Magnetic fields and Morse inequalities for d"-cohomology, *English translation of:*
Champs magnétiques et inégalités de Morse pour la d"-cohomologie
Annales de l'Institut Fourier, tome **35**, n°4 (1985), p. 189–229.

MAGNETIC FIELDS AND MORSE INEQUALITIES FOR d"-COHOMOLOGY

by Jean-Pierre DEMAILLY

0. Introduction.

Let X be a compact complex analytic manifold of dimension n, F a holomorphic vector bundle of rank r, and E a holomorphic line bundle with a Hermitian structure of class $\mathcal{C}^\infty$ on X. Let $D = D' + D''$ be the canonical connection of E and $c(E) = D^2 = D'D'' + D''D'$ be the curvature tensor of this connection. Let us denote by $X(q)$, $0 \leqslant q \leqslant n$, the open subset of points of $x \in X$ that are of index q, i.e. points x at which the $(1,1)$ curvature form $ic(E)(x)$ has exactly q negative and $n-q$ positive eigenvalues. We also put

$$X(\leqslant q) = X(0) \cup X(1) \cup \ldots \cup X(q).$$

We then prove the following Morse inequalities, which bound the dimension of the cohomology groups $H^q(X, E^{\otimes k} \otimes F)$ in terms of integral invariants of the curvature of E.

Theorem 0.1. – *For all degrees $q = 0, 1, \ldots, n$, the following asymptotic inequalities hold when k tends to $+\infty$.*

(a) *Morse inequalities* :

$$\dim H^q(X, E^{\otimes k} \otimes F) \leqslant r\frac{k^n}{n!}\int_{X(q)} (-1)^q \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

(b) *Strong Morse inequalities* :

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(X, E^{\otimes k} \otimes F) \leqslant r\frac{k^n}{n!}\int_{X(\leqslant q)} (-1)^q \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

Keywords : Morse inequalities, d''-cohomology, Hermitian vector bundle, curvature form, magnetic field, Schrödinger operator, Bochner-Kodaira-Nakano identity, Moišezon space.

(c) *Asymptotic Riemann-Roch formula* :

$$\sum_{q=0}^{n}(-1)^q \dim H^q(X, E^{\otimes k}\otimes F) = r\frac{k^n}{n!}\int_X \Big(\frac{i}{2\pi}c(E)\Big)^n + o(k^n).$$

Estimates 0.1 (a), (b) are new as far as we know, even in the case of projective varieties. The asymptotic equality 0.1 (c), on the other hand, is a weakened version of the Riemann-Roch formula, which is itself a special case of Hirzebruch's general formula or, alternatively, of the Atiyah-Singer index theorem [1]. Indeed, the Riemann-Roch formula expresses the Euler-Poincaré characteristic

$$\chi(X, E^{\otimes k}\otimes F) = \sum_{q=0}^{n}(-1)^q \dim H^q(X, E^{\otimes k}\otimes F)$$

under the form

$$\chi(X, E^{\otimes k}\otimes F) = r\frac{k^n}{n!}c_1(E)^n + P_{n-1}(k)\,, \tag{0.2}$$

where $P_{n-1}(k)\in\mathbb{Q}[k]$ is a polynomial of degree $\leqslant n-1$ and $c_1(E)\in H^2(X,\mathbb{Z})$ is the first class of Chern de E, represented in De Rham's cohomology by the closed $(1,1)$-form $\frac{i}{2\pi}c(E)$ (see for example [16]). It can be observed that the numerical constant of inequality 0.1 (a) is optimal, as shown by the example of the line bundle given by the total tensor product $\mathcal{O}(1)^{\boxtimes n-q}\boxtimes\mathcal{O}(-1)^{\boxtimes q}$ over $X=(\mathbb{P}^1(\mathbb{C}))^n$. For this line bundle, one indeed gets $X(q)=X$ and

$$\dim H^q(X, E^{\otimes k}) = (k+1)^{n-q}(k-1)^q,\qquad k\geqslant 1,$$
$$\int_X\Big(\frac{i}{2\pi}c(E)\Big)^n = (-1)^q n!\,.$$

The existence of an inequality of the form 0.1(a) has been conjectured by Y.T. Siu, who has first proved the case where $ic(E)$ is >0 in the complement of a set of measure zero (cf. [16]), and then the case where $ic(E)\geqslant 0$ over X (cf. [17]). A substantial part of the methods used here have been inspired by Siu's work, especially in §3 and §5. The proof of Theorem 0.1 is based on an analytic technique recently introduced by E. Witten [18], [19]. This technique allows, among other things, to reprove the classical Morse inequalities $b_q\leqslant m_q$ on any compact differentiable variety M, where b_q denotes the q-th number of Betti and m_q the number of critical points of index q of a given (arbitrary) Morse function on M. In our situation, the role of the Morse function is held by the choice of the Hermitian metric on E. We also equip X and F with arbitrary Hermitian metrics which, in the end, will only play a role in the expression of the $o(k^n)$ terms in the final estimates. Given a real number $\lambda\geqslant 0$, we consider a sub-complex $\mathcal{H}^\bullet_k(\lambda)$ of the Dolbeault complex $(\mathcal{C}^\infty_{0,\bullet}(X, E^{\otimes k}\otimes F), d'')$, consisting in degree q of the subspace of $(0,q)$-forms of class $\mathcal{C}^\infty$ on X with values in $E^{\otimes k}\otimes F$, that is spanned by the eigenfunctions of the anti-holomorphic Laplacian Δ'', of eigenvalues $\leqslant k\lambda$. The cohomology groups of the $\mathcal{H}^\bullet_k(\lambda)$ complex are then isomorphic to the groups $H^q(X, E^{\otimes k}\otimes F)$ (Proposition 4.1).

As a consequence, it is sufficient to estimate the dimensions of these spaces $\mathcal{H}_k^q(\lambda)$. For this, two tools are used in an essential manner. The first tool consists of a Weitzenböck type formula

$$\frac{2}{k}\int_X \langle \Delta'' u, u\rangle = \int_X \frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle + \frac{1}{k}\langle \Theta u, u\rangle \tag{0.3}$$

prove in §3, and derived from non-Kähler Bochner-Kodaira-Nakano identity given in [6]. Here, ∇_k denotes the natural Hermitian connection on the bundle $\Lambda^{0,q}T^*X \otimes E^{\otimes k} \otimes F$, V is a linear potential of order 0 related to the curvature of the line bundle E, and finally, S and Θ are linear operators of order 0 depending on the torsion of the Hermitian metric on X and on the curvature of F. The study of the spectrum of Δ'' is then reduced to the study of the spectrum of the self-adjoint operator $\nabla_k^*\nabla_k$ associated with the real connection ∇_k. The second fundamental tool precisely consists of a very general spectral theorem relative to elliptic operators of the form $\nabla^*\nabla$. Let (M, g) be a $\mathcal{C}^\infty$ Riemannian manifold of real dimension n, E a complex line bundle over X, equipped with a Hermitian connection ∇. If ∇_k denotes the connection induced by ∇ on $E^{\otimes k}$, one then studies the spectrum of the quadratic form

$$Q_k(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u|^2 - V|u|^2\Big) d\sigma, \qquad u \in L^2(\Omega, E^{\otimes k}) \tag{0.4}$$

for the Dirichlet problem, where Ω is a relatively open-ended compact in M, and where V is a continuous scalar potential on M. From a physical point of view, this is equivalent to studying the spectrum of the operator of Schrödinger $\frac{1}{k}(\nabla_k^*\nabla_k - kV)$ associated with the electric field kV and the magnetic field kB, where $B = -i\nabla^2$ is none other than the curvature 2-form of of the connection ∇. With respect to Witten's arguments explained in [18], [19], our main contribution consists of analyzing the role of the magnetic field; in the case of De Rham cohomology, on the other hand, one can consider the magnetic field to be zero, as a consequence of the fact that $d^2 = 0$.

At any point $x \in X$, let $2s = 2s(x) \leqslant n$ be the rank of the 2-form $B(x)$, and let $B_1(x) \geqslant \ldots \geqslant B_s(x) > 0$ the absolute values of the non-zero eigenvalue of the associated skew-symmetric endomorphism. We define a function $\nu_{B(x)}(\lambda)$ of the pair $(x, \lambda) \in M \times \mathbb{R}$, that is left-continuous in λ, by setting

$$\nu_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2} - s + 1)} B_1 \ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s} \big[\lambda - \sum (2p_j + 1)B_j\big]_+^{\frac{n}{2}-s} \tag{0.5}$$

with the convention that $0^0 = 0$. Finally, if $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ denote the eigenvalues of Q_k (counted with multiplicity), we consider the counting function $N_k(\lambda) = \mathrm{card}\{j\,;\ \lambda_j \leqslant \lambda\}$, for $\lambda \in \mathbb{R}$.

Theorem 0.6. – *If the boundary $\partial\Omega$ is of measure zero, there exists a countable subset $\mathcal{D} \subset \mathbb{R}$ such that*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_k(\lambda) = \int_\Omega \nu_B(V + \lambda)\, d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$.

In order to prove Theorem 0.6, one starts with the simple case where $M = \mathbb{R}^n$, the magnetic field B being constant and $V = 0$. When Ω is a cube, a partial Fourier transform reduces the problem to the classical harmonic oscillator problem in one variable, and one can then compute explicitly the eigenfunctions. The idea of this calculation was strongly inspired by Y. Colin de Verdière's papers [3], [4]. The generalization of this result to the case of a vayring magnetic field elaborates on an idea developped by Siu in [16], consisting of paving Ω by sufficently small cubes. Our method is nevertheless different from the one used in [16], since we work directly on harmonic forms, whereas Siu's argument was based on a use of holomorphic cochains via the Dolbeault isomorphism. In this way, the estimates become substantially more accurate. The side of the cubes must then be ajusted to a value comprised the orders of magnitude $k^{-\frac{1}{2}}$ and $k^{-\frac{1}{4}}$, for instance $k^{-\frac{1}{3}}$: $k^{-\frac{1}{2}}$ is indeed the wavelength of the the first eigenfunction, so that the effect of the magnetic field is not sensible at a lower scale lower ; on the other hand, at a scale larger than $k^{-\frac{1}{4}}$, the oscillation of B becomes too strong. We finally use the minimax principle to compare the eigenvalues of the quadratic form on Ω to the eigenvalues obtained on the small cubes. In the previous method employed in [16] (also reproduced in [7]), the side of the cubes was chosen equal to to $k^{-\frac{1}{2}}$; one can easily see that this choice ss critical to bound the effects of the magnetic field independently of k, but the exact determination of the spectrum then became impossible. The last section of the present paper is devoted to a study of geometric characterizations of Moišezon spaces [13]. Recall that an irreducible compact complex space X is called a Moišezon space if the field $K(X)$ of meromorphic functions on X has a transcendence degree equal to $n = \dim_{\mathbb{C}} X$. The Grauert-Riemenschneider conjecture [10] states that X is Moišezon if and only if there exists a torsion free quasi-positive sheaf $\mathcal{E}$ of rank 1 over X.

By using a desingularization, one is reduced to the case where X is smooth and where $\mathcal{E}$ is the locally free sheaf of sections of a hermitian holomorphic line bundle E of strictly positive curvature on a dense open subset X. Y.T. Siu [17] recently solved the conjecture, and strengthened it by merely assuming that $ic(E)$ is semi-positive and positive in at least one point. The use of theorem 0.1 (b) makes it possible to find even weaker conditions, which do not require the pointwise semi-positivity of $ic(E)$, but only the positivity of a certain integral of the curvature form. For $q = 1$, inequality 0.1 (b) indeed implies a lower bound of the number of holomorphic sections of $E^{\otimes k}$, namely:

$$\dim H^0(X, E^{\otimes k}) \geqslant \frac{k^n}{n!} \int_{X(\leqslant 1)} \Big(\frac{i}{2\pi} c(E)\Big)^n - o(k^n). \tag{0.7}$$

On the other hand, one can show, using a classical arument of Siegel [15] also reinvestigated by [16], that $\dim H^0(X, E^{\otimes k}) \leqslant \mathrm{Const}\cdot k^{n-1}$ if X is not Moišezon (cf. Theorem 5.1). From there, one gets

Theorem 0.8. – *Let X a compact $\mathbb{C}$-analytic manifold of complex dimension n. Then X is Moišezon as soon as it possesses a Hermitian holomorphic line bundle satisfying one of the hypotheses* (a), (b), (c) *below.*

(a) $\int_{X(\leqslant 1)} (ic(E))^n > 0$.

(b) $c_1(E)^n > 0$, *and there is no point where $ic(E)$ has a non zero even index.*

(c) *$ic(E)$ is semi-positive at any point of X, and positive definite in at least one point of X.*

The results of this work have been published in a short Comptes Rendus note [8] with the same title. This paper is an improved version of an earlier work [7], in which techniques closer to Siu's initial approach where employed. In the latter, inequality 0.1 (a) had been proved only up to a certain constant numerical factor, and as a consequence, the estimates 0.1 (b) and (c) remained inaccessible.

The author addresses warm thanks to MM. Gérard Besson, Alain Dufresnoy, Sylvestre Gallot and Yves Colin de Verdière, to whom he is especially indebted, for stimulating discussions that greatly contributed to shape up the ideas involved in this work, especially in section §1.

1. Spectrum of the Schrödinger operator associated with a constant magnetic field.

Let (M, g) be a Riemannian manifold of class $\mathcal{C}^\infty$, of real dimension n, and $E \to M$ a complex line bundle over M, equipped with a $\mathcal{C}^\infty$ Hermitian metric. We denote by $\mathcal{C}^\infty_q(M, E)$ the space of $\mathcal{C}^\infty$ sections of the vector bundle $\Lambda^q T^*M \otimes E$, and $(?|?)$ the canonical sesquilinear pairing

$$\mathcal{C}^\infty_q(M, E) \times \mathcal{C}^\infty_q(M, E) \to \mathcal{C}^\infty_{p+q}(M, \mathbb{C}).$$

We assumed given a smooth Hermitian connection D on E, that is, a linear differential operator of order one

$$D : \mathcal{C}^\infty_q(M, E) \to \mathcal{C}^\infty_{q+1}(M, E), \qquad 0 \leqslant q < n,$$

satisfying identities

$$D(f \wedge u) = df \wedge u + (-1)^m f \wedge Du, \tag{1.1}$$

$$d(u|v) = (Du|v) + (-1)^p (u|Dv), \tag{1.2}$$

for all sections $f \in \mathcal{C}^\infty_m(M, \mathbb{C})$, $u \in \mathcal{C}^\infty_p(M, E)$, $v \in \mathcal{C}^\infty_q(M, E)$. Let us consider an isometric trivialization $\theta : E_{|W} \to W \times \mathbb{C}$ of E over an open $W \subset M$. The Hermitian connections of $E_{|W}$ are then given by a formula of the type

$$Du = du + iA \wedge u,$$

where $u \in \mathcal{C}^\infty_q(W, E) \simeq \mathcal{C}^\infty_q(W, \mathbb{C})$ and $A \in \mathcal{C}^\infty_1(W, \mathbb{R})$ is an arbitrary *real* 1-form. The *magnetic field* (or curvature form) associated with the connection D is the real closed 2-form $B = dA$ such that

$$D^2 u = iB \wedge u$$

for all $u \in \mathcal{C}^\infty_q(M, E)$. Therefore, B depends only on the connection D, but not on trivialization θ that has been chosen. A phase change $u = ve^{i\varphi}$ in θ leads to replace A by $A + d\varphi$. The choice of a trivialization of E and of a 1-form A can be interpreted physically as the choice of a particular "vector potential" of the magnetic field B.

Let us denote by $|u|$ the pointwise norm of any element $u \in \Lambda^q T^\star M \otimes E$ for metric defined as the tensor product of the respective metrics on M and E. If Ω is an open subset of M, we let $L^2(\Omega, E)$ (resp. $L^2_q(\Omega, E)$) be the space of L^2 sections of E (resp. of $\Lambda^q T^\star M \otimes E$) above Ω, equipped with the norm

$$\|u\|^2_\Omega = \int_\Omega |u|^2 d\sigma,$$

where $d\sigma$ is the Riemannian volume element on M.

Let D_k be the connection induced by D on the k-th tensor power $E^{\otimes k}$, and V a scalar potential on M, i.e. a continuous function with real values. Given a relatively compact open subset $\Omega \subset M$, our goal is to find an asymptotic estimate, as k tends to $+\infty$, of the spectrum of the quadratic form

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|D_k u|^2 - V|u|^2\Big) d\sigma \tag{1.3}$$

where $u \in L^2(\Omega, E^{\otimes k})$, with Dirichlet condition $u_{|\partial\Omega} = 0$. The domain of $Q_{\Omega,k}$ is therefore the Sobolev space $W^1_0(\Omega, E^{\otimes k})$ = closure of the space $\mathcal{D}(\Omega, E^{\otimes k})$ of C^∞ sections of $E^{\otimes k}$ with compact support in Ω in the space $W^1(M, E^{\otimes k})$. From a physical point of view, this amounts to study the spectrum of the Schrödinger operator $\frac{1}{k}(D^\star_k D_k - kV)$ associated with the magnetic field kB and the electric field kV, when k tends towards $+\infty$. We refer the reader to the classical article [2] for a general study of the spectrum of Schrödinger operators, and to papers [3], [4], [5], [9], [12] for the study of asymptotic problems that are closely related to the above one.

Definition 1.4. – *Let $N_{\Omega,k}(\lambda)$ denote the number of eigenvalues of the quadratic form $Q_{\Omega,k}$ that do not exceed λ.*

We first study a simple case that will serve as a model for the general case in §2. We consider the following situation : $M = \mathbb{R}^n$ with the constant metric $g = \sum_{j=1}^n dx_j^2$, the open set Ω is the cube of side length r :

$$\Omega = \Big\{(x_1, \ldots, x_n) \in R^n \,;\ |x_j| < \frac{r}{2},\ 1 \leqslant j \leqslant n\Big\},$$

$V = 0$, and finally the magnetic field B is constant, equal to the alternate 2-form of rank $2s$ given by

$$B = \sum_{j=1}^{s} B_j \, dx_j \wedge dx_{j+s},$$

where $B_1 \geqslant B_2 \geqslant \cdots \geqslant B_s > 0$, $s \leqslant \frac{n}{2}$. One can then select a trivialization of E whose vector potential associated to B is

$$A = \sum_{j=1}^{s} B_j x_j \, dx_{j+s}.$$

The connection of $E^{\otimes k}$ can then be written as

$$D_k u = du + ikA \wedge u,$$

and the quadratic form $Q_{\Omega,k}$ is given by

$$Q_{\Omega,k}(u) = \frac{1}{k}\int_\Omega \Big[\sum_{1\leqslant j\leqslant s}\Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + ikB_jx_ju\Big|^2\Big) + \sum_{j>2s}\Big|\frac{\partial u}{\partial x_j}\Big|^2\Big]\,d\mu$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^n$. If we perform the homothety $X_j = \sqrt{k}\,x_j$, we are reduced to study the eigenvalues of the quadratic form

$$\int_{\sqrt{k}\Omega} \Big[\sum_{1\leqslant j\leqslant s}\Big(\Big|\frac{\partial u}{\partial X_j}\Big|^2 + \Big|\frac{\partial u}{\partial X_{j+s}} + iB_jX_ju\Big|^2\Big) + \sum_{j>2s}\Big|\frac{\partial u}{\partial X_j}\Big|^2\Big]\,d\mu$$

on the cubes $\sqrt{k}\,\Omega$ of side $\sqrt{k}\,r$. With the field B, we associate the function of the real variable λ defined by

$$\text{(1.5)}\qquad \nu_B(\lambda) = \frac{2^{s-n}\pi^{-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)}B_1\ldots B_s \sum_{(p_1,\ldots,p_s)\in\mathbb{N}^s}\Big[\lambda - \sum(2p_j+1)B_j\Big]_+^{\frac{n}{2}-s}$$

where we agree to put $\lambda_+^0 = 0$ if $\lambda \leqslant 0$ and $\lambda_+^0 = 1$ if $\lambda > 0$. The function ν_B is then non decreasing and left continuous on $\mathbb{R}$; let us observe that ν_B is actually continuous if $s < \frac{n}{2}$. The spectrum of $Q_{\Omega,k}$ is then described asymptotically by the following theorem, the idea of which was suggested to us by Y. Colin of Verdière [4].

Theorem 1.6. – *Given a real number $R > 0$, we let*

$$P(R) = \Big\{x \in \mathbb{R}^n\,;\ |x_j| < \frac{R}{2}\Big\}$$

be the cube of side R and consider the quadratic form Q_R such that

$$Q_R(u) = \int_{P(R)} \Big[\sum_{1\leqslant j\leqslant s}\Big(\Big|\frac{\partial u}{\partial x_j}\Big|^2 + \Big|\frac{\partial u}{\partial x_{j+s}} + iB_jx_ju\Big|^2\Big) + \sum_{j>2s}\Big|\frac{\partial u}{\partial x_j}\Big|^2\Big]\,d\mu.$$

Finally, we let $N_R(\lambda)$ be the number of eigenvalues $\leqslant \lambda$ of Q_R for the Dirichlet problem. Then for all $\lambda \in \mathbb{R}$, we have

$$\lim_{R\to+\infty} R^{-n}N_R(\lambda) = \nu_B(\lambda).$$

For $s = \frac{n}{2}$, ν_B is a step function. The eigenvalues of Q_R are then grouped in packets around the values $\sum(2p_j+1)B_j$, with approximate multiplicity $(2\pi)^{-s}B_1\ldots B_sR^n$. This can be interpreted physically as a phenomenon of quantification of the eigenstates. Coming back to our initial eigenvalue problem for the quadratic form $Q_{\Omega,k}$, we get

Corollary 1.7. – Asymptotically, $\lim_{k\to+\infty} k^{-\frac{n}{2}}N_{\Omega,k}(\lambda) = r^n\nu_B(\lambda)$. □

Proof of theorem 1.6. – First we seek for an upper bound of $N_R(\lambda)$. In this direction, given $u \in W_0^1(P(R))$, we express u as a partial Fourier series with respect to the variables $x_{s+1}, \dots, x_n$:

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{\ell \in \mathbb{Z}^{n-s}} u_\ell(x') \exp\Big(\frac{2\pi i}{R}\ell \cdot x''\Big),$$

where $u_\ell \in W_0^1(\mathbb{R}^s \cap P(R))$, with the notation

$$\begin{aligned} &x' = (x_1, \dots, x_s), \quad x'' = (x_{s+1}, \dots, x_n),\\ &\ell \cdot x'' = \ell_1 x_{s+1} + \cdots + \ell_{n-s} x_n. \end{aligned}$$

The hypothesis $u \in W_0^1(P(R))$ implies that the series

$$\sum |\ell|^2 |u_\ell(x')|^2$$

is in $L^2(\mathbb{R}^s)$. Let us put $\ell' = (\ell_1, \dots, \ell_s)$, $\ell'' = (\ell_{s+1}, \dots, \ell_{n-s})$. The norm $\|u\|_{P(R)}$ and the quadratic form Q_R are given by

$$\begin{aligned} \|u\|^2_{P(R)} &= \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} |u_\ell(x')|^2 \, d\mu(x'),\\ Q_R(u) &= \sum_{\ell \in \mathbb{Z}^{n-s}} \int_{\mathbb{R}^s} \bigg[\sum_{1 \leqslant j \leqslant s} \Big(\Big|\frac{\partial u_\ell}{\partial x_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |u_\ell|^2 \Big) + \frac{4\pi^2}{R^2} |\ell''|^2 |u_\ell|^2 \bigg] \, d\mu(x'). \end{aligned}$$

We are therefore led to consider a Dirichlet problem with "separate variables" on the cube $\mathbb{R}^s \cap P(R)$. By putting $t = x_j + \frac{2\pi \ell_j}{RB_j}$, we are reduced to studying the spectrum of the quadratic form in one variable

$$q(f) = \int_R \Big(\Big|\frac{df}{dt}\Big|^2 + B_j^2 t^2 |f|^2 \Big) \, dt,$$

with $f \in W_0^1\big(]-\frac{R}{2}\frac{R}{2}[+ \frac{2\pi\ell_j}{RB_j}\big)$. The latter question is the classical harmonic oscillator problem (see for example [14], Vol. I, p. 142). On $\mathbb{R}$, i.e. without support condition for f, the sequence of values of q is the suite $(2m+1)B_j$, $m \in \mathbb{N}$, and the eigenfunctions of q are given by $\Phi_m(\sqrt{B_j}\, t)$ where Φ_0, $\Phi_1, \dots$ are the Hermite functions:

$$\Phi_m(t) = e^{t^2/2} \frac{d^m}{dt^m}(e^{-t^2}).$$

For all $p_j \in \mathbb{N}$, let $\Psi_{p_j,\ell_j}(x_j)$ be the p_j-th eigenfunction of the quadratic form

$$q(f) = \int_R \Big(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}\ell_j + B_j x_j\Big)^2 |f|^2 \Big) \, dx_j \tag{1.8}$$

for $f \in W_0^1(]-\frac{R}{2}\frac{R}{2}[)$, and let λ_{p_j,ℓ_j} be the corresponding eigenvalue. We can then decompose each function u_ℓ as a series of eigenfunctions. Then we see that u can be written under the form

$$u(x) = R^{-\frac{1}{2}(n-s)} \sum_{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}} u_{p,\ell} \Psi_{p,\ell'}(x') \exp\Big(\frac{2\pi i}{R}\ell \cdot x''\Big) \tag{1.9}$$

with

$$u_{p,\ell} \in \mathbb{C}, \qquad \Psi_{p,\ell'}(x') = \prod_{1 \leqslant j \leqslant s} \Psi_{p_j,\ell_j}(x_j).$$

One should pay attention to the fact that $\Psi_{p,\ell'}(x')\exp(\frac{2\pi i}{R}\ell \cdot x'')$ is not not a genuine eigenfunction for the Dirichlet problem, since the exponential term takes non-zero values at the boundary points $x_j = \pm\frac{R}{2}$, $j > s$. Therefore, the coefficients $(u_{p,\ell})$ are not arbitrary for a given function $u \in W^1_0(P(R))$; they must satisfy the relevant vanishing boundary conditions, namely

$$\sum_{t_j \in \mathbb{Z}} (-1)^{\ell_j} u_{p,\ell} = 0 \tag{1.10}$$

for all $j = 1, \ldots, n-s$ and all indices else than ℓ_j fixed :

$$p \in \mathbb{N}^s, \quad \ell_1, \ldots, \ell_{j-1},\ \ell_{j+1}, \ldots, \ell_{n-s} \in \mathbb{Z}.$$

Thanks to formula (1.9), the L^2 norm and the quadratic form Q_R can be expressed under the form

$$\|u\|^2_{P(R)} = \sum |u_{p,\ell}|^2, \qquad Q_R(u) = \sum \Big(\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2\Big)|u_{p,\ell}|^2,$$

where $\lambda_{p,\ell'} = \sum_{1 \leqslant j \leqslant s} \lambda_{p_j,\ell_j}$. The minimax principle 1.20 (b) recalled below implies that

$$N_R(\lambda) \leqslant \mathrm{card}\Big\{(p,\ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}\ ;\ \lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda\Big\}. \tag{1.11}$$

It is therefore sufficient to obtain an appropriate lower bound of λ_{p_j,ℓ_j}.

Lemma 1.12. – *We have an inequality*

$$\lambda_{p_j,\ell_j} \geqslant \max\bigg((2p_j+1)B_j\ ,\ \frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2 + \Big(|\ell_j| - \frac{B_j R^2}{4\pi}\Big)^2_+\Big]\bigg),$$

which moreover is strict if $\ell_j \neq 0$ or if $\Phi_{p_j}(R\sqrt{B_j}/2) \neq 0$.

The lower bound $\lambda_{p_j,\ell_j} \geqslant (2p_j+1)B_j$ follows in fact from the minimax and the fact that the eigenvalues of $q(f)$ on $\mathbb{R}$ are equal to $(2p_j+1)B_j$. In order to obtain the other inequality, we observe that (1.8) dominates the quadratic form

$$\widehat{q}(f) = \int_{x_j|<R/2} \bigg(\Big|\frac{df}{dx_j}\Big|^2 + \Big(\frac{2\pi}{R}|\ell_j| - B_j\frac{R}{2}\Big)^2_+ |f|^2\bigg)dx_j.$$

The eigenfunctions of $\widehat{q}$ are given explicitly by

$$\sin\frac{\pi}{R}(p_j+1)\Big(x_j + \frac{R}{2}\Big), \qquad p_j \in \mathbb{N}\ ;$$

λ_{p_j,t_j} is therefore bounded below by the corresponding eigenvalue

$$\frac{4\pi^2}{R^2}\Big[\Big(\frac{p_j+1}{2}\Big)^2+\Big(|t_j|-\frac{B_jR^2}{4\pi}\Big)_+^2\Big].$$

These inequalities are strict because on the one hand $q(f)>\widehat{q}(f)$ for any $f\neq 0$, and on the other hand, $\Phi_{p_j}(\sqrt{B_j}t)$ can be an eigenfunction of q on $]-R/2,R/2[+2\pi\ell_j/RB_j$ only when

$$\Phi_{p_j}\big(\pm R\sqrt{B_j}/2+2\pi t_j/R\sqrt{B_j}\big)=0.$$

Since the zeros of Φ_{p_j} are algebraic and π is transcendental, this is only possible if

$$\ell_j=0\quad\text{and}\quad\Phi_{p_j}(R\sqrt{B_J}/2)=0.\qquad\square$$

Lemma 1.13. – *Let $\tau_n(\rho)$ be the number of points of $\mathbb{Z}^n$ located in the closed ball $\overline{B}(0,\rho)\subset\mathbb{R}^n$. Then*

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho-\frac{\sqrt{n}}{2}\Big)_+^n\leqslant\tau_n(\rho)\leqslant\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\Big(\rho+\frac{\sqrt{n}}{2}\Big)^n.$$

Indeed, the union of cubes of side 1 centered at integral points $x\in\mathbb{Z}^n$ such that $|x|\leqslant\rho$ is contained in the ball $\overline{B}(0,\rho+\frac{\sqrt{n}}{2})$, and contains the ball $\overline{B}(0,\rho-\frac{\sqrt{n}}{2})$ if $\rho\geqslant\frac{\sqrt{n}}{2}$, where $\frac{\sqrt{n}}{2}$ is half the diagonal of the cube ; the integer $\tau_n(\rho)$ is thus in the interval comprised between the volumes of the balls $\overline{B}(0,\rho\pm\frac{\sqrt{n}}{2})$. $\square$

We now proceed to find an upper bound of $\limsup R^{-n}N_R(\lambda)$ by using (1.11) and lemmas 1.12, 1.13. For $p\in\mathbb{N}^s$ fixed, the inequality $\lambda_{p,\ell'}+\frac{4\pi^2}{R^2}|\ell''|^2\leqslant\lambda$ implies

$$|\ell''|\leqslant\frac{R}{2\pi}\Big(\lambda-\sum(2p_j+1)B_j\Big)_+^{\frac12},\tag{1.14}$$

and the inequality is strict for $R>R_0(p)$ large enough. When $s<n/2$ the number of corresponding multi-indices $\ell''\in\mathbb{Z}^{n-2s}$ is therefore at most

$$\begin{aligned}&\frac{\pi^{\frac{n}{2}-s}}{\Gamma(\frac{n}{2}-s+1)}\Big[\frac{R}{2\pi}\Big(\lambda-\sum(2p_j+1)B_j\Big)_+^{\frac12}+\frac{\sqrt{n}}{2}\Big]^{n-2s}\\ &\underset{R\to+\infty}{\sim}\frac{2^{2s-n}\pi^{s-\frac{n}{2}}}{\Gamma(\frac{n}{2}-s+1)}R^{n-2s}\Big(\lambda-\sum(2p_j+1)B_j\Big)_+^{\frac{n}{2}-s}.\end{aligned}\tag{1.15}$$

When $s=\frac{n}{2}$, this number must be counted as 1 if $\lambda-\sum(2p_j+1)B_j>0$ and 0 otherwise, in conformity with the convention we adopted for the notation λ_+^0. The inequality $\lambda_{p,\ell'}\leqslant\lambda$ implies on the other hand

$$|\ell_j|\leqslant\frac{R}{2\pi}\sqrt{\lambda_+}+\frac{B_jR^2}{4\pi},\qquad 1\leqslant j\leqslant s,\tag{1.16}$$

which asymptotically corresponds to a number of multi-indices $\ell' = (\ell_1, \ldots, \ell_s) \in \mathbb{Z}^s$ equivalent to

$$\prod_{j=1}^{s} \frac{B_j R^2}{2\pi} = 2^{-s}\pi^{-s} B_1 \ldots B_s R^{2s}. \tag{1.17}$$

The upper bound $\limsup R^{-n} N_R(\lambda) \leqslant \nu_B(\lambda)$ is then obtained by multiplying (1.15) by (1.17), and taking the sum over all $p \in \mathbb{N}^s$ (the sum is finite). □

For questions of convergence that will play a role in §2, we will also need an upper bound of $N_R(\lambda)$ that is independent of the magnetic field B. Such a uniform estimate is provided by the following proposition.

Proposition 1.18. – $N_R(\lambda) \leqslant (R\sqrt{\lambda_+} + 1)^n$.

Proof. – For each index j, we bound the number of integers p_j and ℓ_j such that the inequality

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda$$

holds. Lemma 1.12 implies

$$\operatorname{card}\{p_j\} \leqslant \max(p_j+1) \leqslant \min\Big(\frac{\lambda_+}{B_j}\,,\,\frac{R}{\pi}\sqrt{\lambda_+}\Big), \qquad 1 \leqslant j \leqslant s,$$

while (1.16) entails

$$\operatorname{card}\{l_i\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + \frac{B_j R^2}{2\pi} + 1, \qquad 1 \leqslant j \leqslant s.$$

For $1 \leqslant j \leqslant s$, we therefore infer that

$$\operatorname{card}\{(p_j, l_j)\} \leqslant \Big(\frac{R}{\pi}\sqrt{\lambda_+}\Big)^2 + \frac{\lambda_+}{B_j}\cdot\frac{B_j R^2}{2\pi} + \frac{R}{\pi}\sqrt{\lambda_+}\cdot 1 \leqslant \big(R\sqrt{\lambda_+}+1\big)^2.$$

For $s < j \leqslant n - s$, inequality (1.14) gives on the other hand

$$|\ell_j| < \frac{R}{2\pi}\sqrt{\lambda_+},$$

hence $\operatorname{card}\{l_j\} \leqslant \frac{R}{\pi}\sqrt{\lambda_+} + 1$. Proposition 1.18 follows. □

End of the proof of Theorem 1.6 (lower bound of $N_R(\lambda)$).

In order to get a lower bound of $N_R(\lambda)$, it is sufficient by 1.20 (a) to construct a finite dimensional vector space on which $Q_R(u) \leqslant \lambda \|u\|^2_{P(R)}$. We consider for this the vector space $\mathcal{F}_\lambda$ of linear combinations of "eigenfunctions" of the type (1.9), subject to the additional boundary conditions (1.10), for which summations are taken on indices $(p, \ell) \in \mathbb{N}^s \times \mathbb{Z}^{n-s}$ such as

$$\lambda_{p,\ell'} + \frac{4\pi^2}{R^2}|\ell''|^2 \leqslant \lambda.$$

By the arguments used in Proposition 1.18, the number of conditions (1.10) to be realized is bounded above by

$$\sum_{j=1}^{s}\Big[\mathrm{card}\{p_j\}\times\prod_{1\leqslant i\leqslant s,\,i\neq j}\mathrm{card}\{(p_i,\ell_i)\}\times\prod_{s<i\leqslant n-s}\mathrm{card}\{\ell_i\}\Big]$$
$$+\sum_{s<j\leqslant n-s}\Big[\prod_{1\leqslant i\leqslant s}\mathrm{card}\{(p_i,\ell_i)\}\times\prod_{s<i\neq j}\mathrm{card}\{\ell_i\}\Big]\leqslant n(R\sqrt{\lambda_+}+1)^{n-1}.$$

The number $N_R(\lambda)$ is therefore bounded by

$$\dim\mathcal{F}_\lambda\geqslant\mathrm{card}\Big\{(p,\ell)\in\mathbb{N}^s\times\mathbb{Z}^{n-s}\,;\ \lambda_{p,\ell'}+\frac{4\pi^2}{R^2}|\ell''|^2\leqslant\lambda\Big\}-O(R^{n-1}).$$

A combination of Lemma 1.13 with the next lemma below shows that the inequality $\liminf R^{-n}N_R(\lambda)\geqslant\nu_B(\lambda)$ can be obtained from calculations that are similar to those used in the upper bound estimate of $N_R(\lambda)$.

Lemma 1.19. – *Let $p\in\mathbb{N}^s$ be a fixed multi-index. Then there is a constant $C=C(p,B)\geqslant 0$ such that*

$$\lambda_{p,\ell'}\leqslant\Big(1+\frac{C}{R}\Big)\sum_{j=1}^{s}(2p_j+1)B_j$$

when $|\ell_j|\leqslant\frac{B_jR^2}{4\pi}(1-R^{-\frac12})$, $1\leqslant j\leqslant s$.

Proof. – We use again the minimax principle and the fact that the Hermite functions $\Phi_p(\sqrt{B_j}t)$ are good approximations of the eigenfunctions of q on any sufficiently large interval centered at 0. When $|\ell_j|\leqslant\frac{B_jR^2}{4\pi}(1-R^{-\frac12})$ and $x_j\in\,]-\frac{R}{2},\frac{R}{2}[$, the parameter $t=x_j+\frac{2\pi\ell_j}{B_jR}$ that appears in (1.8) indeed runs on an interval containing $]-\frac{\sqrt{R}}{2},\frac{\sqrt{R}}{2}[$. Therefore, we have $\lambda_{p_j,\ell_j}\leqslant\widetilde{\lambda}_{p_j}$ where $(\widetilde{\lambda}_m)_{m\in\mathbb{N}}$ is the sequence of eigenvalues of the quadratic form

$$\widetilde{q}(f)=\int\Big[\Big|\frac{df}{dt}\Big|^2+(B_jt)^2|f|^2\Big]dt,\qquad f\in W_0^1\Big(\Big]-\frac{\sqrt{R}}{2},\frac{\sqrt{R}}{2}\Big[\Big).$$

Let χ_R be a cut-off function with support in $\big[-\frac{\sqrt{R}}{2},\frac{\sqrt{R}}{2}\big]$, equal to 1 on $\big[-\frac{\sqrt{R}}{4},\frac{\sqrt{R}}{4}\big]$, and whose derivative is bounded by $5/\sqrt{R}$. For any linear combination

$$f=\sum_{m\leqslant p_j}c_m\Phi_m(\sqrt{B_j}t),$$

the exponential decay of the Φ_m functions at infinity implies for R large enough an inequality

$$\|f\|\leqslant\Big(1+C_1\exp\Big(-\frac{R}{C_1}\Big)\Big)\|\chi_Rf\|$$

where $C_1 = C_1(p_j, B_j) > 0$. Therefore, we obtain

$$\begin{aligned}
\widetilde{q}(\chi_R f) &\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \left(\frac{10}{\sqrt{R}} \Big| f \frac{df}{dt} \Big| + \frac{25}{R} |f|^2 \right) dt \\
&\leqslant \widetilde{q}(f) + \int_{|t|>\sqrt{R}/4} \left(\frac{1}{R} \Big| \frac{df}{dt} \Big|^2 + 25\Big(1 + \frac{1}{R}\Big) |f|^2 \right) dt \\
&\leqslant \Big(1 + \frac{C_2}{R}\Big) \widetilde{q}(f) \leqslant \Big(1 + \frac{C_2}{R}\Big)(2p_j + 1) B_j \, \|f\|^2 \\
&\leqslant \Big(1 + \frac{C}{R}\Big)(2p_j + 1) B_j \, \|\chi_R f\|^2
\end{aligned}$$

This gives $\lambda_{p_j,\ell_j} \leqslant \widetilde{\lambda}_{p_j} \leqslant \big(1 + \frac{C}{R}\big)(2p_j + 1)B_j$. □

For the reader's convenience, we now state the minimax principle in the exact form it has been applied above.

Proposition 1.20 (minimax principle, see [14], Vol. IV, p. 76 and 78)**. –** *Let Q be a quadratic form with dense domain $D(Q)$ in a Hilbert space $\mathcal{H}$. We assume that Q is bounded from below, i.e. $Q(f) \geqslant -C\|f\|^2$ for $f \in D(Q)$, that $D(Q)$ is complete for the norm $\|f\|_Q = [Q(f) + (C+1)\|f\|^2]^{\frac{1}{2}}$, and finally that the injection morphism $(D(Q), \| \ \|_Q) \hookrightarrow (\mathcal{H}, \| \ \|)$ is compact. Then Q has a discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$, and we have equalities*

(a) $\lambda_p = \min\limits_{F \subset D(Q)} \ \max\limits_{f \in F, \|f\|=1} Q(f),$

where F runs over the set of subspaces of dimension p of $D(Q)$;

(b) $\lambda_{p+1} = \max\limits_{F \subset D(Q)} \ \min\limits_{f \in F, \|f\|=1} Q(f),$

where F runs over the set of Q-closed subspaces of codimension p of $D(Q)$.

2. Asymptotic distribution of the spectrum (case of a variable field).

We consider again the general framework described at the beginning of section §1. Our goal is to study the spectrum of the quadratic form $Q_{\Omega,k}$ (see (1.3)) in the case of arbitrary magnetic field B and electric field V. At any point $a \in M$, let

$$B(a) = \sum_{j=1}^{s} B_j(a) \, dx_j \wedge dx_{j+s} \tag{2.1}$$

be the normalized expression of $B(a)$ in a suitable orthonormal basis $(dx_1, \ldots, dx_n)$ of $T_a^* M$, where $2s = 2s(a) =$ rank of $B(a)$, $2s \leqslant n$, and $B_1(a) \geqslant B_2(a) \geqslant \ldots \geqslant B_s(a) > 0$ are the positive eigenvalues of the associated antisymmetric endomorphism. The defining equality 1.5 allows us to view $\nu_B(\lambda)$ as a function of the pair$(a, \lambda) \in M \times \mathbb{R}$. It will also be useful to consider the function $\overline{\nu}_B(\lambda)$, that is right continuous in λ, defined by

$$\overline{\nu}_B(\lambda) = \lim_{0<\varepsilon\to 0} \nu_B(\lambda + \varepsilon). \tag{2.2}$$

We then prove the following generalization of Corollary 1.7.

Theorem 2.3. *When k tends to $+\infty$, the number $N_{\Omega,k}(\lambda)$ of eigenvalues $\leqslant \lambda$ of $Q_{\Omega,k}$ satisfies the asymptotic inequalities*

$$\int_\Omega \nu_B(V+\lambda)\,d\sigma \leqslant \liminf k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \limsup k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \int_\Omega \overline{\nu}_B(V+\lambda)\,d\sigma.$$

The function $\lambda \mapsto \int_\Omega \nu_B(V+\lambda)\,d\sigma$ is non decreasing and left continuous ; therefore it has at most a countable set $\mathcal{D}$ of points of discontinuity. The set $\mathcal{D}$ is in fact empty when n is odd, since $\nu_B(\lambda)$ is then continuous. From this, we immediately infer

Corollary 2.4. *Assume that $\partial\Omega$ is of measure zero. Then*

$$\lim_{k\to+\infty} k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) = \int_\Omega \nu_B(V+\lambda)\,d\sigma$$

for all $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$, and the spectral density measure $k^{-\frac{n}{2}} \frac{d}{d\lambda} N_{\Omega,k}(\lambda)$ converges weakly on $\mathbb{R}$ to $\frac{d}{d\lambda}\int_\Omega \nu_B(V+\lambda)\,d\sigma$. If n is odd, the limit measure has no atoms. □

The following lemma shows that the integrals involved in Theorem 2.3 are well defined.

Lemma 2.5.

(a) *We have inequalities*

$$\nu_B(\lambda) \leqslant \overline{\nu}_B(\lambda) \leqslant \lambda_+^{n/2}.$$

(b) *$\nu_B(V)$ (resp. $\overline{\nu}_B(V)$) is lower (resp. upper) semi-continuous on M.*

(c) *At any point $x \in M$ where $s(x) < \frac{n}{2}$ we have $\nu_B(V)(x) = \overline{\nu}_B(V)(x)$, and the functions $\nu_B(V)$, $\overline{\nu}_B(V)$ are continuous in x.*

(d) *If n is odd, $\nu_B(V) = \overline{\nu}_B(V)$ is continuous on M.*

Proof. – (a) We always have $\big(\lambda - \sum(2p_j+1)B_j\big)_+^{\frac{n}{2}-s} \leqslant \lambda_+^{\frac{n}{2}-s}$, and the number of integers p_j such that $\lambda - (2p_j+1)B_j$ is $\geqslant 0$ is bounded above by $\frac{\lambda_+}{B_j}$. As the resulting numerical factor occurring in (1.5) is bounded by 1, inequality (a) follows.

(b, c) The rank $s = s(x)$ is a lower semi-continuous function on M, and the eigenvalues $B_1, B_2, \ldots,$ extended by $B_j(x) = 0$ for $j > s(x)$, are continuous on M. As the function $t \mapsto t_+^0$ (resp. $t \mapsto (t+0)_+^0$) is lower (resp. upper) semi-continuous, the semi-continuity of $\nu_B(V)$ and $\overline{\nu}_B(V)$ is a problem only at points $a \in M$ in the neighborhood of which $s(x)$ is not locally constant. At such a point $a \in M$, we necessarily have $s(a) < \frac{n}{2}$, so $\nu_B(V)(a) = \overline{\nu}_B(V)(a)$; we are going to show that $\nu_B(V)$ and $\overline{\nu}_B(V)$ are then continuous at a. The continuity of B_j gives $\lim_{x\to a} B_j(x) = 0$ for $j > s(a)$. When the integers $p_1, \ldots, p_{s(a)}$ are fixed, the summation in (1.5) may be interpreted as a Riemann sum for an integral over $\mathbb{R}^{s(x)-s(a)}$, and we therefore get the equivalent

$$\begin{aligned}
&\sum_{(p_j;\ s(a)<j\leqslant s(x))} \Big(V(x)-\sum(2p_j+1)B_j(x)\Big)_+^{\frac{n}{2}-s(x)}\\
&\qquad\sim \int_{t\in\mathbb{R}^{s(x)-s(a)}} \Big[V(a)-\sum_{j=1}^{s(a)}(2p_j+1)B_j(a)-\sum_{j=s(a)+1}^{s(x)} 2t_jB_j(x)\Big]_+^{\frac{n}{2}-s(x)} dt\\
&\qquad= \frac{2^{s(a)-s(x)}\Big(V(a)-\sum(2p_j+1)B_j(a)\Big)_+^{\frac{n}{2}-s(a)}}{(\frac{n}{2}-s(x)+1)\cdots(\frac{n}{2}-s(a))B_{s(a)+1}(x)\cdots B_{s(x)}(x)}\ .
\end{aligned}$$

Therefore, we find

$$\lim_{x\to a}\nu_B(V)(x)=\nu_B(V)(a)=\lim_{x\to a}\overline{\nu}_B(V)(x).$$

(d) Is a special case of (c). □

The proof of theorem 2.3 is based essentially on two ingredients : firstly, an asymptotic localization principle of the eigenfunctions, which is can be seen through a direct application of the minimax principle (Proposition 2.6) ; secondly, an explicit evaluation of the spectrum of the Schrödinger operator associated with a constant magnetic field (see §1). Indeed, the localization principle reduces the situation to the case of a constant field by using a covering of Ω by small cubes.

Proposition 2.6.

(a) *If* $\Omega_1,\cdots,\Omega_N\subset\Omega$ *are pairwise disjoint open subsets, then*

$$N_{\Omega,k}(\lambda)\geqslant\sum_{j=1}^{N}N_{\Omega_j,k}(\lambda).$$

(b) *Let* $(\Omega'_j)_{1\leqslant j\leqslant \mathrm{N}}$ *an open covering of* $\overline{\Omega}$ *and let* $(\psi_j)_{1\leqslant j\leqslant\mathrm{N}}$ *be a system of functions* $\psi_j\in\mathcal{C}^\infty(\mathbb{R}^n)$ *with support in* Ω'_j*, such that* $\sum\psi_j^2=1$ *on* $\overline{\Omega}$*. We set*

$$C(\psi)=\sup_{\Omega}\sum_{j=1}^{N}|d\psi_j|^2.$$

Then

$$N_{\Omega,k}(\lambda)\leqslant\sum_{j=1}^{N}N_{\Omega'_j,k}\Big(\lambda+\frac{1}{k}C(\psi)\Big).$$

Proof. – (a) Let $\mathcal{F}$ be the $\mathbb{C}$-vector space generated by the collection of all eigenfunctions of the quadratic forms $Q_{\Omega_j,k}$, $1\leqslant j\leqslant N$, corresponding to eigenvalues $\leqslant\lambda$. The space $\mathcal{F}$ is of dimension

$$\dim\mathcal{F}=\sum_{j=1}^{N}N_{\Omega_j,k}(\lambda)$$

and for all $u \in \mathcal{F}$, we have

$$Q_{\Omega,k}(u) = \sum_{j=1}^{N} Q_{\Omega_j,k}(u) \leqslant \sum_{j=1}^{N} \lambda \|u\|^2_{\Omega'_j} = \lambda \|u\|^2_{\Omega}.$$

The minimax principle therefore shows that the eigenvalues of $Q_{\Omega,k}$ of index $\leqslant \dim \mathcal{F}$ are $\leqslant \lambda$, whence inequality (a).

(b) For every $u \in W^1_0(\Omega, E^{\otimes k})$, we have

$$\sum_j |D_k(\psi_j u)|^2 = \sum_j \big|\psi_j D_k u + (d\psi_j)u\big|^2 = |D_k u|^2 + \sum_j |d\psi_j|^2 |u|^2$$

since $2\sum \psi_j d\psi_j = d(\sum \psi_j^2) = 0$. Therefore we get

$$\sum_{j=1}^{N} Q_{\Omega'_j,k}(\psi_j u) = Q_{\Omega,k}(u) + \int_\Omega \frac{1}{k} \sum_{j=1}^{N} |d\psi_j|^2 |u|^2 \, d\sigma \leqslant Q_{\Omega,k}(u) + \frac{1}{k} C(\psi) \|u\|^2_\Omega.$$

If each function $H^q(X, E^{\otimes k} \otimes F)$c is orthogonal to the eigenfunctions of $Q_{\Omega_j,k}$ associated with the eigenvalues $\leqslant \lambda + \frac{1}{k}C(\psi)$, we infer successively

$$\begin{aligned} Q_{\Omega_j,k}(\psi_j u) &> \Big(\lambda + \frac{1}{k} C(\psi)\Big) \|\psi_j u\|^2_{\Omega_j}, \quad \text{if } \psi_j u \neq 0, \\ Q_{\Omega,k}(u) &> \lambda \|u\|^2_\Omega, \quad \text{if } u \neq 0. \end{aligned}$$

The minimax principle 1.20 (b) then shows that $N_{\Omega,k}(\lambda)$ is bounded above by the number of linear constraints that we had to impose on u, namely

$$\sum_{j=1}^{N} N_{\Omega_j,k}\Big(\lambda + \frac{1}{k} C(\psi)\Big). \qquad \square$$

Let $W_1, \ldots, W_N$ be a covering of Ω by open coordinate charts of the manifold M. For any $\varepsilon > 0$, we can find open sets $\Omega_i \subset \Omega'_j$, that are relatively compact in W_j, $1 \leqslant j \leqslant N$, and such that

$$\Omega \supset \bigcup \Omega_j \ \text{ (disjoint)}, \quad \text{and} \quad \mathrm{Vol}(\Omega) = \sum \mathrm{Vol}(\Omega_j), \tag{2.7}$$

$$\overline{\Omega} \subset \bigcup \Omega'_j, \quad \text{and} \quad \sum \mathrm{Vol}(\overline{\Omega}'_j) \leqslant \mathrm{Vol}(\overline{\Omega}) + \varepsilon. \tag{2.8}$$

Proposition 2.6 then reduces the proof of Theorem 2.3 to the case of the open sets Ω_j and Ω'_j (observe for this that the function $\nu_B(V + \lambda)$ is bounded and that the constant $C(\psi)$ is independent of k).

In the end, we can assume that $M = \mathbb{R}^n$, with an arbitrary Riemannian metric g. Since $M = \mathbb{R}^n$ is contractible, the bundle E is then trivial ; let A be a vector potential of the connection D and $B = dA$ the corresponding magnetic field. We first prove the following local version of Theorem 2.3.

Proposition 2.9. – *Let $a \in \mathbb{R}^n$ be a given point, and P_k a sequence of open cubes such that $P_k \ni a$. Denote by r_k the side length of P_k, and assume that*

$$r_k \leqslant 1, \qquad \lim k^{\frac{1}{2}} r_k = +\infty, \qquad \lim k^{\frac{1}{4}} r_k = 0.$$

Then, when k tends to $+\infty$, we have

$$\liminf \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \geqslant \nu_{B(a)}(V(a)+\lambda),$$
$$\limsup \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)} N_{P_k,k}(\lambda) \leqslant \overline{\nu}_{B(a)}(V(a)+\lambda),$$

and for every compact $K \subset \mathbb{R}^n$, $N_{P_k,k}(\lambda)$ admits the upper bound

$$N_{P_k,k}(\lambda) \leqslant C_K\Big(1 + r_k\sqrt{k\big(\lambda_+ + \max_K V_+\big)}\Big)^n$$

uniformly with respect to the point a, as long as $P_k \subset K$.

Proof. – We proceed via a reduction to Theorem 1.6, after applying an homothety of ratio $\sqrt{k}$ to P_k – this is the reason why we had to assume $\lim k^{\frac{1}{2}} r_k = +\infty$. The following lemma measures how much the magnetic field B deviates from the field constant $B(a)$ on each cube P_k.

Lemma 2.10. – *On each cube $\overline{P}_k$, one can choose a vector potential $\widetilde{A}_k$ of the constant field $B(a)$ tel that for all $x \in \overline{P}_k$ we have*

$$|A_k(x) - A(x)| \leqslant C_1 r_k^2,$$

where $C_1 \geqslant 0$ is a constant independent of k (and independent of a when a runs over a compact set $K \subset \mathbb{R}^n$).

The $\mathcal{C}^\infty$ regularity of B indeed leads to an upper bound

$$|B(a) - B(x)| \leqslant C_2 r_k, \qquad x \in \overline{P}_k.$$

Let A'_k be a potential of the field $B(a) - B(x)$ on the cube $\overline{P}_k$, calculated using the usual homotopy formula for starlike open sets. We then get

$$|A'_k(x)| \leqslant C_3 r_k^2,$$

and it is sufficient to take $\widetilde{A}_k = A + A'_k$. □

Let $(x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{R}^n$. Let us take new coordinates $(y_1, \ldots, y_n)$ depending linearly on $x_1, \ldots, x_n$ such that $(dy_1, \ldots, dy_n)$ is an orthonormal basis for the metric g at point a, and such that $B(a)$ admits in this basis a normalized form as in (2.1) :

$$B(a) = \sum_{j=1}^{s} B_j(a)\, dy_j \wedge dy_{j+s}.$$

Let $\widetilde{g}$ be the constant metric

$$\widetilde{g} \equiv g(a) = \sum_{j=1}^{n} dy_j^2.$$

Let us denote by $D_k = d + ikA \wedge ?$, $\widetilde{D}_k = d + ik\widetilde{A}_k \wedge ?$ the connections on $E^{\otimes k}_{|P_k}$ associated with the potentials A, $\widetilde{A}_k$, and by $Q_k = Q_{P_k,k}$, $\widetilde{Q}_k$ the quadratic forms associated respectively with the metrics g, $\widetilde{g}$, and the scalar potentials V, $\widetilde{V} \equiv V(a)$ (formula (1.3)).

Lemma 2.11. – *There exists a sequence ε_k converging to* 0 (*dépending on r_k, but independent of a when a runs over a compact set $K \subset \mathbb{R}^n$), such that the global L^2 norms $\|\ \|_g$, $\|\ \|_{\tilde{g}}$ associated with the metrics g and $\widetilde{g}$ satisfy*

$$(1 - \varepsilon_k)\|u\|_{\tilde{g}}^2 \leqslant \|u\|_g^2 \leqslant (1 + \varepsilon_k)\|u\|_{\tilde{g}}^2,$$
$$(1 - \varepsilon_k)\widetilde{Q}_k(u) - \varepsilon_k\|u\|_{\tilde{g}}^2 \leqslant Q_k(u) \leqslant (1 + \varepsilon_k)\widetilde{Q}_k(u) + \varepsilon_k\|u\|_{\tilde{g}}^2$$

for all $u \in W_0^1(P_k)$.

Indeed, we have on P_k inequalities

$$(1 - C_4 r_k)\,\widetilde{g} \leqslant g \leqslant (1 + C_4 r_k)\,\widetilde{g},$$

and this gives the first double inequality in 2.11. With the notation $A'_k = A_k - A$, we infer from there

$$\begin{aligned} Q_k(u) &= \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_g^2 - V|u|^2\Big)\, d\sigma \\ &\leqslant (1 + C_5 r_k) \int_{P_k} \Big(\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 - V(a)|u|^2\Big)\, d\widetilde{\sigma} + \eta_k\|u\|_{\tilde{g}}^2 \end{aligned}$$

with $\eta_k = \sup_{P_k} |V - V(a)| + C_6 r_k$, a quantity that converges to 0 as k tends to $+\infty$. Using the inequality $(a + b)^2 \leqslant (1 + \alpha)(a^2 + \alpha^{-1}b^2)$, Lemma 2.10 implies on the other hand

$$|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 \leqslant (1 + \alpha)\Big[\,|\widetilde{D}_k u|_{\tilde{g}}^2 + \alpha^{-1} C_1^2 k^2 r_k^4 |u|^2\Big].$$

Let us choose $\alpha = \alpha_k = C_1\sqrt{k}r_k^2$. The sequence α_k tends to 0 by the assumption $\lim k^{\frac{1}{4}} r_k = 0$, and we find

$$\frac{1}{k}|\widetilde{D}_k u - ikA'_k \wedge u|_{\tilde{g}}^2 \leqslant (1 + \alpha_k)\Big[\frac{1}{k}|D_k u|_{\tilde{g}}^2 + \alpha_k|u|^2\Big].$$

This implies the upper bound for Q_k. The lower bound is obtained in the same way by means of the inequality $(a + b)^2 \geqslant (1 - \alpha)(a^2 - \alpha^{-1}b^2)$. $\square$

Lemma 2.11 reduces the proof of Proposition 2.9 to the case where the metric g and the magnetic field B are constant :

$$g = \sum_{j=1}^{n} dy_j^2, \qquad B = \sum_{j=1}^{n} B_j\, dy_j \wedge dy_{j+s}.$$

We can assume moreover that $V \equiv 0$ by performing a translation $\lambda \mapsto \lambda + V(a)$. The only remaining difficulty for applying directly Theorem 1.6 comes from the fact that the cubes P_k become oblique parallelepipeds in the coordinates $(y_1, \ldots, y_n)$; the angles between the different edges of each P_k and the ratios of their side lengths remain however bounded by positive constants. In order to solve this difficulty, it is sufficient to cover each parallelepiped P_k by cubes $P_{k,\alpha}$ whose edges are parallel to the coordinate axes $(y_1, \ldots, y_n)$. Let us fix $\varepsilon \in]0,1[$. For all $\alpha \in \mathbb{Z}^n$, let $(P_{k,\alpha})$, $(P'_{k,\alpha})$ be the open cubes of side lengths εr_k, $\varepsilon(1+\varepsilon) r_k$, and the common center $\varepsilon r_k \alpha$. We can limit ourselves to the consideration of the cubes $P_{k,\alpha}$ contained in P_k and of the cubes $P'_{k,\alpha}$ meeting P_k. Then we get

$$(2.12) \qquad P_k \supset \bigcup_\alpha P_{k,\alpha} \ \ \text{(disjointe)}, \quad \text{and} \quad \frac{\sum_\alpha \mathrm{Vol}(P_{k,\alpha})}{\mathrm{Vol}(P_k)} \geqslant 1 - C_7 \varepsilon,$$

$$(2.13) \qquad P_k \subset \bigcup_\alpha P'_{k,\alpha}, \quad \text{and} \quad \frac{\sum_\alpha \mathrm{Vol}(P'_{k,\alpha})}{\mathrm{Vol}(P_k)} \leqslant 1 + C_7 \varepsilon,$$

where C_7 is a constant independent of k (and also of a, when a runs over a compact set). The number of cubes $P_{k,\alpha}$, $P'_{k,\alpha}$ which are involved in (2.12) or (2.13) is bounded above by $C_8 \varepsilon^{-n}$. As the cubes $P'_{k,\alpha}$ overlap each other on an interval of length $\sim \varepsilon^2 r_k$ when they are contiguous, one can construct a partition of unity $\sum \psi_{k,\alpha}^2 = 1$ on P_k, such that $\operatorname{Supp} \psi_{k,\alpha} \subset P'_{k,\alpha}$ and

$$\sup_{P_k} \sum_\alpha |d\psi_{k,\alpha}|^2 = C(\psi_k) \leqslant C_9 (\varepsilon^2 r_k)^{-2}.$$

The hypothesis $\lim k^{\frac{1}{2}} r_k = +\infty$ actually implies $\lim \frac{1}{k} C(\psi_k) = 0$, and this allows to apply 2.6 (b). On the cubes $P_{k\alpha}$, $P'_{k,\alpha}$ we are now in the situation of Theorem 1.6 : after applying a homothety of ratio $\sqrt{k}$, the side of the homothetic cube $\sqrt{k}\, P_{k,\alpha}$ becomes $R_k = \varepsilon r_k \sqrt{k}$ and this value indeed tends to $+\infty$ by hypothesis. The uniform upper bound of $N_{P_k,k}(\lambda)$ follows from Proposition 1.18 and from the fact that all our constants $C_1, \ldots, C_9$ are uniform. Proposition 2.9 is proved. □

Proof of Theorem 2.3. – According to the remark preceding Proposition 2.9, we can assume $M = \mathbb{R}^n$ and Ω to be a bounded open subset of $\mathbb{R}^n$. The main idea of our argument is to combine Propositions 2.6 and 2.9 with a covering of Ω by cubes of side $r_k = k^{-\frac{1}{3}}$. The actual implementation requires nevertheless some care, in view of the difficulties related to the possible non-uniformity of the $\limsup$ and $\liminf$ involved.

Let us denote by $\Pi_{k,\alpha}$, $\Pi'_{k,\alpha}$, $\alpha \in \mathbb{Z}^n$, the open cubes of respective side lengths

$$k^{-\frac{1}{3}}, \qquad k^{-\frac{1}{3}}(1 + k^{-\frac{1}{8}}) = k^{-\frac{1}{3}} + k^{-\frac{11}{24}}$$

and common center $k^{-\frac{1}{3}}\alpha$. Let $I(k)$ (resp. $I'(k)$) be the set of indices $\alpha \in \mathbb{Z}^n$ such that $\Pi_{k.\alpha} \subset \Omega$ (resp. $\overline{\Pi}'_{k,\alpha} \cap \overline{\Omega} \neq \emptyset$). As in the reasoning of Proposition 2.9, there exists a partition of unity $\sum_{\alpha \in I'(k)} \psi_{k,\alpha}^2 = 1$ on Ω, such that $\operatorname{Supp} \psi_{k,\alpha} \subset \Pi'_{k,\alpha}$ and

$$C(\psi_k) = \sup_\Omega \sum_{\alpha \in I'(k)} |d\psi_{k,\alpha}|^2 \leqslant C_{10} k^{\frac{11}{12}},$$

hence $\lim \frac{1}{k}C(\psi_k) = 0$. We set

$$\Omega_k = \bigcup_{\alpha\in I(k)} \Pi_{k,\alpha}, \qquad \Omega'_k = \bigcup_{\alpha\in I'(k)} \Pi'_{k,\alpha}$$

and consider for every given $\lambda \in \mathbb{R}$, the functions on $\mathbb{R}^n$ defined by

$$\begin{aligned} f_k &= k^{-\frac{n}{2}} \sum_{\alpha\in I(k)} N_{\Pi_{k,\alpha},k}(\lambda)\frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})}\mathbb{1}_{\Pi_{k,\alpha}}, \\ f'_k &= k^{-\frac{n}{2}} \sum_{\alpha\in I'(k)} N_{\Pi'_{k,\alpha},k}\Big(\lambda+\frac{1}{k}C(\psi_k)\Big)\frac{1}{\mathrm{Vol}(\Pi_{k,\alpha})}\mathbb{1}_{\Pi_{k,\alpha}} \end{aligned}$$

where $\mathbb{1}_{\Pi_{k,\alpha}}$ denotes the characteristic function of $\Pi_{k,\alpha}$. Proposition 2.6 implies the inequalities

$$\int_{\mathbb{R}^n} f_k\,d\sigma \leqslant k^{-\frac{n}{2}}N_{\Omega,k}(\lambda) \leqslant \int_{\mathbb{R}^n} f'_k\,d\sigma. \tag{2.14}$$

Let $x \in \mathbb{R}^n$ be a point taken in the complement of the negligible set

$$Z = \bigcup_{k\in\mathbb{N},\,\alpha\in\mathbb{Z}^n} \partial\Pi_{k,\alpha}.$$

Then there exists a unique sequence of indices $\alpha(k) \in \mathbb{Z}^n$ such that $x \in \Pi_{k,\alpha(k)}$. Proposition 2.9 applied to the cubes $P_k = \Pi_{k,\alpha(k)}$ (resp. $P'_k = \Pi'_{k,\alpha(k)}$) with $\mathrm{Vol}(P_k) \sim \mathrm{Vol}\,P'_k$ shows that the pointwise sequences

$$f_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)}N_{P_k,k}(\lambda)\mathbb{1}_{\Omega_k}(x), \qquad f'_k(x) = \frac{k^{-\frac{n}{2}}}{\mathrm{Vol}(P_k)}N_{P'_k,k}(\lambda)\mathbb{1}_{\Omega'_k}(x),$$

satisfy

$$\begin{cases} \liminf f_k(x) \;\geqslant \nu_{B(x)}(V(x)+\lambda)\,\mathbb{1}_{\Omega}(x), \\ \limsup f'_k(x) \leqslant \overline{\nu}_{B(x)}(V(x)+\lambda)\,\mathbb{1}_{\overline{\Omega}}(x). \end{cases} \tag{2.15}$$

On the other hand, the uniform upper bound of Proposition 2.9 implies the existence of constants C_{11}, C_{12} independent of k, x and λ such that

$$f_k(x) \leqslant f'_k(x) \leqslant C_{11}\big(1+\sqrt{\lambda_+ + C_{12}}\,\big)^n.$$

Theorem 2.3 then follows from (2.14), (2.15) and the Fatou lemma. □

In view of applications to complex geometry, we will need below a slight generalization of Theorem 2.3. Consider a Hermitian vector bundle F of rank r and of class $\mathcal{C}^\infty$ over M, equipped with a Hermitian connection ∇, and continuous sections S of the fiber $\Lambda^1_{\mathbb{R}}T^*X \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(F,F)$ and V of the bundle $\mathrm{Herm}(F)$ of Hermitian endomorphisms of F. Let ∇_k be the Hermitian connection on $E^{\otimes k}\otimes F$ induced by the connections D and ∇. For the simplicity of notation, we will still denote by S and V the endomorphisms

$\mathrm{Id}_{E^{\otimes k}} \otimes S$ and $\mathrm{Id}_{E^{\otimes k}} \otimes V$ acting on $E^{\otimes k} \otimes F$. Given a relatively compact open set Ω in M, we consider the quadratic form

$$Q_{\Omega,k}(u) = \int_\Omega \Big(\frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle\Big)\, d\sigma,$$

where $u \in W_0^1(\Omega, E^{\otimes k} \otimes F)$. Let $V_1(x) \leqslant V_2(x) \leqslant \cdots \leqslant V_r(x)$ be the eigenvalues of $V(x)$ at any point $x \in M$. In this setting, the following result holds.

Theorem 2.16. – *The counting function $N_{\Omega,k}(\lambda)$ of the eigenvalues of $Q_{\Omega,k}$ admits for all $\lambda \in \mathbb{R}$ asymptotic estimates*

$$\liminf_{k\to+\infty}\ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \geqslant \sum_{j=1}^{r} \int_\Omega \nu_B(V_j + \lambda)\, d\sigma,$$

$$\limsup_{k\to+\infty}\ k^{-\frac{n}{2}} N_{\Omega,k}(\lambda) \leqslant \sum_{j=1}^{r} \int_\Omega \overline{\nu}_B(V_j + \lambda)\, d\sigma,$$

where B is the magnetic field associated with the connection D on E.

Proof. – The localization principle 2.6 is still valid in the present situation. It is therefore sufficient to prove the inequalities of proposition 2.16 when Ω is small enough. Let $a \in M$ be a fixed point and $(e_1, \ldots, e_r)$ a $\mathcal{C}^\infty$ orthonormal frame of F over a neighborhood W of a, such that $(e_1(a), \ldots, e_r(a))$ diagonalizes the endomorphism $V(a)$. Let us express u under the form

$$u = \sum_{j=1}^{r} u_j \otimes e_j$$

where u_j is a section of $E^{\otimes k}$. For every $\varepsilon > 0$, there is exists a neighborhood $W'_\varepsilon \subset W$ of a on which

$$\sum_{j=1}^{r}(V_j(a) - \varepsilon)|u_j|^2 \leqslant \langle Vu, u\rangle \leqslant \sum_{j=1}^{r}(V_j(a) + \varepsilon)|u_j|^2$$

On the other hand, we have

$$\nabla_k u = \sum_{j=1}^{r} D_k u_j \otimes e_j + u_j \otimes \nabla e_j,$$

and the term $u_j \otimes \nabla e_j$ can be absorbed into Su (this actually brings us back to the case where the connection ∇ is flat). The inequalities

$$(1 - k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 - k^{\frac{1}{2}})|Su|^2 \leqslant |\nabla_k u + Su|^2 \leqslant (1 + k^{-\frac{1}{2}})|\nabla_k u|^2 + (1 + k^{\frac{1}{2}})|Su|^2$$

shows that the term Su only changes $Q_{\Omega,k}$ by a multiplicative factor $1 \pm \varepsilon$ and by an additive factor $\pm\varepsilon\|u\|^2$. As a consequence, for every $\varepsilon > 0$, there exist a neighborhood W_ε of a and an integer $k_0(\varepsilon)$ such that

$$(1 - \varepsilon)\widetilde{Q}_{\Omega.k}(u) - \varepsilon\|u\|^2 \leqslant Q_{\Omega,k}(u) \leqslant (1 + \varepsilon)\widetilde{Q}_{\Omega,k}(u) + \varepsilon\|u\|^2,$$

as soon as $k \geqslant k_0(\varepsilon)$ and $\Omega \subset W_\varepsilon$, where $\widetilde{Q}_{\Omega,k}$ denotes the quadratic form

$$\widetilde{Q}_{\Omega.k}(u) = \sum_{j=1}^{r} \int_\Omega \Big(\frac{1}{k} |D_k u_j|^2 - V_j(a)|u_j|^2 \Big)\, d\sigma.$$

Since $\widetilde{Q}_{\Omega,k}$ is a direct sum of r quadratic forms, the spectrum of $\widetilde{Q}_{\Omega.k}$ is the union (counted with multiplicities) of the spectra of the terms in the sum. Theorem 2.16 follows. □

3. Bochner-Kodaira-Nakano identity in Hermitian geometry.

The goal of the forthcoming sections is to draw consequences of the spectral theorem 2.16 in the study of d''-cohomology of Hermitian holomorphic vector bundles. In this direction, we need to relate the conjugate-holomorphic Laplace-Beltrami operator Δ'' with the Schrödinger operator of a suitable real connection. This is done by means of a Weitzenböck type formula, known in complex differential geometry as the Bochner-Kodaira-Nakano identity.

Let X be a compact complex analytic manifold of dimension n and F a Hermitian holomorphic vector bundle of rank r over X. As is well known, there exists a unique Hermitian connection $D = D' + D''$ on F whose component D'' of type $(0,1)$ coincides with the $\overline{\partial}$ operator of the bundle (this connection is sometimes termed "holomorphic" and called the Chern connection). Let $c(F) = D^2 = D'D'' + D''D'$ be the curvature form of D on F. Let us equip X with an arbitrary Hermitian metric ω of type $(1,1)$ and class $\mathcal{C}^\infty$. The space $\mathcal{C}^\infty_{p,q}(X,F)$ of $\mathcal{C}^\infty$ sections of the bundle $\Lambda^{p,q}T^*X \otimes F$ is then equipped with a natural prehilbertian structure. We let $\delta = \delta' + \delta''$ denote the formal adjoint of D, considered as a differential operator on $\mathcal{C}^\infty(X,F)$, and Λ the adjoint of the Lefschetz operator $L : u \mapsto \omega \wedge u$.

We will use the Hermitian Bochner-Kodaira-Nakano identity in the form it was established in [6], although one could in fact get by with the less precise formula given by P. Griffiths, as Y.T. Siu does in [16], [17]. For differential operators A, B on $\mathcal{C}^\infty(X,F)$, one defines their graded bracket $[A,B]$ by the formula

$$[A,B] = AB - (-1)^{ab} BA$$

where a, b are the respective degrees of A and B. The Laplace-Beltrami operators Δ' and Δ'' are then classically given by

$$\Delta' = [D', \delta'] = D'\delta' + \delta' D', \qquad \Delta'' = [D'', \delta''].$$

With the torsion tensor $d'\omega$, we associate the wedge multiplication operator $u \mapsto d'\omega \wedge u$ on $\mathcal{C}^\infty(X,F)$, type $(2,1)$, which we still denote $d'\omega$, and the operator τ of type $(1,0)$ defined by $\tau = [\Lambda, d'\omega]$. We finally put

$$D'_\tau = D' + \tau, \qquad \delta'_\tau = (D'_\tau)^* = \delta' + \tau^*, \qquad \Delta'_\tau = [D'_\tau, \delta'_\tau].$$

Then the following identity holds. The reader is referred to [6] for a proof.

Proposition 3.1. *We have*

$$\Delta'' = \Delta'_\tau + [ic(F), \Lambda] + T_\omega$$

where T_ω *is the operator of order* 0 *and of type* $(0,0)$ *defined by*

$$T_\omega = \Big[\Lambda, \Big[\Lambda, \frac{i}{2} d' d'' \omega\Big]\Big] - [d'\omega, (d'\omega)^*].$$

According to the Hodge-De Rham theory, the sheaf cohomology group $H^q(X,F)$ is isomorphic to the space of Δ''-harmonic $(0,q)$-forms with values in F. Let $u \in \mathcal{C}^\infty_{p.q}(X,F)$. Proposition 3.1 gives us an equality

$$\text{(3.2)}\quad \int_X |D''u|^2 + |\delta'' u|^2 = \int_X \langle \Delta'' u, u\rangle = \int_X |D'_\tau u|^2 + |\delta'_\tau u|^2 + \langle [ic(F), \Lambda] u, u\rangle + \langle T_\omega u, u\rangle,$$

where integrals are calculated relatively to the volume element $d\sigma = \frac{\omega^n}{n!}$. In particular, if u is of bidegré $(0,q)$, we have $\delta'_\tau u = 0$ by a bidegree consideration. Therefore

$$\text{(3.3)}\qquad \int_X \langle \Delta'' u, u\rangle = \int_X |D'_\tau u|^2 + \langle [ic(F), \Lambda] u, u\rangle + \langle T_\omega u, u\rangle.$$

One can also consider u as a (n,q)-form with values in the vector bundle

$$\widetilde{F} := F \otimes \Lambda^n TX.$$

We will denote by $\widetilde{D} = \widetilde{D}' + \widetilde{D}''$ the Hermitian holomorphic connection on $\widetilde{F}$, and by $\widetilde{u}$ the canonical image of u in $\mathcal{C}^\infty_{n,q}(X,F)$.

Lemma 3.4. *We have commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}^\infty_{0.q}(X,F) & \xrightarrow{D''} & \mathcal{C}^\infty_{0,q+1}(X,F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) & \xrightarrow{\widetilde{D}''} & \mathcal{C}^\infty_{n,q+1}(X,\widetilde{F}), \end{array} \qquad \begin{array}{ccc} \mathcal{C}^\infty_{0,q}(X,F) & \xrightarrow{\Delta''} & \mathcal{C}^\infty_{0,q}(X,F) \\ \sim\downarrow & & \downarrow\sim \\ \mathcal{C}^\infty_{n,q}(X,\widetilde{F}) & \xrightarrow{\widetilde{\Delta}''} & \mathcal{C}^\infty_{n,q}(X,\widetilde{F}), \end{array}$$

where the vertical arrows are the metric isomorphisms $u \mapsto \widetilde{u}$.

Proof. – The commutativity of the left diagram is a consequence of the fact that the line bundle $\Lambda^n TX$ is holomorphic (notice however that the corresponding result for D' and $\widetilde{D}'$ is wrong). Therefore, there is a corresponding commutative diagram for the adjoint operators δ'', $\widetilde{\delta}''$ and for Δ'', $\widetilde{\Delta}''$. □

Lemma 3.4 and identity (3.2) imply

$$\text{(3.5)}\qquad \int_X \langle \Delta'' u, u\rangle = \int_X \langle \widetilde{\Delta}'' \widetilde{u}, \widetilde{u}\rangle = \int_X |\widetilde{\delta}'_\tau \widetilde{u}|^2 + \langle [ic(\widetilde{F}), \Lambda] \widetilde{u}, \widetilde{u}\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\,\rangle.$$

We now slightly transform the expression of (3.3) and (3.5). The holomorphic Hermitian connection on the bundle $\Lambda^q T^*X$ induces on conjugate bundle $\Lambda^{0,q} T^*X$ a connection whose component of type $(1,0)$ coincides with the operator d'. From this we get a natural Hermitian connection ∇ on the tensor product bundle $\Lambda^{0,q}T^*X \otimes F$ (observe however that this vector bundle is in general non holomorphic for $q \neq 0$). Let ∇' and ∇'' be the components of ∇ of type $(1,0)$ and $(0,1)$.

Proposition 3.6. *We have*

$$\nabla' = D' : \mathcal{C}^\infty(\Lambda^{0,q}T^*X \otimes F) \to \mathcal{C}^\infty_{1,0}(\Lambda^{0,q}T^*X \otimes F),$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(X, \Lambda^{0.q}T^*X \otimes F) & \xrightarrow{\nabla''} & \mathcal{C}^\infty_{0,1}(X, \Lambda^{0.q}T^*X \otimes F) \\ \sim \downarrow & & \downarrow \Psi \\ \mathcal{C}^\infty_{n,q}(X, \widetilde{F}) & \xrightarrow{\widetilde{\delta}''} & \mathcal{C}^\infty_{n-1,q}(X, \widetilde{F}), \end{array}$$

where the vertical arrows are isometries, the one on the left being given by $u \mapsto \widetilde{u}$.

Proof. – The equality $\nabla' = D'$ comes from the fact that the $(1,0)$ component of the connection prescribed above on $\Lambda^{0,q}T^*X$ coincides with d'. The diagram requires the vertical arrow Ψ to be defined. Let

$$\{?|?\} : (\Lambda^{p_1,q_1}T^*X \otimes \widetilde{F}) \times (\Lambda^{p_2,q_2}T^*X \otimes \widetilde{F}) \longrightarrow \Lambda^{p_1+q_2,q_1+p_2}T^*X$$

be the sesquilinear canonical pairing induced by the metric on the fibers of F, and let

$$* : \Lambda^{p,q}T^*X \otimes \widetilde{F} \longrightarrow \Lambda^{n-q,n-p}T^*X \otimes \widetilde{F}$$

be the Hodge-De Rham-Poincaré operator defined by

$$\{v| * w\} = \langle v, w\rangle \, d\sigma, \qquad v,\ w \in \Lambda^{p,q}T^*X \otimes \widetilde{F}.$$

By taking the composition, we derive from this an isometry

$$\Psi_0 : \Lambda^{0,1}T^*X \otimes F \xrightarrow{\sim} \Lambda^{n,1}T^*X \otimes \widetilde{F} \xrightarrow{*} \Lambda^{n-1,0}T^*X \otimes \widetilde{F}$$

and the arrow Ψ is obtained by definition by tensorizing $-i^{-n^2}\Psi_0$ with $\Lambda^{0,q}T^*X$. To prove the commutativity, we first assume $q = 0$. Let $u \in \mathcal{C}^\infty(F)$. We have classically

$$\widetilde{\delta}'\widetilde{u} = - * \widetilde{D}'' * \widetilde{u}.$$

Since $\widetilde{u} \in \mathcal{C}^\infty_{n,0}(X, F)$, we get $*\widetilde{u} = i^{-n^2}\widetilde{u}$, hence

$$\widetilde{\delta}'\widetilde{u} = -i^{-n^2} * D''\widetilde{u} = -i^{-n^2} * \sim D''u = -i^{-n^2}\Psi_0(D''u) = \Psi(\nabla''u).$$

In case q is arbitrary, one just needs trivializing $\Lambda^{0,q}T^*X$ in a neighborhood of an arbitrary point x and choosing an orthonormal frame $(e_1, \ldots, e_N)$ of this bundle, such that$\nabla e_1(x) = \cdots = \nabla e_N(x) = 0$. □

We now consider the bundle morphisms

$$S' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{1,0}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$
$$S'' : \Lambda^{0,q}T^*X \otimes F \to \Lambda^{0,1}T^*X \otimes \Lambda^{0,q}T^*X \otimes F$$

where $S' = \tau = [\Lambda, d'\omega]$, and where S'' is the lifting by the isometries $\sim$ and Ψ of the morphism

$$\tau^* = [(d'\omega)^*, L] : \Lambda^{n,q}T^*X \otimes \widetilde{F} \to \Lambda^{n-1.q}T^*X \otimes \widetilde{F}.$$

According to Proposition 3.6, we have

$$|D'_\tau u| = |\nabla' u + S'u|, \qquad |\widetilde{\delta'_\tau}\widetilde{u}| = |\nabla'' u + S'' u|.$$

Putting $S = S' \oplus S''$, the addition of identities (3.3) and (3.5) imply

$$\begin{aligned} 2\int_X \langle \Delta'' u, u\rangle = \int_x |\nabla u + Su|^2 + \int_X \langle [ic(F), \Lambda]u, u\rangle \\ + \int_X \langle [ic(\widetilde{F}), \Lambda]\widetilde{u}, \widetilde{u}\rangle + \langle T_\omega u, u\rangle + \langle T_\omega \widetilde{u}, \widetilde{u}\rangle \end{aligned} \tag{3.7}$$

for all $u \in \mathcal{C}^\infty_{0,q}(X, F)$.

Now, let E be a Hermitian holomorphic fiber of rank 1 over X. For any integer k, we denote by D_k and ∇_k the natural Hermitian connections on the bundles $F_k = E^{\otimes k} \otimes F$ and $\Lambda^{0,q}T^*X \otimes F_k$, and we put $\Delta''_k = [D''_k, \delta''_k]$. The curvature form of F_k (resp. $\widetilde{F}_k$) is given by

$$c(F_k) = c(F) + kc(E) \otimes \mathrm{Id}_F, \quad \text{resp.} \quad c(\widetilde{F}_k) = c(\widetilde{F}) + kc(E) \otimes \mathrm{Id}_{\widetilde{F}}\,. \tag{3.8}$$

Recall, although this will not be used in the sequel, that

$$c(\widetilde{F}) = c(F) + c(\Lambda^n TX) \otimes \mathrm{Id}_F = c(F) + \mathrm{Ricci}(\omega) \otimes \mathrm{Id}_F\,.$$

We thus have to evaluate the terms $[ic(E), \Lambda]$. At any point $x \in X$, let $\alpha_1(x), \alpha_2(x), \ldots,$ $\alpha_n(x)$ be the eigenvalues of $ic(E)(x)$ relatively to the Hermitian metric ω on X. There exists a local coordinate system $(z_1, \ldots, z_n)$ centered at x such that $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$ is an orthonormal basis of $(T_xX, \omega(x))$, and such that

$$\omega(x) = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j,$$
$$ic(E)(x) = \frac{i}{2}\sum_{j=1}^n \alpha_j(x)\, dz_j \wedge d\overline{z}_j.$$

Let $(e_1, \ldots, e_r)$ be an orthonormal frame of the fiber $E^k_x \otimes F_x$. For $v \in \Lambda^{p.q}T^*X \otimes F_k$, we can write

$$v = \sum_{|I|=p, |J|=q,\, \ell} v_{I,J,\ell}\, dz_I \wedge d\overline{z}_J \otimes e_\ell, \qquad |v|^2 = 2^{p+q}\sum_{I,J,\ell} |v_{I,J,\ell}|^2.$$

An elementary calculation, explained for example in [6], gives the formula

$$\langle [ic(E),\Lambda]v,v\rangle = 2^{p+q}\sum_{I,J,\ell}(\alpha_I+\alpha_J-\sum_{j=1}^n\alpha_j)|v_{I,J,\ell}|^2 \tag{3.9}$$

with $\alpha_I=\sum_{j\in I}\alpha_j$. Let $u\in\Lambda^{0,q}T^\star X\otimes F_k$. Put

$$u=\sum_{J,\ell}u_{J,\ell}\,d\overline{z}_J\otimes e_\ell.$$

According to (3.9), we have

$$\langle [ic(E),\Lambda]u,u\rangle = 2^q\sum_{J,\ell}-\alpha_{\complement J}|u_{J,\ell}|^2,$$

$$\langle [ic(E\rangle,\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q\sum_{J,\ell}\alpha_J|u_{J,\ell}|^2.$$

Let V be the Hermitian endomorphism of $\Lambda^{0,q}T^\star X\otimes F_k$ defined by

$$\langle Vu,u\rangle = -\langle [ic(E),\Lambda]u,u\rangle-\langle [ic(E),\Lambda]\widetilde{u},\widetilde{u}\rangle = 2^q\sum_{J,\ell}(\alpha_{\complement J}-\alpha_J)|u_{J,\ell}|^2. \tag{3.10}$$

The eigenvalues of V are therefore the coefficients $\alpha_{\complement J}-\alpha_J$, counted with multiplicity $r=\mathrm{rank}(F)$. Finally, let Θ be the Hermitian endomorphism defined by

$$\langle\Theta u,u\rangle = \langle [ic(F),\Lambda]u,u\rangle+\langle [ic(\widetilde{F}),\Lambda]\widetilde{u},\widetilde{u}\rangle+\langle T_\omega u,u\rangle+\langle T_\omega\widetilde{u},\widetilde{u}\rangle. \tag{3.11}$$

The identities (3.7-11) then imply

$$\frac{2}{k}\int_X\langle\Delta''_k u,u\rangle = \int_X\frac{1}{k}|\nabla_k u+Su|^2-\langle Vu,u\rangle+\frac{1}{k}\langle\Theta u,u\rangle \tag{3.12}$$

where the operators S, V, Θ act merely on the component $\Lambda^{0,q}T^\star X\otimes F$ of $\Lambda^{0,q}T^\star X\otimes F_k$. We can thus use Theorem 2.16 to determine the asymptotic spectral distribution of Δ''_k, because the term $\frac{1}{k}\langle\Theta u,u\rangle$ tends to 0 in norm.

Let $h^q_k(\lambda)$ be the number of eigenvalues $\leqslant k\lambda$ of Δ''_k acting on $\mathcal{C}^\infty_{0,q}(E^{\otimes k}\otimes F)$. The magnetic field B is given here by

$$B=-ic(E)=-\sum_{j=1}^n\alpha_j\,dx_j\wedge dy_j,\qquad z_j=x_j+iy_j. \tag{3.13}$$

Since $\dim_{\mathbb{R}}X=2n$, Theorem 2.16 can be reinterpreted as follows.

Theorem 3.14. *There exists a countable set $\mathcal{D}$ such that for every $q=0,1,\ldots,n$ and $\lambda\in\mathbb{R}\smallsetminus\mathcal{D}$ the asymptotic estimate*

$$h^q_k(\lambda)=rk^n\sum_{|J|=q}\int_X\nu_B(2\lambda+\alpha_{\complement J}-\alpha_J)\,d\sigma+o(k^n)$$

holds when k tends to $+\infty$.

4. Witten complex and Morse inequalities.

E. Witten [18], [19] recently introduced a new analytic method that reproves the standard Morse inequalities for de Rham cohomology. We adapt here his method to the study of d''-cohomology. The main difference lies in the fact that the magnetic field is always zero in the case of the de Rham cohomology (since $d^2 = 0$!), and, in this case, the only thing that plays a role is the “electric field”.

With the notation of §3, let $\mathcal{H}^q_k(\lambda) \subset \mathcal{C}^\infty_{0,q}(X, E^{\otimes k} \otimes F)$ be the direct sum of the eigensubspaces of Δ''_k attached to eigenvalues $\leqslant k\lambda$. The vector space $\mathcal{H}^q_k(\lambda)$ has therefore a finite dimension

$$h^q_k(\lambda) = \dim_{\mathbb{C}} \mathcal{H}^q_k(\lambda).$$

Hodge theory gives us an isomorphism

$$H^q(X, E^{\otimes k} \otimes F) \simeq \mathcal{H}^q_k(0).$$

For brevity of notation, we put

$$h^q_k = \dim H^q(X, E^{\otimes k} \otimes F) = h^q_k(0).$$

Proposition 4.1. – $\mathcal{H}^\bullet_k(\lambda)$ *is a sub-complex of the Dolbeault complex*

$$D''_k : \mathcal{C}^\infty_{0,\bullet}(X, E^{\otimes k} \otimes F).$$

Moreover, the inclusion $\mathcal{H}^\bullet_k(\lambda) \subset \mathcal{C}^\infty_{0,\bullet}(X, E^{\otimes k} \otimes F)$ *and the orthogonal projection*

$$P_\lambda : \mathcal{C}^\infty_{0,\bullet}(X, E^{\otimes k} \otimes F) \to \mathcal{H}^\bullet_k(\lambda)$$

induce inverse isomorphisms in cohomology.

Proof. – The fact that $\mathcal{H}^\bullet_k(\lambda)$ is a sub-complex of $\mathcal{C}^\infty_{0,\bullet}(X, E^{\otimes k} \otimes F)$ comes from the commutativity of operators D''_k and Δ''_k. Now, let

$$G = \int_{\lambda > 0} \frac{1}{\lambda} dP_1$$

be the Green operator of the Laplacian Δ''_k. Since $[P_\lambda, \Delta''_k] = 0$, we have equalities $[G, \Delta''_k] = 0$ and

$$\Delta''_k G + P_0 = \mathrm{Id}\,.$$

Moreover, $[P_\lambda, D''_k] = [G, D''_k] = 0$. Therefore we get

$$\begin{aligned}\mathrm{Id} - P_\lambda &= \Delta''_k G(\mathrm{Id} - P_\lambda) + P_0(\mathrm{Id} - P_\lambda) = \Delta''_k G(\mathrm{Id} - P_\lambda)\\ &= D''_k\big(\delta''_k G(\mathrm{Id} - P_\lambda)\big) + \big(\delta''_k G(\mathrm{Id} - P_\lambda)\big) D''_k,\end{aligned}$$

hence the operator $\delta''_k G(Id - P_\lambda)$ is a homotopy between Id and P_λ. □

We now use a simple and classical lemma of homological algebra.

Lemma 4.2. – *Let*

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

be a complex of vector spaces of finite dimensions c^0, c^1, …, c^n over a field $\mathbb{K}$. Let $h^q = \dim_{\mathbb{K}} H^q(C^\bullet)$. Then we have the following inequalities.

(a) *Morse inequalities :* $h^q \leqslant c^q$, $0 \leqslant q \leqslant n$.

(b) *Equality of Euler-Poincaré characteristics* $\chi(H^\bullet(C^\bullet)) = \chi(C^\bullet)$:

$$h^0 - h^1 + \cdots + (-1)^n h^n = c^0 - c^1 + \cdots + (-1)^n c^n.$$

(c) *Strong Morse inequalities : for all q, $0 \leqslant q \leqslant n$,*

$$h^q - h^{q-1} + \cdots + (-1)^q h^0 \leqslant c^q - c^{q-1} + \cdots + (-1)^q c^0.$$

Proof. – Let $Z^q = \operatorname{Ker} d^q$ and $B^q = \operatorname{Im} d^{q-1}$ have dimensions z^q and b^q. Equality (b) follows from the trivial formulas

$$c^q = z^q + b^{q+1}, \qquad h^q = z^q - b^q,$$

while (c) is a consequence of (b) applied to the complex

$$0 \to C^0 \to C^1 \to \cdots \to C^{q-1} \to Z^q \to 0. \qquad \square$$

If F is a holomorphic vector bundle on X, we define its Euler-Poincaré characteristic by

$$\chi(X, F) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, F).$$

Combining proposition 4.1 and lemma 4.2, we get for all $\lambda \geqslant 0$ and $q \in \mathbb{N}$, $0 \leqslant q \leqslant n$, the inequality

$$h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant h_k^q(\lambda) - h_k^{q-1}(\lambda) + \cdots + (-1)^q h_k^0(\lambda).$$

Now, we evaluate $h_k^q(\lambda)$ by means of Theorem 3.14 and we let $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$ tend to $0+$. We then get the following conclusion.

Corollary 4.3. – *There are asymptotic inequalities*

(a) $h_k^q \leqslant k^n I^q + o(k^n)$,

(b) $\chi(X, E^{\otimes k} \otimes F) = k^n(I^0 - I^1 + \cdots + (-1)^n I^n) + o(k^n)$,

(c) $h_k^q - h_k^{q-1} + \cdots + (-1)^q h_k^0 \leqslant k^n(I^q - I^{q-1} + \cdots + (-1)^q I^0) + o(k^n)$, *where I^q is the curvature integral*

$$I^q = r \sum_{|J|=q} \int_X \overline{\nu}_B(\alpha_{\complement J} - \alpha_J) d\sigma.$$

According to (3.13), the non negative eigenvalues of the magnetic field B are the $|\alpha_j|$, $1 \leqslant j \leqslant n$. For any point $x \in X$, let us arrange these eigenvalues so that

$$|\alpha_1 \geqslant |\alpha_2| \geqslant \cdots \geqslant |\alpha_s| > 0 = |\alpha_{s+1}| = \cdots = |\alpha_n|, \qquad s = s(x).$$

Formula (1.5) gives

$$\overline{\nu}_B(a_{\complement J} - \alpha_J) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} |\alpha_1 \ldots \alpha_s| \sum_{(p_1,\ldots,p_s)} \left\{ \alpha_{\complement J} - \alpha_J - \sum (2p_j+1)|\alpha_j| \right\}_+^{n-s}$$

with the notation $\{\lambda\}_+^0 = 0$ if $\lambda < 0$ and $\{\lambda\}_+^0 = 1$ if $\lambda \geqslant 0$. As the quantity

$$\alpha_{\complement J} - \alpha_J - \sum (2p_j+1)|\alpha_j|$$

is always $\leqslant 0$, $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J)$ can be non zero only for $s = n$. In the latter case

$$\alpha_{\complement J} - \alpha_J - \sum (2p_j+1)|\alpha_j| = 0$$

if and only if $p_1 = \cdots = p_n = 0$ and $\alpha_j < 0$ for $j \in J$, $\alpha_j > 0$ for $j \in \complement J$. This requires the form $ic(E)$ to be non degenerate of index q. For $x \in X(q)$ and $|J| = q$ (see notation from the introduction), we thus get

$$\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = (2\pi)^{-n}|\alpha_1 \ldots \alpha_n| > 0$$

where J is the multi-index $J(x) = \{j\,;\ \alpha_j(x) < 0\}$ and $\overline{\nu}_B(\alpha_{\complement J} - \alpha_J) = 0$ si $J \neq J(x)$. This implies

$$I^q = r \int_{X(q)} (2\pi)^{-n}(-1)^q \alpha_1 \ldots \alpha_n \, d\sigma = \frac{r}{n!} \int_{X(q)} (-1)^q \Big(\frac{i}{2\pi} c(E)\Big)^n.$$

Our fundamental theorem 0.1 is then just a reformulation of Corollary 4.3. The above arguments show that the harmonic forms associated with $H^q(X, E^{\otimes k} \otimes F)$ concentrate asymptotically on $X(q)$, and that, at each point of $X(q)$, their direction tends to be aligned with the q-subspace of TX corresponding to the negative part of $ic(E)$. Moreover, the eigenvalue of minimal energy $p_1 = \cdots = p_n = 0$ of the harmonic oscillator is the only one that is involved in those harmonic forms. For $q = 1$, the Morse inequality 4.3 (c) can be written

$$h_k^1 - h_k^0 \leqslant k^n(I^1 - I^0) + o(k^n).$$

In particular, we find an asymptotic lower bound of the number of holomorphic sections of the vector bundle $E^{\otimes k} \otimes F$.

Theorem 4.4. – *We have*

$$\dim H^0(X, E^{\otimes k} \otimes F) \geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1)}\Big(\frac{i}{2\pi}c(E)\Big)^n - o(k^n).$$

More generally, the addition of the two instances of inequality 4.3 (c) for indices $q+1$ and $q-2$ leads to

$$h_k^{q+1} - h_k^q + h_k^{q-1} \leqslant k^n(I^{q+1} - I^q + I^{q-1}) + o(k^n),$$

hence the lower bound

$$\dim H^q(X, E^{\otimes k} \otimes F) \geqslant r\frac{k^n}{n!}\sum_{j=0,\pm 1}(-1)^q\int_{X(q+j)}\Big(\frac{i}{2\pi}c(E)\Big)^n - o(k^n). \tag{4.5}$$

5. Characterization of Moišezon varieties.

Let X be a compact $\mathbb{C}$-analytic variety of dimension n. The algebraic dimension of X, denoted $a(X)$, is by definition the transcendence degree over $\mathbb{C}$ of the field $K(X)$ of meromorphic functions on X. According to a well-known theorem of Siegel [15], the algebraic dimension of X always satisfies the inequality $0 \leqslant a(X) \leqslant n$. When $a(X) = n$, the variety X is called a Moišezon space. As we recall below, the algebraic dimension of X imposes strong asymptotic bounds for the dimension of spaces of sections of tensor powers of any holomorphic line or vector bundle on X.

Theorem 5.1. – *Let X be a compact complex manifold, a the algebraic dimension of X, F a holomorphic vector bundle of rank r, and E a holomorphic line bundle over X. Then, there exists a constant $C_E \geqslant 0$ that depends only on E, such that*

$$\dim H^0(X, E^{\otimes k} \otimes F) \leqslant C_E r k^a + o(k^a).$$

Proof. – We essentially repeat the arguments detailed by Y.T. Siu in [16]. Let $\{W_\ell\}$ be a covering of X by open coordinate charts $W_\ell \subset \mathbb{C}^n$, and $B_j = B(a_j, R_j)$, $1 \leqslant j \leqslant m$, a family of relatively compact balls in the open sets W_ℓ, such that the concentric balls $B'_j = B(a_j, \frac{1}{7}R_j)$ still cover X. Let us provide E, F with Hermitian metrics, and let $\exp(-\varphi_j)$ be the weight representing the metric of E in a trivialization of E in a neighborhood of $\overline{B}_j$.

Let $s \in H^0(X, E^{\otimes k} \otimes F)$ be a holomorphic section that vanishes at order p at a point $x_j \in B'_j$. The inclusions

$$B'_j \subset B(x_j, \frac{2}{7}R_j) \subset B(x_j, \frac{6}{7}R_j) \subset B_j$$

and the Schwarz lemma applied to the two intermediate balls lead to the inequality

$$\sup_{B'_j}|s| \leqslant \exp(Ak + C_F)3^{-p}\sup_{B_j}|s|, \tag{5.2}$$

where $A = \max_{1\leqslant j\leqslant m} \operatorname{diam} \varphi_j(B_j)$ depends only on of E, and where $C_F \geqslant 0$ is a constant that depends on the metric of F.

Let $\rho \leqslant r = \operatorname{rank}(F)$ be the maximum for all $x \in X$ of the dimension of the subspace of the fiber F_x generated by elements $s(x)$ when s runs over $\bigcup_{k\in\mathbb{N}} H^0(X, E^{\otimes k} \otimes F)$. If $\rho = 0$, then $H^0(X, E^{\otimes k} \otimes F) = 0$ for all k. Let us now distinguish two cases according to whether $\rho = 1$ or $\rho > 1$.

(a) *Assume* $\rho = 1$.

Let $h_k = \dim H^0(X, E^{\otimes k} \otimes F)$, and suppose that $h_k > 0$. Under the hypothesis $\rho = 1$, the global sections of $E^{\otimes k} \otimes F$ define a holomorphic map

$$\Phi_k : X \smallsetminus Z_k \to \mathbb{P}^{h_k-1}(\mathbb{C})$$

where $Z_k \subset X$ is the analytic subset of common zeros. Let d the maximum rank of the differential Φ'_k on $X \smallsetminus Z_k$. We necessarily have $d \leqslant a$, otherwise the field of rational functions of $\mathbb{P}^{h_k-1}(\mathbb{C})$ would induce a field of meromorphic functions on X of transcendence degree $\geqslant d > a$, which is absurd. Let us choose for all $j = 1, \ldots, m$ a point $x_j \in B'_j \smallsetminus Z_k$ such that Φ'_k is of maximum rank $= d$ at x_j, and let $s_0 \in H^0(X, E^{\otimes k} \otimes F)$ be a section that does not vanish at any point x_j. For every $s \in H^0(X, E^{\otimes k} \otimes F)$, the quotient s/s_0 is well defined as a meromorphic function on X, and moreover s/s_0 is a holomorphic function in a neighborhood of x_j, that is constant along the fibers of Φ_k. As Φ_k is a subimmersion in the neighborhood of each point x_j, we can choose a subvariety M_j of dimension d passing through x_j and transverse to the fiber $\Phi_k^{-1}(\Phi_k(x_j))$. The section s will vanish at order p at each point x_j, $1 \leqslant j \leqslant m$, if and only if partial derivatives of order $< p$ of s/s_0 along M_j vanish at x_j. In total, this corresponds to the vanishing of

$$m\binom{p+d-1}{d}$$

derivatives. If we choose $p = [Ak + C_F] + 1$, then inequality (5.2) implies

$$\sup_X |s| \leqslant \left(\frac{e}{3}\right)^p \sup_X |s|,$$

hence $s = 0$. As $d \leqslant a$, we therefore get

$$\dim H^0(X, E^{\otimes k} \otimes F) \leqslant m\binom{p+a-1}{a} \leqslant C_E k^a + o(k^a)$$

with $C_E = mA^a/a!$.

(b) *Assume* $\rho > 1$.

Then there are sections $s_t \in H^0(X, E^{k_t} \otimes F)$, $1 \leqslant t \leqslant \rho$, and a point $x_0 \in X$ such that the vectors $s_1(x_0), \ldots, s_\rho(x_0)$ are linearly independent. By construction, for any $k \in \mathbb{N}$ and any section $s \in H^0(X, E^{\otimes k} \otimes F)$, the line $\mathbb{C}\cdot s(x)$ is contained in the subspace generated by $(s_1(x), \ldots, s_\rho(x))$, except perhaps above the analytic subset $\{x \in X; s_1 \wedge \ldots \wedge s_\rho(x)\} = 0$. Therefore, we get an injective morphism

$$H^0(X, E^{\otimes k} \otimes F) \to \bigoplus_{1\leqslant t\leqslant\rho} H^0(X, E^{k+k_{\hat{t}}} \otimes \Lambda^p F)$$

where $k_{\hat{t}} = (k_1 + \cdots + k_\rho) - k_t$, whose component of index t is given by the morphism $s \to s_1 \wedge \cdots \wedge \widehat{s}_t \wedge \cdots \wedge s_\rho \wedge s$. The image of $H^0(X, E^{\otimes k} \otimes F)$ on each component consists of sections that are colinear at almost any point with $s_1 \wedge \cdots \wedge s_\rho$. We are led to a situation analogous to the one considered in (a), where F is replaced by $E^{k_{\hat{t}}} \otimes \Lambda^\rho F$. Therefore we infer

$$\dim H^0(X, E^{\otimes k} \otimes F) \leqslant C_E \rho k^a + o(k^a), \qquad \rho \leqslant r. \qquad \square$$

Let us choose in particular for F the trivial line bundle $X \times \mathbb{C}$. The combination of Theorems 4.4 and 5.1 leads to the following characterization of Moišezon varieties.

Theorem 5.2. – *Let X be a compact $\mathbb{C}$-analytic manifold of dimension n. Then X is Moišezon as soon as there exists a Hermitian holomorphic line bundle E over X such that*

$$\int_{X(\leqslant 1)} (ic(E))^n > 0. \qquad \square$$

This theorem in turn leads to Theorem 0.8, since 0.8 (c) $\Rightarrow$ 0.8(b) $\Rightarrow$ 0.8(a). This strengthens Y.T. Siu's results [17], [18], and gives in particular a new proof of the Grauert-Riemenschneider conjecture [10].

References

[1] M.F. Atiyah and I.M. Singer, *The index of elliptic operators III*, Ann. of Math., **87** (1978), 546–604.

[2] J. Avron, I. Herbst and B. Simon, *Schrödinger operators with magnetic fields I*; Duke Math. J., **45** (1978), 847–883.

[3] Y. Colin de Verdière, *Calcul du spectre de certaines nilvariétés compactes de dimension* 3 ; Séminaire de Théorie spectrale et Géométrie, Grenoble-Chambéry, 1983-84 (exposé n°5).

[4] Y. Colin de Verdière, *Minorations de sommes de valeurs propres et conjecture de Polya* ; Séminaire de Théorie spectrale et Géométrie, Grenoble-Chambéry, 1984-85.

[5] Y. Colin de Verdière, *L'asymptotique de Weyl pour les bouteilles magnétiques* ; Commun. Math. Phys. **105** (1986), 327–335.

[6] J.-P. Demailly, *Sur l'identité de Bochner-Kodaira-Nakano en géométrie hermitienne* ; Séminaire P. Lelong, P. Dolbeault, H. Skoda (Analyse), 1983-84, Lecture Notes in Math. n°1198, Springer-Verlag, 88–97.

[7] J.-P. Demailly, *Une preuve simple de la conjecture de Grauert-Riemenschneider* ; Séminaire P. Lelong, P. Dolbeault-H. Skoda (Analyse), 1984-1985, Lecture Notes in Math. n°1295, 24–47.

[8] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie* ; Note aux C.R.A.S., Paris, 13 mai 1985.

[9] A. Dufresnoy, *Comportement asymptotique de la distribution des valeurs propres de l'équation de Schrödinger associée à certains champs magnétiques sur un cylindre* ; Séminaire de Théorie spectrale et Géométrie, Grenoble-Chambéry 1983-84 (exposé n°4).

[10] H. Grauert und O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf Komplexen Raüme* ; Invent. Math., **11** (1970), 263–292.

[11] P. Griffiths, *The extension problem in complex analysis II: embedding with positive normal bundle* ; Amer. J. of Math., **88** (1966), 366–446.

[12] F. Michau, *Comportement asymptotique de la distribution des valeurs propres de l' opérateur de Schrödinger en présence d'un champ électrique et d'un champ magnétique* ; Thèse de 3e Cycle, Univ. de Grenoble I (1982).

[13] B. Moišezon, *On n-dimensional compact varieties with n algebraically independent meromorphic functions* ; Amer. Math. Soc. Transl., **63** (1967), 51–177.

[14] M. Reed and B. Simon, *Methods of modern mathematical Physics, vol. I-II-III-IV* ; New York, Academic Press, 1978.

[15] C. L. Siegel, *Meromorphe Funktionen auf kompakten Mannigfaltigkeiten* ; Nachr. Akad. Wiss. Göttingen Math. Phys. K., (1955), n°4, 71–77.

[16] Y.T. Siu, *A vanishing theorem for semipositive line bundles over non-Kähler manifolds* ; J. Diff. Geom., **19** (1984), 431–452.

[17] Y.T. Siu, *Some recent results in complex manifold theory related to vanishing theorems for the semi-positive case* ; Arbeitstagung Bonn, Proceddings of the meeting held by the Max-Planck-Institut für Mathematik, Bonn, June 15-22, 1984, Lecture Notes in Math. n°1111, 169–192.

[18] E. Witten, *Supersymmetry and Morse theory* ; J. Diff. Geom., **17** (1982), 661–692.

[19] E. Witten, *Holomorphic Morse inequalities* ; Teubner-Texte zur Math., band 70, Algebraic and Differential Topology, Ed. G. M. Rassias (1984), 318–333.

Manuscript received May 30, 1985.

Jean-Pierre Demailly
Université de Grenoble I
Institut Fourier
Laboratoire de Mathématiques, B.P. 74
38402 St-Martin d'Hères Cedex