A sharp lower bound for the log canonical threshold

Jean-Pierre Demailly and Phạm Hoàng Hiệp

Institut Fourier, Université de Grenoble I,
Hanoi National University of Education

Abstract. In this note, we prove a sharp lower bound for the log canonical threshold of a plurisubharmonic function φ with an isolated singularity at 0 in an open subset of $\mathbb{C}^n$. This threshold is defined as the supremum of constants $c > 0$ such that $e^{-2c\varphi}$ is integrable on a neighborhood of 0. We relate $c(\varphi)$ to the intermediate multiplicity numbers $e_j(\varphi)$, defined as the Lelong numbers of $(dd^c\varphi)^j$ at 0 (so that in particular $e_0(\varphi) = 1$). Our main result is that $c(\varphi) \geq \sum e_j(\varphi)/e_{j+1}(\varphi)$, $0 \leq j \leq n-1$. This inequality is shown to be sharp; it simultaneously improves the classical result $c(\varphi) \geq 1/e_1(\varphi)$ due to Skoda, as well as the lower estimate $c(\varphi) \geq n/e_n(\varphi)^{1/n}$ which has received crucial applications to birational geometry in recent years. The proof consists in a reduction to the toric case, i.e. singularities arising from monomial ideals.

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1. Notation and main results

Here we put $d^c = \frac{i}{2\pi}(\overline{\partial} - \partial)$, so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. The normalization of the d^c operator is chosen so that we have precisely $(dd^c \log|z|)^n = \delta_0$ for the Monge-Ampère operator in $\mathbb{C}^n$. The Monge-Ampère operator is defined on locally bounded plurisubharmonic functions according to the definition of Bedford-Taylor [BT76, BT82]; it can also be extended to plurisubharmonic functions with isolated or compactly supported poles by [Dem93]. If Ω is an open subset of $\mathbb{C}^n$, we let $\mathrm{PSH}(\Omega)$ (resp. $\mathrm{PSH}^-(\Omega)$) be the set of plurisubharmonic (resp. psh ≤ 0) functions on Ω.

Definition 1.1. *Let Ω be a bounded hyperconvex domain (i.e. a domain possessing a negative psh exhaustion). Following Cegrell [Ce04], we introduce certain classes of psh functions on Ω, in relation with the definition of the Monge-Ampère operator :*

(a) $$\mathcal{E}_0(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < +\infty\},$$

(b) $$\mathcal{F}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\ \mathcal{E}_0(\Omega) \ni \varphi_p \searrow \varphi, \sup_{p\geq 1} \int_\Omega (dd^c\varphi_p)^n < +\infty\},$$

(c) $$\mathcal{E}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\ \varphi_K \in \mathcal{F}(\Omega) \textit{ such that } \varphi_K = \varphi \textit{ on } K,\ \forall K \subset\subset \Omega\}.$$

It is proved in [Ce04] that the class $\mathcal{E}(\Omega)$ is the biggest subset of $PSH^-(\Omega)$ on which the Monge-Ampère operator is well-defined. For a general complex manifold X, after removing the negativity assumption of the functions involved, one can in fact extend the Monge-Ampère operator to the class

$$\widetilde{\mathcal{E}}(X) \subset \mathrm{PSH}(X) \tag{1.2}$$

of psh functions which, on a neighborhood $\Omega \ni x_0$ of an arbitrary point $x_0 \in X$, are equal to a sum $u + v$ with $u \in \mathcal{E}(\Omega)$ and $v \in C^\infty(\Omega)$; again, this is the biggest subclass of functions of $\mathrm{PSH}(X)$ on which the Monge-Ampère operator is locally well defined. It is easy to see that $\widetilde{\mathcal{E}}(X)$ contains the class of psh functions which are locally bounded outside isolated singularities.

For $\varphi \in \mathrm{PSH}(\Omega)$ and $0 \in \Omega$, we introduce the log canonical threshold at 0

$$c(\varphi) = \sup\big\{c > 0 :\ e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } 0\big\}, \tag{1.3}$$

and for $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ we introduce the intersection numbers

$$e_j(\varphi) = \int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c \log \|z\|)^{n-j} \tag{1.4}$$

which can be seen also as the Lelong numbers of $(dd^c\varphi)^j$ at 0. Our main result is the following sharp estimate.

Theorem 1.5. *Let $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ and $0 \in \Omega$. Then $c(\varphi) = +\infty$ if $e_1(\varphi) = 0$, and otherwise*

$$c(\varphi) \geq \sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)}.$$

Remark 1.6. By Lemma 2.1 below, we have $(e_1(\varphi), \dots, e_n(\varphi)) \in D$ where

$$D = \big\{t = (t_1, \dots, t_n) \in [0, +\infty)^n :\ t_1^2 \leq t_2,\ t_j^2 \leq t_{j-1}t_{j+1},\ \forall j = 2, \dots, n-1\big\},$$

i.e. $\log e_j(\varphi)$ is a convex sequence. In particular, we have $e_j(\varphi) \geq e_1(\varphi)^j$, and the denominators do not vanish in 1.5 if $e_1(\varphi) > 0$. On the other hand, a well known inequality due to Skoda [Sko72] tells us that

$$\frac{1}{e_1(\varphi)} \leq c(\varphi) \leq \frac{n}{e_1(\varphi)},$$

hence $c(\varphi) < +\infty$ iff $e_1(\varphi) > 0$. To see that Theorem 1.5 is optimal, let us choose

$$\varphi(z) = \max\big(a_1 \ln|z_1|, \dots, a_n \ln|z_n|\big)$$

with $0 < a_1 \leq a_2 \leq \dots \leq a_n$. Then $e_j(\varphi) = a_1 a_2 \dots a_j$, and a change of variable $z_j = \zeta_j^{1/a_j}$ on $\mathbb{C} \smallsetminus \mathbb{R}_-$ easily shows that

$$c(\varphi) = \sum_{j=1}^{n} \frac{1}{a_j}.$$

Assume that we have a function $f : D \to [0, +\infty)$ such that $c(\varphi) \geq f(e_1(\varphi), \dots, e_n(\varphi))$ for all $\varphi \in \widetilde{\mathcal{E}}(\Omega)$. Then, by the above example, we must have

$$f(a_1, a_1a_2, \dots, a_1 \dots a_n) \leq \sum_{j=1}^{n} \frac{1}{a_j}$$

for all a_j as above. By taking $a_j = t_j/t_{j-1}$, $t_0 = 1$, this implies that

$$f(t_1, \dots, t_n) \leq \frac{1}{t_1} + \frac{t_1}{t_2} + \dots + \frac{t_{n-1}}{t_n}, \qquad \forall t \in D,$$

whence the optimality of our inequality. □

Remark 1.7. Theorem 1.5 is of course stronger than Skoda's lower bound $c(\varphi) \geq 1/e_1(\varphi)$. By the inequality between the arithmetic and geometric means, we infer the main inequality of [dFEM03], [dFEM04] and [Dem09]

$$c(\varphi) \geq \frac{n}{e_n(\varphi)^{1/n}}. \tag{1.8}$$

By applying the arithmetic-geometric inequality for the indices $1 \leq j \leq n-1$ in our summation $\sum e_j(\varphi)/e_{j+1}(\varphi)$, we also infer the stronger inequality

$$c(\varphi) \geq \frac{1}{e_1(\varphi)} + (n-1)\left[\frac{e_1(\varphi)}{e_n(\varphi)}\right]^{\frac{1}{n-1}}. \tag{1.9}$$

2. Log convexity of the multiplicity sequence

The log convexity of the multiplicity sequence can be derived from very elementary integration by parts and the Cauchy-Schwarz inequality, using an argument from [Ce04].

Lemma 2.1. *Let $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ and $0 \in \Omega$. We have $e_j(\varphi)^2 \leq e_{j-1}(\varphi)e_{j+1}(\varphi)$, $\forall j = 1, \ldots, n-1$.*

Proof. Without loss generality, by replacing φ with a sequence of local approximations $\varphi_p(z) = \max(\varphi(z) - C, p\log|z|)$ of $\varphi(z) - C$, $C \gg 1$, we can assume that Ω is the unit ball and $\varphi \in \mathcal{E}_0(\Omega)$. Take also h, $\psi \in \mathcal{E}_0(\Omega)$. Then integration by parts and the Cauchy-Schwarz inequality yield

$$\left[\int_\Omega -h(dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2 = \left[\int_\Omega d\varphi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h\right]^2$$

$$\leq \int_\Omega d\psi \wedge d^c\psi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h \int_\Omega d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j-1} \wedge dd^c h$$

$$= \int_\Omega -h(dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j+1} \int_\Omega -h(dd^c\varphi)^{j+1} \wedge (dd^c\psi)^{n-j-1}.$$

Now, as $p \to +\infty$, take

$$h(z) = h_p(z) = \max\Big(-1, \frac{1}{p}\log\|z\|\Big) \nearrow \begin{cases} 0 & \text{if } z \in \Omega \smallsetminus \{0\} \\ -1 & \text{if } z = 0. \end{cases}$$

By the monotone convergence theorem we get in the limit

$$\left[\int_{\{0\}} (dd^c\varphi)^j \wedge (dd^c\psi)^{n-j}\right]^2 \leq \int_{\{0\}} (dd^c\varphi)^{j-1} \wedge (dd^c\psi)^{n-j+1} \int_{\{0\}} (dd^c\varphi)^{j+1} \wedge (dd^c\psi)^{n-j-1}.$$

For $\psi(z) = \ln\|z\|$, this is the desired estimate. □

Corollary 2.2. *Let $\varphi \in \widetilde{\mathcal{E}}(\Omega)$ and $0 \in \Omega$. We have the inequalities*

$$\begin{aligned} e_j(\varphi) &\geq e_1(\varphi)^j, & \forall j = 0, 1, \ldots \leq n \\ e_k(\varphi) &\leq e_j(\varphi)^{\frac{l-k}{l-j}} e_l(\varphi)^{\frac{k-j}{l-j}}, & \forall 0 \leq j < k < l \leq n. \end{aligned}$$

In particular $e_1(\varphi) = 0$ implies $e_k(\varphi) = 0$ for $k = 2, \ldots, n-1$ if $n \geq 3$.

Proof. If $e_j(\varphi) > 0$ for all j, Lemma 2.1 implies that $j \mapsto e_j(\varphi)/e_{j-1}(\varphi)$ is increasing, at least equal to $e_1(\varphi)/e_0(\varphi) = e_1(\varphi)$, and the inequalities follow from the log convexity. The general case can be proved by considering $\varphi_\varepsilon(z) = \varphi(z) + \varepsilon\log\|z\|$, since $0 < \varepsilon^j \leq e_j(\varphi_\varepsilon) \to e_j(\varphi)$ when $\varepsilon \to 0$. The last statement is obtained by taking $j = 1$ and $l = n$. □

3. Proof of the main theorem

We start with a monotonicity statement.

Lemma 3.1. *Let $\varphi, \psi \in \widetilde{\mathcal{E}}(\Omega)$ be such that $\varphi \le \psi$ (i.e. φ is "more singular" than ψ). Then*
$$\sum_{j=0}^{n-1} \frac{e_j(\varphi)}{e_{j+1}(\varphi)} \le \sum_{j=0}^{n-1} \frac{e_j(\psi)}{e_{j+1}(\psi)}.$$

Proof. As in Remark 1.6, we set
$$D = \{t = (t_1, \ldots, t_n) \in [0, +\infty)^n : \ t_1^2 \le t_2,\ t_j^2 \le t_{j-1}t_{j+1},\ \forall j = 2, \ldots, n-1\}.$$
Then D is a convex set in $\mathbb{R}^n$, as can be checked by a straightforward application of the Cauchy-Schwarz inequality. We consider the function $f : \operatorname{int} D \to [0, +\infty)$
$$f(t_1, \ldots, t_n) = \frac{1}{t_1} + \frac{t_1}{t_2} \ldots + \frac{t_{n-1}}{t_n}. \tag{3.2}$$
We have
$$\frac{\partial f}{\partial t_j}(t) = -\frac{t_{j-1}}{t_j^2} + \frac{1}{t_{j+1}} \le 0, \qquad \forall t \in D.$$
For $a, b \in \operatorname{int} D$ such that $a_j \ge b_j$, $\forall j = 1, \ldots, n$, $[0,1] \ni \lambda \to f(b+\lambda(a-b))$ is thus a decreasing function. This implies that $f(a) \le f(b)$ for $a, b \in \operatorname{int} D$, $a_j \ge b_j$, $\forall j = 1, \ldots, n$. On the other hand, the hypothesis $\varphi \le \psi$ implies $e_j(\varphi) \ge e_j(\psi)$, $\forall j = 1, \ldots, n$, by the comparison principle (see e.g. [Dem87]). Therefore $f(e_1(\varphi), \ldots, e_n(\varphi)) \le f(e_1(\psi), \ldots, e_n(\psi))$. □

(3.3) Proof of the main theorem in the "toric case".

It will be convenient here to introduce Kiselman's refined Lelong numbers (cf. [Kis87], [Kis94a]):

Definition 3.4. *Let $\varphi \in PSH(\Omega)$. Then the function*
$$\nu_\varphi(x) = \lim_{t \to -\infty} \frac{\max\{\varphi(z) : |z_1| = e^{x_1 t}, \ldots, |z_n| = e^{x_n t}\}}{t}$$
is called the refined Lelong number of φ at 0. This function is increasing in each variable x_j and concave on $\mathbb{R}_+^n$.

By "toric case", we mean that $\varphi(z_1, \ldots, z_n) = \varphi(|z_1|, \ldots, |z_n|)$ depends only on $|z_j|$ for all j; then φ is psh if and only if $(t_1, \ldots, t_n) \mapsto \varphi(e^{t_1}, \ldots, e^{t_n})$ is increasing in each t_j and convex. By replacing φ with $\varphi(\lambda z) - \varphi(\lambda, ..., \lambda)$, $0 < \lambda \ll 1$, we can assume that $\Omega = \Delta^n$ is the unit polydisk, $\varphi(1, \ldots, 1) = 0$ (so that $\varphi \le 0$ on Ω), and we have $e_1(\varphi) = n\, \nu_\varphi(\frac{1}{n}, \ldots, \frac{1}{n})$. By convexity, the slope $\frac{\max\{\varphi(z) : |z_j| = e^{x_j t}\}}{t}$ is increasing in t for $t < 0$. Therefore, by taking $t = -1$ we get
$$\nu_\varphi(-\ln|z_1|, \ldots, -\ln|z_n|) \le -\varphi(z_1, \ldots, z_n).$$
Notice also that $\nu_\varphi(x)$ satisfies the 1-homogeneity property $\nu_\varphi(\lambda x) = \lambda \nu_\varphi(x)$ for $\lambda \in \mathbb{R}_+$. As a consequence, ν_φ is entirely characterized by its restriction to the set
$$\Sigma = \Big\{x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n : \ \sum_{j=1}^n x_j = 1\Big\}.$$
We choose $x^0 = (x_1^0, \ldots, x_n^0) \in \Sigma$ such that
$$\nu_\varphi(x^0) = \max\{\nu_\varphi(x) : \ x \in \Sigma\} \in \Big[\frac{e_1(\varphi)}{n}, e_1(\varphi)\Big].$$

By Theorem 5.8 in [Kis94a] we have the formula

$$c(\varphi) = \frac{1}{\nu_\varphi(x^0)}.$$

Set

$$\zeta(x) = \nu_\varphi(x^0) \min\left(\frac{x_1}{x_1^0}, \dots, \frac{x_n}{x_n^0}\right), \qquad \forall x \in \mathbb{R}_+^n.$$

Then ζ is the smallest nonnegative concave 1-homogeneous function on $\mathbb{R}_+^n$ that is increasing in each variable x_j and such that $\zeta(x^0) = \nu_\varphi(x^0)$. Therefore we have $\zeta \le \nu_\varphi$, hence

$$\begin{aligned} \varphi(z_1, \dots, z_n) &\le -\nu_\varphi(-\ln|z_1|, \dots, -\ln|z_n|) \\ &\le -\zeta(-\ln|z_1|, \dots, -\ln|z_n|) \\ &\le \nu_\varphi(x^0) \max\left(\frac{\ln|z_1|}{x_1^0}, \dots, \frac{\ln|z_n|}{x_n^0}\right) := \psi(z_1, \dots, z_n). \end{aligned}$$

By Lemma 3.1 and Remark 1.6 we get

$$f(e_1(\varphi), \dots, e_n(\varphi)) \le f(e_1(\psi), \dots, e_n(\psi)) = c(\psi) = \frac{1}{\nu_\varphi(x^0)} = c(\varphi).$$

(3.5) Reduction to the case of psh functions with analytic singularities.
In the second step, we reduce the proof to the case $\varphi = \log(|f_1|^2 + \ldots + |f_N|^2)$, where $f_1, \dots, f_N$ are germs of holomorphic functions at 0. Let $\mathcal{H}_{m\varphi}(\Omega)$ be the Hilbert space of holomorphic functions f on Ω such that

$$\int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty,$$

and let $\psi_m = \frac{1}{2m} \log \sum |g_{m,k}|^2$ where $\{g_{m,k}\}_{k\ge1}$ is an orthonormal basis of $\mathcal{H}_{m\varphi}(\Omega)$. From Theorem 4.2 in [DK00], there are constants $C_1, C_2 > 0$ independent of m such that

$$\varphi(z) - \frac{C_1}{m} \le \psi_m(z) \le \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$ and

$$\nu(\varphi) - \frac{n}{m} \le \nu(\psi_m) \le \nu(\varphi),$$

$$\frac{1}{c(\varphi)} - \frac{1}{m} \le \frac{1}{c(\psi_m)} \le \frac{1}{c(\varphi)}.$$

By Lemma 3.1, we have

$$f(e_1(\varphi), \dots, e_n(\varphi)) \le f(e_1(\psi_m), \dots, e_n(\psi_m)), \qquad \forall m \ge 1.$$

The above inequalities show that in order to prove the lower bound of $c(\varphi)$ in Theorem 1.5, we only need to prove it for $c(\psi_m)$ and let m tend to infinity. Also notice that since the Lelong numbers of a function $\varphi \in \mathcal{E}(\Omega)$ occur only on a discrete set, the same is true for the functions ψ_m.

(3.6) Reduction of the main theorem to the case of monomial ideals.
The final step consists of proving the theorem for $\varphi = \log(|f_1|^2 + \ldots + |f_N|^2)$, where $f_1, \dots, f_N$ are germs of holomorphic functions at 0 [this is because the ideals $(g_{m,k})_{k\in\mathbb{N}}$ in the Noetherian ring $\mathcal{O}_{\mathbb{C}^n,0}$ are always finitely generated]. Set $\mathcal{J} = (f_1, \dots, f_N)$, $c(\mathcal{J}) = c(\varphi)$, $e_j(\mathcal{J}) = e_j(\varphi)$, $\forall j = 0, \dots, n$. By the final observation of 3.5, we can assume that $\mathcal{J}$ has an isolated zero at 0. Now, by fixing a multiplicative order on the monomials $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ (see [Eis95] Chap. 15 and [dFEM04]), it is well known that one can construct a flat family $(\mathcal{J}_s)_{s\in\mathbb{C}}$ of

ideals of $\mathcal{O}_{\mathbb{C}^n,0}$ depending on a complex parameter $s \in \mathbb{C}$, such that $\mathcal{J}_0$ is a monomial ideal, $\mathcal{J}_1 = \mathcal{J}$ and $\dim(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_s^t) = \dim(\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}^t)$ for all s and $t \in \mathbb{N}$; in fact $\mathcal{J}_0$ is just the initial ideal associated to $\mathcal{J}$ with respect to the monomial order. Moreover, we can arrange by a generic rotation of coordinates $\mathbb{C}^p \subset \mathbb{C}^n$ that the family of ideals $\mathcal{J}_{s\,|\,\mathbb{C}^p}$ is also flat, and that the dimensions

$$\dim(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J}_{s\,|\,\mathbb{C}^p})^t) = \dim(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J}_{|\,\mathbb{C}^p})^t)$$

compute the intermediate multiplicities

$$e_p(\mathcal{J}_s) = \lim_{t\to+\infty} \frac{p!}{t^p} \dim(\mathcal{O}_{\mathbb{C}^p,0}/(\mathcal{J}_{s\,|\,\mathbb{C}^p})^t) = e_p(\mathcal{J})$$

(notice, in the analytic setting, that the Lelong number of the (p,p)-current $(dd^c\varphi)^p$ at 0 is the Lelong number of its slice on a generic $\mathbb{C}^p \subset \mathbb{C}^n$); in particular $e_p(\mathcal{J}_0) = e_p(\mathcal{J})$ for all p. The semicontinuity property of the log canonical threshold (see for example [DK00]) now implies that $c(\mathcal{J}_0) \le c(\mathcal{J}_s)$ for s small. As $c(\mathcal{J}_s) = c(\mathcal{J})$ for $s \ne 0$ ($\mathcal{J}_s$ being a pull-back of $\mathcal{J}$ by a biholomorphism, in other words $\mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}_s \simeq \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{J}$ as rings, see again [Eis95], chap. 15), the lower bound is valid for $c(\mathcal{J})$ if it is valid for $c(\mathcal{J}_0)$.

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Jean-Pierre Demailly
Université de Grenoble I, Département de Mathématiques
Institut Fourier, 38402 Saint-Martin d'Hères, France
e-mail: demailly@fourier.ujf-grenoble.fr

Phạm Hoàng Hiệp
Department of Mathematics, National University of Education
136-Xuan Thuy, Cau Giay, Hanoi, Vietnam
and Institut Fourier (on a Post-Doctoral grant from Univ. Grenoble I)
e-mail: phhiep_vn@yahoo.com