VALUE DISTRIBUTION THEORY FOR PARABOLIC RIEMANN SURFACES

MIHAI PĂUN, NESSIM SIBONY

Abstract. A conjecture by Green-Griffiths states that if X is a projective manifold of general type, then there exists an algebraic proper subvariety of X which contains the image of all holomorphic curves from the complex plane to X. To our knowledge, the general case is far from being settled. We question here the choice of the complex plane as a source space.

Let $\mathcal{Y}$ be a parabolic Riemann surface, i.e bounded subharmonic functions defined on $\mathcal{Y}$ are constant. The results of Nevanlinna's theory for holomorphic maps f from $\mathcal{Y}$ to the projective line are parallel to the classical case when $\mathcal{Y}$ is the complex line except for a term involving a weighted Euler characteristic. Parabolic Riemann surfaces could be hyperbolic in the Kobayashi sense.

Let X be a manifold of general type, and let A be an ample line bundle on X. It is known that there exists a holomorphic jet differential P (of order k) with values in the dual of A. If the map f has infinite area and if $\mathcal{Y}$ has finite Euler characteristic, then f satisfies the differential relation induced by P. As a consequence, we obtain a generalization of Bloch Theorem concerning the Zariski closure of maps f with values in a complex torus. We then study the degree of Nevanlinna's currents $T[f]$ associated to a parabolic leaf of a foliation $\mathcal{F}$ by Riemann surfaces on a compact complex manifold. We show that the degree of $T[f]$ on the tangent bundle of the foliation is bounded from below in terms of the counting function of f with respect to the singularities of $\mathcal{F}$, and the Euler characteristic of $\mathcal{Y}$. In the case of complex surfaces of general type, we obtain a complete analogue of McQuillan's result: a parabolic curve of infinite area and finite Euler characteristic tangent to $\mathcal{F}$ is not Zariski dense.That requires some analysis of the dynamics of foliations by Riemann Surfaces.

1. Introduction

Let X be a compact complex manifold. S. Kobayashi introduced a pseudo-distance, determined by the complex structure of X. We recall here its infinitesimal version, cf. [17].

Given a point $x \in X$ and a tangent vector $v \in T_{X,x}$ at X in x, the length of v with respect to the *Kobayashi-Royden pseudo-metric* is the following quantity

$$\mathbf{k}_{X,x}(v) := \inf\{\lambda > 0; \exists f : \mathbb{D} \to X, f(0) = x, \lambda f'(0) = v\},$$

where $\mathbb{D} \subset \mathbb{C}$ is the unit disk, and f is a holomorphic map.

We remark that it may very well happen that $\mathbf{k}_{X,x}(v) = 0$; however, thanks to Brody re-parametrization lemma, this situation has a geometric counterpart, as follows. If there exists a couple (x, v) as above such that $v \neq 0$ and such that $\mathbf{k}_{X,x}(v) = 0$, then one can construct a holomorphic non-constant map $f : \mathbb{C} \to X$. The point x is not necessarily in the image of f.

In conclusion, if any entire curve drawn on X is constant, then the pseudo-distance defined above is a distance, and we say that X is *Brody hyperbolic*, or simply *hyperbolic* (since most of the time we will be concerned with compact manifolds).

As a starting point for the questions with which we will be concerned with in this article, we have the following result.

Proposition 1.1. *Let X be a hyperbolic manifold, and let $\mathcal{C}$ be a Riemann surface. Let $E \subset \mathcal{C}$ be a closed, countable set. Then any holomorphic map $f : \mathcal{C} \setminus E \to X$ admits a (holomorphic) extension to the surface $\mathcal{C}$.*

In particular, in the case of the complex plane we infer that any holomorphic map $f : \mathbb{C} \setminus E \to X$ must be constant (under the hypothesis of Proposition 1.1). We will give a proof and discuss some related statements and questions in the first paragraph of this paper. Observe however that if the cardinal of E is at least 2, then $\mathbb{C} \setminus E$ is Kobayashi hyperbolic.

Our next remark is that the surface $\mathbb{C} \setminus E$ is a particular case of a *parabolic Riemann surface*; we recall here the definition. A Riemann surface $\mathcal{Y}$ is parabolic if any bounded subharmonic function defined on $\mathcal{Y}$ is constant. This is a large class of surfaces, including e.g. $Y \setminus \Lambda$, where Y is a compact Riemann surface of arbitrary genus and $\Lambda \subset Y$ is any closed polar set. It is well-known (cf. [1]) that a Riemann surface $\mathcal{Y}$ is *parabolic* if and only if it admits a smooth exhaustion function

$$\sigma : \mathcal{Y} \to [1, \infty[$$

such that:

• σ is strictly subharmonic in the complement of a compact set;

• $\tau := \log \sigma$ is harmonic in the complement of a compact set of $\mathcal{Y}$. Moreover, we impose the normalization

$$\int_{\mathcal{Y}} dd^c \log \sigma = 1, \tag{1}$$

where the operator d^c is defined as follows

$$d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial).$$

On the boundary $S(r) := (\sigma = r)$ of the parabolic ball of radius r we have the induced measure

$$d\mu_r := d^c \log \sigma|_{S(r)}.$$

The measure $d\mu_r$ has total mass equal to 1, by the relation (1) combined with Stokes formula.

Since we are dealing with general parabolic surfaces, the growth of the Euler characteristic of the balls $\mathbb{B}(r) = (\sigma < r)$ will appear very often in our estimates. We introduce the following notion.

Definition 1.2. *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface, together with an exhaustion function as above. For each $t \geq 1$ such that $S(t)$ is non-singular we denote by $\chi_\sigma(t)$ the Euler characteristic of the domain $\mathbb{B}(t)$, and let*

$$\mathfrak{X}_\sigma(r) := \int_1^r \left|\chi_\sigma(t)\right| \frac{dt}{t}$$

be the (weighted) mean Euler characteristic of the ball of radius r.

If $\mathcal{Y} = \mathbb{C}$, then $\mathfrak{X}_\sigma(r)$ is bounded by $\log r$. The same type of bound is verified if $\mathcal{Y}$ is the complement of a finite number of points in $\mathbb{C}$. If $\mathcal{Y} = \mathbb{C} \setminus E$ where E is a closed polar set of infinite cardinality, then things are more subtle, depending on the density of the distribution of the points of E in the complex plane. However, an immediate observation is that the surface $\mathcal{Y}$ has finite Euler characteristic if and only if

$$\mathfrak{X}_\sigma(r) = \mathcal{O}(\log r). \tag{2}$$

In the first part of this article we will extend a few classical results in hyperbolicity theory to the context of parabolic Riemann surfaces, as follows.

We will review the so-called "first main theorem" and the logarithmic derivative lemma for maps $f : \mathcal{Y} \to X$, where X is a compact complex manifold. We also give a version of the first main theorem with respect to an ideal $\mathcal{J} \subset \mathcal{O}_X$. This will be a convenient language when studying foliations with singularities.
As a consequence, we derive a vanishing result for jet differentials, similar to the one obtained in case $\mathcal{Y} = \mathbb{C}$, as follows.

Let $\mathcal{P}$ be a jet differential of order k and degree m on X, with values in the dual of an ample bundle (see [12]; we recall a few basic facts about this notion in the next section). Then we prove the following result.

Theorem 1.3. *Let $\mathcal{Y}$ be a parabolic Riemann surface. We consider a holomorphic map $f : \mathcal{Y} \to X$, such that the area of $f(\mathcal{Y}) \subset X$ (counted with multiplicities) is infinite. Let $\mathcal{P}$ be an invariant jet differential of order k and degree m, with values in the dual of an ample line bundle. Then we have*

$$\mathcal{P}\left(j_k(f)\right) = 0$$

identically on $\mathcal{Y}$, provided that

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)} = 0$$

For example, the requirement above is satisfied if $\mathcal{Y}$ has finite Euler characteristic. In the previous statement we denote by $j_k(f)$ the $k^{\rm th}$ jet associated to the map f. If $\mathcal{Y} = \mathbb{C}$, then this result is well-known, starting with the seminal work of A. Bloch (cf. [4]; see also [11], [30] and the references therein), and it is extremely useful in the investigation of the hyperbolicity properties of projective manifolds. In this context, the above result says that the vanishing result still holds in the context of Riemann surfaces of (eventually) infinite Euler characteristic, provided that the growth of this topological invariant is slow when compared to $T_{f,\omega}(r)$.

As a consequence of Theorem 1.3 we obtain the following result (see section 4, Corollary 4.6). Let X be a projective manifold, and let $D = Y_1 + \cdots + Y_N$ be an effective snc divisor. We assume that there exists a logarithmic jet differential $\mathcal{P}$ on (X, D) with values in a bundle A^{-1}, where A is ample. Let $f : \mathbb{C} \to X$ be an entire curve which do not satisfies the differential equation defined by $\mathcal{P}$. Then we obtain a lower bound for the number of intersection points of $f(\mathbb{D}_r)$ with D as $r \to \infty$, where $\mathbb{D}_r \subset \mathbb{C}$ is the disk of radius r.

Concerning the existence of jet differentials, we recall Theorem 0.1 in [13], see also [19].

Theorem 1.4. *Let X be a manifold of general type. Then there exist a couple of integers $m \gg k \gg 0$ and a (non-zero) holomorphic invariant jet differential $\mathcal{P}$ of order k and degree m with values in the dual of an ample line bundle A.*

Thus, our result 1.3 can be used in the context of the general type manifolds.

As a consequence of Theorem 1.3, we obtain the following analogue of Bloch's theorem. It does not seem to be possible to derive this result by using e.g. Ahlfors-Schwarz negative curvature arguments. Observe also that we cannot use a Brody-Green type argument, because the Brody reparametrization lemma is not available in our context.

Theorem 1.5. *Let $\mathbb{C}^N/\Lambda$ be a complex torus, and let $\mathcal{Y}$ be a parabolic Riemann surface of finite Euler characteristic. Then the smallest analytic subset containing the closure of a holomorphic map*

$$f : \mathcal{Y} \to \mathbb{C}^N/\Lambda$$

of infinite area is a translate of a sub-torus in $\mathbb{C}^N/\Lambda$.

In the second part of this paper our aim is to recast some of the work of M. McQuillan and M. Brunella concerning the Green-Griffiths conjecture in the parabolic setting. We first recall the statement of this problem.

Conjecture 1.6. ([14]) *Let X be a projective manifold of general type. Then there exists an algebraic subvariety $W \subsetneq X$ which contains the image of all holomorphic curves $f : \mathbb{C} \to X$.*

It is hard to believe that this conjecture is correct for manifolds X of dimension ≥ 3. On the other hand, it is very likely that this holds true for surfaces (i.e. $\dim X = 2$), on the behalf of the results available in this case.

Given a map $f : \mathcal{Y} \to X$ defined on a parabolic Riemann surface $\mathcal{Y}$, we can associate a Nevanlinna-type closed positive current $T[f]$. If X is surface of general type and if $\mathcal{Y}$ has finite Euler characteristic, then there exists an integer k such that the k-jet of f satisfies an algebraic relation. As a consequence, there exists a foliation $\mathcal{F}$ by Riemann surfaces on the space of k-jets X_k of X_0, such that the lift of f is tangent to $\mathcal{F}$. In conclusion we are naturally led to consider the pairs $(X, \mathcal{F})$, where X is a compact manifold, and $\mathcal{F}$ is a foliation by curves on X. We denote by $T_{\mathcal{F}}$ the so-called tangent bundle of $\mathcal{F}$.

We derive a lower bound of the intersection number $\displaystyle\int_X T[f] \wedge c_1(T_{\mathcal{F}})$ in terms of a Nevanlinna-type counting function of the intersection of f with the singular points of $\mathcal{F}$. As a consequence, if X is a complex surface and $\mathcal{F}$ has reduced singularities, we show that $\displaystyle\int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq 0$. For this part we follow closely the original argument of [21].

When combined with a result by Y. Miyaoka, the preceding inequality shows that the classes $\{T[f]\}$ and $c_1(T_{\mathcal{F}})$ are orthogonal. Since the class of the current $T[f]$ is nef, we show by a direct argument that we have $\displaystyle\int_X \{T[f]\}^2 = 0$, and from this we infer that the Lelong numbers of the diffuse part of $\{T[f]\}$, say R are equal to zero at each point of X.

This regularity property of R is crucial, since it allows to show –via the Baum-Bott formula and an elementary fact from dynamics– that we have $\displaystyle\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0$, where $N_{\mathcal{F}}$ is the normal bundle of the foliation, and $c_1(N_{\mathcal{F}})$ is the first Chern class of $N_{\mathcal{F}}$.

We then obtain the next result, in the spirit of [21].

Theorem 1.7. *Let X be a surface of general type, and consider a holomorphic map $f : \mathcal{Y} \to X$, where $\mathcal{Y}$ is a parabolic Riemann surface of* *finite Euler characteristic. We assume that f is tangent to a holomorphic foliation $\mathcal{F}$; then the dimension of the Zariski closure of $f(\mathcal{Y})$ is at most 1.*

In the last section of our survey we give a short proof of M. Brunella index theorem [7]. Furthermore, we show that that this important result admits the following generalization.

Let L be a line bundle on a complex surface X, such that $S^m T^\star_X \otimes L$ has a non-identically zero section u. Let f be a holomorphic map from a parabolic Riemann surface $\mathcal{Y}$ to X, directed by the multi-foliation $\mathcal{F}$ defined by u, i.e. we have $u\left((f')^{\otimes m}\right) = 0$ Then we show that we have

$$\int_X c_1(L) \wedge T[f] \geq 0.$$

Brunella's theorem corresponds to the case $m = 1$ and $L = N_\mathcal{F}$: indeed, a foliation on X can be seen as a section of $T^\star_X \otimes N_\mathcal{F}$ (or in a dual manner, as a section of $T_X \otimes T^\star_\mathcal{F}$).

If X is a minimal surface of general type, such that $c_1^2 > c_2$, we see that this implies Theorem 1.7 directly, i.e. without considering the $T[f]$-degree of the tangent of $\mathcal{F}$. In particular, we do not need to invoke Miyaoka's generic semi-positivity theorem, nor the blow-up procedure of McQuillan.

It is a very interesting problem to generalize the inequality above in the framework of higher order jet differentials, cf. section 7 for a precise statement.

2. Preliminaries

2.1. **Motivation: an extension result.** We first give the proof of Proposition 1.1. We refer to [24] for further results in this direction.

Proof (of Proposition 1.1) A first observation is that it is enough to deal with the case where E is a single point. Indeed, assume that this case is settled. We consider the set $E_0 \subset E$ such that the map f does not extends across E_0; our goal is to prove that we have $E_0 = \emptyset$. If this is not the case, then we remark that E_0 contains at least an isolated point –since it is countable, closed and non-empty–, and thus we obtain a contradiction.

Thus we can assume that we have a holomorphic map

$$f : \mathbb{D}^\star \to X$$

where $\mathbb{D}^\star$ is the pointed unit disc. Let $g_\mathcal{P}$ be the Kobayashi metric on X; we remark that by hypothesis, $g_\mathcal{P}$ is non-degenerate. By the *distance decreasing property* of this metric we infer that

$$|f'(t)|^2_{g_\mathcal{P}} \leq \frac{1}{|t|^2 \log^2 |t|^2} \tag{3}$$

for any $t \in \mathbb{D}^\star$. This is a crucial information, since now we can argue as follows. The inequality (3) implies that the area of the graph associated to our map $\Gamma_f^0 \subset \mathbb{D}^\star \times X$ defined by

$$\Gamma_f^0 := \{(t, x) \in \mathbb{D}^\star \times X : f(t) = x\}$$

is finite. By the theorem of Bishop-Skoda (cf. [29] and the references therein) this implies that there exists an analytic subset $\Gamma \subset \mathbb{D} \times X$ whose restriction to $\mathbb{D}^\star \times X$ is precisely Γ_f^0. Hence we infer that the fiber of the projection $\Gamma \to \mathbb{D}$ on the second factor is a point. Indeed, if this is not the case, then the area of the image (via f) of the disk of radius ε is bounded from below by a constant independent of $\varepsilon > 0$. This of course cannot happen, as one can see by integrating the inequality (3) over the disk of radius ε.

□

In connection with this result, we recall the following conjecture proposed in [24].

Conjecture 2.1. *Let X be a Kobayashi hyperbolic compact manifold of dimension n. We denote by $\mathbb{B}$ the unit ball in $\mathbb{C}^p$, and let E be a closed pluripolar subset of $\mathbb{B}$. Then any holomorphic map*

$$f : \mathbb{B} \setminus E \to X$$

extends across E.

In the case where X is the quotient of a bounded domain in $\mathbb{C}^n$, a proof of this conjecture was proposed by M. Suzuki in [33]. In general, even if we assume that the holomorphic bisectional curvature of X is bounded from above by -1, the conjecture above seems to be open.

2.2. **Jet spaces.** We will recall here a few basic facts concerning the jet spaces associated to complex manifolds; we refer to [12], [17] for a more complete overview.

Let X be an l-dimensional complex space; we denote by $J^k(X)$ the space of k-jets of holomorphic discs, described as follows. Let f and g be two germs of analytic discs $(\mathbb{C}, 0) \to (X, x)$, we say that they define the same k jet at x if their derivatives at zero coincide up to order k, i. e.

$$f^{(j)}(0) = g^{(j)}(0)$$

for $j = 0, ...k$. The equivalence classes defined by this equivalence relation is denoted by $J^k(X, x)$; as a set, $J^k(X)$ is the union of $J^k(X, x)$ for all $x \in X$. We remark that if $x \in X_{\text{reg}}$ is a non-singular point of X, then $J^k(X, x)$ is isomorphic to $\mathbb{C}^{kl}$, via the identification

$$f \to \big(f'(0), ..., f^{(k)}(0)\big).$$

This map is not intrinsic, it depends on the choice of some local coordinate system needed to express the derivatives above; at a global level the projection map

$$J^k(X_{\rm reg}) \to X_{\rm reg}$$

is a holomorphic fiber bundle (which is not a vector bundle in general, since the transition functions are polynomial instead of linear).

If $k = 1$, and $x \in X_{\rm reg}$ is a regular point, then $J^1(X, x)$ is the tangent space of X at x. We also mention here that the structure of the analytic space $J^k(X)$ at a singular point of X is far more complicated.

We assume that X is a subset of a complex manifold M; then for each positive integer k we have a natural inclusion

$$J^k(X) \subset J^k(M)$$

and one can see that the space $J^k(X)$ is the Zariski closure of the analytic space $J^k(X_{\rm reg})$ in the complex manifold $J^k(M)$ (note that this coincides with the topological closure)

Next we recall the definition of the main geometric objects we will use in the analysis of the structure of the subvarieties of complex tori which are Zariski closure of some parabolic image.

As before, let $x \in X_{\rm reg}$ be a regular point of X; we consider a coordinate system $(x^1, ..., x^l)$ of X centered at x. We consider the symbols

$$dx^1, ..., dx^l, d^2x^1, ..., d^2x^l, ..., d^kx^1, ..., d^kx^l$$

and we say that the weight of the symbol d^px^r is equal to p, for any $r = 1, ..., l$. A *jet differential* of order k and degree m at x is a homogeneous polynomial of degree m in $(d^px^r)_{p=1,...,k,r=1,...,l}$; we denote by $E^{\rm GG}_{k,m}(X, x)$ the vector space of all such polynomials, and then the set

$$E^{\rm GG}_{k,m}\big(X_{\rm reg}\big) := \cup_{x\in X_{\rm reg}} E^{\rm GG}_{k,m}(X, x)$$

has a structure of vector bundle, whose global sections are called jet differentials of weight m and order k. A global section $\mathcal{P}$ of the bundle $E^{\rm GG}_{k,m}(X)$ can be written locally as

$$\mathcal{P} = \sum_{|\alpha_1|+\cdots+k|\alpha_k|=m} a_\alpha (dx)^{\alpha_1} \dots (d^kx)^{\alpha_k};$$

here we use the standard multi-index notation.

Let $f : (\mathbb{C}, 0) \to (X, x)$ be a k-jet at x. The group $\mathbb{G}_k$ of k–jets of biholomorphisms of $(\mathbb{C}, 0)$ acts on $J_k(X)$, and we say that the operator $\mathcal{P}$ is *invariant* if

$$\mathcal{P}\big((f\circ\varphi)', \dots, (f\circ\varphi)^{(k)}\big) = \varphi'^m \mathcal{P}\big(f', \dots, f^{(k)}\big).$$

The bundle of invariant jet differentials is denoted by $E_{k,m}(X)$; we will recall next an alternative description of this bundle, which will be very useful in what follows.

Along the next few lines, we indicate a compactification of the quotient $J_k^{reg}(X)/\mathbb{G}_k$ following [12], where $J_k^{reg}(X)$ denote the space of non-constants jets.

We start with the pair (X,V), where $V \subset T_X$ is a subbundle of the tangent space of X. Then we define $X_1 := \mathbb{P}(T_X)$, and the bundle $V_1 \subset T_{X_1}$ is defined fiberwise by

$$V_{1,(x,[v])} := \{\xi \in T_{X_1(x,[v])} : d\pi(\xi) \in \mathbb{C}v\}$$

where $\pi : X_1 \to X$ is the canonical projection and $v \in V$. It is easy to see that we have the following alternative description of V_1: consider a non-constant disk $u : (\mathbb{C}, 0) \to (X, x)$. We can lift it to X_1 and denote the resulting germ by u_1. Then the derivative of u_1 belongs to the V_1 directions.

In a more formal manner, we have the exact sequence

$$0 \to T_{X_1/X} \to V_1 \to \mathcal{O}_{X_1}(-1) \to 0$$

where $\mathcal{O}_{X_1}(-1)$ is the tautological bundle on X_1, and $T_{X_1/X}$ is the relative tangent bundle corresponding to the fibration π. This shows that the rank of V_1 is equal to the rank of V.

Inductively by this procedure we get a tower of manifolds (X_k, V_k), starting from (X, T_X) and it turns out that we have an embedding $J_k^{reg}/\mathbb{G}_k \to X_k$. On each manifold X_k, we have a tautological bundle $\mathcal{O}_{X_k}(-1)$, and the positivity of its dual plays an important role here.

We denote by $\pi_k : X_k \to X$ the projection, and consider the direct image sheaf $\pi_{k*}\big(\mathcal{O}_{X_k}(m)\big)$. The result is a vector bundle $E_{k,m}(X)$ whose sections are precisely the invariant jet differentials considered above.

The fiber of π_k at a non-singular point of X is denoted by $\mathcal{R}_{n,k}$; it is a rational manifold, and it is a compactification of the quotient $\mathbb{C}^{nk} \setminus 0/\mathbb{G}_k$.

The articles [4], [12], [30] (to quote only a few) show that the existence of jet differentials are crucial in the analysis of the entire maps $f : \mathbb{C} \to X$. As we will see in the next sections, they play a similar role in the study of the images of the parabolic Riemann surfaces.

3. Basics of Nevanlinna Theory for Parabolic Riemann Surfaces

Let $\mathcal{Y}$ be a parabolic Riemann surface; as we have recalled in the introduction, this means that there exists a non-singular exhaustion function

$$\sigma : \mathcal{Y} \to [1, \infty[$$

such that:

- σ is strictly psh in the complement of the subset $(\sigma < r_0) \subset \mathcal{Y}$;
- The function $\tau := \log \sigma$ is harmonic in the complement of a compact subset of $\mathcal{Y}$, and we have $\int_{\mathcal{Y}} dd^c\tau = 1$.

We denote by $\mathbb{B}(r) \subset \mathcal{Y}$ the parabolic ball of radius r, that is to say

$$\mathbb{B}(r) := \{y \in \mathcal{Y} : \sigma(y) \leq r\}.$$

For almost every value $r \in \mathbb{R}$, the sphere $S(r) := \partial\mathbb{B}(r)$ is a smooth curve drawn on $\mathcal{Y}$. The induced length measure on $S(r)$ is equal to

$$d\mu_r := d^c \log \sigma|_{S(r)}.$$

Let $v : \mathcal{Y} \to [-\infty, \infty[$ be a function defined on $\mathcal{Y}$, such that locally near every point of $\mathcal{Y}$ it can be written as a difference of two subharmonic functions, i.e. $dd^c v$ is of order zero.

Then we recall here the following formula, which will be very useful in what follows.

Proposition 3.1. (Jensen formula) *For every $r \geq 1$ large enough we have*

$$\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c v = \int_{S(r)} v d\mu_r - \int_{\mathbb{B}(r_0)} v dd^c \tau =$$
$$= \int_{S(r)} v d\mu_r + \mathcal{O}(1)$$

where we have $\tau = \log \sigma$.

Proof. The arguments are standard. To start with, we remark that for each regular value r of σ the function v is integrable with respect to the measure $d\mu_r$ over the sphere $S(r)$. Next, we have

$$\int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c v = \int_{\mathbb{B}(r)} \big(\log r - \log \sigma\big) dd^c v =$$
$$= \int \log^+ \frac{r}{\sigma} dd^c v = \int v dd^c \big(\log^+ \frac{r}{\sigma}\big)$$
$$= \int_{S(r)} v d\mu_r - \int_{\mathbb{B}(r)} v dd^c \tau.$$

□

Remark 3.2. As we can see, the Jensen formula above holds true even without the assumption that the function τ is harmonic outside a compact set. The only difference is eventually as $r \to \infty$, since the term $\int_{\mathbb{B}(r)} v dd^c \tau$ may tend to infinity.

We reformulate next the notion of mean Euler characteristic in analytic terms. To this end, we first recall that the tangent bundle $T_{\mathcal{Y}}$ of a non-compact parabolic surface admits a trivializing global holomorphic section $v \in H^0(\mathcal{Y}, T_{\mathcal{Y}})$, cf. [15] (actually, any such Riemann surface admits a submersion into $\mathbb{C}$). Using the Poincaré-Hopf index theorem, we obtain the following result.

Proposition 3.3. *Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface, so that $\log \sigma$ is harmonic in the complement of a compact set. Then we have*

$$\mathfrak{X}_\sigma(r) = \frac{1}{2}\int_{S(r)} \log^+ |d\sigma(v)|^2 d\mu_r + \mathcal{O}(\log r)$$

for any $r \geq 1$.

Proof. Since v is a vector of type (1,0), Jensen formula gives

$$\begin{aligned}\int_{S(r)} \log^+ |d\sigma(v)|^2 d\mu_r =& \int_{S(r)} \log^+ |\partial\sigma(v)|^2 d\mu_r \\ =& 2\log r + \int_{S(r)} \log^+ |\partial \log \sigma(v)|^2 d\mu_r \\ =& \int_1^r \frac{dt}{t}\int_{\mathbb{B}(t)} dd^c \log^+ |\partial \log \sigma(v)|^2 + 2\log r + \mathcal{O}(1).\end{aligned}$$

Observe that we are using the fact that the function $\partial \log \sigma(v)$ is holomorphic in the complement of a compact set, so $\log^+ |\partial \log \sigma(v)|^2$ is subharmonic.

The term $\mathcal{O}(\log r)$ above depends on the exhaustion function σ and on the fixed vector field v, but the quantity

$$\int_1^r \frac{dt}{t}\int_{\mathbb{B}(t)} dd^c \log |\partial \log \sigma(v)|^2$$

is equal to the weighted Euler characteristic $\mathfrak{X}_\sigma(r)$ of the domains $\mathbb{B}(r)$, in particular it is independent of v, up to a bounded term. This can be seen as a consequence of the Poincaré-Hopf index theorem, combined with the fact that the function $\partial_v \log \sigma$ is holomorphic, so $dd^c \log |\partial_v \log \sigma|^2$ count the critical points of $\partial_v \sigma$. □

We obtain next the *first main theorem* and the *logarithmic derivative lemma* of Nevanlinna theory in the parabolic setting. The results are variation on well-known techniques (see [32] and the references therein). But for the convenience of the reader, we will reproduce here the arguments.

Let X be a compact complex manifold, and let $L \to X$ be a line bundle on X, endowed with a smooth metric h. We make no particular assumptions concerning the curvature form $\Theta_h(L)$. Let s be a non-trivial section of L normalized such that $\sup_X |s| = 1$, and let $f : \mathcal{Y} \to X$ be a holomorphic map, where $\mathcal{Y}$ is parabolic.

We define the usual characteristic function of f with respect to $\Theta_h(L)$ as follows

$$T_{f,\Theta_h(L)}(r) := \int_{r_0}^r \frac{dt}{t}\int_{\mathbb{B}(t)} f^\star \Theta_h(L).$$

If the form $\Theta_h(L)$ is positive definite, then precisely as in the classical case $\mathcal{Y} = \mathbb{C}$, the area of the image of f will be finite if and only if $T_{f,\Theta_h(L)}(r) = \mathcal{O}(\log r)$ as $r \to \infty$.

Let

$$N_{f,s}(r) := \int_{r_0}^{r} n_{f,s}(t)\frac{dt}{t}$$

be the counting function, where $n_{f,s}(t)$ is the number of zeroes of $s \circ f$ in the parabolic ball of radius t (counted with multiplicities). Hence we assume implicitly that the image of f is not contained in the set $(s = 0)$. Moreover, in our context the proximity function becomes

$$m_{f,s}(r) := \frac{1}{2\pi}\int_{S(r)} \log \frac{1}{|s \circ f|_h} d\mu_r.$$

In the important case of a (meromorphic) function $F : \mathcal{Y} \to \mathbb{P}^1$, one usually takes $L := \mathcal{O}(1)$ –hence $\Theta_h(L)$ is the Fubini-Study metric– and s the section vanishing at infinity; the proximity function becomes

$$m_{f,\infty}(r) := \frac{1}{2\pi}\int_{S(r)} \log_+ |f| d\mu_r, \tag{4}$$

where $F := [f_0 : f_1]$, $f = f_1/f_0$ and $\log_+ := \max(\log, 0)$.

As a consequence of Jensen formula, we derive the next result.

Theorem 3.4. *With the above notations, we have*

$$T_{f,\Theta_h(L)}(r) = N_{f,s}(r) + m_{f,s}(r) + \mathcal{O}(1) \tag{5}$$

as $r \to \infty$.

Proof. The argument is similar to the usual one: we apply the Jensen formula cf. Proposition 3.1 to the function $v := \log |s \circ f|_h$. Recall that that the Poincaré-Lelong equation gives $dd^c \log |s|_h^2 = [s = 0] - \Theta_h(L)$, which implies

$$dd^c v = \sum_j m_j \delta_{a_j} - f^\star(\Theta_h(L))$$

so by integration we obtain (5). □

Remark 3.5. If the measure $dd^c\tau$ does not have a compact support, then the term $\mathcal{O}(1)$ in the equality (5) is to be replaced by

$$\int_{\mathbb{B}(r)} \log |s \circ f|_h^2 dd^c\tau. \tag{6}$$

We observe that, thanks to the normalization condition we impose to s, the term (6) is negative. In particular we infer that

$$T_{f,\Theta_h(L)}(r) \geq N_{f,s}(r) + \int_{\mathbb{B}(r)} \log |s \circ f|_h^2 dd^c\tau \tag{7}$$

for any $r \geq r_0$.

We will discuss now a version of Theorem 3.4 which will be very useful in dealing with singular foliations. Let $\mathcal{J} \subset \mathcal{O}_X$ be a coherent ideal of holomorphic functions. We consider a finite covering of X with coordinate open sets $(U_\alpha)_{\alpha\in\Lambda}$, such that on U_α the ideal $\mathcal{J}$ is generated by the holomorphic functions $(g_{\alpha i})_{i=1\dots N_\alpha}$.

Then we can construct a function $\psi_{\mathcal{J}}$, such that for each $\alpha \in \Lambda$ the difference

$$\psi_{\mathcal{J}} - \log(\sum_i |g_{\alpha i}|^2) \tag{8}$$

is bounded on U_α. Indeed, let ρ_α be a partition of unity subordinated to $(U_\alpha)_{\alpha\in\Lambda}$. We define the function $\psi_{\mathcal{J}}$ as follows

$$\psi_{\mathcal{J}} := \sum_\alpha \rho_\alpha \log(\sum_i |g_{\alpha i}|^2) \tag{9}$$

and the boundedness condition (8) is verified, since there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\sum_i |g_{\alpha i}|^2}{\sum_i |g_{\beta i}|^2} \leq C$$

holds on $U_\alpha \cap U_\beta$, for each pair of indexes α, β. In the preceding context, the function $\psi_{\mathcal{J}}$ corresponds to $\log |s|_h^2$.

We can define a counting function and a proximity function for a holomorphic map $f : \mathcal{Y} \to X$ with respect to the analytic set defined by $\mathcal{J}$, as follows. Let $(t_j) \subset \mathcal{Y}$ be the set of solutions of the equation

$$\exp\big(\psi_{\mathcal{J}} \circ f(y)\big) = 0.$$

For each $r > 0$ we can write

$$\psi_{\mathcal{J}} \circ f(y)|_{\mathbb{B}(r)} = \sum_{\sigma(t_j)<r} \nu_j \log |y - t_j|^2 + \mathcal{O}(1)$$

for a set of multiplicities ν_j, and then the counting function is defined as follows

$$N_{f,\mathcal{J}}(r) := \sum_j \nu_j \log \frac{r}{\sigma(t_j)}. \tag{10}$$

In a similar way, the proximity function is defined as follows

$$m_{f,\mathcal{J}}(r) := -\int_{S(r)} \psi_{\mathcal{J}} \circ f d\mu_r. \tag{11}$$

Since $\mathcal{J}$ is coherent, the principalization theorem (cf. e.g. [18]), there exists a non-singular manifold $\widehat{X}$ together with a birational map $p : \widehat{X} \to X$ such that the inverse image of the ideal $\mathcal{J}$ is equal to $\mathcal{O}_{\widehat{X}}(-D)$, where $D := \sum_j e_j W_j$ is a simple normal crossing divisor on $\widehat{X}$. Recall that $\mathcal{O}_{\widehat{X}}(-D) \subset \mathcal{O}_{\widehat{X}}$ is the sheaf of holomorphic functions vanishing on D. In terms of the function $\psi_{\mathcal{J}}$ associated to the ideal $\mathcal{J}$, this can be expressed as follows

$$\psi_{\mathcal{J}} \circ p = \sum_j e_j \log |s_j|^2_{h_j} + \theta \tag{12}$$

where $W_j = (s_j = 0)$, the metric h_j on $\mathcal{O}(W_j)$ is arbitrary (and non-singular), and where θ is a bounded function on $\widehat{X}$.
Since we assume that the image of the map f is not contained in the zero set of the ideal $\mathcal{J}$, we can define the lift $\widehat{f} : \mathcal{Y} \to \widehat{X}$ of f to $\widehat{X}$ such that $p \circ \widehat{f} = f$. We have the next result.

Theorem 3.6. *Let $\mathcal{O}(D)$ be the line bundle associated to the divisor D; we endow it with the metric induced by (h_j), and let Θ_D be the associated curvature form. Then we have*

$$T_{\widehat{f},\Theta_D}(r) = N_{f,\mathcal{J}}(r) + m_{f,\mathcal{J}}(r) + \mathcal{O}(1)$$

as $r \to \infty$.

The argument is completely similar to the one given for Theorem 3.4 (by using the relation (12)). We basically apply the first main theorem for each s_j and add up the contributions. □

We will treat now another important result in Nevanlinna theory, namely the *logarithmic derivative lemma* in the parabolic context. To this end, we will suppose that as part of the data we are given a vector field

$$\xi \in H^0(\mathcal{Y}, T_{\mathcal{Y}})$$

which is nowhere vanishing –hence it trivializes the tangent bundle of our surface $\mathcal{Y}$. We denote by f' the section $df(\xi)$ of the bundle $f^\star T_X$. For example, if $\mathcal{Y} = \mathbb{C}$, then we can take $\xi = \dfrac{\partial}{\partial z}$.

In the proof of the next result, we will need the following form of the co-area formula. Let ψ be a 1-form defined on the surface $\mathcal{Y}$; then we have

$$\int_{\mathbb{B}(r)} d\sigma \wedge \psi = \int_0^r dt \int_{S(t)} \psi \tag{13}$$

for any $r > 1$. In particular, we get

$$\frac{d}{dr} \int_{\mathbb{B}(r)} d\sigma \wedge \psi = \int_{S(r)} \psi \tag{14}$$

We have the following version of the classical logarithmic derivative lemma.

Theorem 3.7. *Let $f : \mathcal{Y} \to \mathbb{P}^1$ be a meromorphic map defined on a parabolic Riemann surface $\mathcal{Y}$. The inequality*

$$m_{f'/f,\infty}(r) \leq C\big(\log T_f(r) + \log r\big) + \mathfrak{X}_\sigma(r)$$

holds true for all r outside a set of finite Lebesgue measure. We also get a similar estimate for higher order derivatives.

Proof. Within the framework of Nevanlinna theory, this kind of results can be derived in many ways if $\mathcal{Y}$ is the complex plane; the proof presented here follows an argument due to Selberg in [27].

On the complex plane $\mathbb{C} \subset \mathbb{P}^1$ we consider the coordinate w corresponding to $[1 : w]$ in homogeneous coordinates on $\mathbb{P}^1$. The form

$$\Omega := \frac{1}{|w|^2(1+\log^4|w|)}\frac{\sqrt{-1}}{2\pi}dw \wedge d\overline{w} \tag{15}$$

on $\mathbb{C}$ has finite volume, as one can easily check by a direct computation. In what follows, we will use the same letter to denote the expression of the meromorphic function $\mathcal{Y} \to \mathbb{C}$ induced by f. For each $t \geq 0$, we denote by $n(t, f, w)$ the number of zeroes of the function $z \to f(z) - w$ in the parabolic ball $\mathbb{B}(t)$ and we have

$$\int_{\mathbb{B}(t)} f^\star\Omega = \int_{\mathbb{C}} n(t, f, w)\Omega \tag{16}$$

by the change of variables formula.

Next, by integrating the relation (16) above and using Theorem 3.4, we infer the following

$$\int_1^r \frac{dt}{t}\int_{\mathbb{B}(t)} f^\star\Omega = \int_{\mathbb{C}} N_{f-w,\infty}(r)\Omega \leq \int_{\mathbb{C}} T_f(r)\Omega + \int_{\mathbb{C}} \log^+|w|\Omega; \tag{17}$$

by Remark 3.5. This last quantity is smaller than $C_0T_f(r)$, where C_0 is a positive constant. Thus, we have

$$\int_1^r \frac{dt}{t}\int_{\mathbb{B}(t)} f^\star\Omega \leq C_0T_f(r). \tag{18}$$

A simple algebraic computation shows the next inequality

$$\log\Big(1+\frac{|df(\xi)|^2}{|f(z)|^2}\Big) \leq \log\Big(1+\frac{|df(\xi)|^2}{|f(z)|^2(1+\log^2|f(z)|^2)|d\sigma(\xi)|^2}\Big)+ \\ +\log(1+\log^2|f(z)|^2)+\log(1+|d\sigma(\xi)|^2)$$

and therefore we get

$$\begin{aligned} m_{f'/f,\infty}(r) \leq & \frac{1}{4\pi}\int_{S(r)} \log\Big(1+\frac{|df(\xi)|^2}{|f(z)|^2}\Big)d\mu_r \leq \\ & \leq \log^+\int_{S(r)} \frac{|df(\xi)|^2}{|f(z)|^2(1+\log^2|f(z)|)}\frac{1}{|d\sigma_z(\xi)|^2}d\mu_r+ \\ & +\int_{S(r)} \log(1+\log^2|f(z)|^2)d\mu_r+ \\ & +\frac{1}{4\pi}\int_{S(r)} \log(1+|d\sigma_z(\xi)|^2)d\mu_r + C_1, \end{aligned}$$

where C_1 is a positive constant; here we used the concavity of the log function.

By the formula (14) we obtain

$$\int_{S(r)} \frac{|df(\xi)|^2}{|f(z)|^2(1+\log^2|f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\mu_r =$$
$$= \frac{1}{r}\frac{d}{dr}\int_{\mathbb{B}(r)} \frac{|df(\xi)|^2}{|f(z)|^2(1+\log^2|f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\sigma\wedge d^c\sigma.$$

Next we show that we have

$$\frac{|df(\xi)|^2}{|f(z)|^2(1+\log^2|f(z)|)} \frac{1}{|d\sigma_z(\xi)|^2} d\sigma\wedge d^c\sigma = f^\star\Omega.$$

Indeed this is clear, since we can choose a local coordinate z such that $\xi = \dfrac{\partial}{\partial z}$ and moreover we have $d\sigma\wedge d^c\sigma = \dfrac{\sqrt{-1}}{2\pi}\left|\dfrac{\partial\sigma}{\partial z}\right|^2 dz\wedge d\overline{z}$ (we are using here the fact that f is holomorphic).

Let H be a positive, strictly increasing function defined on $(0,\infty)$. It is immediate to check that the set of numbers $s\in\mathbb{R}_+$ such that the inequality

$$H'(s)\leq H^{1+\delta}(s)$$

is not verified, is of finite Lebesgue measure. By applying this calculus lemma to the function $H(r) := \displaystyle\int_1^r \frac{dt}{t}\int_{\mathbb{B}(t)} f^\star\Omega$ we obtain

$$\begin{aligned}\log^+\frac{1}{r}\frac{d}{dr}\int_{\mathbb{B}(r)} f^\star\Omega &\leq \log^+\frac{1}{r}\Big(\int_{\mathbb{B}(r)} f^\star\Omega\Big)^{1+\delta} + \mathcal{O}(1)\\ &\leq \log^+\left(r^\delta\Big[\frac{d}{dr}\int_1^r\frac{dt}{t}\int_{\mathbb{B}(t)} f^\star\Omega\Big]^{1+\delta}\right) + \mathcal{O}(1)\\ &\leq \delta\log r + \frac{1}{2}\log^+\Big(\int_1^r\frac{dt}{t}\int_{\mathbb{B}(t)} f^\star\Omega\Big)^{(1+\delta)^2} + \mathcal{O}(1)\end{aligned}$$

for all r outside a set of finite measure.

The term

$$\int_{S(r)} \log(1+\log^2|f(z)|^2)d\mu_r$$

is bounded -up to a constant- by $\log T_f(r)$; combined with Proposition 3.3, this implies the desired inequality. □

It is a simple matter to deduce the so-called *second main theorem* of Nevanlinna theory starting from the logarithmic derivative lemma (cf. e.g. [11]). The parabolic version of this result can be stated as follows.

Theorem 3.8. *Let $f:(\mathcal{Y},\sigma)\to\mathbb{P}^1$ be a meromorphic function. We denote by $N_{R_f}(r)$ the Nevanlinna counting function for the ramification* *divisor associated to f. Then for any set of distinct points $(a_j)_{1\leq j\leq p}$ in $\mathbb{P}^1$ there exists a set $\Lambda\subset\mathbb{R}_+$ of finite Lebesgue measure such that*

$$N_{R_f}(r)+\sum_{j=1}^{p} m_{f,a_j}(r)\leq 2T_{f,\omega}(r)+\mathfrak{X}_\sigma(r)+\mathcal{O}\big(\log r+\log^+ T_{f,\omega}(r)\big)+$$

for all $r\in\mathbb{R}_+\setminus\Lambda$.

For the proof we refer e.g. to [11]; as we have already mentioned, it is a direct consequence of the logarithmic derivative lemma. As a consequence, we have

$$\sum_{j=1}^{p}\delta_f(a_j)\leq 2+\overline{\lim}_{r\to\infty}\frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)}$$

provided that $T_{f,\omega}(r)\gg\log r$. In the inequality above we use the classical notation $\delta_f(a):=\underline{\lim}_r\dfrac{m_{f,a}(r)}{T_{f,\omega}(r)}=1-\overline{\lim}_r\dfrac{N_a(r)}{T_{f,\omega}(r)}$.

4. The Vanishing Theorem

Let $\mathcal{P}$ be an invariant jet differential of order k and degree m. We assume that it has values in the dual of an ample line bundle, that is to say

$$\mathcal{P}\in H^0(X,E_{k,m}T_X^\star\otimes A^{-1})$$

where A is an ample line bundle on X.

Let $f:\mathcal{Y}\to X$ be a parabolic curve on X; assume that we are given the exhaustion function σ and a vector field ξ such that we have

$$\overline{\lim}_{r\to\infty}\frac{\mathfrak{X}_\sigma(r)}{T_{f,\omega}(r)}=0 \tag{19}$$

on $\mathcal{Y}$. As we have already recalled in the preliminaries, the operator $\mathcal{P}$ can be seen as section of $\mathcal{O}_{X_k}(m)$ on X_k.

On the other hand, the curve f admits a canonical lift to X_k as follows. One first observes that the derivative $df:T_\mathcal{Y}\to f^\star T_X$ induces a map

$$f_1:\mathcal{Y}\to\mathbb{P}(T_X);$$

we remark that to do so we do not need any supplementary data, since $df(v_1)$ and $df(v_2)$ are proportional, provided that $v_j\in T_{\mathcal{Y},t}$ are tangent vectors at the same point. Using the notations of section two, it turns out that the curve f_1 is tangent to $V_1\subset T_{X_1}$, so that we can continue this procedure and define inductively $f_k:\mathcal{Y}\to X_k$.

We prove next the following result.

Theorem 4.1. *We assume that the curve f has infinite area, and that condition (19) holds. Then the image of f_k is contained in the zero set of the section of $\mathcal{O}_{X_k}(m)\otimes A^{-1}$ defined by the jet differential $\mathcal{P}$.*

Proof. We observe that we have

$$df_{k-1}(\xi) : \mathcal{Y} \to f_k^\star\big(\mathcal{O}_{X_k}(-1)\big)$$

that is to say, the derivative of f_{k-1} is a section of the inverse image of the tautological bundle. Thus the quantity

$$\mathcal{P}\big(df_{k-1}(\xi)^{\otimes m}\big)$$

is a section of $f_k^\star(A^{-1})$, where the above notation means that we are evaluating $\mathcal{P}$ at the point $f_k(t)$ on the m^{th} power of the section above at $t \in \mathcal{Y}$.

As a consequence, if ω_A is the curvature form of A, we have

$$\sqrt{-1}\partial\overline{\partial} \log |\mathcal{P}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 \geq f_k^\star(\omega_A). \tag{20}$$

The missing term involves the Dirac masses at the critical points of f. We observe that the positivity of the bundle A is fully used at this point: we obtain an upper bound for the characteristic function of f. By integrating and using Jensen formula, we infer that we have

$$\int_{S(r)} \log |\mathcal{P}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 d\mu_r \geq T_{f,\omega_A}(r) + \mathcal{O}(1)$$

as $r \to \infty$.

Now we follow the arguments in [11]: there exists a finite set of rational functions $u_j : X_k \to \mathbb{P}^1$ and a positive constant C such that we have

$$\log^+ |\mathcal{P}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 \leq C \sum_j \log^+ \frac{|d(u_j \circ f_{k-1})(\xi)|^2}{|u_j \circ f_{k-1}|^2}$$

pointwise on $\mathcal{Y}$. Indeed, we can use the meromorphic functions u_j as local coordinates on X, and then we can write the jet differential $\mathcal{P}$ as $Q\big(f, d^p(\log u_j \circ f)\big)$, hence the previous inequality. We invoke next the logarithmic derivative lemma (Theorem 3.7) established in the previous section, and so we infer that we have

$$\int_{S(r)} \log |\mathcal{P}\big(df_{k-1}(\xi)^{\otimes m}\big)|^2 d\mu_r \leq C(\log T_{f_{k-1}(r)} + \log r + \mathfrak{X}_\sigma(r)) \tag{21}$$

out of a set of finite Lebesgue measure. It is not difficult to see that the characteristic function corresponding to f_{k-1} is the smaller than $CT_f(r)$, for some constant C; by combining the relations (20) and (21) we have

$$T_f(r) \leq C\big(\log T_f(r) + \log r + \mathfrak{X}_\sigma(r)\big)$$

in the complement of a set of finite Lebesgue measure. Since the area of f is infinite, and since by assumption $\mathfrak{X}_\sigma(r) = o\big(T_f(r)\big)$, we will also have

$$\lim_r \frac{\log r}{T_f(r)} = 0.$$

So we get a contradiction. Therefore, if the image of f_k is not contained in the zero set of $\mathcal{P}$, then the area of f is finite. $\square$

Let $E \subset \mathbb{C}$ be a polar subset of the complex plane. In the case

$$\mathcal{Y} = \mathbb{C} \setminus E,$$

we show that we have the following version of the previous result in the context of arbitrary jet differentials.

Theorem 4.2. *Let $f : \mathbb{C} \setminus E \to X$ be a holomorphic curve; we assume that the area of f is infinite, and that condition (19) is satisfied. Then $\mathcal{P}(f', \dots, f^{(k)}) \equiv 0$ for any holomorphic jet differential $\mathcal{P}$ of degree m and order k with values in the dual of an ample line bundle.*

Proof. The argument is similar to the proof of the preceding Theorem 4.1, except that we use the pointwise inequality

$$\log^+ |\mathcal{P}(f', \dots, f^{(k)})| \leq C \sum_j \sum_{l=1}^{k} \log |d^l \log(u_j \circ f)|$$

combined with Theorem 3.7 in order to derive a contradiction. □

The following statement is an immediate consequence of Theorem 4.1.

Corollary 4.3. *Let X be a projective manifold whose cotangent bundle is ample. Then X does not admits any holomorphic curve $f : \mathcal{Y} \to X$ with infinite area such that that the condition (19) is satisfied.*

We recall that in the articles [5], [36], [6] it is shown that any generic complete intersection X of sufficiently high degree and codimension in $\mathbb{P}^n$ satisfies the hypothesis of Theorem 4.3.

To state our next result, we consider the following data. Let X be a non-singular, projective manifold and let $D = Y_1 + \dots + Y_l$ be an effective divisor, such that the pair (X, D) is log-smooth (this last condition means that the hypersurfaces Y_j are non-singular, and that they have transverse intersections). In some cases we have

$$H^0\big(X, E_{k,m} T^\star_X \langle D \rangle \otimes A^{-1}\big) \neq 0 \tag{22}$$

where $E_{k,m} T^\star_X \langle D \rangle$ is the log version of the space of invariant jet differentials of order k and degree m. Roughly speaking, the sections of the bundle in (22) are homogeneous polynomials in

$$d^p \log z_1, \dots, d^p \log z_d, d^p z_{d+1}, \dots, d^p z_n$$

where $p = 1, \dots k$ and $z_1 z_2 \dots z_d = 0$ is a local equation of the divisor D.

We have the following result, which is a more general version of Theorem 4.2.

Theorem 4.4. *Let $f : \mathbb{C} \setminus E \to X \setminus D$ be a non-algebraic, holomorphic map. If the parabolic Riemann surface $\mathcal{Y} := \mathbb{C} \setminus E$ verifies*

$$\mathfrak{X}_\sigma(r) = o\big(T_f(r)\big)$$

then for any invariant log-jet differential

$$\mathcal{P} \in H^0\big(X, E_{k,m}T_X^\star\langle D\rangle \otimes A^{-1}\big)$$

we have $\mathcal{P}(f', \dots, f^{(k)}) \equiv 0$.

Proof. We only have to notice that the "logarithmic derivative lemma" type argument used in the proof of the vanishing theorem is still valid in our context, despite of the fact that the jet differential has poles along D (see [25] for a complete treatment). Thus, the result follows as above. □

4.1. **A few examples.** At the end of this paragraph we will discuss some examples of parabolic surfaces; we will try to emphasize in particular the properties of the function $\mathfrak{X}_\sigma(r)$.

(1) Let $E \subset \mathbb{C}$ be a finite subset of the complex plane. Define

$$\log\sigma := \log^+|z| + \sum_{a\in E}\log^+\frac{r}{|z-a|}$$

for $r > 0$ small enough. Then clearly we have $\mathfrak{X}_\sigma(r) = \mathcal{O}(\log r)$ for $\mathcal{Y} := \mathbb{C}\setminus E$.

(2) We treat next the case of $\mathcal{Y} := \mathbb{C}\setminus E$, where $E = (a_j)_{j\geq 1}$ is a closed, countable set of points in $\mathbb{C}$. As we will see, in this case it is natural to use the Jensen formula without assuming that the support of the measure $dd^c\log\sigma$ is compact, see Remark 3.2.

Let $(r_j)_{j\geq 1}$ be a sequence of positive real numbers, such that the Euclidean disks $\mathbb{D}(a_j, r_j)$ are disjoints. As in the preceding example, we define the exhaustion function σ such that

$$\log\sigma = \log^+|z| + \sum_{j\geq 1}\log^+\frac{r_j}{|z-a_j|};$$

the difference here is that $d\mu_r$ is no longer a probability measure. However, we have the following inequality

$$\frac{1}{\sigma}\Big|\frac{\partial\sigma}{\partial z}\Big| \leq \Big(\sum_{j\geq 1}\frac{1}{|z-a_j|}\chi_{\mathbb{D}(a_i,1)} + \frac{1}{|z|}\chi_{(|z|>1)}\Big)$$

which holds true on $\mathcal{Y}$. On the parabolic sphere $S(r)$ we have $|z| < r$ and $|z-a_j| > \dfrac{r_j}{r}$ so that we obtain

$$\mathfrak{X}_\sigma(r) \leq \log r + \sum_{\sigma(a_j+r_je^{i\theta_j})<r}\log\frac{r}{a_j} \leq$$

$$\leq \log r + \sum_{|a_j|<r}\log\frac{r}{a_j} := \log r + N_{(a_j)_{j\geq 1}}(r).$$

Therefore, we can bound the Euler characteristic by the counting function for $(a_j)_{j\geq 1}$.

Remark 4.5. Let $\mathcal{Y}$ be a parabolic Riemann surface, with an exhaustion σ normalized as in (1). In more abstract terms, the quantity $\mathfrak{X}_\sigma(r)$ can be estimated along the following lines, by using the same kind of techniques as in the proof of the logarithmic derivative lemma.

$$\begin{aligned}
\mathfrak{X}_\sigma(r) = & \int_{S(r)} \log^+ \Big|\frac{\partial\sigma}{\partial z}\Big|^2 d\mu_r \leq \\
\leq & \frac{2}{\varepsilon}\log^+ \int_{S(r)} \Big|\frac{\partial\sigma}{\partial z}\Big|^\varepsilon d^c \log\sigma = \frac{2}{\varepsilon}\log^+ \frac{1}{r}\int_{S(r)} \Big|\frac{\partial\sigma}{\partial z}\Big|^\varepsilon d^c\sigma \leq \\
\leq & \frac{2}{\varepsilon}\log^+ \frac{1}{r}\frac{d}{dr}\int_{B(r)} \Big|\frac{\partial\sigma}{\partial z}\Big|^\varepsilon d\sigma\wedge d^c\sigma \leq \\
\leq & \frac{2}{\varepsilon}\log^+ \frac{1}{r}\Big(\int_{B(r)} \Big|\frac{\partial\sigma}{\partial z}\Big|^{2+\varepsilon} dz\wedge d\overline{z}\Big)^{1+\delta} \leq \\
\leq & \frac{2}{\varepsilon}\log^+ r^\delta\Big(\int_{B(r)} \frac{1}{r}\Big|\frac{\partial\sigma}{\partial z}\Big|^{2+\varepsilon} dz\wedge d\overline{z}\Big)^{1+\delta} \leq \\
\leq & \frac{2\delta}{\varepsilon}\log^+ r + \frac{(1+\delta)^2}{\varepsilon}\log^+\Big(\int_1^r \frac{dt}{t}\int_{B(t)} \Big|\frac{\partial\sigma}{\partial z}\Big|^{2+\varepsilon} dz\wedge d\overline{z}\Big).
\end{aligned}$$

On the other hand, we remark that we have

$$\int_{\mathbb{B}(t)} d\sigma\wedge d^c\sigma = t^2; \tag{24}$$

this can be verified e.g. by considering the derivative of the left hand side of the expression (24) above. However, this does not means that the surface $\mathcal{Y}$ is of finite mean Euler characteristic, since we cannot take $\varepsilon = 0$ in our previous computations–and as the example (2) above shows it, there is a good reason to that.

As a corollary of Theorem 4.4, we obtain the following statement.

Corollary 4.6. *Let (X, D) be a pair as above, and let $f : \mathbb{C} \to X$ be a non-algebraic, holomorphic map; we define $E := f^{-1}(D)$. Moreover, we assume that there exists an invariant log-jet differential $\mathcal{P}$ with values in A^{-1} such that $\mathcal{P}(f', \dots, f^{(k)})$ is not identically zero. We denote by $N_E(r)$ the Nevanlinna counting function associated to E, i.e. $N_E(r) = \int_0^r \frac{dt}{t}\mathrm{card}\big(E\cap(\sigma < r)\big)$. Then there exists a constant $C > 0$ such that we have*

$$\liminf_r \frac{N_E(r)}{T_f(r)} > C. \tag{23}$$

Moreover, the constant C is independent of f.

Remark 4.7. Observe that *we are not* in the situation of Theorem 4.3, since the image of f can intersect the support of the divisor D. Actually the main point in the previous statement is to analyze the intersection of f with D.

Proof. Let σ denote the parabolic exhaustion associated to E as in example 2 above, i.e. By the proof of the vanishing theorem we infer the existence of a constant $C > 0$ such that the inequality

$$T_f(r) \leq C^{-1}\big(\log T_f(r) + \mathcal{O}(\log r) + \mathfrak{X}_\sigma(r)\big)$$

holds as $r \to \infty$ for any map f such that $\mathcal{P}(f', \dots, f^{(k)})$ is not identically zero. On the other hand, we have $\mathfrak{X}_\sigma(r) = N_E(r)$, so the relation (23) is verified. The fact that C is independent of f can be seen directly from the previous inequality. □

5. Bloch Theorem

Let T be a complex torus, and let $X \subset T$ be an irreducible analytic set. In some sense, the birational geometry of X was completely understood since the work of Ueno [35]. He has established the following result.

Theorem 5.1. [35] *Let X be a subvariety of a complex torus T. Then there exist a complex torus $T_1 \subset T$, a projective variety W and an abelian variety A such that*

1. *We have $W \subset A$ and W is a variety of general type;*
2. *There exists a dominant (reduction) map $\mathcal{R} : X \to W$ whose general fiber is isomorphic to T_1.*

Thus, via this theorem the study of an arbitrary submanifold X of a torus is reduced to the case where X is of general type.

We analyze next the case where X is the Zariski closure of a *parabolic Riemann surface of finite Euler characteristic*. The next result can be seen as the complete analogue of the classical theorem of Bloch [4]

Theorem 5.2. *Let $X \subset T$ be a submanifold of a complex torus T. We assume that X is the Zariski closure of a parabolic Riemann surface of finite mean Euler characteristic, whose area is infinite. Then up to a translation, X is a torus.*

Proof. We will follow the approach in [4], [30]; starting with $V := T_X$, we consider the tower of directed manifolds $(T_k, V_k)_{k\geq 0}$, whose construction was recalled in the 1st part of this article. Since T is flat, we have

$$T_k = T \times \mathcal{R}_{n,k}$$

where we recall that here $\mathcal{R}_{n,k}$ is the "universal" rational homogeneous variety $\mathbb{C}^{nk} \setminus \{0\}/\mathbb{G}_k$, cf. [12]. The curve $f : \mathcal{Y} \to T$ lifts to T_k, as already explained; we denote by

$$f_k : \mathcal{Y} \to T_k$$

the lift of f.

Let X_k be the Zariski closure of the image of f_k, and let

$$\tau_k : X_k \to \mathcal{R}_{n,k}$$

be the composition of the injection $X_k \mapsto T_k$ with the projection on the second factor $T_k \to \mathcal{R}_{n,k}$.
Then either the dimension of the generic fiber of τ_k is strictly positive for all k, or there exists a value of k for which the map τ_k is generically finite. The geometric counterpart of each of these eventualities in analyzed along the two following statements, due to A. Bloch. After this is done, we will show that the proof of 5.2 follows almost immediately.

Proposition 5.3. *We assume that for each $k \geq 1$ the fibers of τ_k are positive dimensional. Then the dimension of the subgroup A_X of T defined by*

$$A_X := \{a \in T : a + X = X\}$$

is strictly positive.

Proof. We fix a point $x_0 \in \mathcal{Y}$ such that $f_k(x_0)$ is a regular point of X_k, for each $k \geq 1$. It is clear that such a point exists, since the image of the curve f_k is dense in X_k, hence the inverse image of the singular set of X_k by f_k is at most countable. By the same argument, we can assume that the fiber of the map τ_k through $f_k(x_0)$ is "generic", meaning that it is positive dimensional.

By this choice of the point x_0 the fiber

$$F_{k,0} := \tau_k^{-1}\tau_k\big(f_k(x_0)\big)$$

is positive dimensional, for each $k \geq 1$. Thus, there exists a curve $\gamma_k : (\mathbb{C}, 0) \to X_k$ such that

$$\gamma_k(t) = \big(z(t), \lambda\big)$$

such that $f_k(x_0)$ corresponds to the couple $\big(z(0), \lambda\big)$ according to the decomposition of T_k and λ is fixed (independently of t). In particular, this shows that the dimension of the analytic set

$$\Xi_k := \{a \in T : f_k(x_0) \in J^k(X) \cap J^k(a + X)\}$$

is strictly positive. This is so because for each t close enough to zero we can define an element a_t by the equality

$$f_k(x_0) = a_t + z(t)$$

and thus $a_t \in \Xi_k$ since the curve $t \to z(t)$ lies on X.

Next we see that we have the sequence of inclusions

$$\Xi_1 \supset \Xi_2 \supset ... \supset \Xi_k \supset \Xi_{k+1} \supset ...$$

therefore by Noetherian induction there exists a large enough positive integer k_0 such that

$$\Xi_k = \Xi_{k+1}$$

for each $k \geq k_0$, and such that the dimension of Ξ_k is strictly positive. But this means that for every $a \in \Xi_{k_0}$, the image of the curve f belongs to the translation $a+X$ of the set X (because this curve is tangent to an infinite order to the a-translation of X). Thus we will have $X = X + a$ given that the curve f is Zariski dense.The proposition is proved. □

In the following statement we rule out the other possibility.

Proposition 5.4. *Let k be a positive integer such that the map*

$$\tau_k : X_k \to \mathcal{R}_{n,k}$$

has finite generic fibers. Then there exists a jet differential $\mathcal{P}$ of order k with values in the dual of an ample line bundle, and whose restriction to X_k is non-identically zero.

Proof. The proof relies on the following claim: *the restriction to X_k of the tautological bundle $\mathcal{O}_k(1)$ associated to T_k is big*. This condition is equivalent to the fact that for $m \gg 0$ large enough we have

$$H^0\big(X_k, \mathcal{O}_k(m) \otimes A^{-1}\big) \neq 0.$$

Indeed, in the first place we know that $\mathcal{O}_k(1) = \tau_k^\star \mathcal{O}(1)$ where $\mathcal{O}(1)$ is the tautological bundle on $\mathcal{R}_{n,k}$. It also turns out that $\mathcal{O}(1)$ is big (cf. [12]). Since the generic fibers of τ_k are of dimension zero, the inverse image of $\mathcal{O}(1)$ is big –indeed, the growth of the space of sections of $\tau_k^\star \mathcal{O}(m)$ as $m \to \infty$ obtained by pull-back proves it. The claim is therefore established. □

We show next that Theorem 5.2 follows from the two statements above: let

$$\varphi : \mathcal{Y} \to T$$

be a non-constant holomorphic map from a parabolic surface $\mathcal{Y}$ of finite mean Euler characteristic into a complex torus T. We denote by X the Zariski closure of its image. Thanks to the result of Ueno we can consider the reduction map $R : X \to W$ associated to X. We claim that under the hypothesis of Theorem 5.2 the base W is reduced to a point.

If this is not the case, then we can assume that X is of general type. By the vanishing theorem 4.1 we see that the hypothesis of Proposition 5.4 will never be verified, for any $k \geq 1$ (otherwise X_k would not be the Zariski closure of the lift of the curve). Hence the hypothesis of the Proposition 5.3 are verified, and so X will be invariant by a positive dimensional sub-torus of T. Since X is assumed to be a manifold of general type, its automorphisms group is finite, so this cannot happen. Our result is proved. □

Remark 5.5. The same arguments show that Theorem 5.2 still holds in a more general context: it is enough to assume that we have $\mathfrak{X}_\sigma(r) = o(T_f(r))$. We leave the details to the interested reader.

6. Parabolic Curves Tangent to Holomorphic Foliations

6.1. **Nevanlinna's Currents Associated to a Parabolic Riemann Surface.**

Let $(\mathcal{Y},\sigma)$ be a parabolic Riemann surface. We fix a Kähler metric ω on X, and let $\varphi : \mathcal{Y} \to X$ be holomorphic map. For an open set $S \subset \mathcal{Y}$ with smooth boundary we denote by

$$\|\varphi(S)\| := \int_S \varphi^\star \omega \tag{25}$$

We define the (normalized) integration current

$$T_S := \frac{[\varphi_\star(S)]}{\|\varphi(S)\|}$$

which has bidimension (1,1) and total mass equal to 1. In general, the current T_S is positive but not closed. However, it may happen that for some accumulation point

$$T_\infty = \lim_k T_{S_k}$$

is a closed current, when $S_k \to \mathcal{Y}$.

Theorem 6.1. [10] *Let $(\mathcal{Y},\sigma)$ be a parabolic Riemann surface; we denote by $S_j := \mathbb{B}(r_j)$ the parabolic balls of radius r_j, where $(r_j)_{j\geq 1}$ is a sequence of real numbers such that $r_j \to \infty$. We consider a holomorphic map $\varphi : \mathcal{Y} \to X$ of infinite area. Then there exists at least one accumulation point of the sequence of currents*

$$T_j := \frac{[\varphi_\star(S_j)]}{\|\varphi(S_j)\|}$$

which is a closed (positive) current, denoted by T_∞.

Proof. The arguments presented here are a quantitative version of the ones in [10].

We denote by $u := \log\sigma$ the log of the exhaustion function; by hypothesis, the measure $dd^c u$ has compact support. We define a function H on $\mathcal{Y}$ by the equality

$$\varphi^\star\omega := H du \wedge d^c u.$$

Let $A(t) := \int_{(u<t)} \varphi^\star\omega$ be the area of the image of parabolic ball of radius e^t with respect to the metric ω, and let

$$L(t) := \int_{(u=t)} \sqrt{H} d^c u;$$

geometrically, it represents the length of the parabolic sphere of radius e^t measured with respect to the metric induced by $H du \wedge d^c u$ (or the length of the image with respect to ω).

By Cauchy-Schwarz inequality we have

$$L(t)^2 \leq \int_{(u=t)} H d^c u \int_{\mathbb{B}(t)} dd^c u = \int_{(u=t)} H d^c u \tag{26}$$

because $\int_{\mathbb{B}(t)} dd^c u = 1$. On the other hand we have

$$\frac{d}{dt}A(t) = \int_{(u=t)} H d^c u \tag{27}$$

thus combining the inequalities (26) and (27) we obtain

$$L(t)^2 \leq \frac{d}{dt}A(t).$$

For every positive ε we have $\frac{d}{dt}A(t) \leq A^{1+2\varepsilon}(t)$ for any t belonging to the complement of a set Λ_ε of finite measure; as a result we infer that the inequality

$$L(t) \leq A(t)^{1/2+\varepsilon} \tag{28}$$

holds true for any $t \in \mathbb{R}_+ \setminus \Lambda_\varepsilon$. In particular, this implies the existence of a current as an accumulation point of T_j, and the Theorem 6.1 is proved. □

We consider next the case where $dd^c u$ is not necessarily with compact support. We see that the previous statement admits the following version. Let

$$\rho(t) := \int_{\mathbb{B}(t)} dd^c u$$

be the mass of the measure $dd^c u$ on the ball of radius t. The inequalities (26) and (27) above show that we have

$$L(t)^2 \leq \rho(t)\frac{d}{dt}A(t).$$

As already seen, we have $\frac{d}{dt}A(t) \leq A^{1+2\varepsilon}$ for any t in the complement of a set Λ_ε of finite measure. Thus we have

$$L(t) \leq (\rho(t))^{1/2} A(t)^{1/2+\varepsilon},$$

and we see that we get a closed current as soon as there exists a constant $c > 0$ such that we have

$$\rho(t) \leq cA(t)^{1-\varepsilon'}$$

for some positive ε'.

By using similar arguments, combined with results by B. Kleiner [16] and B. Saleur [26] we obtain a result in the direction of the conjecture in paragraph 1. Let $E \subset \mathbb{D}$ be a polar subset of the unit disk, and let $f : \mathbb{D} \setminus E \to M$ be a holomorphic map with values in a compact, Kobayashi hyperbolic manifold M. As it is well-known (cf [34]) the set $\mathbb{D} \setminus E$ carries a local exhaustion function σ such that $u := \log \sigma$ is harmonic. Let $\chi(t)$ be the Euler characteristic of the domain $(\sigma < t)$; we define $A(t) := \int_{\sigma < t} f^\star \omega$.

Corollary 6.2. *If we have* $\dfrac{|\chi(t)|}{A(t)} \to 0$ *as* $t \to \infty$*, then the map* f *admits an extension through* E*.*

Proof. The argument relies heavily on the following result, which uses a technique due to B. Kleiner [16].

Theorem 6.3. [26] *Let* (M, ω) *be a compact Kobayashi hyperbolic manifold. There exist two constants* C_1, C_2 *such that for every holomorphic map* $f : \Sigma \to M$ *defined on a Riemann surface with smooth boundary* $\partial\Sigma$ *we have*

$$\mathrm{Area}\big(f(\Sigma)\big) \leq C_1|\chi(\Sigma)| + C_2 \mathrm{L}\big(f(\partial\Sigma)\big). \tag{29}$$

Coming back to the domains $(\sigma < t)$, inequality (28) shows that we have

$$L(t) \leq A(t)^{1/2+\varepsilon}.$$

When combined with the inequality (29) of the preceding theorem, we get

$$A(t) \leq C_1|\chi(t)| + C_2 A(t)^{1/2+\varepsilon}.$$

Given the hypothesis concerning the growth of the Euler characteristic, we infer the existence of a constant C_3 such that

$$A(t) \leq C_3$$

and for the rest of the proof we will follow the argument given in Proposition 1.1. Indeed, the current associated to the graph $\Gamma_f \subset \mathbb{D} \setminus E \times M$ of the map f has finite mass near the polar set $E \times M \subset \mathbb{D} \times M$. By using Skoda-ElMir extension theorem (for a simple proof, see [29]), the current $[\Gamma]$ extends to $\mathbb{D} \times M$ with no mass on $E \times M$. But this implies that the graph Γ extends as an analytic subset of $\mathbb{D} \times M$; in other words, f extends as a meromorphic map. It follows that in fact f is holomorphic, since f is defined on a 1-dimensional disk. □

Remark 6.4. We can also consider a version of the current T_∞ in the above statement. For each $r > 0$, the expression

$$T_r := \frac{1}{T_f(r)} \int_0^r \frac{dt}{t} [\varphi_\star(\mathbb{B}_t)]$$

defines a positive current on X. One can show that there exists a sequence r_k such that the limit points of (T_{r_k}) are positive and closed. Any such limit will be called a Nevanlinna current associated to φ, and will be denoted by $T[f]$. If we consider the lift of f to $\mathbb{P}(T_X)$, we get (with the same construction) a current denoted $T[f']$. Let $\pi : \mathbb{P}(T_X) \to X$ be the projection; then we can assume that $\pi_\star(T[f']) = T[f]$, as we will see later.

The preceding considerations apply e.g. to maps $f : \mathbb{D} \to X$ defined on the unit disk $\mathbb{D} \subset \mathbb{C}$; in this case we have $u := \log \dfrac{1}{1-|z|}$, so f will define a closed positive current provided that its area grows fast enough; this is the content of the next statement.

Corollary 6.5. *Let $f : \mathbb{D} \to X$ be a holomorphic map. We denote by $T(r)$ the Nevanlinna characteristic of f, and we assume that we have*

$$\frac{T(r)}{\log \frac{1}{1-r}} \to \infty$$

as $r \to 1$. Then any limit of $T_r(f)$ is a closed positive current.

6.2. **Metrics on the tangent bundle of a holomorphic foliation by disks.** Let $\mathcal{F}$ be a 1-dimensional holomorphic foliation (possibly with singularities) on a manifold X. This means that we are given a finite open covering $(U_\alpha)_\alpha$ of X with coordinates charts, and a family of associated vector fields $v_\alpha \in H^0(U_\alpha, T_X|_{U_\alpha})$ such that there exists $g_{\alpha\beta} \in \mathcal{O}^\star(U_\alpha \cap U_\beta)$ with the property that

$$v_\alpha = g_{\alpha\beta} d\pi_{\alpha\beta}(v_\beta)$$

on the intersection of U_α and U_β; here we denote by $(\pi_{\alpha\beta})$ the transition functions of X, corresponding to the covering (U_α). The (analytic) set of zeros of (v_α) is supposed to have codimension at least two, and it is denoted by $\mathcal{F}_{\text{sing}}$. The functions $(g_{\alpha\beta})$ verify the cocycle property, and they define the cotangent bundle corresponding to the foliation $\mathcal{F}$, denoted by $T^\star_{\mathcal{F}}$.

From the global point of view, the family of vector fields $(v_\alpha)_\alpha$ corresponds to a section V of the vector bundle $T_X \otimes T^\star_{\mathcal{F}}$.

Let ω be a metric on X (which is allowed to be singular). We will show that ω induces a metric h_s on the tangent bundle $T_{\mathcal{F}}$ (as we will see, the induced metric may be singular even if the reference metric ω is smooth).

Let $x \in X$, and let $\xi \in T_{\mathcal{F},x}$ be an element of the fiber at x of the tangent bundle corresponding to $\mathcal{F}$. Then we define its norm as follows

$$|\xi|^2_{h_s} := |V_x(\xi)|^2_\omega. \tag{30}$$

The local weights of the metric h_s on the set U_α are described as follows. Let $z^1, \dots z^n$ be local coordinates on X centered at x. We write

$$v_\alpha = \sum_{i=1}^n a^i_\alpha \frac{\partial}{\partial z^i}$$

where a^i_α are holomorphic functions defined on U_α; we assume that their common zero set has codimension at least 2 in X.

The local weight ϕ_α of the metric h_s is given by the expression

$$\phi_\alpha = -\log \sum_{i,j} a^i_\alpha \overline{a^j_\alpha} \omega_{i\bar{j}}$$

where $\omega_{i\bar{j}}$ are the coefficients of the metric ω with respect to the local coordinates $(z^j)_{j=1,\dots n}$. Indeed, let θ be a local trivialization of the bundle $T_{\mathcal{F}}$. Then according to the formula (30) we have

$$|\xi|^2_{h_s} = \big(\sum_{i,j} a^i_\alpha \overline{a^j_\alpha} \omega_{i\bar{j}}\big)|\theta(\xi)|^2, \tag{31}$$

which clarifies the formula for the local weight of h_s.

In some cases, the previous construction can be further refined, as follows.

Let $B = \sum_{j=1}^N W_j$ be a divisor on X. We assume that the following requirements are fulfilled.

(a) At each point of $x \in \mathrm{Supp}(B)$ the local equations of the analytic sets

$$(W_j, x)_{j=1,\dots,k}$$

can be completed to a local coordinate system centered at x. Here we denote by k the number of hypersurfaces in the support of B containing the point x (and we make a slight abuse of notation). In the language of algebraic geometry, one calls such a pair (X, B) *log-smooth*.

(b) We assume that each component W_j of $\mathrm{Supp}(B)$ is invariant by the foliation $\mathcal{F}$.

If the condition (a) above is verified, then we recall that the *logarithmic tangent bundle* of (X, B) is the subsheaf of $\mathcal{O}(T_X)$ defined locally as follows.

Let $U \subset X$ be a coordinate open set. We assume that we have a coordinate system $z_1, \dots, z_n$ on U, such that

$$\mathrm{Supp}(B) \cap U = \big(z_1 z_2 \dots z_k = 0\big).$$

Then the logarithmic tangent bundle $T_X\langle B\rangle$ corresponding to the pair (X, B) is the subsheaf of T_X whose local sections on U are given by

$$v = \sum_{j=1}^k v_j z_j \frac{\partial}{\partial z_j} + \sum_{p=k+1}^n v_p \frac{\partial}{\partial z_p}.$$

In other words, the local sections of $T_X\langle B\rangle|_U$ are the vector fields of $T_X|_U$ which are tangent to B when restricted to B. We note that the $T_X\langle B\rangle$ is a vector bundle of rank n, and the local model of a hermitian metric on it is given by

$$\omega_U \equiv \sqrt{-1} \sum_{j=1}^k \frac{dz_j \wedge d\overline{z}_j}{|z_j|^2} + \sum_{j=k+1}^n dz_j \wedge d\overline{z}_j$$

i.e. a metric with logarithmic poles along B. So, we have

$$|v|^2_{\omega_U} = \sum_j |v_j|^2.$$

From a global point of view, a hermitian metric $\omega_{X,B}$ on $T_X\langle B\rangle$ can be written as

$$\begin{aligned}\omega_{X,B}|_U =& \sqrt{-1}\sum_{j,i=1}^{k}\omega_{j\bar{i}}\frac{dz_j\wedge d\overline{z}_i}{z_j\overline{z}_i} + 2\mathrm{Re}\sqrt{-1}\sum_{j>k\geq i}\omega_{j\bar{i}}\frac{dz_j\wedge d\overline{z}_i}{\overline{z}_i}+\\ &+\sqrt{-1}\sum_{j,i\geq k+1}\omega_{j\bar{i}}dz_j\wedge d\overline{z}_i\end{aligned}$$

where the Hermitian matrix $(\omega_{j\bar{i}})$ is positive definite.

If moreover the condition (b) is fulfilled, then the family of vector fields v_α defining the foliation $\mathcal{F}$ can be seen as a global section V_B of the bundle

$$T_X\langle B\rangle \otimes T^\star_{\mathcal{F}}$$

and we have the following version of the metric constructed above. For each vector $\xi \in T_{\mathcal{F},x}$ we define

$$\|\xi\|^2_{h_{s,B}} := |V_{B,x}(\xi)|^2_{\omega_{X,B}}.$$

As in the case discussed before, we can give the local expression of the metric on $T_{\mathcal{F}}$, as follows. Let

$$v_\alpha = \sum_{i=1}^{k} a^i_\alpha z_i\frac{\partial}{\partial z^i} + \sum_{i=k+1}^{n} a^i_\alpha\frac{\partial}{\partial z^i}$$

be a logarithmic vector field trivializing the tangent bundle of the foliation on a coordinate set U_α. Then the local weight $\phi_{\alpha,B}$ of the metric $h_{s,B}$ induced by the metric $\omega_{X,B}$ is given by the expression

$$\phi_{\alpha,B} = -\log\sum_{i,j} a^i_\alpha\overline{a^j_\alpha}\omega_{i\bar{j}}. \tag{32}$$

In particular we see that this weight is *less singular than* the one in the expression (31). This will be crucial in the applications.

As far as the curvature current is concerned, the metric h_s as well as its logarithmic variant $h_{s,B}$ seem useless: given the definition above, its associated curvature is neither positive nor negative. Indeed, ϕ_α may tend to infinity along the singular set of the foliation $\mathcal{F}$, and it may tend to minus infinity along the singularities of the metric ω. However, we will present a few applications of this construction in the next paragraphs.

6.3. **Degree of currents associated to parabolic Riemann surfaces on the tangent bundle of foliations.** Let (X, ω) be a compact complex hermitian manifold, and let $\mathcal{F}$ be a holomorphic foliation on X of dimension 1. Let $f : \mathcal{Y} \to X$ be a holomorphic map, where $(\mathcal{Y}, \sigma)$ is a parabolic Riemann surface tangent to $\mathcal{F}$, and let

$$T[f] := \lim_r T_r[f]$$

be a Nevanlinna current associated to it.
In this section we will derive a lower bound in arbitrary dimension for the quantity

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}),$$

in the same spirit as [21], [7]. Prior to this, we introduce a few useful notations.

Let $\mathcal{J}_{\mathcal{F}_{\mathrm{s}}}$ be the coherent ideal associated to the singularities of $\mathcal{F}$; this means that locally on U_α the generators of $\mathcal{J}_{\mathcal{F}_{\mathrm{s}}}$ are precisely the coefficients (a_α) of the vector v_α defining $\mathcal{F}$, i.e.

$$v_\alpha = \sum_{i=1}^{n} a_\alpha^i \frac{\partial}{\partial z^i}.$$

As we have already mentioned in paragraph 3, there exists a function ψ_{sing} defined on X and having the property that locally on each open set U_α we have

$$\psi_{\mathrm{sing}} \equiv \log |v_\alpha|^2_\omega$$

modulo a bounded function.

Let $B = \sum_j W_j$ be a divisor on X, such that the pair (X, B) satisfies the requirements (a) and (b) in the preceding paragraph. Then the local generator of $T_{\mathcal{F}}$ can be written in this case as

$$v_{\alpha,B} = \sum_{i=1}^{k} a_\alpha^i z_i \frac{\partial}{\partial z^i} + \sum_{i=k+1}^{n} a_\alpha^i \frac{\partial}{\partial z^i}. \tag{33}$$

We denote by $\mathcal{J}_{\mathcal{F}_{\mathrm{s,B}}}$ the coherent ideal defined by the functions (a_α^i) in (33). Then we have

$$\mathcal{J}_{\mathcal{F}_{\mathrm{s}}} \subset \mathcal{J}_{\mathcal{F}_{\mathrm{s,B}}},$$

and the inclusion may be strict. We denote by $\psi_{\mathrm{sing,B}}$ the associated function.

The counting function with respect to the ideal defined by $\mathcal{F}_{\mathrm{sing}}$ will be denoted

$$N_{f,\mathcal{J}_{\mathcal{F}_{\mathrm{s}}}}(r) = \sum_{0<\sigma(t_j)<r} \nu_j \log \frac{r}{\sigma(t_j)} = \int_0^r \frac{dt}{t} \int_{B(t)} \left(dd^c \psi_{\mathrm{sing}} \circ f\right)_s,$$

with $f(t_j) \in \mathrm{Supp}(\mathcal{F}_{\mathrm{sing}})$, and the subscript s above denotes the singular part of the considered measure. Its normalized expression will be written as

$$\nu^T(f,\mathcal{F}_{\rm sing})(r) := \frac{1}{T_f(r)} N_{f,\mathcal{J}_{\mathcal{F}_{\rm s}}}(r). \tag{34}$$

The upper limit of the expression above will be denoted by

$$\nu^T(f,\mathcal{F}_{\rm sing}) := \overline{\lim}_r \nu^T(f,\mathcal{F}_{\rm sing})(r).$$

If Ξ is an arbitrary analytic subset of X, we will denote by $\nu^T(f,\Xi)$ the quantity defined in a similar manner by using the function ψ_Ξ instead of $\psi_{\rm sing}$.

We define the counting function with respect to $\mathcal{F}_{\rm s,B}$ as

$$N_{f,\mathcal{J}_{\mathcal{F}_{\rm s,B}}}(r) = \sum_{0<\sigma(t_j)<r} \nu_j \log\frac{r}{\sigma(t_j)},$$

with $f(t_j)\in \mathrm{Supp}(\mathcal{F}_{\rm s,B})$, together with it normalized expression

$$\nu^T(f,\mathcal{F}_{\rm sing,B})(r) := \frac{1}{T_f(r)} N_{f,\mathcal{J}_{\mathcal{F}_{\rm s,B}}}(r). \tag{35}$$

The following truncated counting function will appear in our next computations:

$$N^{(1)}_{f,\mathcal{J}_{\mathcal{F}_{\rm s}}\cap B}(r) = \sum_{0<\sigma(t_j)<r, f(t_j)\in B} \log\frac{r}{\sigma(t_j)}. \tag{36}$$

and let $\nu_1^T(f,\mathcal{F}_{\rm sing}\cap B)$ be its normalized upper limit.

We also recall the definition of

$$m^T(f,\mathcal{F}_{\rm sing}) := \overline{\lim}_r \frac{1}{T_f(r)}\int_{S(r)} -\psi_{\rm sing}\circ f d\mu_r$$

which is the (normalized) asymptotic proximity function for f with respect to the ideal $\mathcal{J}_{\mathcal{F}_{\rm s}}$, together with its logarithmic variant

$$m^T(f,\mathcal{F}_{\rm sing,B}) := \overline{\lim}_r \frac{1}{T_f(r)}\int_{S(r)} -\psi_{\rm sing,B}\circ f d\mu_r.$$

The ramification function corresponding to f is

$$R_f(r) = \sum_{0<\sigma(t'_j)<r} \mu_j \log\frac{r}{\sigma(t'_j)}; \tag{37}$$

so that μ_j is the vanishing order of f' at t'_j. The curve f is tangent to $\mathcal{F}$, therefore for each open set $\Omega\subset\mathcal{Y}$ such that $f(\Omega)\subset U_\alpha$ for some index α we can write

$$f'(t) = \lambda(t) v_{\alpha,f(t)}$$

for some function λ which is holomorphic on $\Omega\setminus f^{-1}(\mathcal{F}_{\rm sing})$. We remark that if $f(t_j)\notin\mathcal{F}_{\rm sing}$, then the multiplicities μ_j above coincide with the vanishing order of λ evaluated at the critical points of f. It will be useful in what follows to have the decomposition

$$R_f(r) := M_f(r) + N_f(\mathrm{Ram}, r)$$

according to the possibility that the critical value $f(t'_j)$ of f belongs to the set $\mathcal{F}_{\rm sing}$ or not. We are using the notations

$$N_f(\mathrm{Ram}, r) := \sum_{0<\sigma(t'_j)<r, f(t'_j)\notin \mathcal{F}_{\rm sing}} \mu_j \log \frac{r}{\sigma(t'_j)},$$

and

$$M_f(r) := \sum_{0<\sigma(t'_j)<r, f(t'_j)\in \mathcal{F}_{\rm sing}} \mu_j \log \frac{r}{\sigma(t'_j)}.$$

Finally, the asymptotic normalized ramification of f is denoted by

$$\overline{\nu}(\mathrm{Ram}, f) := \overline{\lim}_r \frac{1}{T_f(r)} R_f(r).$$

We establish next the following general result, which gives an estimate of the quantity $\int_X T[f]\wedge c_1(T_\mathcal{F})$ in terms of the intersection of f with the singularities of the foliation. Our statement is a quantitative expression of the fact that the derivative of f can be seen as a meromorphic section of $f^\star T_\mathcal{F}$.

Theorem 6.6. *Let $(X, \mathcal{F})$ be a compact complex manifold endowed with a holomorphic 1-dimensional foliation $\mathcal{F}$. Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface of finite Euler characteristic, and let $f : \mathcal{Y} \to X$ be a holomorphic map whose image is tangent to $\mathcal{F}$. We assume that the image of f is Zariski dense. Then we have*

$$(\star) \qquad \int_X T[f]\wedge c_1(T_\mathcal{F}) \geq -\nu^T(f, \mathcal{F}_{\rm sing}) - m^T(f, \mathcal{F}_{\rm sing}) + \overline{\nu}(\mathrm{Ram}, f)$$

Proof. Let ω be a smooth metric on X, and let h_s be the metric induced on $T_\mathcal{F}$ by the procedure described in the preceding sub-section.

Let $r > t > 0$; we begin by evaluating the quantity

$$\int_X T_r[f]\wedge \Theta_{h_s}(T_\mathcal{F})$$

and to this end we introduce the notations

$$\mathbb{B}_{\alpha,\varepsilon}(t) := \{z \in \mathbb{B}(t) : f(z) \in U_\alpha, \text{ and } d_\omega\big(f(z), \mathcal{F}_{\rm sing}\big) \geq \varepsilon\}$$

as well as its complement set inside the parabolic ball of radius t

$$\mathbb{B}^c_{\alpha,\varepsilon}(t) := \mathbb{B}_\alpha(t) \setminus \mathbb{B}_{\alpha,\varepsilon}(t)$$

where $\mathbb{B}_\alpha(t) := \mathbb{B}(t)\cap f^{-1}(U_\alpha)$. Let (ρ_α) be a partition of unit corresponding to the cover (U_α).

In the definition of the metric h_s we use the smooth Kähler metric ω we have fixed on X, and we have.

$$\int_X T_r[f]\wedge \Theta_{h_s}(T_\mathcal{F}) = -\sum_\alpha \frac{1}{T_f(r)}\int_1^r \frac{dt}{t}\int_{B^c_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega$$
$$-\sum_\alpha \frac{1}{T_f(r)}\int_1^r \frac{dt}{t}\int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega.$$

We remark that for each $t < r$ and for each index α we have

$$\int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega = \int_{B_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega - \sum_{0<\sigma(t'_j)<t, f(t'_j)\notin \mathcal{F}_{\rm sing}} \rho_\alpha\big(f(t'_j)\big)\mu_j \delta_{t'_j}.$$

The equality in the formula above is due to the fact that locally at each point in the complement of the set $\mathcal{F}_{\rm sing}$ we have

$$f'(t) = \lambda v_{f(t)}$$

for some holomorphic function λ. We also remark that the relation above is valid for any $\varepsilon > 0$, and if we let $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega = \int_1^r \frac{dt}{t} \int_{\mathbb{B}_\alpha(t)} \rho_\alpha(f) dd^c \log |f'|^2_\omega - \sum_{f(t'_j)\in \mathcal{F}_{\rm sing}} \rho_\alpha\big(f(t'_j)\big)\mu_j \log \frac{r}{\sigma(t'_j)}$$

as well as

$$\lim_{\varepsilon\to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}^c_{\alpha,\varepsilon}(t)} \rho_\alpha(f) f^\star dd^c \log |v_\alpha|^2_\omega = \sum_{t_j\in \mathbb{B}_\alpha(t), f(t_j)\in \mathcal{F}_{\rm sing}} \rho_\alpha\big(f(t_j)\big)\nu_j \log \frac{r}{\sigma(t_j)}.$$

Therefore we obtain

$$\begin{aligned}\langle T_r[f], \Theta_{h_s}(T_\mathcal{F})\rangle \geq & -\nu^T(f, \mathcal{F}_{\rm sing})(r) + \frac{1}{T_f(r)} M_f(r) \\ & -\frac{1}{T_f(r)} \int_1^r \frac{dt}{t} \int_{\mathbb{B}(t)} dd^c \log |f'|^2_\omega + \frac{1}{T_f(r)} N_f(\mathrm{Ram}, r) = \\ & = -\nu^T(f, \mathcal{F}_{\rm sing})(r) + \frac{1}{T_f(r)} R_f(r) - \\ & \frac{1}{T_f(r)} \int_{S(r)} \log |f'|^2_\omega d\mu_r.\end{aligned}$$

Let $h := h_s \exp(-\psi_{\rm sing})$; it is a metric with *bounded* weights of $T_\mathcal{F}$, hence we can use it in order to compute the quantity we are interested in, namely

$$\int_X T[f] \wedge c_1(T_\mathcal{F}) = \lim_r \langle T_r[f], \Theta_h(T_\mathcal{F})\rangle.$$

We recall that we have the formula

$$\Theta_h(T_\mathcal{F}) = \Theta_{h_s}(T_\mathcal{F}) + dd^c \psi_{\rm sing},$$

so as a consequence we infer that we have

$$\int_X T_r[f] \wedge \Theta_h(T_\mathcal{F}) = \langle T_r[f] \wedge \Theta_{h_s}(T_\mathcal{F})\rangle + \frac{1}{T_f(r)} \int_{S(r)} \psi_{\rm sing} \circ f d\mu_r + o(1).$$

By combining the relations above we infer that we have

$$\int_X T_r[f]\wedge\Theta_h(T_{\mathcal F})\geq -\,\nu^T(f,\mathcal F_{\rm sing})(r)+\frac{1}{T_f(r)}\int_{S(r)}\psi_{\rm sing}\circ f d\mu_r$$
$$+\frac{1}{T_f(r)}R_f(r)-\frac{1}{T_f(r)}\int_{S(r)}\log|f'|^2_\omega d\mu_r+o(1).$$

By the logarithmic derivative lemma (or rather by an estimate as in Theorem 4.2) the last term of the preceding relation tends to a positive value, as $r\to\infty$, hence we obtain

$$\int_X T_r[f]\wedge\Theta_h(T_{\mathcal F})\geq -\nu^T(f,\mathcal F_{\rm sing})-m^T(f,\mathcal F_{\rm sing})+\overline{\nu}(\mathrm{Ram},f)$$

and Theorem 6.6 is proved. □

Before stating a version of Theorem 6.6, we note the following observations. The lower bound obtained in Theorem 6.6 admits an easy interpretation, as follows.

Let $\mathcal J$ be the ideal sheaf defined by the scheme $\mathcal F_{\rm sing}$; locally, this ideal is generated by the coefficients of the vectors (v_α) defining the foliation $\mathcal F$. Let $p:\widehat X\to X$ be a principalization of $\mathcal J$, so that $p^\star(\mathcal J)=\mathcal O(-D)$ for some (normal crossing) effective divisor D on $\widehat X$. According to Theorem 3.5, we have

$$T_{\widehat f,\Theta_D}(r)=N_{\widehat f,\mathcal J}(r)+m_{\widehat f,\mathcal J}(r)+\mathcal O(1)$$

where $\widehat f$ is the lift of the map f to $\widehat X$. As a consequence, we infer the relation

$$\int_{\widehat X}T[\widehat f]\wedge c_1(D)\geq\nu^T(\widehat f,\mathcal F_{\rm sing})+m^T(\widehat f,\mathcal F_{\rm sing})$$

and therefore Theorem 6.6 applied to $\widehat f,\widehat{\mathcal F}$ can be restated as follows.

Corollary 6.7. *We have the inequality*

$$\int_{\widehat X}T[\widehat f]\wedge\big(c_1(T_{\widehat{\mathcal F}})+c_1(D)\big)\geq 0. \tag{38}$$

In a similar framework, we note the following "tautological" inequality in parabolic context.

Lemma 6.8. *Let $f:(\mathcal Y,\sigma)\to X$ be a holomorphic curve, where $\mathcal Y$ is a parabolic Riemann surface. We denote by $f_1:\mathcal Y\to\mathbb P(T_X)$ the lift of f. Let h be a hermitian metric on X; we denote by the same letter the metric induced on the bundle $\mathcal O(-1)\to\mathbb P(T_X)$. Then we have*

$$\langle T_r[f_1],\Theta_h\big(\mathcal O(-1)\big)\rangle\geq -C\big(\log T_f(r)+\log r+\mathfrak X_\sigma(r)\big)$$

where the positive constant C above depends on (X, h). In particular, if $\frac{\mathfrak{X}_\sigma(r) + \log r}{T_f(r)} \to 0$ *as* $r \to \infty$, *then we can construct a Nevanlinna current* $T[f_1]$ *associated to* f_1 *such that* $\pi_\star T[f_1] = T[f]$, *and we infer*

$$\langle T[f_1], \Theta_h\big(\mathcal{O}(-1)\big)\rangle \geq 0.$$

Proof. We note that the derivative $f' := df(\xi)$ can be seen as a section of the bundle $f_1^\star \mathcal{O}(-1)$. So the lemma follows from Jensen formula combined with logarithmic derivative lemma by an argument already used in Theorem 4.1 for sections of $\mathcal{O}_{X_k}(m) \otimes A^{-1}$; we offer no further details. □

We turn next to the logarithmic version of Theorem 6.6.

Theorem 6.9. *Let (X, B) be a log-smooth pair, such that every component of the support of B is invariant by the foliation $\mathcal{F}$. Let $(\mathcal{Y}, \sigma)$ be a parabolic Riemann surface of finite characteristic, and let $f : \mathcal{Y} \to X$ be a holomorphic map whose image is tangent to $\mathcal{F}$, and it is not contained in the set* Supp(B). *Then we have the inequality*

$(\star_{\rm B})$

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq -\nu^T(f, \mathcal{F}_{\rm sing,B}) - \nu_1^T(f, \mathcal{F}_{\rm sing} \cap B) - m^T(f, \mathcal{F}_{\rm sing,B}).$$

The proof of Theorem 6.9 follows from the arguments we have used for 6.6. The additional negative term in the statement is due to the singularities of the metric $\omega_{X,B}$:

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_1^r \frac{dt}{t} \int_{\mathbb{B}_{\alpha,\varepsilon}(t)} \rho_\alpha(f) dd^c \log |f'|^2_{\omega_{X,B}} &= \int_1^r \frac{dt}{t} \int_{\mathbb{B}_\alpha(t)} \rho_\alpha(f) dd^c \log |f'|^2_{\omega_{X,B}} \\ &- \sum_{f(t'_j) \in \mathcal{F}_{\rm sing} \setminus B} \rho_\alpha\big(f(t'_j)\big)\nu_j \log \frac{r}{\sigma(t'_j)} \\ &+ \sum_{f(t''_j) \in \mathcal{F}_{\rm sing} \cap B} \rho_\alpha\big(f(t''_j)\big) \log \frac{r}{\sigma(t''_j)}.\end{aligned}$$

where we denote by t'_j the critical points of f. The points t''_j appearing in the last expression above are not necessarily critical. We remark (as in Theorem 4.2) that the limit

$$\overline{\lim}_r \frac{1}{T_f(r)} \int_{S(r)} \log |f'|^2_{\omega_{X,B}} d\mu_r$$

is non-positive, as it follows from the logarithmic derivative lemma, i.e. this term is not affected by the poles of $\omega_{X,B}$.

In conclusion, the presence of a log-smooth divisor on X invariant by $\mathcal{F}$ improves substantially the lower bound we have obtained in Theorem 6.6, since the main negative terms are defined by $\mathcal{J}_{\mathcal{F}_s,B}$. □

6.4. **Foliations with reduced singularities on surfaces.** In this subsection we assume that the dimension of X is equal to $n = 2$. As an application of the results in the preceding paragraph, we obtain here a complete analogue of some results originally due to Michael McQuillan [21].

The twisted vector field V defining the foliation $\mathcal{F}$ is locally given by the expression

$$v_\alpha = a_{\alpha 1}\frac{\partial}{\partial z} + a_{\alpha 2}\frac{\partial}{\partial w}.$$

In this paragraph we will assume that *the singularities of $\mathcal{F}$ are reduced*, i.e. the linearization of the vector field at a singular point has at least a non-zero eigenvalue. A result by Seidenberg implies that this situation can be achieved after finitely many monoidal transformations. We extract next the following important consequences from the classification theory of foliations with reduced singularities in dimension two [28].

(s_1) *A singular point x_0 of $\mathcal{F}$ is called non-degenerate if we have*

$$C^{-1} \leq \frac{|a_{\alpha 1}(z,w)|^2 + |a_{\alpha 2}(z,w)|^2}{|z|^2 + |w|^2} \leq C$$

for some coordinate system (z, w) centered at x_0.

(s_2) *A singular point x_1 of $\mathcal{F}$ is called degenerate if we have*

$$C^{-1} \leq \frac{|a_{\alpha 1}(z,w)|^2 + |a_{\alpha 2}(z,w)|^2}{|z|^2 + |w|^{2k}} \leq C$$

for some coordinate system (z, w) centered at x_1, where $k \geq 2$ is an integer.

(s_3) *Any singular point of $\mathcal{F}$ is either non-degenerate or degenerate.*

(s_4) *For any blow-up $p : \widehat{X} \to X$ of a point x_0 on the surface X we denote by $\widehat{\mathcal{F}} := p^{-1}(\mathcal{F})$ the induced foliation on $\widehat{X}$. Then we have*

$$p^\star T_{\mathcal{F}} = T_{\widehat{\mathcal{F}}}$$

and moreover, the foliations $\mathcal{F}$ and $\widehat{\mathcal{F}}$ have the same number of degenerate singular points. In addition, the number "k" appearing in the inequality (s_2) is invariant.

For a proof of the preceding claims (s_1)–(s_3) we refer to the paper [28]. As for the property (s_4), we can verify it by an explicit computation, as follows.

Locally near the point x_0 the equations of the blow-up map p are given by

$$(x, y) \to (x, xy) \quad \text{or } (x, y) \to (xy, x)$$

corresponding to the two charts covering $\mathbb{P}^1$. Then the expression of the vector field defining the foliation $\widehat{\mathcal{F}}$ on the first chart is as follows

$$a_{\alpha 1}(x, xy)\frac{\partial}{\partial x} + \Big(\frac{a_{\alpha 2}(x, xy)}{x} - y\frac{a_{\alpha 1}(x, xy)}{x}\Big)\frac{\partial}{\partial y}.$$

We denote by $A_{\alpha 1}$ and $A_{\alpha 2}$ the coefficient of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in the expression above, respectively. They are holomorphic functions, and the set of their common zeroes is discrete. Indeed, this is clear if the singularity x_0 is non-degenerate. If x_0 is a degenerate singularity of $\mathcal{F}$, then this can be verified by an explicit computation, given the normal form [28]

$$v_\alpha = \big(z(1 + \tau w^k) + wF(z, w)\big)\frac{\partial}{\partial z} + w^{k+1}\frac{\partial}{\partial w},$$

where F is a function vanishing to order k, and τ is a complex number. In conclusion, the transition functions for the tangent bundle of $\widehat{\mathcal{F}}$ are the same as the ones corresponding to $\mathcal{F}$, modulo composition with the blow-up map p, so our statement is proved.

6.4.1. *Intersection with the tangent bundle.* In the context of foliations with reduced singularities, the lower bound obtained in Theorem 6.6 can be improved substantially, as follows.

Theorem 6.10. *Let $(X, \mathcal{F})$ be a non-singular compact complex surface, endowed with a foliation. Let $f : \mathcal{Y} \to X$ be a holomorphic map tangent to $\mathcal{F}$; here $(\mathcal{Y}, \sigma)$ is a parabolic Riemann surface not necessarily of finite Euler characteristic. If the singularities of $\mathcal{F}$ are reduced, then we have*

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) \geq 0. \tag{39}$$

Proof. If $\mathcal{Y} = \mathbb{C}$, then Theorem 6.10 is one of the key result established in [21]. The original arguments in this article can be adapted to the parabolic setting we are interested in, as we will sketch next.

• Let $x_0 \in \mathcal{F}_{\text{sing}}$ be a singular point of the foliation. We will assume for simplicity that p is non-degenerate; the general case is a little bit more complicated technically, but the main ideas are the same. We denote by $\pi : X_1 \to X$ the blow-up of X at x_0, and let E_1 be the corresponding exceptional divisor. Let $\mathcal{F}_1$ be the foliation $\pi^\star\mathcal{F}$ on X_1; then E_1 is an invariant curve. The foliation $\mathcal{F}_1$ has two singularities on E_1, say $\widehat{x}_1$ and $\widehat{y}_1$, both non-degenerate.

We repeat this procedure, and blow-up $\widehat{x}_1$ and $\widehat{y}_1$. On the surface X_2 obtained in this way, the inverse image of x_0 is equal to

$$B := \widehat{E}_1 + E_2 + E_3,$$

where $\widehat{E}_1$ is the proper transform of E_1 and E_2, E_3 are the exceptional divisors. In the new configuration, we have 4 singular points of the induced foliation $\mathcal{F}_2$. Two of them belong to $\widehat{E}_1 \cap E_2$ and $\widehat{E}_1 \cap E_3$, respectively, and we denote $\widehat{x}_2$ and $\widehat{y}_2$ the other ones.

We have the injection of sheaves

$$0 \to T_{\widehat{\mathcal{F}}} \to T_{\widehat{X}}\langle B\rangle \tag{40}$$

that is to say, the tangent bundle of $\mathcal{F}_2$ is a subsheaf of the logarithmic tangent bundle of (X, B). The metric on $T_{\widehat{\mathcal{F}}}$ induced by the morphism above is *non-singular* at each of the four singular points above, and moreover, the image of the lift of the transcendental curve to X_2 do not intersect $\widehat{E}_1 \cap E_2$ or $\widehat{E}_1 \cap E_3$. Indeed, these singularities of the foliation $\widehat{\mathcal{F}}$ have the property that both separatrices containing them are algebraic sets. So if one of these points belong to the image of the curve, then the curve is automatically contained in $\widehat{E}_1, E_2$ or E_3 (since they are leaves of the foliation).

Therefore, by Theorem 6.9 we do not have any negative contribution in $(\star_B)$ due to these two points; the only term we have to understand is

$$\nu_1^T(f, \widehat{x}_2) + \nu_1^T(f, \widehat{y}_2) \tag{41}$$

i.e. the truncated counting function corresponding to $\widehat{x}_2$ and $\widehat{y}_2$.

• Let $T[f]$ be a Nevanlinna current associated to a Zariski-dense parabolic curve on the surface X. Let $\pi : \widehat{X} \to X$ be the blow-up of X at x; we denote by $\widehat{f}$ the lift of f. Then there exists $T[\widehat{f}]$ a Nevanlinna current associated to $\widehat{f}$ such that we have

$$\pi^\star T[f] = T[\widehat{f}] + \rho[E] \tag{42}$$

where $\rho = \displaystyle\int_{\widehat{X}} T[\widehat{f}] \wedge c_1(E)$ is a positive number, in general smaller than the Lelong number of T at p. We remark that we have

$$\rho^2 + \int_{\widehat{X}} \{T[\widehat{f}]\}^2 = \int_X \{T[f]\}^2$$

where we denote by $\{T[\widehat{f}]\}$ the cohomology class of $T[\widehat{f}]$. Indeed we have $\displaystyle\int_{\widehat{X}} \pi^\star\{T[f]\}^2 = \int_{\widehat{X}} \big(\{T[\widehat{f}]\} + \rho c_1(E)\big)^2 = \int_{\widehat{X}} \{T[\widehat{f}]\}^2 - \rho^2 + 2\rho \int_{\widehat{X}} \{T[\widehat{f}]\} \wedge c_1(E)$, from which the above equality follows, because $\displaystyle\int_{\widehat{X}} \{T[\widehat{f}]\} \wedge c_1(E) = \rho$.

• If we iterate the blow-up procedure, the quantities ρ_j we obtain as in (42) verify

$$\sum_j \rho_j^2 \leq \int_X \{T[f]\}^2, \tag{43}$$

that is to say, the preceding sum is convergent.

• Let $p \in \mathcal{F}_{\rm sing}$ be a singular point of the foliation $\mathcal{F}$. We blow-up the points x_2 and y_2 and we obtain x_3 and y_3, plus two singular points at the intersection of rational curves. After iterating k times the blow-up procedure described above, the only negative factor we have to deal with is

$$-\nu_1^T(f, \widehat{x}_k) - \nu_1^T(f, \widehat{y}_k) \tag{44}$$

where X_k is the surface obtained after iterating k times the procedure described above, f_k is the induced parabolic curve and $\mathcal{F}_k$ is the induced folia tion. We emphasize that even if the number of singular points of the induced foliation has increased, the corresponding negative terms we have to take into account in $(\star_B)$ remains the same, i.e. the algebraic intersection of the lift of f with the two "extremal" singular points. By using the notations in (44) above, the convergence of the sum (43) implies that we have

$$\sum_{k\geq 1} \nu_1^T(f_k, \widehat{x}_k)^2 + \nu_1^T(f_k, \widehat{y}_k)^2 < \infty$$

since $\nu_1^T(f_k, \widehat{x}_k) = \displaystyle\int_{X_k} T[f_k] \wedge c_1(E_k)$. Hence the algebraic multiplicity term (44) tends to zero as $k \to \infty$.

• We therefore have, by using (38)

$$\int_{X_k} T[f_k] \wedge c_1(T_{\mathcal{F}_k}) \geq -\varepsilon_k$$

for $\varepsilon_k \to 0$ as $k \to \infty$. Since the singularities of $\mathcal{F}$ are reduced, we use the property s_4 and we have

$$\int_{X_k} T[f_k] \wedge c_1(T_{\mathcal{F}_k}) = \int_{X_k} T[f_k] \wedge \pi_k^\star c_1(T_{\mathcal{F}})$$

and this last term is simply

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}})$$

by the projection formula. Hence it is non-negative. □

The theorem above is particularly interesting when coupled with the following very special case of a result due to Y. Miyaoka, cf. [20]

Theorem 6.11. *Let X be a projective surface, whose canonical bundle K_X is big. Let $L \to X$ be a line bundle such that $H^0\big(X, S^m T_X \otimes L\big) \neq 0$. Then L is pseudo-effective.*

We apply this result for $L = T^\star_{\mathcal{F}}$, so we infer that $T^\star_{\mathcal{F}}$ is pseudo-effective (indeed, the bundle $T_X \otimes T^\star_{\mathcal{F}}$ has a non-trivial section).

Combined with Theorem 6.10, we obtain

$$\int_X T[f] \wedge c_1(T_{\mathcal{F}}) = 0. \tag{45}$$

We derive the following consequence.

Theorem 6.12. *We consider the data $(X, \mathcal{F})$ and $f : \mathcal{Y} \to X$ as in Theorem 6.10; in addition, we assume that K_X is big. Then we have*

$$\int_X \{T[f]\}^2 = 0. \tag{46}$$

If R denotes the diffuse part of $T[f]$, then we have $\int_X \{R\}^2 = 0$. In particular, since $T[f]$ is already nef, we infer that $\nu(R, x) = 0$ at each point $x \in X$.

Proof. If the equality (46) does not hold, we remark that the class $\{T[f]\}$ contains a Kähler current, i.e. there exists $S \in \{T[f]\}$ such that $S \geq \delta\omega$ for some positive constant δ. Unfortunately we cannot use this representative directly in order to conclude that the intersection number $\int_X T[f] \wedge c_1(T_{\mathcal{F}})$ is strictly negative (and obtain in this way a contradiction), because of the possible singularities of S and of the positively defined representatives of $c_1(T_{\mathcal{F}})$.

We will proceed therefore in a different manner, and show next that if the conclusion of our theorem does not hold, then we have $\int_X T[f] \wedge c_1(T_{\mathcal{F}}) < 0$, contradicting the relation (45).
We will only discuss here the case K_X ample; the general case (i.e. K_X big) is obtained in a similar manner.

Let ω be a metric on X, such that $\mathrm{Ricci}_\omega \leq -\varepsilon_0\omega$. We have a sequence of Kähler classes $(\alpha_k)_{k\geq 1}$ whose limit is $\{T[f]\}$.
We recall the following particular case of Yau's theorem [37]

Theorem 6.13. [37] *Let α be a Kähler class on a compact complex surface X. Let ω be a Kähler form on X such that*

$$\int_X \alpha^2 = \int_X \omega^2.$$

Then there exists a Kähler form $\omega_1 \in \alpha$ such that the equality

$$\omega_1^2 = \omega^2$$

holds at each point of X.

We infer that there exists a sequence of Kähler metrics $(\omega_k)_{k\geq 1}$, such that $\omega_k \in \alpha_k$ for each $k \geq 1$, and such that we have

$$\omega_k^2 = \lambda_k \omega^2.$$

The sequence of normalizing constants λ_k is bounded from above, and also from below *away from zero*, by our assumption $\int_X \{T[f]\}^2 > 0$.

We recall now a basic fact from complex differential geometry. Let (E, h) be a Hermitian vector bundle, and let ξ be a holomorphic section of E. Then we have

$$\sqrt{-1}\partial\overline{\partial}\log|\xi|_h^2 \geq -\frac{\langle\Theta_h(E)\xi, \xi\rangle}{|\xi|_h^2}$$

where the curvature form $\Theta_h(E)$ corresponds to the Chern connection of (E, h). We detail here the computation showing the previous inequality.

$$\overline{\partial}\log|\xi|_h^2 = \frac{\langle\xi, D'\xi\rangle}{|\xi|_h^2}$$

where D' is the (1,0) part of the Chern connection. It follows that

$$\partial\overline{\partial}\log|\xi|_h^2 = \frac{\langle D'\xi, D'\xi\rangle}{|\xi|_h^2} - \frac{\langle D'\xi, \xi\rangle \wedge \langle\xi, D'\xi\rangle}{|\xi|_h^4} + \frac{\langle\overline{\partial}D'\xi, \xi\rangle}{|\xi|_h^2}.$$

By Legendre inequality we know that

$$\sqrt{-1}\frac{\langle D'\xi, D'\xi\rangle}{|\xi|_h^2} - \sqrt{-1}\frac{\langle D'\xi, \xi\rangle \wedge \langle\xi, D'\xi\rangle}{|\xi|_h^4} \geq 0.$$

As for the remaining term, we have the equality

$$\overline{\partial}D'\xi = -\Theta_h(E)\xi$$

so the inequality above is established.

We apply this for V the section of $T_X \otimes T_{\mathcal{F}}^\star$ which defined the foliation. The formula above gives

$$\sqrt{-1}\partial\overline{\partial}\log|V|_k^2 \geq \Theta_h(T_{\mathcal{F}}) - \frac{\langle\Theta_{\omega_k}(T_X)V, V\rangle}{|V|_k^2}$$

where we denote by $|V|_k$ the norm of V measured with respect to ω_k and an arbitrary metric h on $T_{\mathcal{F}}$.

By considering the wedge product with ω_k and integrating over X the above inequality we obtain

$$\int_X \omega_k \wedge c_1(T_{\mathcal{F}}) \leq \int_X \frac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}\omega_k^2.$$

We remark that the expression $\frac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}$ under the integral sign is well-defined, even if V is only a vector field with values in $T_{\mathcal{F}}^\star$. Given the Monge-Ampère equation satisfied by ω_k we have

$$\int_X \frac{\mathrm{Ricci}_{\omega_k}(V, \overline{V})}{|V|_{\omega_k}^2}\omega_k^2 = \lambda_k \int_X \frac{\mathrm{Ricci}_{\omega}(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2.$$

Since Ricci_ω is negative definite, we have

$$\int_X \frac{\mathrm{Ricci}_{\omega}(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2 \leq \int_{U_k} \frac{\mathrm{Ricci}_{\omega}(V, \overline{V})}{|V|_{\omega_k}^2}\omega^2$$

for any open set $U_k \subset X$.

We have $\displaystyle\int_X \omega_k \wedge \omega \leq C$ for some constant C independent of k. So there exists an open set U_k of large volume (with respect to ω), on which the trace of ω_k with respect to ω is bounded uniformly with respect to k, i.e. there exists a constant C_1 such that

$$\omega_k|_{U_k} \leq C_1 \omega|_{U_k}$$

for any $k \geq 1$. With this choice of U_k, we will have

$$\int_X \alpha_k \wedge c_1(T_{\mathcal{F}}) \leq \int_{U_k} \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|^2_{\omega_k}} \omega^2 \leq C_2 \int_{U_k} \frac{\mathrm{Ricci}_\omega(V, \overline{V})}{|V|^2_{\omega}} \omega^2 < -\delta$$

for some strictly positive quantity δ, and the first part of Theorem 6.12 is established by taking $k \to \infty$.

We write next the Siu decomposition of $T[f]$

$$T[f] = \sum_j \tau_j [C_j] + R \tag{46}$$

and we remark that the class of the current R is nef. Indeed, using Demailly approximation theorem we have $R = \lim R_\varepsilon$, where for each $\varepsilon > 0$ the current R_ε is closed, positive, and non-singular in the complement of a finite set of points. Hence R_ε^2 is well-defined and positive as a current; in particular, $\int_X \{R\}^2 \geq 0$. Then we have $\displaystyle\int_X \{T[f]\}^2 \geq \int_X \{R\}^2$, and it follows that $\int_X \{R\}^2 = 0$. Thus using a local computation with potential of the current R we infer that

$$\nu(R, x) = 0 \tag{47}$$

for any $x \in X$, i.e. the Lelong number of R at each point of X is equal to zero. □

6.4.2. *Intersection with the normal bundle,* I. Our aim in this part is to evaluate the degree of the curve f on the normal bundle of the foliation $\mathcal{F}$. To this end we follow [7], [21] up to a certain point; their approach relies on the Baum-Bott formula [2], which is what we will survey next.

Actually, we will first give the precise expression of a representative of the Chern class

$$c_1\big(N_{\mathcal{F},B}\big) \tag{48}$$

in arbitrary dimension, where $B = B_1 + \cdots + B_N$ is a divisor on X, such that (X, B) verifies the conditions (a) and (b) in subsection 6.2. The vector bundle in (48) is defined by the exact sequence

$$0 \to T_{\mathcal{F}} \to T_X\langle B\rangle \to N_{\mathcal{F},B} \to 0. \tag{49}$$

We remark that in the case of surfaces, this bundle equals $N_{\mathcal{F}}\otimes\mathcal{O}(-B)$.

Let $U_\alpha \subset X$ be an open coordinate set, and let $(z_1, \dots, z_n)$ be a coordinate system on U_α. We assume that

$$\mathrm{Supp}(B) \cap U_\alpha = (z_1 \dots z_p = 0).$$

Since B is invariant by $\mathcal{F}$, the vector field giving the local trivialization of $T_{\mathcal{F}}$ is written as

$$v_\alpha = \sum_{j=1}^{p} z_j F_{j\alpha}(z)\frac{\partial}{\partial z_j} + \sum_{i=p+1}^{n} F_{i\alpha}(z)\frac{\partial}{\partial z_i} \tag{50}$$

where $F_{j\alpha}$ are holomorphic functions defined on U_α.

With respect to the coordinates (z) on U_α, the canonical bundle associated to (X, B) is locally trivialized by

$$\Omega_\alpha := \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \dots \wedge dz_n, \tag{51}$$

and therefore we obtain a local trivialization of the bundle $\det\left(N^\star_{\mathcal{F},B}\right)$ by contracting the form (51) with the vector field (50), i.e.

$$\omega_\alpha := \Lambda_{v_\alpha}(\Omega_\alpha), \tag{52}$$

that is to say

$$\omega_\alpha = \sum_{j=1}^{p}(-1)^{j+1} F_{j\alpha}(z)\frac{dz_1}{z_1} \wedge \cdots \widehat{\frac{dz_j}{z_j}} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \dots \wedge dz_n$$
$$+ \sum_{i=p+1}^{n}(-1)^{i+1} F_{i\alpha}(z)\frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge dz_{p+1} \wedge \dots \wedge \widehat{dz_i} \wedge \cdots dz_n$$

We define the differential 1-form ξ_α on U_α by the formula

$$\xi_{\alpha,\varepsilon} := \frac{F^{(1)}_\alpha}{\|G_\alpha\|^2} \sum_j \overline{G}_{j\alpha} dz_j \tag{53}$$

where we have used the following notations. If the index j is smaller than p, then $G_{j\alpha} := z_j F_{j\alpha}$, and if $j > p$, then we define $G_{j\alpha} := F_{j\alpha}$. Also, we write

$$F^{(1)}_\alpha(z) := \sum_{j=1}^{p} z_j F_{j\alpha, z_j}(z) + \sum_{i=p+1}^{n} F_{i\alpha, z_i}(z)$$

where $F_{k\alpha, z_r}$ is the partial derivative of $F_{k\alpha}$ with respect to z_r, and

$$\|G_\alpha\|^2 := \sum_i |G_{i\alpha}|^2.$$

Then we have the equality

$$d\omega_\alpha = \xi_\alpha \wedge \omega_\alpha \tag{54}$$

as one can check by a direct computation.

Let α, β be two indexes such that $U_\alpha \cap U_\beta \neq \emptyset$. Then we have

$$\omega_\alpha = g_{\alpha\beta}\omega_\beta \tag{55}$$

on $U_\alpha \cap U_\beta$; here $g_{\alpha\beta}$ are the transition functions for the determinant bundle of $N^\star_{\mathcal{F}}(B)$. By differentiating the relation (55), we obtain

$$d\omega_\alpha = d\log g_{\alpha\beta} \wedge \omega_\alpha + g_{\alpha\beta} d\omega_\beta \tag{56}$$

hence from the relation (54) we get

$$\big(\xi_\alpha - \xi_\beta - d\log g_{\alpha\beta}\big) \wedge \omega_\alpha = 0 \tag{57}$$

on $U_\alpha \cap U_\beta$.

For the rest of this section, we assume that the singularities of $\mathcal{F}$ are *isolated and non-degenerate.*

Then we can find a family of non-singular 1-forms γ_α so that we have

$$\xi_\alpha - \xi_\beta - d\log g_{\alpha\beta} = \gamma_\alpha - \gamma_\beta; \tag{58}$$

for example, we can take

$$\gamma_\alpha := \sum_\beta \rho_\beta \big(\xi_\alpha - \xi_\beta - d\log g_{\alpha\beta}\big), \tag{59}$$

where (ρ_α) is a partition of unit subordinate to the covering (U_α). Here we assume that each U_α contains at most one singular point of $\mathcal{F}$. Then the global 2-form whose restriction to U_α is

$$\Theta|_{U\alpha} := \frac{1}{2\pi\sqrt{-1}} d(\xi_\alpha - \gamma_\alpha) \tag{60}$$

represents the first Chern class of $N_{\mathcal{F},B}$.

Let T be a closed positive current of type $(n-1, n-1)$ on X; we assume that T is *diffuse and directed by the foliation* $\mathcal{F}$. This implies that for each index α there exists a positive measure τ_α on U_α such that

$$T|_{U_\alpha} = \tau_\alpha \omega_\alpha \wedge \overline{\omega}_\alpha. \tag{61}$$

We intend to use the representative (60) in order to evaluate the degree of the current T on the normal bundle of the foliation. Prior to this, we have to regularize the forms ξ_α. This is done simply by replacing ξ_α with $h_\alpha \xi_\alpha$, where h_α is equal to 0 in a open set containing the singularity of $\mathcal{F}$, and it is equal to one out of a slightly bigger open set. A specific choice of such functions will be made shortly; we remark that the effect of the multiplication with h_α is that the equality (54) is only verified in the complement of an open set containing the singular point. The formula for the curvature becomes $\Theta|_{U\alpha} := \dfrac{1}{2\pi\sqrt{-1}} d(h_\alpha \xi_\alpha - \gamma_\alpha)$.

As a consequence, we have the next statement, which is the main result of this subsection.

Lemma 6.14. *Let $\mathcal{F}$ be a foliation by curves with isolated singularities on X. We consider a closed positive current T of bidimension (1,1), which is directed by the foliation $\mathcal{F}$. If $\mathcal{F}_{\mathrm{sing}} \cap \mathrm{Supp}(T)$ consists in non-degenerate points only, then we have*

$$\int_X T \wedge \Theta \geq -C \sup_{x \in \mathcal{F}_{\mathrm{sing}}} \nu(T, x) \tag{62}$$

for some positive constant C.

Proof. We first observe that we have

$$d\gamma_\alpha \wedge T = 0. \tag{63}$$

Indeed, ω_α is holomorphic and γ_α is a (1,0) form, so we have

$$d\gamma_\alpha \wedge T = \tau_\alpha \overline{\partial}(\gamma_\alpha \wedge \omega_\alpha) \wedge \overline{\omega}_\alpha. \tag{64}$$

The equality (63) is therefore a consequence of (57)-(59).

In order to compute the other term of (62), we use the relation (54), and we infer that we have

$$\overline{\partial}(h_\alpha \xi_\alpha) \wedge \omega_\alpha = \overline{\partial} h_\alpha \wedge \xi_\alpha \wedge \omega_\alpha \tag{65}$$

for any choice of h_α.

In what follows, we will drop the index α, and concentrate around a single singular point namely the origin of the coordinate system. Let χ be a smooth function vanishing near zero, and which equals 1 for $x \geq 1/2$. We consider

$$h_r(z) := \chi\big(\|z\|^r\big) \tag{66}$$

for each $r > 0$. We have then $0 \leq h_r \leq 1$, and as $r \to 0$, h_r converges to the the characteristic function of $U \setminus 0$.
If moreover the singular point $z = 0$ is non-degenerate, in the sense that we have

$$C^{-1} < \frac{\|G(z)\|}{\|z\|} < C$$

for some constant $C > 0$, then we have to bound from above the mass of the measure

$$r \frac{\chi'\big(\|z\|^r\big)}{\|z\|^{2-r}} \sigma_T$$

as $r \to 0$; here σ_T is the trace measure of T. In other words we have to obtain an upper bound for the integral

$$r \int_{C_1 < \|z\|^r < C_2} \|z\|^{r-2} \sigma_T. \tag{67}$$

We observe that we have

$$\int_{C_1 < \|z\|^r < C_2} \|z\|^{r-2} \sigma_T = \int_0^\infty \sigma_T\Big(\big(\|z\|^{r-2} > s\big) \cap \big(C_1 < \|z\|^r < C_2\big)\Big) ds;$$

and up to a $\mathcal{O}(r)$ term, the quantity we have to evaluate is smaller than

$$r\int_1^{C_1^{-\frac{2-r}{r}}} \sigma_T\big(\|z\| < s^{\frac{1}{r-2}}\big)ds. \tag{68}$$

If ν is Lelong number of T at 0, then we have $\sigma_T\big(\|z\| < \tau\big) \leq 2\nu\tau^2$, as soon as τ is small enough. Therefore, a quick computation shows that the integral (68) is bounded by $C\nu$, for some positive constant $C > 0$. □

As one can see from the proof, this kind of arguments can only be used if the singular point is non-degenerate. However, this is sufficient for our purposes, given the next result.

Theorem 6.15. *Let X be a projective surface endowed with a holomorphic foliation $\mathcal{F}$. We assume that the singularities of $\mathcal{F}$ are reduced. Let R be a diffuse closed positive (1,1) current on X, directed by $\mathcal{F}$. Let $\{x_1, \dots x_d\}$ be the set of degenerate singularities of $\mathcal{F}$. Then we have*

$$\mathrm{Supp}(R) \subset X \setminus \{x_1, \dots x_d\}.$$

Proof. Suppose $(0,0)$ is a degenerate reduced singularity of $\mathcal{F}$. It is well-known that the holonomy map h at such a point is tangent to identity. We assume that the separatrix is given by the z-axis, and that the w-axis is transversal to $\mathcal{F}$.

The argument is based on the fact that the dynamics of such a map near the origin is well-understood. It is the Léau "flower theorem". There is a neighborhood of zero, say U, such that

$$U \setminus 0 = \bigcup_{1\leq j\leq p} P_j^+ \cup P_j^-.$$

If the point z belongs to the petals P_j^+, then we have $h^n(z) \to 0$, where we denote by h^n the n^{th} iteration of h. If z belongs to the petals P_j^-, then we have $h^{-n}(z) \to 0$.

Suppose that the current R has mass in a small open Ξ set near 0, which can be assumed to have the form $\Xi = P_1^+ \times \Delta$, where Δ is a disk. Let $x_0 := (0, w_0)$ be a point in $U \cap \mathrm{Supp}(R)$, such that $w_0 \neq 0$.

The current R can be written on Ξ as

$$R = \int_{P_1^+} [V_w] d\mu(w),$$

where V_w is the plaque through w.

We show now that μ has no mass out of the origin. Let $W \subset P_1^+$ be an open set containing w_0, such that $W \cap (w = 0) = \emptyset$. We construct a new open set $W_1 \subset P_1^+$ via the holonomy map, as follows. Without loss of generality we can assume that the petal P_1^+ is invariant by h.

We define $W_1 := h^{n_1}(W)$ where n_1 is large enough in order to have $W_1 \cap W = \emptyset$.

We can restart this procedure with W_1, obtaining $W_2, \ldots, W_k$. In conclusion, we get a sequence of open sets $(W_k) \subset P_1^+$ such that $W_i \cap W_j = \emptyset$ if $i \neq j$. The mass of each set with respect to the transverse measure is preserved, since the measure is invariant by the holonomy map. This is equivalent to the fact that the current R is closed and directed by $\mathcal{F}$. Therefore we obtain a contradiction. □

In conclusion, we obtain the next result.

Theorem 6.16. *Let X be a projective surface endowed with a holomorphic foliation $\mathcal{F}$. We assume that the singularities of $\mathcal{F}$ are reduced. Let R be a diffuse closed positive (1,1) current on X, directed by $\mathcal{F}$. Then there exists a constant $C > 0$ such that*

$$\int_X R \wedge c_1\big(N_{\mathcal{F}}(B)\big) \geq -C \sup_{x \in \mathcal{F}_{\text{sing}}} \nu(R, x). \tag{69}$$

In particular, if we have $\nu(R, x) = 0$ for every $x \in \mathcal{F}_{\text{sing}}$, then we have $\int_X R \wedge c_1\big(N_{\mathcal{F}}(B)\big) \geq 0$.

Proof. By Theorem 6.15, R has no mass near degenerate points. In the case of a non-degenerate singularity, we use Lemma 6.12, since Θ is a representative of $c_1(N_{\mathcal{F}}(-B))$. The theorem is thus proved. □

As a conclusion of the considerations in the preceding subsections, we infer the following "parabolic version" of the result in [21].

Corollary 6.17. *Let $(X, \mathcal{F})$ be a surface of general type endowed with a holomorphic foliation $\mathcal{F}$, and let $f : \mathcal{Y} \to X$ be a holomorphic curve, where $\mathcal{Y}$ is a parabolic Riemann surface of finite Euler characteristic. If f is tangent to $\mathcal{F}$, then the dimension of the Zariski closure of the image of f is at most 1.*

Proof. The first thing to remark is that the hypothesis is stable by blow-up, hence we can assume that the singularities of $\mathcal{F}$ are reduced. We can also assume that the current associated to f can be written as

$$T[f] = \sum_j \nu_j [C_j] + R$$

where $C := C_1 + \ldots C_N$ verifies (a) and (b), and the Lelong numbers of R may be positive eventually at a finite subset of X. We argue by contradiction, so we add the hypothesis that the image of f is Zariski dense.

If so, by Theorem 6.10 we infer that $\displaystyle\int_X T[f] \wedge c_1(T_{\mathcal{F}}) = 0$, as well as $\displaystyle\int_X \{T[f]\}^2 = 0$. This implies that the Lelong numbers of the diffuse part R of $T[f]$ are equal to zero; by Theorem 6.16, we infer that

$$\int_X R \wedge c_1(N_{\mathcal{F}}(-C)) \geq 0.$$

The next step is to show the inequality

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq \int_X R \wedge c_1(N_{\mathcal{F}}(-C)).$$

We follow the presentation in [7], and we first show that we have $\int_X (T[f] - R) \wedge c_1(N_{\mathcal{F}}) \geq \int_X (T[f] - R) \wedge C$. We recall the formula

$$\int_{C_j} c_1(N_{\mathcal{F}}) = \int_X c_1(C_j)^2 + Z(C_j, \mathcal{F})$$

where $Z(C_j, \mathcal{F})$ is the multiplicity of the singularities of $\mathcal{F}$ along the curve C_j. We clearly have

$$Z(C_j, \mathcal{F}) \geq \sum_{j \neq k} c_1(C_j) \wedge c_1(C_k)$$

and thus we obtain

$$\sum_j \int_X \nu_j c_1(C_j) \wedge c_1(N_{\mathcal{F}}) \geq \sum_{j,k} \int_X \nu_j c_1(C_j) \wedge c_1(C_k).$$

The preceding inequality implies that

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq \int_X R \wedge c_1(N_{\mathcal{F}}) + \int_X (T[f] - R) \wedge c_1(C)$$

and given that $T[f]$ is nef, we infer that

$$\int_X R \wedge c_1(N_{\mathcal{F}}) + \int_X (T[f] - R) \wedge c_1(C) \geq \int_X R \wedge c_1(N_{\mathcal{F}}(-C)).$$

As a consequence we obtain

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0.$$

Since $c_1(N_{\mathcal{F}}) + c_1(T_{\mathcal{F}}) = c_1(X)$, the inequalities above imply that $\int_X T[f] \wedge c_1(K_X) \leq 0$. This is absurd since K_X is big and $T[f]$ is nef. □

7. Brunella index theorem

We will survey here a large part of the proof of the following beautiful result, due to M. Brunella [7]. The context is as follows: X is a compact complex surface, endowed with a holomorphic foliation $\mathcal{F}$ and f is a parabolic curve tangent to $\mathcal{F}$.

Theorem 7.1. [7] *If the curve $f : \mathcal{Y} \to X$ has a Zariski dense image, then we have*

$$\int_X T[f] \wedge c_1(N_{\mathcal{F}}) \geq 0. \tag{70}$$

We remark that a similar statement appeared in the preceding subsection, but it was obtained in a very indirect way, under the assumption that the canonical bundle of X is big (and as part of an argument by contradiction...).

M. Brunella's arguments involve many considerations from dynamics, including a study at Siegel points of $\mathcal{F}_{\rm sing}$. They seem difficult to accommodate to higher dimensional case; in the next lemma we treat the Siegel points by using the following elegant lemma for which we are indebted to M. McQuillan, cf. [22].

Lemma 7.2. [22] *Let T be a closed positive current on a surface X. We assume that T is diffuse and directed by a foliation $\mathcal{F}$. Let $x_0 \in \mathcal{F}_{\rm sing}$ be a singular point of $\mathcal{F}$. We assume that there exists a coordinate open set $U \subset X$ centered at x_0 and local coordinates (z, w) on U such that the restriction $\mathcal{F}|_U$ is given by $adw - bdz = 0$, where*

$$a(z, w) = z(1 + o(z, w)), \quad b(z, w) = \lambda w(1 + o(z, w))$$

where $\lambda \in \mathbb{R}$ is a strictly negative real number. Then $\nu(T, x_0) = 0$, i.e. the Lelong number of T at x_0 vanishes.

Proof. Let $\varepsilon > 0$ be a positive number, and let Ω_ε be the open set $(|z| < \varepsilon) \times (|w| < \varepsilon) \subset \mathbb{C}^2$. We denote by $\partial\Omega_\varepsilon$ the boundary of this domain. We proceed in two steps, as follows.

(I) There exists a constant $C > 0$ such that

$$\nu(T, x_0) \leq C \int_{\partial\Omega_\varepsilon} d^c \log |zw|^2 \wedge T.$$

(II) We have $\displaystyle\lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon} d^c \log |zw|^2 \wedge T = 0.$

The lemma follows from the two statements (I) and (II) above; we will show that they hold true in what follows.

Proof of (I). Let $\varepsilon > \delta > 0$ be two positive real numbers; we have

$$\begin{aligned}
\nu(T, x_0) \leq & \int_{|z|^2+|w|^2<\delta^2} dd^c \log\left(|z|^2 + |w|^2\right) \wedge T \\
=& \frac{1}{\delta^2} \int_{|z|^2+|w|^2<\delta^2} dd^c \left(|z|^2 + |w|^2\right) \wedge T \\
\leq& \frac{1}{\delta^2} \int_{|z|<\delta, |w|<\delta} dd^c \left(|z|^2 + |w|^2\right) \wedge T \\
=& \frac{1}{\delta^2} \left(\int_{|z|=\delta, |w|<\delta} d^c|z|^2 \wedge T + \int_{|z|<\delta, |w|=\delta} d^c|w|^2 \wedge T \right) \\
+& \frac{1}{\delta^2} \left(\int_{|z|=\delta, |w|<\delta} d^c|w|^2 \wedge T + \int_{|z|<\delta, |w|=\delta} d^c|z|^2 \wedge T \right).
\end{aligned}$$

By using the relation

$$(\star) \qquad d^c|w|^2 \wedge T = |w|^2(\lambda^{-1} + o(z, w)) d^c \log|z|^2 \wedge T$$

(which is a consequence of the hypothesis), we have

$$\int_{|z|=\delta, |w|<\delta} d^c|w|^2 \wedge T = \int_{|z|=\delta, |w|<\delta} |w|^2(\lambda^{-1} + o(z, w)) d^c \log|z|^2 \wedge T$$

so in particular the limit as $\delta \to 0$ of the term

$$\frac{1}{\delta^2} \left(\int_{|z|=\delta, |w|<\delta} d^c|w|^2 \wedge T + \int_{|z|<\delta, |w|=\delta} d^c|z|^2 \wedge T \right)$$

exists, and it is bounded by the limit of

$$I_\delta := \frac{1}{\delta^2} \left(\int_{|z|=\delta, |w|<\delta} d^c|z|^2 \wedge T + \int_{|z|<\delta, |w|=\delta} d^c|w|^2 \wedge T \right)$$

as $\delta \to 0$. We have

$$\begin{aligned}
I_\delta =& \int_{|z|=\delta, |w|<\delta} d^c \log|z|^2 \wedge T + \int_{|z|<\delta, |w|=\delta} d^c \log|w|^2 \wedge T \\
\leq& \int_{|z|=\delta, |w|<\varepsilon} d^c \log|z|^2 \wedge T + \int_{|z|<\varepsilon, |w|=\delta} d^c \log|w|^2 \wedge T \\
:=& I_\delta(\varepsilon, 1) + I_\delta(\varepsilon, 2).
\end{aligned}$$

By Stokes theorem we obtain

$$I_\delta(\varepsilon, 1) = \int_{|z|=\varepsilon, |w|<\varepsilon} d^c \log|z|^2 \wedge T + \int_{\delta<|z|<\varepsilon, |w|=\varepsilon} d^c \log|z|^2 \wedge T$$

and thus we infer that

$$\lim_{\delta \to 0} \left(I_\delta(\varepsilon, 1) + I_\delta(\varepsilon, 2)\right) \leq C \int_{\partial\Omega_\varepsilon} d^c \log|zw|^2 \wedge T.$$

The proof of the point (I) is finished by combining the relations above.

Proof of (II). We use the relation

$$\frac{dw}{w} \wedge T = (\lambda + O(z,w))\frac{dz}{z} \wedge T$$

and we obtain

$$\int_{\partial\Omega_\varepsilon} d^c \log |w|^2 \wedge T = \lambda \int_{\partial\Omega_\varepsilon} d^c \log |z|^2 \wedge T + O(\varepsilon)$$

so we have

$$\int_{\partial\Omega_\varepsilon} d^c \log |zw|^2 \wedge T = \lambda \int_{\partial\Omega_\varepsilon} d^c \log |z|^2 \wedge T + \lambda^{-1} \int_{\partial\Omega_\varepsilon} d^c \log |w|^2 \wedge T + O(\varepsilon).$$

The conclusion follows, since λ is a negative real number and the left hand side term of the preceding equality is positive. □

For the rest of the proof of Theorem 7.1 we refer to [7]: it consists in a case-by-case analysis, as follows. By the proof of Corollary 6.17 it is enough to show that we have

$$\int_X R \wedge c_1\left(N_{\mathcal{F}}(-B)\right) \geq 0,$$

where R is the diffuse part of $T[f]$. Theorem 6.16 shows that this quantity is bounded from below by the Lelong numbers of R at the singular points of the foliation $\mathcal{F}$. If $x_0 \in \mathcal{F}_{\text{sing}}$ is a linearizable singularity, then his contribution to the quantity $\int_X R \wedge c_1\left(N_{\mathcal{F}}(B)\right)$ can be seen to be positive by a direct computation. If x_0 is degenerate, then we have nothing to do, thanks to Theorem 6.15. The remaining case follows from Lemma 7.2: since the Lelong number is equal to zero, the result follows. □

We will establish here a generalization of of Theorem 7.1 in the context of symmetric differentials.

Corollary 7.3. *Let L be a line bundle on X, such that*

$$H^0(X, S^m T_X^\star \otimes L) \neq 0.$$

Then we have

$$\int_X T[f] \wedge c_1(L) \geq 0. \tag{71}$$

Proof. Observe that Brunella's result corresponds to the case $m = 1$ and $L = N_{\mathcal{F}}$. Indeed, if the inequality (71) above does not hold, then given a section u of $S^m T_X^\star \otimes L$ we have $u(f'^{\otimes m}) \equiv 0$, by the vanishing theorem. Let $\Gamma \subset \mathbb{P}(T_X)$ be the set of zeros of u, interpreted as a section of the bundle $\mathcal{O}(m) \otimes \pi^\star(L)$, where $\pi : \mathbb{P}(T_X) \to X$ is the natural projection. The natural lift of the curve f is contained in one of the irreducible components of Γ; let X_m be a desingularization of the component containing the image of f.

In conclusion, we have a projective surface X_m, endowed with a holomorphic foliation $\mathcal{F}_m$, a generically finite morphism $p : X_m \to X$, and a curve $f_m : \mathcal{Y} \to X_m$, such that $f = p \circ f_m$.

Let $u_m := p^\star(u)$ be the section of $S^m T^\star_{X_m} \otimes p^\star(L)$ obtained by taking the inverse image of u. We decompose the set $Z_m := (u_m = 0) \subset \mathbb{P}(T_{X_m})$ as follows

$$Z_m = \sum_j m_j \Gamma_j \tag{72}$$

where $\Gamma_j \subset \mathbb{P}(T_{X_m})$ are irreducible hypersurfaces, and m_j are positive integers.

The image of the lift of f_m to $\mathbb{P}(T_{X_m})$ is contained in Z_m, and it is equally contained in the graph of $\mathcal{F}_m$. Since the curve f_m is supposed to be Zariski dense, its lift can be contained in at most one hypersurface Γ_j. We will henceforth assume that the graph of the foliation $\mathcal{F}_m$ coincides with Γ_1.

Numerically, we have

$$\Gamma_j \equiv \nu_j \mathcal{O}(1) + \pi_m^\star(L_j) \tag{73}$$

where $\pi_m : \mathbb{P}(T_{X_m}) \to X_m$ is the projection, ν_j are positive integers, and L_j are line bundles on X_m. This is a consequence of the structure of the Picard group of $\mathbb{P}(T_{X_m})$.

The relations (72), (73) show that we have

$$p^\star(L) \equiv \sum m_j L_j.$$

Since the lift of the curve f_m is not contained in any Γ_j for $j \geq 2$, we have $\displaystyle\int_X T[f_m'] \wedge c_1(\Gamma_j) \geq 0$. The tautological inequality 6.8 states that $\displaystyle\int_X T[f_m'] \wedge c_1\big(\mathcal{O}(1)\big) \leq 0$ so we obtain from (73)

$$\int_{X_m} T[f_m] \wedge c_1(L_j) \geq 0 \tag{74}$$

for each index $j \geq 2$.

But we have assumed that (71) does not hold, so we infer that

$$\int_{X_m} T[f_m] \wedge c_1(L_1) < 0 \tag{75}$$

which contradicts Theorem 7.1 (because L_1 is the normal bundle of $\mathcal{F}_m$). □

In particular, this gives a proof of the Green-Griffiths conjecture for minimal general type surfaces X such that $c_1^2(X) > c_2(X)$ without using any consideration about the tangent bundles of foliations! Indeed, under this hypothesis we have

$$H^0(X, S^m T_X^\star \otimes A^{-1}) \neq 0$$

where A is ample, and $m \gg 0$ is a positive integer (by a result due to F. Bogomolov, [3]; it is at this point that we are using the hypothesis about the Chern classes of X). So if f is Zariski dense, we obtain a contradiction by the corollary above, given the strict positivity of A.

Given the aforementioned consequence of Brunella's theorem, it is very tempting to formulate the following problem.

Conjecture 7.4. *Let k, m be two positive integers, and let $L \to X$ be a line bundle, such that*

$$H^0\big(X, E_{k,m}T^\star_X \otimes L\big) \neq 0.$$

If $f : \mathcal{Y} \to X$ is a Zariski dense parabolic curve of finite Euler characteristic, then we have

$$\int_X T[f] \wedge c_1(L) \geq 0. \tag{76}$$

This would imply the Green-Griffiths conjecture for surfaces of general type, since in this case it is known that for any ample line bundle A there exists $k \gg m \gg 0$ such that $H^0\big(X, E_{k,m}T^\star_X \otimes A^{-1}\big) \neq 0$.

References

[1] Ahlfors, L. V., Sario, L. : *Riemann surfaces,* Princeton Mathematical Series, No. 26, Princeton 1960.

[2] Baum, P., Bott, R. : *On the zeroes of meromorphic vector fields.* Essais en l'honneur de de Rham (1970).

[3] Bogomolov, F.A.: *Holomorphic tensors and vector bundles on projective varieties.* Math. USSR Izvestija 13/3 (1979), 499–555.

[4] Bloch, André : *Sur les systèmes de fonctions holomorphes à variétés linéaires* Ann. Sci. Ecole Norm. sup 43 (1926) 309-362.

[5] Brotbek, Damian : *Hyperbolicity related problems for complete intersection varieties.* Compos. Math. 150 (2014), no. 3, 369395.

[6] Brotbek, Damian; Darondeau, Lionel : *Complete intersection varieties with ample cotangent bundles.* arXiv:1511.04709.

[7] Brunella, Marco : *Courbes entières et feuilletages holomorphes.* Enseign. Math. (2) 45 (1999), no. 1-2, 195–216.

[8] Brunella, Marco : *Courbes entières dans les surfaces algébriques complexes (d'après McQuillan, Demailly-El Goul et al). (French) [Entire curves in complex algebraic surfaces (following McQuillan, Demailly-El Goul, et al.)]* Séminaire Bourbaki, Vol. 2000/2001. Astérisque No. 282 (2002), Exp. No. 881, vii, 39–61.

[9] Brunella, Marco : *Birational geometry of foliations* Publicações Matemáticas do IMPA, Instituto de Matemtica Pura e Aplicada (IMPA), Rio de Janeiro, 2004.

[10] Burns, Dan; Sibony, Nessim : *Limit currents and value distribution of holomorphic maps.* Ann. Inst. Fourier (Grenoble) 62 (2012), no. 1, 145–176.

[11] Demailly, J.-P. : *Variétés projectives hyperboliques et équations différentielles algébriques. (French) [Hyperbolic projective varieties and algebraic differential equations]* Journée en l'Honneur de Henri Cartan, 3–17,SMF Journ. Annu., 1997, Soc. Math. France, Paris, 1997.

[12] Demailly, J.-P. : *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials.* Algebraic geometry, Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.
[13] Demailly, J.-P. : *Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture.* Pure Appl. Math. Q. 7 (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1165–1207.
[14] Green, Mark; Griffiths, Phillip: *Two applications of algebraic geometry to entire holomorphic mappings.* The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 41–74, Springer, New York-Berlin, 1980.
[15] Gunning, R. C.; Narasimhan, R. : *Immersion of open Riemann surfaces.* Math. Ann. 174 1967 103–108.
[16] Kleiner, B. : *Hyperbolicity using minimal surfaces.* unpublished manuscript.
[17] Kobayashi, S. : *Hyperbolic Complex Spaces.* Grundlehren der math. Wissenschaften 318. Springer, Berlin (1998).
[18] Kollár, János : *Lectures on resolution of singularities.* Annals of Mathematics Studies, 166. Princeton University Press, Princeton, NJ, 2007.
[19] Merker, J. : *Complex projective hypersurfaces of general type: towards a conjecture of Green and Griffiths.* preprint ArXiv:1005.0405.
[20] Miyaoka, Y. : *Deformation of a morphism along a foliation and applications.* Algebraic Geometry (Bowdoin, 1985), Proc. Symp. Pure Math. 46 (1987).
[21] McQuillan, M : *Diophantine approximations and foliations.* Inst. Hautes Etudes Sci. Publ. Math. (1998), no. 87, p. 121–174.
[22] McQuillan, M : *Personal communication.* April 2016.
[23] Nevanlinna, R. : *Analytic functions.* Springer, Berlin (1970).
[24] Noguchi, Junjiro : *Some problems in value distribution and hyperbolic manifolds.* International Symposium "Holomorphic Mappings, Diophantine Geometry and Related Topics" (Kyoto, 1992).
[25] Ru, Min : *Nevanlinna theory and its relation to Diophantine approximation.* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
[26] Saleur, Benoît : *Inégalité d'Ahlfors en dimension supérieure. (French. English summary) [An Ahlfors inequality in higher dimension]* Math. Ann. 344 (2009), no. 4, 847–852.
[27] Selberg, H.L. : *Algebroide Funktionen und Umkehlfunktionen Abelscher Integrale.* Avh.Norske Vid Akad. Oslo8, (1934)1-72.
[28] Seidenberg, A. : *Reduction of singularities of the differential equation Ady=Bdx.* Amer. J. Math. 90 1968 248–269.
[29] Sibony, N. : *Quelques problèmes de prolongement de courants en analyse complexe. (French) [Some extension problems for currents in complex analysis]* Duke Math. J. 52 (1985), no. 1, 157–197.
[30] Siu, Y.-T. *Recent techniques in hyperbolicity problems.* Several complex variables (Berkeley, CA, 1995–1996), 429–508, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
[31] Siu, Y.-T. *Hyperbolicity of Generic High-Degree Hypersurfaces in Complex Projective Spaces* arXiv:1209.2723.
[32] Stoll, Wilhelm : *The Ahlfors-Weyl theory of meromorphic maps on parabolic manifolds.*Value distribution theory (Joensuu, 1981), 101–219, Lecture Notes in Math., 981, Springer, Berlin, 1983.
[33] Suzuki, Masakazu : *Comportement des applications holomorphes autour d'un ensemble polaire. II. (French) [Behavior of holomorphic mappings around a polar set. II]* C. R. Acad. Sci. Paris Sér. I Math. 306 (1988), no. 13, 535–538.
[34] Tsuji, M. : *Potential theory in modern function theory.* Maruzen Co.,Tokyo,1959.

[35] Ueno, Kenji : *Classification theory of algebraic varieties and compact complex spaces.* Notes written in collaboration with P. Cherenack; Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. xix+278 pp.

[36] Xie, Song-Yan : *Generalized Brotbek's symmetric differential forms and applications.* arXiv:1601.05133.

[37] Yau, S.-T. : *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*, Comm. Pure Appl. Math. 31 (1978).

Mihai Păun
Korea Institute for Advanced Study,
Seoul, 130-722 South KOREA
E-mail address: paun@kias.re.kr

Nessim Sibony
Orsay University, Paris, France
and
Korea Institute for Advanced Study,
Seoul, 130-722 South KOREA
E-mail address: nessim.sibony@math.u-psud.fr