parameter λ_n and write $\sigma_n^j(\lambda)=\sigma_{n-1}^j(\lambda)+\sigma_{n-1}^{j-1}(\lambda)\,\lambda_n$). □

We apply here Lemma 9.45 with α,β replaced by

$$\alpha_k=g_k^\gamma(z,x,u),\qquad \beta_k=\beta_k^{(s),\gamma}=\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)(\Theta_{A,h_A}+r\gamma),$$

which are both semipositive by our assumptions. Then (9.32) leads to

$$\begin{aligned}&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta^{n+kr-1}_{L_{A,k},\Psi^*_{h,p,\varepsilon}}\\ &\quad=\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}1\!\!1_{\alpha_k-\beta_k,\leqslant 1}\,(\alpha_k-\beta_k)^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\quad\geqslant\frac{(n+kr-1)!}{n!(k!)^r(kr-1)!}\int_{z\in X}\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\big(\alpha_k^n-n\alpha_k^{n-1}\wedge\beta_k\big)\,d\nu_{k,r}(x)\,d\mu(u).\end{aligned}$$

The resulting integral now produces a "closed formula" which can be expressed solely in terms of Chern classes (at least if we assume that γ is the Chern form of some semipositive line bundle). It is just a matter of routine to find a sufficient condition for the positivity of the integral. One can first observe that α_k is bounded from above by taking the trace of $(c^{(s,\gamma)}_{ij\alpha\beta})$, in this way we get

$$0\leqslant\alpha_k\leqslant\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\big(\Theta_{\det V^*\langle\Delta^{(s)}\rangle}+r\gamma\big)$$

where the right hand side no longer depends on $u\in(S^{2r-1})^k$. Also, $\alpha_k=g_k^\gamma$ can be written as a sum of semipositive $(1,1)$-forms

$$g_k^\gamma=\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\theta^{s,\gamma}(u_s),\qquad \theta^{s,\gamma}(u)=\sum_{i,j,\alpha,\beta}c^{(s,\gamma)}_{ij\alpha\beta}u_\alpha\overline{u}_\beta\,dz_i\wedge d\overline{z}_j,$$

hence for $k\geqslant n$ we have

$$\alpha_k^n=(g_k^\gamma)^n\geqslant n!\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{x_{s_1}\ldots x_{s_n}}{s_1\ldots s_n}\,\theta^{s_1,\gamma}(u_{s_1})\wedge\theta^{s_2,\gamma}(u_{s_2})\wedge\ldots\wedge\theta^{s_n,\gamma)}(u_{s_n}).$$

Since $\int_{S^{2r-1}}\theta^{s,\gamma}(u)\,d\mu(u)=\frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*\langle\Delta^{(s)}\rangle}+\gamma\otimes\mathrm{Id})=\frac{1}{r}(\Theta_{\det V^*\langle\Delta^{(s)}\rangle}+r\gamma)$, we infer from this

$$\begin{aligned}&\int_{(x,u)\in\Delta_{k-1}\times(S^{2r-1})^k}\alpha_k^n\,d\nu_{k,r}(x)\,d\mu(u)\\ &\qquad\geqslant\frac{n!}{r^n}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\int_{\Delta_{k-1}}x_1\ldots x_n\,d\nu_{k,r}(x)\bigwedge_{\ell=1}^{n}\big(\Theta_{\det V^*\langle\Delta^{(s_\ell)}\rangle}+r\gamma\big).\end{aligned}$$

By putting everything together, we conclude:

9.46. Theorem. *Assume that the curvature of the orbifold bundles satisfy the lower bounds $\Theta_{V^*\langle\Delta^{(s)}\rangle} \geqslant -\gamma\otimes \mathrm{Id}_{V^*}$ (in the sense of Griffiths) with a semipositive $(1,1)$-form γ on X, for all $s=1,2,\ldots,k$. Also assume that their determinants admit an upper bound $\Theta_{\det V^*\langle\Delta^{(s)}\rangle} \leq \delta$ with another semipositive $(1,1)$-form δ on X. Then the Morse integral of the orbifold line bundle*

$$L_{A,k} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes \pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)A\Big), \qquad A\geqslant 0$$

satisfies for $k\geqslant n$ the inequality

$$\begin{aligned}\frac{1}{(n+kr-1)!}&\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)} \Theta^{n+kr-1}_{L_{A,k},\Psi^*_{h,p,\varepsilon}}\\ &\geqslant \frac{1}{n!(k!)^r(kr-1)!}\int_X c_{n,r,k}\sum_{1\leqslant s_1<\ldots<s_n\leqslant k}\frac{1}{s_1\ldots s_n}\bigwedge_{\ell=1}^{n}\big(\Theta_{\det V^*\langle\Delta^{(s_\ell)}\rangle}+r\gamma\big)\\ &\qquad\qquad - c'_{n,r,k}\big(\delta+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)\end{aligned}$$

where

$$\begin{aligned} c_{n,r,k} &= \frac{n!}{r^n}\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x),\\ c'_{n,r,k} &= \frac{n}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)\int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1}d\nu_{k,r}(x).\end{aligned}$$

Especially we have a lot of sections in $H^0(X_k^{\mathrm{GG}}, mL_{A,k})$, $m\gg 1$, as soon as the difference occurring in $(*)$ is positive.

The statement is also true for $k<n$, but then the term in factor of $c_{n,r,k}$ vanishes and the lower bound cannot be positive. By Corollary 9.11, it still provides a non trivial lower bound for $h^0(X_k^{\mathrm{GG}}, mL_{A,k}) - h^1(X_k^{\mathrm{GG}}, mL_{A,k})$, though. For $k\geqslant n$ we have $c_{n,r,k}>0$ and $(*)$ will be positive if $\Theta_{\det V^*}$ is large enough. By formulas (9.20) and (9.31) we get

$$\int_{\Delta_{k-1}} x_1\ldots x_n\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\,\frac{r!^n(r-1)!^{k-n}}{(n+rk-1)!} = \frac{(kr-1)!\,r^n}{(n+rk-1)!},$$

hence

$$c_{n,r,k} = \frac{n!\,(kr-1)!}{(n+kr-1)!} \tag{9.47}$$

(with equality for $k=n$). For every multi-index $\beta=(\beta_1,\ldots,\beta_k)\in\mathbb{N}^k$ with $\sum\beta_s = n-1$, we also find

$$\int_{\Delta_{k-1}} x_1^{\beta_1}\ldots x_k^{\beta_k}\,d\nu_{k,r}(x) = \frac{(kr-1)!}{(r-1)!^k}\,\frac{\prod_{s=1}^k(r+\beta_s-1)!}{(n+kr-2)!}\;\begin{cases}\leq \dfrac{(kr-1)!(n+r-2)!}{(r-1)!(n+kr-2)!},\\[2ex] \geq \dfrac{(kr-1)!\,r^{n-1}}{(n+kr-2)!},\end{cases}$$

because the maximum is attained for the length $n-1$ multi-index $\beta=(n-1,0,\ldots,0)$, and the minimum for $\beta=(1,\ldots,1,0,\ldots,0)$ (or any permutation). An expansion of $\big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\big)^{n-1}$ by means of the multinomial formula then yields

$$\tag{9.48} \int_{\Delta_{k-1}}\Big(\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\Big)^{n-1}d\nu_{k,r}(x)\ \begin{cases}\displaystyle\le\frac{(kr-1)!(n+r-2)!}{(r-1)!(n+kr-2)!}\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}\Big)^{n-1},\\[2ex] \displaystyle\ge\frac{(kr-1)!\,r^{n-1}}{(n+kr-2)!}\Big(\sum_{1\leqslant s\leqslant k}\frac{1}{s}\Big)^{n-1}.\end{cases}$$

The ratio between the upper bound and the lower bound is $\frac{(n+r-2)!}{r^{n-1}(r-1)!}$ which, for $r=n$ is $\sim 2^{-3/2}(4/e)^n$ by the Stirling formula; thus, when taking the upper bound, the error factor is at most exponential. From (9.47) and (9.48) and the fact that $\frac{n+kr-1}{k}=r+\frac{n-1}{k}\le n+r-1$, we infer

$$\tag{9.49} \begin{aligned} c''_{n,r,k}:=\frac{c'_{n,r,k}}{c_{n,r,k}/n!}&\le\frac{n(n+kr-1)}{kr}\,\frac{(n+r-2)!}{(r-1)!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\\ &\le\frac{n\,(n+r-1)!}{r!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n. \end{aligned}$$

The right hand side of (9.49) increases with r. For $r\le n$, the Stirling formula and the standard integral upper bound $1+\log k$ for the harmonic series partial sum yield

$$\tag{9.50} c''_{n,r,k}\le\frac{(2n)!}{2\,n!}(1+\log k)^n\le\frac{\sqrt{2n}\,(\frac{2n}{e})^{2n}}{2\sqrt{n}\,(\frac{n}{e})^n}(1+\log k)^n\le\frac{1}{\sqrt{2}}\big(4e^{-1}\,n(1+\log k)\big)^n.$$

9.51. Corollary. *Under the assumptions of Theorem 9.46, we have an inequality*

$$\begin{aligned}&\frac{1}{(n+kr-1)!}\int_{X_k^{\mathrm{GG}}(L_k,\leqslant 1)}\Theta^{n+kr-1}_{L_{A,k},\Psi^*_{h,p,\varepsilon}}\\ &\quad\geqslant\frac{1}{n!(k!)^r(n+kr-1)!}\int_X\bigwedge_{\ell=1}^{n}\big(\Theta_{\det V^*\langle\Delta^{(\ell)}\rangle}+r\gamma\big)-c''_{n,r,k}\big(\delta+r\gamma\big)^{n-1}\wedge\big(\Theta_{A,h_A}+r\gamma\big)\end{aligned}$$

with

$$c''_{n,r,k}\le\frac{n(n+kr-1)}{k}\,\frac{(n+r-2)!}{r!}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)^n\le\frac{1}{\sqrt{2}}\big(4e^{-1}\,n(1+\log k)\big)^n.$$

A less refined estimate is $c''_{n,r,k}\le(3n\log k)^n$ for $k\ge n\ge 2$. In view of concrete applications of these estimates, one can rely on the following lemma.

9.52. Lemma. *Let $X\subset\mathbb{P}^N$ be a projective variety and (X,V,Δ) an orbifold directed structure where Δ is a normal crossing divisor in X transverse to V. Then the orbifold vector bundle $V\langle D\rangle$ possesses a smooth hermitian metric such that the induced curvature tensor of $V^*\langle D\rangle\otimes\mathcal{O}_X(2)$ is Griffiths semi-positive.*

Proof. Very straightforward calculation, for the obvious metric induced by the Fubini-Study metric on $\mathbb{P}^N$!!

A very special case is the case when $X=\mathbb{P}^n$, $V=T_X$ and $\Delta=\sum(1-\frac{1}{\rho_j})\Delta_j$ is a normal crossing divisor, with components Δ_j of degree δ_j. Then

$$\det V^*\langle\Delta^{(s)}\rangle=\mathcal{O}_{\mathbb{P}^n}\Big(-n-1+\sum(1-s/\rho_j)_+\delta_j\Big).$$

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