Bollettino dell'Unione Matematica Italiana

Morse cohomology estimates for jet differential operators

--Manuscript Draft--

Manuscript Number:	BUMI-D-18-00079
Full Title:	Morse cohomology estimates for jet differential operators
Article Type:	Original Research
Funding Information:	European Research Council (ALKAGE 670846) — Mr Jean-Pierre Demailly
Corresponding Author:	Jean-Pierre Demailly, Ph.D Université Grenoble Alpes Gières, FRANCE
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Université Grenoble Alpes
Corresponding Author's Secondary Institution:	
First Author:	Jean-Pierre Demailly, Ph.D
First Author Secondary Information:	
Order of Authors:	Jean-Pierre Demailly, Ph.D
	Mohammad Reza Rahmati, Ph.D.
Order of Authors Secondary Information:	
Author Comments:	Submission to the Volume in memory of Paolo di Bartolomeis

MORSE COHOMOLOGY ESTIMATES FOR JET DIFFERENTIAL OPERATORS

JEAN-PIERRE DEMAILLY*, MOHAMMAD REZA RAHMATI**

*Institut Fourier, Université Grenoble Alpes, Gières, France
Institute of Algebraic Geometry, Leibniz Universität Hannover, Germany

Dedicated to the memory of Professor Paolo de Bartolomeis

Abstract. We provide detailed holomorphic Morse estimates for the cohomology of sheaves of jet differentials and their dual sheaves. These estimates apply on arbitrary directed varieties, and a special attention has been given to the analysis of the singular situation. As a consequence, we obtain existence results for global jet differentials and global differential operators under positivity conditions for the canonical or anticanonical sheaf of the directed structure.

Key words: Directed variety, jet bundle, jet differential, jet metric, holomorphic Morse inequalities, canonical sheaf.

MSC Classification (2010): 32H30, 32L10, 14J17, 14J40, 53C55

Contents

0. Introduction

Recent developments in the theory of Kobayashi hyperbolicity have shown that a very convenient framework is the category of directed varieties. By definition, a directed variety is a pair (X, V) where X is a complex manifold or variety, and $V \subset T_X$ a complex analytic linear subspace that may itself possess singularities. Such a structure is defined alternatively as a saturated coherent subsheaf $\mathcal{V}$ of $\mathcal{O}(T_X)$. One can then introduce a natural concept of canonical sheaf sequence $K_V^{[\bullet]}$ that generalizes the usual canonical sheaf $\det(V^*)$ in the case where V is nonsingular; one says that (X, V) is of general type if $K_V^{[\bullet]}$ is big (see section 2 for details).

Following classical ideas of A. Bloch [Blo26a, 26b], Green-Griffiths [GG79] greatly advanced the study of Kobayashi hyperbolicity through a powerful use of jet bundles; as an application, they obtained a new geometric proof of the Bloch conjecture. In this vein, to any directed variety (X, V)

Both authors are supported by the ERC project ALKAGE, contract 670846 from September 2015.

one can associate a bundle J^kV of k-jets of holomorphic curves $f:(\mathbb{C},0)\to X$ tangent to V. This gives rise to tautological rank 1 sheaves $\mathcal{O}_{X_k^{\rm GG}}(m)$ on the weighted projectivized bundles

$$\pi_k : X_k^{\rm GG}\to X,\qquad X_k^{\rm GG}:=(J^kV\smallsetminus\{0\})/\mathbb{C}^*. \tag{0.1}$$

By taking direct images (cf. [GG79] and [Dem95], see also section 3), one gets sheaves of jet differentials

$$\mathcal{O}(E_{k,m}V^*)=(\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m). \tag{0.2}$$

The goal of this work is to obtain precise estimates for the dimensions of the cohomology groups

$$H^q\big(X,\mathcal{O}(E_{k,m}V^*)\big)\quad\text{and}\quad H^q\big(X,\mathcal{O}((E_{k,m}V^*)^*)\big), \tag{0.3}$$

along the lines of [Dem11]. The crucial technical argument is an application of holomorphic Morse inequalities to the rank one sheaf $\mathcal{O}_{X_k^{\rm GG}}(m)$, suitably twisted. The main contribution of the present exposition is to analyze the role of singularities in a more systematic way than in our previous papers [Dem11, 12, 15, 18]. The method to cope with singularities is to introduce ad hoc sheaves of holomorphic sections that are bounded near singular points. In a quite general setting, these sheaves interact very well with holomorphic Morse inequalities, and allow us to extend the estimates of the nonsingular case in a simple manner. Another new idea is to introduce the dual sheaf $\mathcal{O}((E_{k,m}V^*)^*)$, which can be seen as a sheaf of differential operators acting on functions of J^kV. In this direction, we prove an existence theorem for twisted differential operators. It somehow extends the existence theorem of slanted vector fields on jet spaces, as established by Siu [Siu04], Păun [Pau08] and Merker [Mer09]. The positivity conditions needed to get the existence of twisted differential operators are much weaker than those needed for the existence of twisted vector fields, but the draw-back of the present approach is that we get a priori no information on the degeneration loci.

Paolo de Bartolomeis was one of the great world experts of deformation theory. In the present context, it would be interesting to investigate the deformation theory of directed varieties, in the smooth and singular contexts as well. For instance, for a deformation $(\mathcal{X},\mathcal{V})\to S$ of directed structures over a base S, a natural question is the invariance of "directed plurigenera" $h^0(X_t,{}^bK_{V_t}^m)$ (where ${}^bK_{V_t}^m$ refers to the sheaf of bounded sections, see § 2), or at least the invariance of the volume of K_{V_t} along the fibers X_t of $\mathcal{X}\to S$.

The first author wishes to thank Mihai Păun for raising several important issues that play a very significant role in this work.

1. Category of directed varieties

We first recall the main definitions concerning the category of directed varieties. We start with the nonsingular case and then explain in detail the additional concepts and requirements that we introduce in the presence of singularities.

1.1. Definition. *A (complex, nonsingular)* directed variety *is a pair (X,V) consisting of a n-dimensional complex manifold X equipped with a holomorphic subbundle $V\subset T_X$. A morphism $\Phi:(X,V)\to(Y,W)$ in the category of directed varieties is a holomorphic map such that $d\Phi_x(V_x)\subset W_{\Phi(x)}$ for every point $x\in X$.*

The absolute situation is the case $V=T_X$ and the relative situation is the case when $V=T_{X/S}$ is the relative tangent space to a smooth holomorphic map $X\to S$. In general, we can associate to V a sheaf $\mathcal{V}=\mathcal{O}(V)\subset\mathcal{O}(T_X)$ of holomorphic sections. No assumption need be made on the Lie bracket tensor $[\bullet,\bullet]:\mathcal{V}\times\mathcal{V}\to\mathcal{O}(T_X)/\mathcal{V}$, i.e. we do not assume any sort of integrability for $\mathcal{V}$; if this happens, then $\mathcal{V}$ defines a holomorphic foliation on X.

1.2. Complement to the definition (singular case). Usually we are interested in questions that are birationally invariant; in such cases one can always blow-up X and reduce the situation to the case when X is nonsingular. Even then, the tangent bundle T_X need not have many holomorphic subbundles, but it always possesses many analytic subsheaves, thus it is important to allow singularities for V. When defining a directed structure V on X, we assume that there exists a dense Zariski open set $X' = X \smallsetminus Y \subset X$ such that $V_{|X'}$ is a subbundle of $T_{X|X'}$ and the closure $\overline{V_{|X'}}$ in the total space of T_X is an analytic subset. The rank $r \in \{0, 1, \ldots, n\}$ of V is by definition the dimension of V_x at points $x \in X'$; the dimension may be larger at points $x \in Y$. This happens e.g. on $X = \mathbb{C}^n$ for the rank 1 linear space V generated by the Euler vector field $\varepsilon(z) = \mathbb{C}\sum_{1\leqslant j\leqslant n} z_j \frac{\partial}{\partial z_j}$: then $V_z = \mathbb{C}\varepsilon(z)$ for $z \neq 0$, and $V_0 = \mathbb{C}^n$. The singular set $\mathrm{Sing}(V)$ is by definition the complement of the largest open subset on which V is a subbundle of T_X; it is equal to the indeterminacy set of the meromorphic map $X \dashrightarrow \mathrm{Gr}(T_X, r)$ into the Grassmannian bundle of r-dimensional subspaces of T_X, hence $\mathrm{codim}(\mathrm{Sing}(V)) \geqslant 2$. The category of directed varieties (X, V) is obtained by allowing X to be an arbitrary reduced complex space; if $X \hookrightarrow Z$ is a local embedding in a smooth ambient variety Z, we assume that there exists a Zariski open set $X'' \subset X' = X_{\mathrm{reg}}$ on which $V_{|X''}$ is a subbundle of $T_{X'}$, and that $\overline{V_{|X''}}$ is a closed analytic subset of T_Z (similarly T_X is defined to be the closure of $T_{X'}$ in T_Z).

2. Pluricanonical sheaves of a directed variety

Let (X, V) be a directed projective manifold where V is possibly singular, and let $r = \mathrm{rank}\, V$. If $\mu : \widehat{X} \to X$ is a proper modification (a composition of blow-ups with smooth centers, say), we get a directed manifold $(\widehat{X}, \widehat{V})$ by taking $\widehat{V}$ to be the closure of $\mu_*^{-1}(V')$, where $V' = V_{|X'}$ is the restriction of V over a Zariski open set $X' \subset X \smallsetminus \mathrm{Sing}(V)$ such that $\mu : \mu^{-1}(X') \to X'$ is a biholomorphism. We say that $(\widehat{X}, \widehat{V})$ is a *modification* of (X, V) and write $\widehat{V} = \mu^* V$.

We will be interested in taking modifications realized by iterated blow-ups of certain nonsingular subvarieties of the singular set $\mathrm{Sing}(V)$, so as to eventually "improve" the singularities of V; outside of $\mathrm{Sing}(V)$ the effect of blowing-up is irrelevant. The canonical sheaf K_V, resp. the pluricanonical sheaf sequence $K_V^{[m]}$, is defined by using the concept of bounded pluricanonical forms that was already introduced in [Dem11].

2.1. Definition. *For a directed pair (X, V) with X nonsingular, we define ${}^bK_V$, resp. ${}^bK_V^{[m]}$, for any integer $m \geqslant 0$, to be the rank 1 analytic sheaves such that*

$$ {}^bK_V(U) = \textit{sheaf of locally bounded sections of } \mathcal{O}_X\big(\Lambda^r V'^*\big)(U \cap X') $$
$$ {}^bK_V^{[m]}(U) = \textit{sheaf of locally bounded sections of } \mathcal{O}_X\big((\Lambda^r V'^*)^{\otimes m}\big)(U \cap X') $$

where $r = \mathit{rank}(V)$, $X' = X \smallsetminus \mathrm{Sing}(V)$, $V' = V_{|X'}$, and "locally bounded" means bounded with respect to a smooth hermitian metric h_X on T_X, on every set $U_c \cap X'$ such that U_c is relatively compact in U.

The above definition of ${}^bK_V^{[m]}$ may look like an analytic one, but it can easily be turned into an equivalent algebraic definition (cf. [Dem18]). Let us recall that, given a coherent ideal sheaf $\mathcal{J} = (g_1, \ldots, g_N)$ and a positive rational (or even real) number p, one defines the p-th integral closure, denoted formally $\overline{\mathcal{J}^p}$, to be the sheaf of holomorphic functions f that satisfy locally an inequality $|f| \leqslant C\big(\sum |g_j|\big)^p$; this is a coherent sheaf that can be identified with the sheaf of functions satisfying an integral equation $f^d + a_1 f^{d-1} + \ldots + a_d = 0$ where $a_s \in \mathcal{J}^{\lceil ps \rceil}$ for some $d \geqslant 1$ (see e.g. [agbook]).

2.2. Proposition. *Consider the natural morphism $\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*)$ where $r = \mathrm{rank}\, V$ and $\mathcal{O}(\Lambda^r V^*)$ is defined as the quotient of $\mathcal{O}(\Lambda^r T_X^*)$ by r-forms that have zero restrictions to $\mathcal{O}(\Lambda^r V^*)$ on $X \smallsetminus \mathrm{Sing}(V)$. The bidual $\mathcal{L}_V = \mathcal{O}(\Lambda^r V^*)^{**}$ is an invertible sheaf, and our natural morphism can be written*

$$ \mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) = \mathcal{L}_V \otimes \mathcal{J}_V \subset \mathcal{L}_V \qquad (2.2\,\mathrm{a}) $$

where $\mathcal{J}_V$ is a certain ideal sheaf of $\mathcal{O}_X$ whose zero set is contained in $\mathrm{Sing}(V)$ *and the arrow on the left is surjective by definition. Then*

$$ {}^bK_V^{[m]} = \mathcal{L}_V^{\otimes m} \otimes \overline{\mathcal{J}_V^m} \tag{2.2 b} $$

where $\overline{\mathcal{J}_V^m}$ is the integral closure of $\mathcal{J}_V^m$ in $\mathcal{O}_X$. In particular, ${}^bK_V^{[m]}$ is always a coherent sheaf.

A typical example of what may happen is the Euler vector field linear space $V \subset T_{\mathbb{C}^n}$: then the sheaf of holomorphic sections $\mathcal{O}(V)$ is trivial and generated by ε, i.e. $\mathcal{O}(V) = \mathcal{O}_{\mathbb{C}^n}\varepsilon$, hence its sheaf theoretic dual is $\mathcal{O}(V)^* = \mathcal{O}_{\mathbb{C}^n}\varepsilon^*$ where ε^* is the (unbounded) 1-form such that $\varepsilon^* \cdot \varepsilon = 1$. With the notation of Proposition 2.2, we have

$$\begin{aligned} \mathcal{O}(V^*) &= \mathcal{O}(\Lambda^1 V^*) = {}^bK_V = \mathfrak{m}_0\mathcal{O}_{\mathbb{C}^n}\varepsilon^* = {}^bK_V, \\ \mathcal{L}_V &= \mathcal{O}(V^*)^{**} = \mathcal{O}_{\mathbb{C}^n}\varepsilon^* = \mathcal{O}(V)^*, \\ \mathcal{J}_V &= \mathfrak{m}_0, \qquad \overline{\mathcal{J}_V^m} = \mathcal{J}_V^m = \mathfrak{m}_0^m. \end{aligned}$$

It is equally important to understand the effect of modifications on the sheaves ${}^bK_V^{[m]}$.

2.3. Proposition. *For any modification $\mu : (\widehat{X}, \widehat{V}) \to (X, V)$, there are always well defined injective natural morphisms of rank* 1 *sheaves*

$$ {}^bK_V^{[m]} \hookrightarrow \mu_*\big({}^bK_{\widehat{V}}^{[m]}\big) \hookrightarrow \mathcal{L}_V^{\otimes m} \tag{2.3 a} $$

and the direct image $\mu_*\big({}^bK_{\widehat{V}}^{[m]}\big)$ may only increase when μ is replaced by a "higher" modification $\widetilde{\mu} = \mu' \circ \mu : \widetilde{X} \to \widehat{X} \to X$ and $\widehat{V} = \mu^* V$ by $\widetilde{V} = \widetilde{\mu}^* V$, i.e. there are injections

$$ \mu_*\big({}^bK_{\widehat{V}}^{[m]}\big) \hookrightarrow \widetilde{\mu}_*\big({}^bK_{\widetilde{V}}^{[m]}\big) \hookrightarrow \mathcal{L}_V^{\otimes m}. \tag{2.3 b} $$

We refer to this property as the monotonicity principle.

Proof. (a) The existence of the first arrow is seen as follows: the differential $\mu_* = d\mu : \widehat{V} \to \mu^* V$ is smooth, hence bounded with respect to ambient hermitian metrics on X and $\widehat{X}$, and going to the duals reverses the arrows while preserving boundedness with respect to the metrics. We thus get an arrow

$$ \mu^*({}^bV^\star) \hookrightarrow {}^b\widehat{V}^\star. $$

By taking the top exterior power, followed by the m-th tensor product and the integral closure of the ideals involved, we get an injective arrow $\mu^*\big({}^bK_V^{[m]}\big) \hookrightarrow {}^bK_{\widehat{V}}^{[m]}$. Finally we apply the direct image fonctor μ_* and the canonical morphism $\mathcal{F} \to \mu_*\mu^*\mathcal{F}$ to get the first inclusion morphism. The second arrow comes from the fact that ${}^bK_V^{[m]}$ coincides with $\mathcal{L}_V^{\otimes m}$ (and with $\det(V^*)^{\otimes m}$) on the complement of the codimension 2 set $S = \mathrm{Sing}(V) \cup \mu(\mathrm{Exc}(\mu))$, and the fact that for every open set $U \subset X$, sections of $\mathcal{L}_V$ defined on $U \smallsetminus S$ automatically extend to U by the Riemann extension theorem, even without any boundedness assumption.

(b) Given $\mu' : \widetilde{X} \to \widehat{X}$, we argue as in (a) that there is a bounded morphism $d\mu' : \widetilde{V} \to \widehat{V}$. □

By the monotonicity principle and the strong Noetherian property of coherent sheaves, we infer that there exists a maximal direct image when $\mu : \widehat{X} \to X$ runs over all nonsingular modifications of X. The following definition is thus legitimate.

2.4. Definition. *We define the pluricanonical sheaf $K_V^{[m]}$ of (X, V) to be the inductive limit*

$$ K_V^{[m]} := \varinjlim_{\mu} \mu_*\big({}^bK_{\widehat{V}}^{[m]}\big) = \max_{\mu} \mu_*\big({}^bK_{\widehat{V}}^{[m]}\big) $$

taken over the family of all modifications $\mu : (\widehat{X}, \widehat{V}) \to (X, V)$, with the trivial (filtering) partial order. The canonical sheaf K_V itself is defined to be the same as $K_V^{[1]}$. By construction, we have *for every $m \geqslant 0$ inclusions*

$$ {}^bK_V^{[m]} \hookrightarrow K_V^{[m]} \hookrightarrow \mathcal{L}_V^{\otimes m}, $$

and $K_V^{[m]} = \mathcal{J}_V^{[m]} \cdot \mathcal{L}_V^{\otimes m}$ for a certain sequence of integrally closed ideals $\mathcal{J}_V^{[m]} \subset \mathcal{O}_X$.

It is clear from this construction that $K_V^{[m]}$ is birationally invariant in the sense that we have $K_V^{[m]} = \mu_*\big(K_{V'}^{[m]}\big)$ for every modification $\mu : (X', V') \to (X, V)$. One of the most central conjectures in the theory is the

2.5. Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed variety. Assume that (X, V) is of "general type" in the sense that there exists $m \geqslant 1$ such that the invertible sheaf $\mu_m^*(K_V^{[m]})$ is a big line bundle when one takes a log-resolution μ_m of the ideal $\mathcal{J}_V^{[m]}$. Then there should exist a proper algebraic subvariety $Y \subsetneq X$ containing the images $f(\mathbb{C})$ of all entire curves $f : \mathbb{C} \to X$ tangent to V.*

The main reason for incorporating the ideals $\mathcal{J}_V^{[m]}$ in the definition of $K_V^{[m]}$ is that the above conjecture would otherwise be trivially false: for instance, it is easy to see that a pencil of conics passing through 4 points in general position in $\mathbb{P}^2_{\mathbb{C}}$ are tangent to a rank 1 subspace $V \subset T_{\mathbb{P}^2}$ such that $\mathcal{L}_V = \mathcal{O}(V)^* \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ is ample; however, all leaves are rational curves. Here, we are in fact more interested here in the dual situation where a positivity assumption is made for the line bundle $\big(\mu_m^*(K_V^{[m]})\big)^*$.

3. Jet bundles and jet differentials

3.A. Nonsingular case

Following Green-Griffiths [GrGr79], we consider the bundle $J_kX \to X$ of k-jets of germs of parametrized curves in X, i.e., the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \leqslant j \leqslant k$, when computed in some local coordinate system of X near x. The projection map $J_kX \to X$ is simply $f \mapsto f(0)$. If $(z_1, \ldots, z_n)$ are local holomorphic coordinates on an open set $\Omega \subset X$, the elements f of any fiber J_kX_x, $x \in \Omega$, can be seen as $\mathbb{C}^n$-valued maps

$$ f = (f_1, \ldots, f_n) : (\mathbb{C}, 0) \to \Omega \subset \mathbb{C}^n, $$

and they are completetely determined by their Taylor expansion of order k at $t = 0$

$$ f(t) = x + t\, f'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1}). $$

In these coordinates, the fiber J_kX_x can thus be identified with the set of k-tuples of vectors $(\xi_1, \ldots, \xi_k) = (f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_kX is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X. However, J_kX is not a vector bundle for $k \geqslant 2$, because of the nonlinearity of coordinate changes: a coordinate change $z \mapsto w = \Psi(z)$ on X induces a polynomial transition automorphism on the fibers of J_kX, given by a formula

$$ (3.1) \qquad (\Psi \circ f)^{(j)} = \Psi'(f) \cdot f^{(j)} + \sum_{s=2}^{s=j} \sum_{j_1+j_2+\cdots+j_s=j} c_{j_1\ldots j_s} \Psi^{(s)}(f) \cdot (f^{(j_1)}, \ldots, f^{(j_s)}) $$

with suitable integer constants $c_{j_1\ldots j_s}$ (this is easily checked by induction on s). According to the above philosophy, we introduce the concept of jet bundle in the general situation of complex directed manifolds, assuming V nonsingular to start with.

3.2. Definition. *Let (X, V) be a complex directed manifold. We define $J_kV \to X$ to be the bundle of k-jets of curves $f : (\mathbb{C}, 0) \to X$ that are tangent to V, i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f \mapsto f(0)$ onto X.*

For any point $x_0 \in X$, there are local coordinates $(z_1, \ldots, z_n)$ on a neighborhood Ω of x_0 such that the fibers $(V_z)_{z\in\Omega}$ can be defined by linear equations

$$V_z = \Big\{v = \sum_{1\leqslant j\leqslant n} v_j \frac{\partial}{\partial z_j}\,;\, v_j = \sum_{1\leqslant k\leqslant r} a_{jk}(z)v_k \text{ for } j = r+1, \ldots, n\Big\}, \tag{3.3}$$

where (a_{jk}) is a holomorphic $(n-r)\times r$ matrix. Let $f : D(0,R) \to X$ be a curve tangent to V such that $f(D(0,R)) \in \Omega$, and let $(f_1, \ldots, f_n)$ be the components of f in the coordinates The curve f is uniquely determined by its initial value $x = f(0)$ and by the first r components $(f_1, \ldots, f_r)$. Indeed, as $f'(t) \in V_{f(t)}$, we can recover the other components by integrating the system of ordinary differential equations

$$f_j'(t) = \sum_{1\leqslant k\leqslant r} a_{jk}(f(t))f_k'(t), \qquad r+1 \leqslant j \leqslant n, \tag{3.4}$$

on a neighborhood of 0, with initial data $f(0) = x$. As a consequence, J_kV is actually a subbundle of J_kX. In fact, by using (3.4), we see that the fibers J_kV_x are parametrized by

$$\big((f_1'(0), \ldots, f_r'(0)); (f_1''(0), \ldots, f_r''(0)); \ldots; (f_1^{(k)}(0), \ldots, f_r^{(k)}(0))\big) \in (\mathbb{C}^r)^k$$

for all $x \in \Omega$, hence J_kV is a locally trivial $(\mathbb{C}^r)^k$-subbundle of J_kX. Alternatively, we can pick a local holomorphic connection ∇ on V such that for any germs $w = \sum_{1\leqslant j\leqslant n} w_j \frac{\partial}{\partial z_j} \in \mathcal{O}(T_{X,x})$ and $v = \sum_{1\leqslant \ell\leqslant r} v_\ell e_\ell \in \mathcal{O}(V)_x$ in a local trivializing frame $(e_1, \ldots, e_r)$ of $V_{\restriction\Omega}$ we have

$$\nabla_w v(x) = \sum_{1\leqslant j\leqslant n,\,1\leqslant \ell\leqslant r} w_j \frac{\partial v_\ell}{\partial z_j} e_\ell(x) + \sum_{1\leqslant j\leqslant n,\,1\leqslant \ell,\mu\leqslant r} \Gamma^\mu_{j\ell}(x) w_j v_\ell\, e_\mu(x). \tag{3.5}$$

We can of course take the (unique) frame $(e_\ell)_{1\leqslant \ell\leqslant r}$ in V such that $\partial/\partial z_\ell$ is the projection of e_ℓ on the first r coordinates, and the trivial connection ∇^0 given by the zero Christoffel symbolds $\Gamma = 0$ with respect to this frame, but any other holomorphic connection ∇ is acceptable. One then obtains a trivialization $J^kV_{\restriction\Omega} \simeq V^{\oplus k}_{\restriction\Omega}$ by considering

$$J_kV_x \ni f \mapsto (\xi_1, \xi_2, \ldots, \xi_k) = (\nabla f(0), \nabla^2 f(0), \ldots, \nabla^k f(0)) \in V_x^{\oplus k} \tag{3.6}$$

and computing inductively the successive derivatives $\nabla f(t) = f'(t)$ and $\nabla^s f(t)$ via

$$\nabla^s f = (f^*\nabla)_{d/dt}(\nabla^{s-1} f) = \sum_{1\leqslant \ell\leqslant r} \frac{d}{dt}\Big(\nabla^{s-1} f\Big)_\ell e_\ell(f) + \sum_{1\leqslant j\leqslant n,\,1\leqslant \ell,\mu\leqslant r} \Gamma^\mu_{j\ell}(f) f_j' \Big(\nabla^{s-1} f\Big)_\ell e_\mu(f).$$

This identification depends of course on the choice of ∇ and cannot be defined globally in general (unless we are in the rare situation where V has a global holomorphic connection). Now, we consider the natural $\mathbb{C}^*$-action on J^kV that maps a k-jet $t \mapsto f(t)$ to $\lambda \cdot f(t) := f(\lambda t)$, $\lambda \in \mathbb{C}^*$. Since $\nabla^s(\lambda \cdot f)(t) = \lambda^s \nabla^s f(t)$, the $\mathbb{C}^*$ action is described in coordinates by

$$\lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k), \quad \xi_s = \nabla^s f(0). \tag{3.7}$$

Following [GrGr79], we introduce the bundle $E^{\mathrm{GG}}_{k,m}V^* \to X$ of polynomials $P(x\,; \xi_1, \ldots, \xi_k)$ that are homogeneous on the fibers of J_kV of weighted degree m with respect to the $\mathbb{C}^*$ action, i.e.

$$P(x\,; \lambda\xi_1, \ldots, \lambda^k\xi_k) = \lambda^m P(x\,; \xi_1, \ldots \xi_k), \tag{3.8}$$

in other words they are polynomials of the form

$$P(x\,; \xi_1, \ldots \xi_k) = \sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(x)\, \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_k^{\alpha_k} \tag{3.9}$$

where $\xi_s = (\xi_{s,1}, \ldots, \xi_{s,r}) \in \mathbb{C}^r \simeq V_x$ and $\xi_s^{\alpha_s} = \xi_{s,1}^{\alpha_{s,1}} \ldots \xi_{s,r}^{\alpha_{s,r}}$, $|\alpha_s| = \sum_{1\leqslant j\leqslant r} \alpha_{s,j}$. Sections of the sheaf $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$ can also be viewed as algebraic differential operators acting on germs of curves

$f:(\mathbb{C},0)\to X$ tangent to V, by putting

$$P(f)(t)=\sum_{|\alpha_1|+2|\alpha_2|+\cdots+k|\alpha_k|=m} a_{\alpha_1\ldots\alpha_k}(f(t))\,(\nabla f(t))^{\alpha_1}(\nabla^2 f(t))^{\alpha_2}\cdots(\nabla^k f(t))^{\alpha_k} \tag{3.9'}$$

where the $a_{\alpha_1\ldots\alpha_k}(x)$ are holomorphic in x. With the graded algebra bundle $E^{\rm GG}_{k,\bullet}V^*=\bigoplus_m E^{\rm GG}_{k,m}V^*$ we associate an analytic fiber bundle

$$X^{\rm GG}_k:=\mathrm{Proj}(E^{\rm GG}_{k,\bullet}V^*)=(J_kV\smallsetminus\{0\})/\mathbb{C}^* \tag{3.10}$$

over X, which has weighted projective spaces $\mathbb{P}(1^{[r]},2^{[r]},\ldots,k^{[r]})$ as fibers; here $J_kV\smallsetminus\{0\}$ is the set of nonconstant jets of order k. As such, it possesses a tautological sheaf $\mathcal{O}_{X^{\rm GG}_k}(1)$ [the reader should observe however that $\mathcal{O}_{X^{\rm GG}_k}(m)$ is invertible only when m is a multiple of $\mathrm{lcm}(1,2,\ldots,k)$].

3.11. Proposition. *By construction, if $\pi_k:X^{\rm GG}_k\to X$ is the natural projection, we have the direct image formula*

$$(\pi_k)_*\mathcal{O}_{X^{\rm GG}_k}(m)=\mathcal{O}(E^{\rm GG}_{k,m}V^*)$$

for all k and m.

3.B. Singular case

When V has singularities and X is nonsingular, we simply consider the inclusion morphism $(X,V)\to(X,T_X)$ into the absolute directed structure. This yields a morphism $J_kV_{|X'}\to J_kT_{X|X'}$ in restriction to the Zariski open set $X'=X\smallsetminus\mathrm{Sing}(V)$, and we define $J_kV=\overline{J_kV_{|X'}}$ to be the closure of $J_kV_{|X'}$ in J_kT_X. It is then easy to see that J_kV is an analytic subset of J_kT_X, hence we get an inclusion morphism $J_kV\hookrightarrow J_kT_X$ over X, which also induces an inclusion

$$X^{\rm GG}_k\hookrightarrow X^{\rm abs,GG}_k \tag{3.12}$$

of $X^{\rm GG}_k=(J_kV\smallsetminus\{0\})/\mathbb{C}^*$ into the absolute Green-Griffiths bundle $X^{\rm abs,GG}_k=(J_kT_X\smallsetminus\{0\})/\mathbb{C}^*$. In analogy with our concept of canonical sheaf, it is natural to introduce the following definitions.

3.13. Theorem and definition. *Let $p_k:J^kV\smallsetminus\{0\}\to X^{\rm GG}_k$ be the natural projection. The sheaf ${}^b\mathcal{O}_{X^{\rm GG}_k}(m)$ (here, the b means "locally bounded" sections) is defined as follows: for any open set $U\subset X^{\rm GG}_k$, the space of sections ${}^b\mathcal{O}_{X^{\rm GG}_k}(m)(U)$ consists of holomorphic operators $F(x\,;\xi_1,\ldots,\xi_k)$ on the conical open set*

$$p_k^{-1}(U)\cap J^kV_{|X'}\subset J^kV_{|X'}\subset J^kT_X$$

that are homogeneous of degree m with respect to the $\mathbb{C}^*$-action, and are locally bounded with respect to a smooth hermitian metric h_X on T_X. Namely, if ∇ is a local holomorphic connection on $T_{X|\Omega}$ and $(\xi_1,\ldots,\xi_k)$ are the components of a k-jet computed with respect to ∇, we require that

$$|F(x\,;\xi_1,\ldots,\xi_k)|\leqslant C(U_c)\bigg(\sum_{1\leqslant s\leqslant k}\|\xi_s\|^{1/s}_{h_X}\bigg)^m \tag{3.13*}$$

on $p_k^{-1}(U_c)$, for every relatively compact open subset $U_c\Subset U\cap\pi_k^{-1}(\Omega)$. Then ${}^b\mathcal{O}_{X^{\rm GG}_k}(m)$ is a rank 1 coherent analytic sheaf, and is independent of the choice of h_X and ∇.

3.14. Definition. *With the above notation, the sheaf ${}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)$ is the analytic sheaf on X whose spaces of sections are polynomial differential operators $P(x\,;\xi_1,\ldots,\xi_k)$ in*

$${}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)(U)\subset\mathcal{O}(E^{\rm GG}_{k,m}V^*)(U\cap X'),$$

i.e. polynomial functions $P=F$ that satisfy inequality (3.13) on $\pi_k^{-1}(U_c\cap X')$, for all open sets $U_c\Subset U\subset X$. In other words, we have*

$$(\pi_k)_*{}^b\mathcal{O}_{X^{\rm GG}_k}(m)={}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*). \tag{3.14*}$$

Proof. That homogeneous functions on $J^kV_{|U}$ must be polynomials on the fibers is a trivial fact (using e.g. power series expansions in terms of $(\xi_1,\ldots,\xi_k)$). The coherence of ${}^b\mathcal{O}_{X_k^{\rm GG}}(m)$ is a simple consequence of the fact that we have a restriction morphism

$$\mathcal{O}_{X_k^{\rm abs,GG}}(m)_{|X_k^{\rm GG}} \longrightarrow {}^b\mathcal{O}_{X_k^{\rm GG}}(m) \tag{3.15}$$

and that, almost by definition, $\bigoplus_m {}^b\mathcal{O}_{X_k^{\rm GG}}(m)$ consists of taking the normalization of the image of the graded ring of sections; this shows again that our concepts are purely algebraic, in spite of the analytic definition that was given in 3.13. The coherence of ${}^b\mathcal{O}(E_{k,m}^{\rm GG}V^*)$ then follows by the direct image theorem. It is very important to observe that in condition (3.13*) one must refer to a metric h_X and a connection ∇ on the ambient bundle T_X, and not to a holomorphic connection on V, since any such connection would be unbounded near $\mathrm{Sing}(V)$, leading to the failure of (3.15). □

It is also clear form the definitions that our sheaf of jet differentials coincides with the already defined vector bundle on $X' = X \smallsetminus \mathrm{Sing}(V)$:

$$\Big({}^b\mathcal{O}(E_{k,m}^{\rm GG}V^*)\Big)_{|X\smallsetminus\mathrm{Sing}(V)} = \mathcal{O}(E_{k,m}^{\rm GG}V^*_{|X\smallsetminus\mathrm{Sing}(V)}). \tag{3.16}$$

When X itself has singularities, one can locally embed X in a smooth ambient variety Z and refer to smooth hermitian metrics on Z, taking bounded sections on a Zariski open set where both X and V are smooth. We will not consider this case much further and leave details to the reader.

4. Morse inequalities, in the smooth and singular contexts

4.A. Smooth case

The main purpose of holomorphic Morse inequalities is to provide estimates of cohomology groups with values in high tensor powers of a given line bundle L, once a hermitian metric h on L is given. We denote by $\Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ the $(1,1)$-curvature form of h, and for any $(1,1)$-form $u(z) = i\sum_{1\leqslant j,k\leqslant n} u_{jk}(z)\,dz_j\wedge d\overline{z}_k$, we define its q-index set $X(u,q)$ to be the open set

$$X(u,q) = \big\{z\in X\,;\ u(z)\ \text{has signature}\ \ (n-q,q)\big\} \tag{4.1}$$

(so that q is the number of negative eigenvalues of $u(z)$). The following statement summarizes the main results of [Dem85].

4.2. Holomorphic Morse inequalities for smooth metrics. *Let X be a compact complex n-dimensional manifold, $E\to X$ a holomorphic vector bundle of rank r, and L a holomorphic line bundle equipped with a smooth hermitian metric h. The dimensions $h^q(X,E\otimes L^m)$ of cohomology groups of the tensor powers $E\otimes L^m$ satisfy the following asymptotic estimates as $m\to+\infty$:*

(4.2 WM) *Weak Morse inequalities*:

$$h^q(X,E\otimes L^m) \leqslant r\frac{m^n}{n!}\int_{X(\Theta_{L,h},q)}(-1)^q\Theta_{L,h}^n + o(m^n)\ .$$

(4.2 SM) *Strong Morse inequalities*:

$$\sum_{0\leqslant j\leqslant q}(-1)^{q-j}h^j(X,E\otimes L^m) \leqslant r\frac{m^n}{n!}\sum_{0\leqslant j\leqslant q}\int_{X(\Theta_{L,h},j)}(-1)^{q-j}\Theta_{L,h}^n + o(m^n)\ .$$

(4.2 RR) *Asymptotic Riemann-Roch formula*:

$$\chi(X,E\otimes L^m) := \sum_{0\leqslant j\leqslant n}(-1)^j h^j(X,E\otimes L^m) = r\frac{m^n}{n!}\int_X\Theta_{L,h}^n + o(m^n)\ .$$

In fact, the strong Morse inequality implies the weak form (by adding the inequalities for q and $q-1$), and the asymptotic Riemann-Roch formula (by taking $q=n$ and $q=n+1$). By adding the strong Morse inequalities for $q+1$ and $q-2$, one also gets the lower bound

$$\begin{aligned} h^q(X,E\otimes L^m) &\geqslant h^q(X,E\otimes L^m) - h^{q-1}(X,E\otimes L^m) - h^{q+1}(X,E\otimes L^m)\\ &\geqslant r\frac{m^n}{n!}\sum_{q-1\leqslant j\leqslant q+1}\int_{X(\Theta_{L,h},j)}(-1)^{q-j}\Theta_{L,h}^n - o(m^n),\end{aligned} \tag{4.3$_q$}$$

and especially, for the important case $q=0$, the lower bound

$$h^0(X,E\otimes L^m) \geqslant \int_{X(\Theta_{L,h},0)}\Theta_{L,h}^n - \int_{X(\Theta_{L,h},1)}\Theta_{L,h}^n - o(m^n). \tag{4.3$_0$}$$

4.B. Case of metrics with $\mathbb{Q}$-analytic singularities

The above estimates are volume estimates, and therefore are not sensitive to modifications. In particular, we have the following easy lemma.

4.4. Lemma. *Let $\mathcal{E}$ be a coherent analytic sheaf and (L,h) a hermitian line bundle on a reduced irreducible compact complex space Z. Then*

(i) *If $Y=\mathrm{Supp}(\mathcal{E})$ has at most p-dimensional irreducible components, we have*

$$h^q(Z,\mathcal{E}\otimes\mathcal{O}(L^m)) = O(m^p).$$

(ii) *If $n=\dim Z$ and r is the generic rank of $\mathcal{E}$, the Morse inequalities are still valid with X replaced by Z and the locally free sheaf $\mathcal{O}(E)$ replaced by $\mathcal{E}$.*

Proof. We prove 4.4 $(\mathrm{i})_{p\leqslant N}$ and 4.4 $(\mathrm{ii})_{n\leqslant N}$ by induction on N, everything being obvious for $N=0$.

If Y is not irreducible and $Y=\bigcup Y_j$ is the decomposition in irreducible components, let $\mathcal{E}_j$ be the sheaf of sections of $\mathcal{E}$ that vanish on all components Y_k, $k\neq j$. Then we have an exact sequence

$$0\to\bigoplus\mathcal{E}_j\to\mathcal{E}\to\mathcal{F}\to 0$$

where $\mathcal{F}$ is supported on $Y'=\bigcup_{j\neq k}Y_j\cap Y_k$, $\dim Y\leqslant p-1$. If we use the corresponding exact sequence and argue by induction on p, we see that it is sufficient to check 4.4 $(\mathrm{i})_p$ when Y is irreducible. If $\mathcal{I}_Y$ is the reduced ideal sheaf of Y, we have $\mathcal{I}_Y^k\mathcal{E}=0$ for some $k\in\mathbb{N}^*$, and we thus get a decreasing filtration $\mathcal{E}_\ell=\mathcal{I}_Y^\ell\mathcal{E}$ of $\mathcal{E}$ such that $\mathcal{E}_\ell/\mathcal{E}_{\ell+1}$ can be viewed as a coherent sheaf on the reduced space Y_{red}, whose structure sheaf is $\mathcal{O}_X/\mathcal{I}_Y$. Then, by taking $Z=Y$ and exploiting the exact sequences $0\to\mathcal{E}_{\ell+1}\to\mathcal{E}_\ell\to\mathcal{E}_\ell/\mathcal{E}_{\ell+1}\to 0$, we see that 4.4 $(\mathrm{ii})_{n\leqslant N}$ implies 4.4 $(\mathrm{i})_{p\leqslant N}$. The last part of the proof consists in showing that Theorem 4.2$_{n\leqslant N}$ and 4.4 $(\mathrm{i})_{p\leqslant N-1}$ imply 4.4 $(\mathrm{ii})_{n\leqslant N}$, and for this, it is enough to consider the case where $\dim Z=n=N$.

In that case, the Hironaka desingularization theorem implies that there exists a modification $\mu:X\to Z$ such that X is nonsingular and $\mathcal{F}=\mu^*\mathcal{E}$ is locally free modulo torsion, so that we have an exact sequence

$$0\to\mathcal{F}_{\mathrm{tors}}\to\mathcal{F}\to\mathcal{F}/\mathcal{F}_{\mathrm{tors}}\to 0$$

where $\mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is locally free (i.e. a vector bundle on X). Therefore, holomorphic inequalities 4.2 can be applied to the groups $H^q(X,\mathcal{F}/\mathcal{F}_{\mathrm{tors}}\otimes\mathcal{O}(\mu^*L^m))$, and since $Y=\mathrm{Supp}(\mathcal{F}_{\mathrm{tors}})$ has dimension $p\leqslant N-1$, part 4.4 $(\mathrm{i})_{p\leqslant N-1}$ of the Lemma shows that the groups $H^q(X,\mathcal{F}\otimes\mathcal{O}(\mu^*L^m))$ also satisfy holomorphic Morse inequalities. Finally, we use the Leray spectral sequence. It yields a convergent spectral sequence

$$H^p(Z,R^q\mu_*(\mathcal{F})\otimes\mathcal{O}(L^m))\Rightarrow H^{p+q}(X,\mathcal{F}\otimes\mathcal{O}(\mu^*L^m))$$

and we know that $R^q\mu_*(\mathcal{F})$ is supported for $q\geqslant 1$ on an analytic subset $Y'\subsetneq Z$, and that the morphism $\mathcal{E}\to\mu_*\mu^*\mathcal{E}=R^0\mu_*(\mathcal{F})$ is an isomorphism outside of codimension 1. From this we conclude that holomorphic Morse inequalities for the $H^q(X,\mathcal{F}\otimes\mathcal{O}(\mu^*L^m))$ (which we already know), imply holomorphic Morse inequalities for $H^q(Z,\mathcal{E}\otimes\mathcal{O}(L^m))$, thanks to 4.4 $(\mathrm{i})_{n\leqslant N-1}$ applied on Z. □

In his PhD thesis, Bonavero [Bon93] extended Morse inequalities to the case of singular hermitian metrics. We state here a variant that allows more general singularities (but in fact, everything can be reduced to the smooth case 4.2 by means of a desingularization, thus all versions are in fact equivalent).

4.5. Definition. *Let Z be an irreducible and reduced complex space and $\mathcal{L}$ a torsion free rank 1 sheaf on Z. A hermitian metric h on $\mathcal{L}$ with $\mathbb{Q}$-analytic singularities is a hermitian metric h defined on a dense open set $Z' \subset Z_{\rm reg}$ where $\mathcal{L}$ is invertible, with the following property: there exists a smooth modification $\mu : X \to Z$ such that $\mu^*\mathcal{L}$ is invertible and the pull-back metric μ^*h has normal crossing $\mathbb{Q}$-divisorial singularities, i.e. if g is a local generator of $\mu^*\mathcal{L}$ on a small open set $U \subset X$, we have*

$$\log |g|^2_{\mu^*h} = \psi - \sum \lambda_j \log |z_j|^2$$

for some holomorphic coordinates (z_j) on U, $\lambda_j \in \mathbb{Q} \smallsetminus \{0\}$ and $\psi \in C^\infty(U)$. We will say that h has $\mathbb{Q}_+$-analytic singularities if one can take all $\lambda_j \in \mathbb{Q}_+ \smallsetminus \{0\}$ (or an empty sum).

(Of course the normal crossing hypothesis is not necessary, since it can always be achieved a posteriori by an application of the Hironaka desingularization theorem). We define the singular set $S = \mathrm{Sing}(\mu^*h)$ to be the common zero set $\{z_j = 0\}$ for the coordinates z_j involved above, and $\mathrm{Sing}(h)$ to be the union of $\mu(S)$ and of the closed analytic subset of Z where $\mathcal{L}$ is not invertible.

4.6. Definition. *Let Z be an irreducible and reduced complex space, and $\mathcal{L}$ a torsion free rank 1 sheaf on Z equipped with a hermitian metric h with $\mathbb{Q}$-analytic singularities. We define the sheaf of h-bounded sections ${}^b\mathcal{O}(\mathcal{L}^m)_h$ to be the sheaf of germs of meromorphic sections σ of $\mathcal{L}^m$ such that $|\sigma|_h$ is bounded (just consider the restriction to $U \cap Z'$ where U is a neighborhood of a given point $z_0 \in Z$ and $Z' = Z \smallsetminus \mathrm{Sing}(h)$).*

It follows from the definitions that ${}^b\mathcal{O}(\mathcal{L}^m)_h$ is a coherent sheaf of rank 1; in fact, with the notation of Definition 4.5, it is the direct image by μ of $(\mu^*\mathcal{L})^m \otimes \mathcal{O}_X(-\lceil mD \rceil)$ where $D = \sum \lambda_j D_j$ is the $\mathbb{Q}$-divisor of X given by $D_j = \{z_j = 0\}$, and $\lceil mD \rceil = \sum \lceil m\lambda_j \rceil D_j$ is the round up. Especially, if Z is normal, $\mathcal{L}$ invertible and h has $\mathbb{Q}_+$-analytic singularities (i.e. $D \geqslant 0$), then we have ${}^b\mathcal{O}(\mathcal{L}^m)_h \subset \mathcal{O}(\mathcal{L}^m)$. Otherwise, we may get some meromorphic sections in ${}^b\mathcal{O}(\mathcal{L}^m)_h$ that are actually not holomorphic.

4.7. Holomorphic Morse inequalities for singular metrics. *Let Z be an irreducible and reduced complex space, $\mathcal{L}$ a torsion free rank 1 sheaf on Z equipped with a hermitian metric h with $\mathbb{Q}$-analytic singularities, and $\mathcal{E}$ be a coherent sheaf of generic rank r. Then, for $Z' = Z \smallsetminus \mathrm{Sing}(h)$ and all $q = 0, 1, \ldots, n = \dim Z$, we have estimates*

$$\sum_{j=q-1,q,q+1} r\frac{m^n}{n!}\int_{Z'(\Theta_{L,h},j)} (-1)^{q-j}\Theta^n_{\mathcal{L},h} - o(m^n)$$

$$\leqslant h^q(Z, \mathcal{E} \otimes {}^b\mathcal{O}(\mathcal{L}^m)_h) \leqslant r\frac{m^n}{n!}\int_{Z'(\Theta_{L,h},q)} (-1)^q\Theta^n_{\mathcal{L},h} + o(m^n).$$

Proof. We use a smooth modification $\mu : X \to Z$ such that $\mu^*\mathcal{L}$ satisfies the requirements of Definition 4.5 and $\mu^*\mathcal{E}$ is locally free modulo torsion. The proof of Lemma 4.4 then shows that the torsion of $\mathcal{F} = \mu^*\mathcal{E}$ can be neglected. Again, the conclusion follows from the Leray spectral sequence and the smooth case of Morse inequalities applied to the sequence of invertible line bundles

$$p \mapsto \mathcal{G} \otimes \widetilde{\mathcal{L}}^p \quad \text{on } X,$$

where

$$\mathcal{G} = \mathcal{F}/\mathcal{F}_{\rm tors} \otimes \mu^*\mathcal{L}^r \otimes \mathcal{O}_X(-\lceil rD \rceil), \quad \widetilde{\mathcal{L}} = \mu_\star\mathcal{L}^a \otimes \mathcal{O}_X(-\lceil aD \rceil),$$

$a \in \mathbb{N}^*$ is a denominator of the $\mathbb{Q}$-divisor D and $m = ap + r$, $0 \leqslant r < a$. It follows that there is a morphism

$$\mathcal{E} \otimes {}^b\mathcal{O}(\mathcal{L}^m)_h) \to \mu_*(\mathcal{G} \otimes \widetilde{\mathcal{L}}^p)$$

whose kernel and cokernel are supported on subvarieties $Y, Y' \subsetneq Z$, and by definition μ^*h^a induces a smooth metric on $\widetilde{\mathcal{L}}$. As a consequence $\Theta_{\widetilde{\mathcal{L}},\mu^*h^a} = a\,\mu^*\Theta_{\mathcal{L},h}$ on $\mu^{-1}(Z \smallsetminus \mathrm{Sing}(h))$ and

$$\int_{X(\Theta_{\widetilde{\mathcal{L}},\mu^*h^a},q)} (-1)^q \Theta^n_{\widetilde{\mathcal{L}},\mu^*h^a} = a^n \int_{Z'(\Theta_{L,h},q)} (-1)^q \Theta^n_{\mathcal{L},h}. \qquad \square$$

4.8. Remark. In Bonavero's thesis [Bon93], Z is a manifold, $\mathcal{E} = \mathcal{O}_Z(E)$ and $\mathcal{L} = \mathcal{O}_Z(L)$ are assumed to be locally free, $h = e^{-\varphi}$ is a singular hermitian metric with $\mathbb{Q}_+$-analytic singularities and the cohomology groups involved are the groups

$$H^q(Z, \mathcal{O}_Z(E \otimes L^m) \otimes \mathcal{I}(h^m))$$

where $\mathcal{I}(h^m)$ are the L^2 multiplier ideal sheaves

$$\mathcal{I}(h^m) = \mathcal{I}(k\varphi) = \big\{ f \in \mathcal{O}_{Z,x},\ \ \exists V \ni x,\ \int_V |f(z)|^2 e^{-m\varphi(z)} d\lambda(z) < +\infty \big\}.$$

Here, we have in fact replaced the L^2 condition by a L^∞ condition. When pulling-back to X via a modification $\mu : X \to Z$ resolving the singularities of h into divisorial singularities, the difference is just a twist by the relative canonical bundle $K_{X/Z}$ and the use of the round down $\lfloor mD \rfloor$ instead of $\lceil mD \rceil$. These differences are "bounded" and thus do not make any change in the estimates produced by the Morse inequalities. For the same reason, we could even allow the singularity D of μ^*h to be a $\mathbb{R}$-divisor, although periodicity would be lost in the round up process; one then needs the fact that Morse inequalities are "uniform" when E remains in a bounded family of vector bundles, which follows easily from the analytic proof.

5. Cohomology estimates for sheaves of jet differentials and their duals

On a directed variety (X, V), is may be necessary to allow V to have singular hermitian metrics, even when (X, V) is smooth: indeed, this leads to more comprehensive curvature conditions, since one can e.g. relax the ampleness conditions and consider instead big line bundles. However, it is also useful to consider situations where V is singular. According to the philosophy of § 4, we have to explain what are the sheaves involved in the presence of such singularities.

5.1. Definition. *Let (X, V) be a directed variety, X being nonsingular.*

(a) *A singular hermitian metric on a linear subspace $V \subset T_X$ is a metric h on the fibers of V such that the function $\log h : \xi \mapsto \log |\xi|_h^2$ is locally integrable on the total space of V.*

(b) *A singular metric h on V will be said to have $\mathbb{Q}$-analytic singularities if h can be written as $h = e^{\varphi}(h_X)_{|V}$ where h_X is a smooth positive definite hermitian metric on T_X and φ is a weight with $\mathbb{Q}$-analytic singularities, i.e. one can find a modification $\mu : \widetilde{X} \to X$ such that locally $\varphi \circ \mu = \sum \lambda_j \log |g_j(z)| + \psi(z)$, where $\lambda_j \in \mathbb{Q} \smallsetminus \{0\}$, $\psi \in C^\infty$ and the g_j are holomorphic functions; one can then further assume that $D = \sum \lambda_j D_j$, $D_j = \{g_j = 0\}$, is a simple normal crossing divisor on $\widetilde{X}$. We define $\mathrm{Sing}(h)$ to be the union $\mu(\mathrm{Supp}(D)) \cup \mathrm{Sing}(V)$.*

A singular metric h on V can also be viewed as a singular hermitian metric on the tautological line bundle $\mathcal{O}_{P(V)}(-1)$ on the projectivized bundle $P(V) = V \smallsetminus \{0\}/\mathbb{C}^*$, and therefore its dual metric h^{-1} defines a curvature current $\Theta_{\mathcal{O}_{P(V)}(1),h^{-1}}$ of type $(1,1)$ on $P(V) \subset P(T_X)$, such that

$$p^*\Theta_{\mathcal{O}_{P(V)}(1),h^{-1}} = \frac{i}{2\pi}\partial\overline{\partial}\log h, \qquad \text{where } p : V \smallsetminus \{0\} \to P(V). \tag{5.2}$$

5.3. Remark. In [Dem11], [Dem12], [Dem15], we introduced the concept of an admissible metric h on V, which is closely related to the concept of a metric with $\mathbb{Q}_+$-analytic singularities (in the sense that the divisor of singularities D is nonnegative); then $\log h$ is quasi-plurisubharmonic (i.e. psh modulo addition of a smooth function) on the total space of V, hence

$$\Theta_{\mathcal{O}_{P(V)}(1),h^{-1}} \geqslant -C\omega \tag{5.3 a}$$

for some smooth positive $(1,1)$-form ω on $P(V)$ and some constant $C > 0$; if h has $\mathbb{Q}$-analytic singularities, we can choose a smooth modification $\widetilde{X} \to X$ such that $\mathcal{O}(\mu^* V)$ is locally free, the injection $\mathcal{O}(\mu^* \det V) \hookrightarrow \mathcal{O}(\mu^* \Lambda^r T_X)$ vanishes along an invertible ideal $\mathcal{O}(-\Delta) \subset \mathcal{O}_{\widetilde{X}}$ and $\mu^* \log(h/h_X)$ has divisorial singularities given by a $\mathbb{Q}$-divisor D. Then

$$\Theta_{\mathcal{O}_{P(\mu^* V)}(1),\mu^* h^{-1}} = [D + \Delta] + \beta \tag{5.3 b}$$

for some smooth $(1,1)$-form β on $P(\mu^* V)$. Hence if $D + \Delta \geqslant 0$ (and especially if $D \geqslant 0$), we still have

$$\Theta_{\mathcal{O}_{P(\mu^* V)}(1),\mu^* h^{-1}} \geqslant -C\widetilde{\omega} \tag{5.3 c}$$

for any smooth positive $(1,1)$-form $\widetilde{\omega}$ on $P(\mu^* V)$.

5.4. Bounded sections. *If h is a singular metric with $\mathbb{Q}$-analytic singularities on (X, V) and h^* the corresponding dual metric on V^*, we consider the Zariski open set $X' = X \smallsetminus (\mathrm{Sing}(V) \cup \mathrm{Sing}(h))$ and define:*

(a) ${}^b\mathcal{O}(V^*)_{h^*}$ *to be the sheaf of germs of holomorphic sections of $V^*_{|X'}$ which are h^*-bounded near every point of X ;*

(b) *the h-bounded pluricanonical sequence to be*

$${}^bK_{V,h}^{[m]} = \textit{sheaf of germs of holomorphic sections of } (\det V^*_{|X'})^{\otimes m} = (\Lambda^r V^*_{|X'})^{\otimes m} \textit{ which are } \det h^* \textit{-bounded,}$$

so that ${}^bK_V^{[m]} := {}^bK_{V,h_X}^{[m]}$ according to Def. 2.1.

(c) *for $U \subset X$ open, the space of sections ${}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*_h)(U)$ of ${}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*_h)$ consists of functions $P(x\,;\xi_1,\ldots,\xi_k)$ that are holomorphic in $x \in U \cap X'$ and polynomial in the ξ_j's, satisfying upper bounds of the form*

$$\big|P(z\,;\,\xi_1,\ldots,\xi_k)\big| \leqslant C(U_c)\bigg(\sum_{1\leqslant s\leqslant k} \|\xi_s\|_h^{1/s}\bigg)^m, \quad x \in U_c \Subset U.$$

We then have a direct image formula

$${}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*_h) = (\pi_k)_* {}^b\mathcal{O}_{X_k^{\mathrm{GG}}}(m)_h. \tag{d}$$

where ${}^b\mathcal{O}_{X_k^{\mathrm{GG}}}(m)_h$ denotes the sheaf of h-bounded sections of ${}^b\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$.

If the divisor D of singularities of h is $\geqslant 0$, we have $h = e^{\varphi} h_X \leqslant C h_X$ locally, hence

$${}^bK_{V,h}^{[m]} \subset {}^bK_V^{[m]} \quad \text{and} \quad {}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*_h) \subset {}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m} V^*). \tag{5.5}$$

On the other hand, if $D \leqslant 0$, reversed inequalities and inclusions hold, e.g. ${}^bK_{V,h}^{[m]} \supset {}^bK_V^{[m]}$, therefore

$$({}^bK_{V,h}^{[m]})^* \subset ({}^bK_V^{[m]})^* \quad \text{and} \quad \mathcal{O}({}^bE^{\mathrm{GG}}_{k,m} V^*_h)^* \subset \mathcal{O}({}^bE^{\mathrm{GG}}_{k,m} V^*)^* \tag{5.5*}$$

for the dual $\mathcal{O}_X$-modules.

5.6. Morse integral estimates. *Let (X, V) be a directed variety, where X is a nonsingular compact complex manifold. Fix a singular hermitian metric h_V on V and denote by*

$$\Theta_{V,h_V} = \frac{i}{2\pi} \sum_{1\leqslant i,j\leqslant n, 1\leqslant\alpha,\beta\leqslant r} c_{ij\alpha\beta}(z)\, dz_i \wedge d\overline{z}_j \otimes e_\alpha^* \otimes e_\beta$$

the curvature tensor of V with respect to an h_V-orthonormal frame (e_α), in the sense of currents. Let (F, h_F) be a singular $\mathbb{Q}$ hermitian line bundle on X, and let

$$\eta(z) = \Theta_{\det(V^*),\det h_V^*} + \Theta_{F,h_F} = -\operatorname{Tr}_{\mathrm{End}(V)} \Theta_{V,h_V} + \Theta_{F,h_F}. \tag{5.6 a}$$

Assume that $h = e^{\varphi}(h_X)_{|V}$ and h_F are metrics with $\mathbb{Q}$-analytic singularities and let Σ be their joint singular set $\Sigma = \mathrm{Sing}(h_V) \cup \mathrm{Sing}(h_F)$. Finally, equip the tautological bundle $\mathcal{O}_{X_k^{\mathrm{GG}}}(-1)$ with the induced metric $h_{V,k,\varepsilon}$ such that

$$\|\xi\|^2_{h_{V,k,\varepsilon}} = e^{\varphi}\Bigg(\sum_{1\leqslant s\leqslant k} \varepsilon_s|\xi_s|^{2p/s}_{h_X}\Bigg)^{1/p}, \quad p = \mathrm{lcm}(1,2,\ldots,k), \tag{5.6 b}$$

where $\xi = f_{[k]}(0)$ is the k-jet of an integral germ of curve f in (X,V), $\xi_s = (\nabla_0)^s f(0)$ with respect to a global smooth connection ∇_0 on T_X and $1 = \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_k > 0$. We consider on X_k^{GG} the rank 1 sheaf

$$\mathcal{L}_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\bigg(\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\bigg) \tag{5.6 c}$$

(it is just a rank 1 "$\mathbb{Q}$-sheaf", and we actually have to take a power $\mathcal{L}_k^m$ with m sufficiently divisible to get a genuine rank 1 sheaf), equipped with the metric $h_{V,F,k,\varepsilon}$ induced by $(h_{V,k,\varepsilon})^{-1}$ and h_F. Then, for $m \gg \varepsilon_k^{-1} \gg k \gg 1$, the q-index Morse integral of $(\mathcal{L}_k, h_{V,F,k,\varepsilon})$ on X_k^{GG} is given by

$$\int_{X_k^{\mathrm{GG}}(\Theta_{\mathcal{L}_k,h_{V,F,k,\varepsilon}},q)\smallsetminus\pi_k^{-1}(\Sigma)} \Theta^{n+kr-1}_{\mathcal{L}_k,h_{V,F,k,\varepsilon}} = \frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,q)\smallsetminus\Sigma} \mathbb{1}_{\eta,q}\eta^n + O((\log k)^{-1})\bigg) \tag{5.6 d}$$

for all $q = 0,1,\ldots,n$, and the error term $O((\log k)^{-1})$ can be bounded explicitly in terms of $\Theta_{V,h}$ and Θ_{F,h_F}. Moreover, the left hand side is identically zero for $q > n$.

Proof. We refer to [Dem11] which contains all details in the smooth case, i.e. when $V \subset T_X$ is a subbundle, and when h_V, h_F are smooth. The main argument is that the $(1,1)$ curvature form of $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ can be expressed explicitly, modulo small error terms. It is convenient to use polar coordinates

$$\xi_s = \varepsilon_s^{-1/2p} x_s^{1/p} u_s, \tag{5.7}$$

where $u_s \in SV$ is in the unit sphere bundle of V ($|u_s|_h = 1$) and $x_s \geqslant 0$. Then at any point $(z,\xi) \in X_k^{\mathrm{GG}}$, a straightforward calculation shows that the curvature of $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is given by

$$\Theta_{\mathcal{O}_{X_k^{\mathrm{GG}}}(1),h_{V,k,\varepsilon}}(z,\xi) = \omega_{p,\mathrm{FS}}(\xi) - \frac{i}{2\pi}\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u_{s\beta}}\, dz_i \wedge d\overline{z}_j + O(\varepsilon) \tag{5.8}$$

where

$$\omega_{p,\mathrm{FS}}(\xi) = \frac{1}{p}\frac{i}{2\pi}\partial\overline{\partial}\log\sum_{1\leqslant s\leqslant k}|\xi_s|^{2p/s}$$

is the weighted Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$. As $k \to +\infty$, the double summation of (5.8) can be seen as a Monte-Carlo evaluation of the curvature tensor $u \mapsto \langle\Theta_{V,h_V}u,u\rangle$ on the sphere bundle SV. It thus exhibits a probabilistic convergence, and is on average equivalent to a quantity essentially independent of ξ, namely

$$-\frac{1}{r}\sum_{1\leqslant s\leqslant k}\frac{x_s}{s}\,\mathrm{Tr}_{\mathrm{End}(V)}\,\Theta_{V,h_V}$$

proportional to $\Theta_{\det(V^*),\det(h^*)}$ (the integral of a quadratic form on a sphere is just its trace!). Then (5.6 d) follows by analyzing the error terms and computing fiber integrals (the latter depend only on the weighted Fubini-Study metric and can be easily evaluated). One just needs to add Θ_{F,h_F} to estimate the Morse integral of $\mathcal{L}_k$. The corresponding integrals vanish for $q > n$ because $\mathcal{L}_k$ is semi-positive (and generically strictly positive) along the fibers of $\pi_k : X_k^{\mathrm{GG}} \to X$, and the only negative eigenvalues are those coming from $\eta(z)$ in the dz_i's. One important point is that the rescaling factor ε_s in (5.8) makes the metric (5.6 b) essentially independent of the connection ∇_0, up to errors $O(\varepsilon)$ (cf. Lemma 2.12 in [Dem11]). Therefore, one can use a local holomorphic connection ∇

of V (say, a trivial flat one) to perform calculations – this is helpful to ensure that the ξ_s are actually holomorphic coordinates. In the singular case, we use a smooth modification $\mu : \widetilde{X} \to X$ that takes V to a locally free sheaf μ^*V and converts h_V, h_F into metrics with $\mathbb{Q}$-divisorial singularities on $\widetilde{X}$, in such a way that we are reduced to the smooth case, after removing the corresponding divisors and computing the Morse integrals in the complement of Σ. However, it must be observed that we still need in (5.6 b) to use a *smooth* (or local holomorphic) connection ∇_0 on T_X, and not a holomorphic connection ∇_V on V, because such a connection would "explode" near the singularities. Far away from the singularities, there is uniform convergence to what would be obtained from a holomorphic connection ∇_V on V. Near $\mathrm{Sing}(V)$, the argument is that the curvature of Θ_{V,h_X} is controlled by the second fundamental form of V (the relevant term being negative), but the total volume of any exterior power is convergent and therefore goes to 0 if we pick a sufficiently small neighborhood of the singular set $\mathrm{Sing}(V)$. This would not work if we had chosen a holomorphic connection on V near the singularities! Also notice that the singular factor e^φ of $h = e^\varphi (h_X)_{|V}$ does not interfere as it is purely scalar and "diagonal". □

Our general Morse inequalities 4.7 then yield directly

5.9. Morse inequalities for tautological sheaves. *Let (X, V) be a directed variety, where X is a nonsingular compact complex manifold, let F be a $\mathbb{Q}$-line bundle and $\mathcal{G}$ a coherent sheaf of rank ρ on X. Assume that V and F are equipped with metrics h_V and h_F with $\mathbb{Q}$-analytic singularities and let $\eta = \Theta_{\det(V^*),\det(h^*)} + \Theta_{F,h_F}$, $X' = X \smallsetminus (\mathrm{Sing}(h_V) \cup \mathrm{Sing}(h_F))$. Then for $m \gg k \gg 1$ we have estimates*

$$\rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\sum_{j=q-1,q,q+1}\int_{X'(\eta,j)}(-1)^{q-j}\eta^n - O((\log k)^{-1})\bigg),$$

$$\leqslant h^q\Big(X_k^{\mathrm{GG}}, {}^b\mathcal{O}_{X_k^{\mathrm{GG}}}(m)_{h_{k,\varepsilon}^{-m}} \otimes \pi_k^*\Big({}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)_{h_F} \otimes \mathcal{G}\Big)\Big)$$

$$\leqslant \rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\,\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X'(\eta,q)}(-1)^q\eta^n + O((\log k)^{-1})\bigg).$$

Proof. Apply 4.7 on $Z = X_k^{\mathrm{GG}}$ to the m-th power of the rank 1 sheaf

$$\mathcal{L}_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\Big(\mathcal{O}_X\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)\Big)$$

equipped with the metric induced by $(h_{V,k,\varepsilon})^{-1}$ and h_F, and evaluate the resulting cohomology of the sheaf of bounded sections of $\mathcal{L}_k^m \otimes \pi_k^*\mathcal{G}$. □

Notice that there is in fact no loss of generality to take $h_V = (h_X)_{|V}$ where h_X is a smooth metric on T_X, since any additional weight e^φ with $\mathbb{Q}$-analytic singularities can in fact be moved to the factor F. In this case, we simply denote by ${}^b\mathcal{O}_{X_k^{\mathrm{GG}}}(m)$ the sheaf of bounded sections and do not specify the metric on V. By taking the direct image via $\pi_k : X_k^{\mathrm{GG}} \to X$ and applying the Leray spectral sequence we obtain the same bounds as above for the cohomology groups

$$H^q\Big(X, {}^b\mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*) \otimes {}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)_{h_F} \otimes \mathcal{G}\Big),$$

putting now $\eta = \Theta_{\det(V^*),\det(h_X^*)} + \Theta_{F,h_F}$. By Serre duality, we infer similar bounds for the "dual" cohomology groups

$$H^{n-q}\Big(X, \big({}^b\mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)\big)^* \otimes {}^b\mathcal{O}\Big(-\frac{m}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big)_{h_F} \otimes \mathcal{G}'\Big)$$

with $\mathcal{G}' = K_X \otimes \mathcal{G}^*$, but the duality holds only when $\mathcal{G}$ and ${}^b\mathcal{O}(E_{k,m}^{\mathrm{GG}}V^*)$ are locally free, which is the case if V is nonsingular. In fact, there is no change for the dominant term of the bounds if $\mathcal{G}$ is not locally free, because we can replace $\mathcal{G}$ by $K_X\otimes\mathcal{G}^*$ and then $\mathcal{G}'$ equals $\mathcal{G}^{**}$, which coincides with $\mathcal{G}$ outside of codimension 1 (cf. the proof of Lemma 4.4), and anyway the dominant term depends only on the generic rank $\rho=\operatorname{rank}\mathcal{G}$. When V is singular, we have by definition an injection $J^kV\hookrightarrow J^kT_X$. It yields a restriction morphism

$$\mathcal{O}(E^{\rm GG}_{k,m}T^*_X)={}^b\mathcal{O}(E^{\rm GG}_{k,m}T^*_X)\to{}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*),$$

and by duality an injection

$$\big({}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\big)^*\hookrightarrow\mathcal{O}(E^{\rm GG}_{k,m}T^*_X)^* \tag{5.10}$$

where the right hand side is a vector bundle. In order to deal properly with the duality, one way is use a modification $\mu:\widetilde X\to X$ such that $\mu^*({}^b\mathcal{O}(E^{\rm GG}_{k,m}T^*_X))$ is locally free modulo torsion. This modification can be chosen independent of m because at any point x, $\bigoplus_m{}^b\mathcal{O}_{X,x}(E^{\rm GG}_{k,m}V^*)$ is a finitely generated graded algebra over $\mathcal{O}_{X,x}$ (equal to the integral closure of the image of the finitely generated algebra $\bigoplus_m\mathcal{O}_{X,x}(E^{\rm GG}_{k,m}T^*_X)$ in its field of quotients). Once this is done, one can instead pull-back to $\widetilde X$ and apply Serre duality on $\widetilde X$. Again, the dominant term is unchanged as the replacement of K_X by $K_{\widetilde X}$ leaves it unaffected. This is especially interesting for $q=n$ since we then obtain an estimate of holomorphic sections of $\big(\mu^*\,{}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\big)^*$ on $\widetilde X$, and therefore of $\mu_*\big(\big(\mu^*\,{}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\big)^*\big)$ on X. If we reinterpret these operations in terms of metrics and bounded sections, we obtain precisely the sheaf of bounded sections ${}^b\mathcal{O}\big((E^{\rm GG}_{k,m}V^*)^*\big)$, defined as the space of sections of $(E^{\rm GG}_{k,m}V^*)^*$ on $X\smallsetminus\operatorname{Sing}(V)$ that are bounded with respect to a smooth metric on $(E^{\rm GG}_{k,m}T^*_X)^*$, and we see that we have in fact

$${}^b\mathcal{O}\big((E^{\rm GG}_{k,m}V^*)^*\big)=\mu_*\big(\big(\mu^*\,{}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\big)^*\big)=\big({}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\big)^*.$$

From these considerations we infer

5.11. Morse inequalities for holomorphic jet differentials and their duals. *Let (X,V) be a directed variety, where X is a nonsingular compact complex manifold, let F be a $\mathbb{Q}$-line bundle and $\mathcal{G}$ a coherent sheaf of rank ρ on X. Assume that F is equipped with a metric h_F with $\mathbb{Q}$-analytic singularities and let h_X be a smooth metric on T_X. For $m\gg k\gg 1$ we have the following estimates.*

(a) *Let $\eta=\Theta_{\det(V^*),\det(h^*_X)}+\Theta_{F,h_F}$ and $X'=X\smallsetminus(\operatorname{Sing}(V)\cup\operatorname{Sing}(h_F))$. Then*

$$\begin{aligned}\rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}&\bigg(\sum_{j=q-1,q,q+1}\int_{X'(\eta,j)}(-1)^{q-j}\eta^n-O((\log k)^{-1})\bigg),\\ &\leqslant h^q\Big(X,{}^b\mathcal{O}(E^{\rm GG}_{k,m}V^*)\otimes{}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)F\Big)_{h_F}\otimes\mathcal{G}\Big)\\ &\leqslant\rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X'(\eta,q)}(-1)^q\eta^n+O((\log k)^{-1})\bigg).\end{aligned}$$

(b) *Let $\eta^*=\Theta_{\det(V),\det(h_X)}+\Theta_{F,h_F}$ and $X'=X\smallsetminus(\operatorname{Sing}(V)\cup\operatorname{Sing}(h_F))$. Then*

$$\begin{aligned}\rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}&\bigg(\sum_{j=q-1,q,q+1}\int_{X'(\eta^*,j)}(-1)^{q-j}(\eta^*)^n-O((\log k)^{-1})\bigg),\\ &\leqslant h^q\Big(X,{}^b\mathcal{O}((E^{\rm GG}_{k,m}V^*)^*)\otimes{}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1+\frac12+\ldots+\frac1k\Big)F\Big)_{h_F}\otimes\mathcal{G}\Big)\\ &\leqslant\rho\,\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X'(\eta^*,q)}(-1)^q(\eta^*)^n+O((\log k)^{-1})\bigg).\end{aligned}$$

Proof. (a) is a consequence of our direct image argument. part (b) follows by duality if we observe that $X(\eta, n-q) = X(q, -\eta)$ and change F into $-F$; this has the effect of replacing $-\eta$ by η^* and $(-1)^{n-q}\eta^n$ by $(-1)^q(\eta^*)^n$. □

5.12. Definition. *Let $\mathcal{L}$ be a rank one sheaf equipped with a hermitian metric $h_{\mathcal{L}}$ with $\mathbb{Q}$-analytic singularities. We say for short that ${}^b\mathcal{L}$ is big if there exists $m \in \mathbb{N}^*$ and a log resolution $\mu : \widetilde{X} \to X$ of the sheaf ${}^b\mathcal{L}^m = {}^b\mathcal{L}^m_{h_{\mathcal{L}}}$ of bounded sections such that $\mu^*({}^b\mathcal{L}^m)$ is a big line bundle. When taking the product $\mathcal{L} \otimes \mathcal{F}$ with an invertible sheaf and speaking of the bigness of $\mathcal{L} \otimes \mathcal{F}$, we agree implicitly to take the tensor product $h_{\mathcal{L}} \otimes h_{\mathcal{F}}$ with a smooth metric on $\mathcal{F}$, so that ${}^b(\mathcal{L} \otimes \mathcal{F})^m = ({}^b\mathcal{L}^m) \otimes \mathcal{F}^m$.*

With this terminology in mind, we can now state an important application of our Morse estimates, in relation with positivity properties of the canonical or anticanonical sheaf of a directed variety.

5.13. Corollary. *Let X be a projective n-dimensional manifold, (X, V) a directed structure, and F an invertible sheaf on X. We consider here bounded sections with respect to a smooth hermitian metric h_X on T_X.*

(a) *If ${}^bK_V \otimes F$ is big, there are many sections in*

$$H^0\Big(X, {}^b\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*) \otimes {}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big) \quad \textit{for } m \gg k \gg 1.$$

(b) *If ${}^b\mathcal{O}(\det(V)) \otimes F$ is big, there are many sections in*

$$H^0\Big(X, {}^b\mathcal{O}\big((E^{\mathrm{GG}}_{k,m}V^*)^*\big) \otimes {}^b\mathcal{O}\Big(\frac{m}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)F\Big)\Big) \quad \textit{for } m \gg k \gg 1.$$

The asymptotic growth is of the form $c\,m^{n+kr-1}(\log k)^n/((n+kr-1)!\,(k!)^r)$ where $r = \operatorname{rank} V$ and the constant $c > 0$ depends only on X, V and F.

Proof. In case (a), by [Dem90], we can find a singular metric h_F on F such that

$$\eta = \Theta_{K_V, \det(h_X)^*)} + \Theta_{F, h_F}$$

is a Kähler current, i.e. $\eta \geqslant c\omega$ for some Kähler metric ω on X and some small constant $c > 0$. In fact, we can work with the invertible sheaf $\mu^*({}^bK_V \otimes F)$ on $\widetilde{X}$ and observe that the push forward of a Kähler current on $\widetilde{X}$ is a Kähler current on X. Then all chambers $X'(\eta, q)$ are empty except $X'(\eta, 0)$ which yields a strictly positive Morse integral, whence the result. Part (b) is entirely similar, using η^* instead of η. Notice that this does not necessarily imply that the higher cohomology groups vanish, only that $h^q_{m,k} = O((\log k)^{-1})h^0_{m,k}$ as $m \gg k \gg 1$. □

5.14. Remark. Since $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$ is the sheaf of homogeneous polynomials of degree m on J^kV, its dual $\mathcal{O}((E^{\mathrm{GG}}_{k,m}V^*)^*)$ can be thought of as a sheaf of differential operators of degree m on J^kV. Hence, our result (5.13 b) can be seen as an extension of the results of Siu [Siu04], Păun [Pau08], Merker [Mer09] on the existence of slanted vector fields on jet bundles.

References

[Blo26a] Bloch, A.: *Sur les systèmes de fonctions uniformes satisfaisant à l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension.* J. de Math., **5** (1926), 19–66.

[Blo26b] Bloch, A.: *Sur les systèmes de fonctions holomorphes à variétés linéaires lacunaires.* Ann. Ecole Normale, **43** (1926), 309–362.

[Bon93] Bonavero, L.: *Inégalités de Morse holomorphes singulières.* C. R. Acad. Sci. Paris Sér. I Math. **317** (1993) 1163–1166.

[Dem85] Demailly, J.-P.: *Champs magnétiques et inégalités de Morse pour la d''-cohomologie.* Ann. Inst. Fourier (Grenoble) **35** (1985) 189–229.

[Dem90] Demailly, J.-P.: *Singular hermitian metrics on positive line bundles.* Proceedings of the Bayreuth conference "Complex algebraic varieties", April 2–6, 1990, edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math. n° 1507, Springer-Verlag (1992), 87–104.

[Dem11] Demailly, J.-P.: *Holomorphic Morse Inequalities and the Green-Griffiths-Lang Conjecture.* Pure and Applied Math. Quarterly **7** (2011), 1165–1208.

[Dem12] Demailly, J.-P.: *Hyperbolic algebraic varieties and holomorphic differential equations.* expanded version of the lectures given at the annual meeting of VIASM, Acta Math. Vietnam. **37** (2012), 441-–512.

[Dem15] Demailly, J.-P.: *Towards the Green-Griffiths-Lang conjecture,* Conference "Analysis and Geometry", Tunis, March 2014, in honor of Mohammed Salah Baouendi, ed. by A. Baklouti, A. El Kacimi, S. Kallel, N. Mir, Springer (2015), 141–159.

[Dem18] Demailly, J.-P.: *Recent results on the Kobayashi and Green-Griffiths-Lang conjectures,* arXiv: Math.AG/1801.04765, Contribution to the 16th Takagi lectures in celebration of the 100th anniversary of K.Kodaira's birth, November 2015, to appear in the Japanese Journal of Mathematics.

[GrGr79] Green, M., Griffiths, P.: *Two applications of algebraic geometry to entire holomorphic mappings.* The Chern Symposium 1979, Proc. Internal. Sympos. Berkeley, CA, 1979, Springer-Verlag, New York (1980), 41–74.

[Lang86] Lang, S.: *Hyperbolic and Diophantine analysis.* Bull. Amer. Math. Soc. **14** (1986), 159–205.

[Mer09] Merker, J.: *Low pole order frames on vertical jets of the universal hypersurface* Ann. Inst. Fourier (Grenoble), **59** (2009), 1077–1104.

[Pau08] Păun, M.: *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity.* Math. Ann. **340** (2008) 875–892.

[Siu04] Siu, Y.T.: *Hyperbolicity in complex geometry.* The legacy of Niels Henrik Abel, Springer, Berlin (2004) 543–566.

(version of June 25, 2018, printed on September 11, 2018, 18:14)

Jean-Pierre Demailly
Université de Grenoble-Alpes, Institut Fourier (Mathématiques)
UMR 5582 du C.N.R.S., 100 rue des Maths, 38610 Gières, France
e-mail: jean-pierre.demailly@univ-grenoble-alpes.fr

Mohammad Reza Rahmati
Institute of Algebraic Geometry, Gottfried Wilhelm Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
e-mail: mrahmati.mrr@gmail.com