1. Lifting of sections of the Green–Griffiths graded bundle.

In this short section, we outline a nice geometric interpretation by Benoît Cadorel of the arguments leading to [VIASM, Prop. 9.28]. We then give an alternative elementary proof.

1.1. Morse inequalities and semi-continuity.

1.1. Morse inequalities and semi-continuity. A simple observation is that for a sheaf $E \to X$ Morse inequalities ([dem]) are about not $h^q(X, E)$ but about alternate sums

$$h^q(X,E) - h^{q-1}(X,E) + ... + (-1)^q h^0(X,E).$$

Now, let $\mathcal{X} \to S$ be a proper and flat morphism of reduced complex spaces, and let $(X_t)_{t\in s}$ be the fibres. Given a sheaf $\mathcal{E}$ over $\mathcal{X}$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules, inducing on the fibres a family of sheaves $(E_t \to X_t)_{t\in S}$, the following semicontinuity property holds ([CRAS]): For every $q \geq 0$, the alternate sum

$$h^q(X_t,E_t) - h^{q-1}(X_t,E_t) + ... + (-1)^q h^0(X_t,E_t)$$

is upper semicontinuous for the (analytic) Zariski topology.

This gives a very simple geometric argument to control dimensions of spaces of global sections using holomorphic Morse inequalities on deformations.

1.2. Rees deformation construction (after Cadorel). Recall after [Cad, Prop. 4.2, Prop. 4.5], that Rees deformation construction allows one to construct natural deformations of Green–Griffiths jets spaces to weighted projectivized bundles.

Let (X, Δ) be a smooth orbifold pair, and $\pi\colon Y \to (X, \Delta)$ be an adapted covering. For $k \in \mathbb{Z}_+$, recall that the Green-Griffiths jet bundle of graded algebras $E_{k,\bullet}\Omega_{(\pi,\Delta)} \to Y$ admits a natural filtration, the *Green–Griffiths filtration* ([GG,CDR]) , with associated graded bundle (of graded algebras):

$$E^{\mathrm{lin}}_{k,\bullet}\Omega_{(\pi,\Delta)} = \bigoplus_{N\geq 1}\ \bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=N} \mathrm{Sym}^{\ell_1}(\Omega_{\pi,\Delta^{(1)}}) \otimes \cdots \otimes \mathrm{Sym}^{\ell_1}(\Omega_{\pi,\Delta^{(k)}}).$$

Applying the Proj functor, one gets a weighted projective bundle:

$$\mathbb{P}_{(1,\cdots,k)}\left(\Omega_{\pi,\Delta^{(1)}} \oplus \cdots \oplus \Omega_{\pi,\Delta^{(k)}}\right) = \mathbf{Proj}\Big(E^{\mathrm{lin}}_{k,\bullet}\Omega_{(\pi,\Delta)}\Big) \xrightarrow{\rho_k} Y,$$

Then, following mutadis mutandus the arguments of Cadorel, one constructs a family $Y \xleftarrow{\rho_k} \mathcal{Y}_k \to \mathbb{C}$ parametrized by $\mathbb{C}$, with a canonical line bundle $\mathcal{O}_{\mathcal{Y}_k}(1)$ such that:

- the central fiber $\mathcal{Y}_{k,0}$ is $\mathbb{P}_{(1,\cdots,k)}\left(\Omega_{\pi,\Delta^{(1)}} \oplus \cdots \oplus \Omega_{\pi,\Delta^{(k)}}\right)$ and the restriction of $\mathcal{O}_{\mathcal{Y}k}(1)$ conincide with the canonical line bundle of this weighted projective bundle. Hence $\rho_{k*}\mathcal{O}_{\mathcal{Y}_{k,0}}(N) = E^{\mathrm{lin}}_{k,N}\Omega_{(\pi,\Delta)}$.
- other fibers $\mathcal{Y}_{k,\lambda}$ are isomorphic to the singular variety $J_k(\pi,\Delta)/\mathbb{C}^*$, for the natural $\mathbb{C}^*$-action by homotheties, and $\rho_{k*}\mathcal{O}_{\mathcal{Y}_{k,\lambda}}(N) \simeq E_{k,N}\Omega_{(\pi,\Delta)}$.

Applying the semicontinuity result of Demailly, and working with holomorphic inequalities, we obtain a control about dimensions of cohomology spaces of $E_{k,N}\Omega$ in terms of dimensions of cohomology spaces of the much simpler $E^{\mathrm{lin}}_{k,N}\Omega$. In particular, one can work directly on the weighted projective space with symmetric differentials, and it is not necessary to define higher order jet metrics.

Lemma 1 (Cadorel). *For any $j \in \mathbb{Z}_+$:*

$$(-1)^j \sum_{i=0}^{j} (-1)^i h^i(Y, E^{lin}_{k,N}\Omega_{(\pi,\Delta)}) \geq (-1)^j \sum_{i=0}^{j} (-1)^i h^i(Y, E_{k,N}\Omega_{(\pi,\Delta)}).$$

In particular:

$$h^0(Y, E_{k,N}\Omega_{(\pi,\Delta)}) \geq h^0(Y, E^{lin}_{k,N}\Omega_{(\pi,\Delta)}) - h^1(Y, E^{lin}_{k,N}\Omega_{(\pi,\Delta)}).$$

1.3. **Alternative proof.** The previous considerations give a nice geometric interpretation of the argumentation already present in [VIASM]. Of course, once stated, the result can be easily directly recovered for multifiltrations of vector bundles by much more elementary arguments, as follows.

Lemma 2. *Let $E \to X$ be a vector bundle over a variety, and let $\Sigma \to X$ be the graded bundle associated to a multifiltration of E:*

$$(-1)^j \sum_{i=0}^{j} (-1)^i h^i(X, \Sigma) \geq (-1)^j \sum_{i=0}^{j} (-1)^i h^i(X, E).$$

Proof. By transitivity of inequalities, it is sufficient to prove the result for simple filtrations. Then, by induction on the length of filtrations, it is sufficient to prove the result for exact sequences $0 \to S \to E \to Q \to 0$ of vector bundles on X. Consider the associated (truncated) long exact sequence in cohomology:

$$0 \to H^0(S) \to H^0(E) \to H^0(Q) \xrightarrow{\delta_1} \dots$$
$$\xrightarrow{\delta_{j-1}} H^j(S) \to H^j(E) \to H^j(Q) \xrightarrow{\delta_j} \mathrm{Im}(\delta_j) \to 0.$$

By the rank theorem of linear algebra,

$$0 \leq \mathrm{rk}(\delta_j) = (-1)^j \sum_{i=0}^{j} (-1)^i (h^i(X, Q) - h^i(X, E) + h^i(X, S)).$$

The result follows, since here $h^i(X, \Sigma) = h^i(X, Q) + h^i(X, S)$. □

[Cad] "Jet differentials on toroidal compactifications of ball quotients" Benoît Cadorel.

[CRAS] JP Demailly "Proprits de semi-continuit de la cohomologie et de la dimension de Kodaira-Iitaka" C. R. Acad. Sci. Paris Sr. I Math. 320 (1995), 341-346