MCQUILLAN

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Abstract. fff

Contents

1. Introduction

Let X be a smooth projective variety. If X is hyperbolic, then there is no, non constant analytic map $f : \mathbf{C} \to X$. If it is also known that hyperbolicity implies algebraic hyperbolicity. Thus if X is hyperbolic and L is an ample line bundle on it, we can find a constant A for which the following holds: for every smooth projective curve C of genus g and morphism $f : C \to X$ we have $\deg(f^*(L)) \leq A(2g-2)$.

For a smooth projective surface of general type we cannot expect that an inequality like the one above holds. This is, for instance, due to the fact that one can find rational or elliptic curves on it. But a conjecture of this kind may probably hold:

Conjecture 1.1. *Let X be a smooth projective surface of general type. Let L an ample line bundle on it. Then there exist constants A and B for which the following holds: for every*

smooth projective curve C of genus g and morphism $f : C \to X$ we have

$$\deg(f^*(L)) \leq Ag + B. \tag{1.1}$$

This conjecture is very deep and in particular it implies that on a surface of general type there are only finitely many rational or elliptic curves. More specifically the conjecture implies the following:

Conjecture 1.2. *Let X be a smooth surface of general type. Then there exists a proper Zariski closed set $Z \subset X$ for which the following holds: If C is a rational or an elliptic curve and $f : C \to X$ is a non constant map, then f factorise through Z.*

The Green–Griffiths conjecture is a generalization of this last conjecture:

Conjecture 1.3. *(Green–Gritffiths Conjecture) Let X be a smooth surface of general type. Then there exists a proper Zariski closed set $Z \subset X$ for which the following holds: every analytic map $f : \mathbf{C} \to X$ factorizes through Z.*

If, in the possibility that Conjecture 1.2 is false, we may weaken the conjecture 1.3 by requiring only that the closed set Z depends only on f (but it should be one dimensional).

In this chapter we will discuss some advances on these conjectures and a strategy, due to McQuillan, to attack conjecture 1.3 in general.

2. Curves of bounded genus on surfaces with big cotangent bundle

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Definition 2.1. *Let X be a smooth projective variety and E be a vector bundle over it. We will say that E is ample, big, nef ..., if the tautological bundle $\mathcal{O}(1)$ is ample, big, nef, ... on the projective bundle $\mathbf{P}(E)$ respectively.*

We start this section by showing how to prove algebraically a strong version of conjecture 1.1 on varieties with ample cotangent bundle. Remark that if a variety X has ample cotangent bundle, then his canonical line bundle K_X is ample too.

Theorem 2.2. *Let X be a smooth projective variety with ample cotangent bundle. Then there exists a constant A with the following property: for every smooth curve C of genus g and every non constant map $f : C \to X$ we have*

$$\deg(f^*(K_X)) \leq A(2g - 2). \tag{2.1}$$

Proof. Consider the structure morphism $p : \mathbf{P}(\Omega^1_X) \to X$. Since, by hypothesis, $\mathcal{O}(1)$ is ample on $\mathbf{P}(E)$, there is an integer N for which the line bundle $M_N := \mathcal{O}(N) \otimes p^*(K_X^{-1})$ is ample on it.

Let C be a smooth projective curve and $f : X \to X$ be a non constant map. The natural map $f^*(\Omega^1_X) \to \Omega^1_C$ gives, by functoriality a map $f' : C \to \mathbf{P}(\Omega^1_X)$ such that $f = p \circ f'$.

By construction $f'^*(\mathcal{O}(1)) \hookrightarrow \Omega^1_C$. Thus $\deg(f'^*(\mathcal{O}(1)) \leq 2g - 2$.

Since M_N is ample on $\mathbf{P}(\Omega^1_X))$ we have that $\deg(f'^*(M_N)) \geq 0$. Consequenty $\deg(f^*(K_X)) \leq N(2g - 2)$. $\square$

We observe that, in particular, we obtain that such a variety do not contain rational or elliptic curves.

Bogomolov theorem 2.3 generalize theorem 2.2 to surfaces whose cotangent bundle is big.

Theorem 2.3. *(Bogomolov) Let X be a smooth projective surface with big cotangent bundle. Then there exist constants A_1 and A_2 with the following property: for every smooth curve C of genus g and every non constant map $f : C \to X$ we have*

$$\deg(f^*(K_X)) \leq A_1(2g-2) + A_2. \tag{2.2}$$

Observe that a sufficient condition for the cotangent bundle to be big is that $c_1(X)^2 > c_2(X)$ (exercise).

Let's remark an interesting corollary:

Corollary 2.4. *Let X be a surface with big cotangent bundle. Then X contains only finitely many rational or elliptic curves.*

Proof. Indeed, curves of bounded genus in such a surface are a bounded family and surfaces of general type are not covered by rational or elliptic curves. □

We now prove theorem 2.3.

Proof. As before we fix an ample line bundle L on X. and the natural morphism $p : \mathbf{P}(\Omega^1_X) \to X$. If M is a line bundle on $\mathbf{P}(\Omega^1_X)$, we denote by $Bs(M)$ the base locus of it; it is a Zariski closed set which coincides with $\mathbf{P}(\Omega^1_X)$ if and only if $H^0(X, M) = \{0\}$.

For every positive integers n and m, consider the closed set $B_{n,m} := Bs((\mathscr{O}(m) \otimes p^*(L^{-1}))^{\otimes n}) \subset \mathbf{P}(\Omega^1_X)$. Let $B = \bigcap_{n,m} B_{n,m}$.

Since $\mathscr{O}(1)$ is big, we have that $B \neq \mathbf{P}(\Omega^1_X)$. By Noetherianity we may suppose that there exists n_0 and m_0 such that $B = B_{n_0,m_0}$.

Let C be a smooth projective curve of genus g and $f : C \to X$ be a non constant map. Suppose that the lift $f' : C \to \mathbf{P}(\Omega^1_X$ of f *do not factor through* B. This implies that there is a global section $s \in H^0(\mathbf{P}(\Omega^1_X); (\mathscr{O}(N_0) \otimes p^*(L^-1))^{m_0})$ which do not vanish identically along $f'(C)$. Consequently $\deg(f'^{(}\mathscr{O}(n_0)) \otimes f^*(L^{-1})) \geq 0$. Thus

$$\deg(f^*(L)) \leq n_0(2g-2). \tag{2.3}$$

Thus, for these curves it suffices to take A_2 bigger then n_0.

We must now deal with the case when the morphism f' factor through B. Consider an irreducible component of B. By abuse of notation we will denote it again by B.

If $p(B)$ is of dimension at most one the conclusion of the theorem easily follows: $f(C)$ can only be a finite list of curves inside X, the genus and the degree of which can be absorbed by the constant A_2.

Suppose that $p : B \to X$ is dominant. In this case, since $B \neq \mathbf{P}(\Omega^1_X)$, the dimension of B is two.

We will now show that the image of C in B is leaf of a natural foliation on it.

Lemma 2.5. *there exists a smooth projective surface $\widetilde{B}$, a birational morphism $\widetilde{B} \to B$ and a natural algebraic foliation $\mathscr{F}$ on $\widetilde{B}$ with the following property: Let $f_1 : C \to \widetilde{B}$ be a lift of f'. Then, either C belongs to a finite list or $f_1(C)$ is leaf of the foliation $\mathscr{F}$.*

Before we start the proof of the lemma, we recall some basic definitions of foliations on surfaces.

2.1. Standard fact about algebraic foliations on surfaces.

Definition 2.6. *Let Y be a smooth algebraic surface. An algebraic foliation $\mathscr{F}$ is a sub sheaf N de Ω^1_Y which is locally free of rank one and et such that the quotient Ω^1_Y/N is torsion free.*

We recall the following facts:

a) The line bundle N is usually called *the conormal sheaf of the foliation.* The line bundle $\det(\Omega^1_Y/N)$ is usually called *the canonical sheaf of $\mathscr{F}$* and denoted by $K_{\mathscr{F}}$.

b) There is a zero dimensional subschema $Z \subset Y$ (in general non reduced) which is called *the singular locus of the foliation* and an exact sequence

$$0 \longrightarrow N \longrightarrow \Omega^1_Y \longrightarrow I_Z \otimes K_{\mathscr{F}} \longrightarrow 0 \tag{2.4}$$

where I_Z is the ideal sheaf of Z. Points in the support of Z are called *singular points of the foliation*. Points outside Z are called *regular, or smooth, points for the foliation*.

c) Consequently we have $K_Y = N \otimes K_{\mathscr{F}}$.

d) Let M be a Riemann surface (not necessarily compact nor algebraic). A morphism $\iota : M \to Y$ is said to be *a leaf of the foliation* if:

d.1) There is a discrete set of points $P \subset M$ such that $\iota|_{M\setminus P} : M \setminus P \to T$ is an embedding;

d.2) the natural map $\iota^*(N) \to \iota^*(\Omega^1_Y) \to \Omega^1_M$ is the zero map.

e) If $z \in Y$ is a regular point for the foliation, then there is a unique leaf of the foliation passing through z.

f) Denote by Δ the one dimensional unit disk. If $z \in Y$ is a regular point for the foliation, then there is an *analytic* neighborhood $z \in U \subset Y$ isomorphic to $\Delta \times \Delta$ with coordinates (z_1, z_2) and the restriction of the exact sequence 2.4 to U is the exact sequence

$$0 \longrightarrow \mathscr{O}_U dz_1 \longrightarrow \mathscr{O}_U dz_1 \oplus \mathscr{O}_U dz_2 \longrightarrow \mathscr{O}_U dz_2 \longrightarrow 0. \tag{2.5}$$

Consequently the leaves of the foliation passing through U are given by the equations $z_1 = c$ ($c \in \Delta$).

Point (f) above explain a bit the geometry of a foliation on the open set of regular points: we can cover the regular locus of the foliation by open sets such that the restriction of the foliation to each of them is just a product. On the other side, near the singular locus of the foliation, the structure of the leaves may be much more complicated.

g) Suppose that Y_1 is a smooth projective surface and $p : Y_1 \to Y$ is a dominant morphism. We have an inclusion $p^*(N) \to p^*(\Omega^1_Y) \to \Omega^1_{Y_1}$. The saturation N_1 of N inside $\Omega^1_{Y_1}$ is then a foliation on Y_1. It will be called *the pull back of the foliation $\mathscr{F}$ to Y_1 via p* and denoted by $p^*(\mathscr{F})$.

One should be aware that, in general, the conormal sheaf of $p^*(\mathscr{F})$ *is not* $p^*(N)$ and, in general, $K_{p^*(\mathscr{F})} \neq p^*(K_{\mathscr{F}})$.

h) A leaf M of the foliation is said to be *algebraic* if the Zariski closure of it in Y is an algebraic curve.

i) Suppose that S is a smooth algebraic curve and $f : Y \to S$ is a dominant morphism. The natural exact sequence

$$0 \longrightarrow f^*(\Omega^1_S) \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0 \tag{2.6}$$

induces a foliation $\mathscr{F}_f$ on Y. The leaves of this foliation are the fibres of f thus they are all compact. Observe that in general $f^*(\Omega^1_S)$ *is not* the conormal sheaf of $\mathscr{F}_f$. The conormal sheaf of $\mathscr{F}_f$ will be the saturation of $f^*(\Omega^1_S)$ in Ω^1_Y.

j) A foliation $\mathscr{F}$ is said to be *a fibration* if there is a a birational morphism $p : Y_1 \to Y$ and a dominant morphism $f : Y_1 \to S$ where S is a smooth projective curve such that $p^*(\mathscr{F}) = \mathscr{F}_f$. All the leaves of a fibration are algebraic.

The following theorem characterizes foliations with "many" algebraic leaves:

Theorem 2.7. *(Jouanoulou) Let $\mathscr{F}$ be a foliation on a smooth projective surface. Then $\mathscr{F}$ has infinitely many algebraic leaves if and only if it is a fibration (and thus all the leaves are algebraic).*

Let's prove Lemma 2.5:

Proof. Over $\mathbf{P}(\Omega^1_X)$ we have the tautological exact sequence

$$0 \longrightarrow \Omega^1_{\mathbf{P}(\Omega^1_X/X}(1) \longrightarrow p^*(\Omega^1_X) \longrightarrow \mathscr{O}(1) \longrightarrow 0. \tag{2.7}$$

Observe that if $f : C \to X$ is a morphism and $f' : C \to \mathbf{P}(\Omega^1_X)$ is the natural lift, then the natural map $f'^*(\Omega^1_{\mathbf{P}(\Omega^1_X/X}(1)) \to f^*(\Omega^1_X) \to \Omega^1_C$ is the zero map.

Let B_1 be a desingularization of B. denote by $p : B_1 \to B$ and by $h : B_1 \to X$ the natural projections. By construction we have an inclusion $p*(\Omega^1_{\mathbf{P}(\Omega^1_X/X}(1))|_B) \to h^*(\Omega^1_X) \to \Omega^1_{B_1}$. Let $N \subset \Omega^1_{B_1}$ be the saturation of $p*(\Omega^1_{\mathbf{P}(\Omega^1_X/X}(1))|_B)$. Taking a blow up $\widetilde{B}$ of B_1 if necessary, we may suppose that N is locally free of rank one and thus it defines a foliation $\mathscr{F}$ on $\widetilde{B}$.

Suppose now that $f'(C)$ intersects the smooth locus of B. Thus we can lift f' to a morphism $f_1 : C \to \widetilde{B}$. By construction, and the observation above, the natural map $f_1^*(N) \to \Omega^1_C$ is the zero map. Thus $f_1(C)$ is a leaf of the foliation $\mathscr{F}$. Since there are only finitely many curves contained in the singular locus of B the conclusion of the Lemma follows. $\square$

Now the conclusion of Theorem 2.3 easily follows from Jouanoulou Theorem 2.7: If $f(C)$ factors through B then either it belongs to a finite list (thus his degree and genus may be absorbed by the constant A_2 or it is leaf of an algebraic foliation $\mathscr{F}$ over $\widetilde{B}$.

If the foliation is a fibration then all the leaves (up to finitely many) are algebraically equivalent. in particular they will be of fixed genus and degree. Thus an inequality as the one proposed by the theorem holds, up to rise the constant A_1 if necessary

If the foliation is not a fibration then there will be only finitely many algebraic leaves and their degree and genus may be absorbed in the constant A_2. $\square$

3. Entire curves on surfaces with big cotangent bundle

In this section, the core of the chapter, we will se how a strategy which is essentially similar to the proof of Bogomolov theorem, may prove Green–Griffiths conjecture 1.3 for surfaces with big cotangent bundle. The proof, even if it follows the same philosophy, is much more involved and requires stronger technical tools. In the meanwhile we will see that the proof will also provide a very easy proof of Theorem 2.2.

The first thing we remark in the proof of Theorem 2.3 is that there is a extended use of intersection theory. We we deal with analytic maps of $\mathbf{C}$ inside a variety we cannot use it but Nevanlinna theory can provide a substitute of it.

3.1. **Review of Nevanlinna Theory.** Let X be a smooth projective variety and ω a positive $(1,1)$ form on it. Let $f : \mathbf{C} \to X$ an analytic map. For every positive real number r we define the *characteristic function*

$$T_f(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^*(\omega). \tag{3.1}$$

and more in general, for every closed $(1,1)$ form α on X we define

$$T_f(r,\alpha) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^*(\alpha). \tag{3.2}$$

We will show now how the counting function may play the role of an intersection number.

Remark 3.1. *It is known that $T_f(r) = O(\log(r))$ if and only if the map f is algebraic. To avoid degenerate cases, we suppose from now on, that the entire maps we consider are not algebraic.*

Similarly, if β is a $(1,1)$ form on $\mathbf{C}$ we can define $T(r,\beta)$.

If $D = \sum_i n_i P_i$ is a (possibly infinite) formal sum of points of $\mathbf{C}$ (if the sum is infinite, we suppose that there is no accumulating point), we define the *counting function* of D by

$$N(D,r) := \sum_{|P_i|<r} n_i \log \frac{r}{|P_i|}. \tag{3.3}$$

If L is an hermitian line bundle on $\mathbf{C}$ and $s \in H^0(\mathbf{C}, L)$, then we define *the proximity function* of s by

$$m(s,r); = \int_0^{2\pi} \log \|s\|(re^{i\theta}) \frac{d\theta}{2\pi r}. \tag{3.4}$$

With this notation, we have the *Nevanlinna First Main Theorem*:

Theorem 3.2. *Let L be an hermitian line bundle on $\mathbf{C}$ and $s \in H^0(\mathbf{C}, L)$. Then there exists a constant C, independent on r such that*

$$T(r, c_1(L)) + m(s, r) = N(\mathrm{div}(s), r) + C. \tag{3.5}$$

Where $c_1(L)$ is the $(1,1)$ form associated to L.

In order to relate the characteristic function with intersection theory, we show now that, given a smooth projective variety X and an entire map $f : \mathbf{C} \to X$, the *positive* $(1,1)$ *form* $T_f(r, -)$ is "very small" on exact forms. We fix an ample line bundle L equipped with a positive hermitian metric.

Lemma 3.3. *Let α be a smooth exact $(1,1)$ form on X. Then there exists a constant C, depending on α, such that, for every $r \gg 0$*

$$|T_f(r, \alpha)| \le C\, (T_f(r, c_1(L) \log(r)))^{1/2}. \tag{3.6}$$

Proof. We put $g(z) := \frac{1}{2} \log |z|^2$. We may suppose that $\alpha = \overline{\partial}\beta$ for a $(1,0)$ form β on X. By Stokes theorem we have

$$\begin{aligned} T_f(r; \alpha) = & \int_0^r \frac{dt}{t} \int_{|z|=t} f^*(\beta) \\ = & \int_0^r dg \wedge f^*(\beta) \\ = & \; C_1 + \int_{r_0}^r dg \wedge f^*(\beta). \end{aligned}$$

(We fix $r_0 > 0$).

By Cauchy–Schwartz inequality,

$$\left| \int_{r_0}^r dg \wedge f^*(\beta) \right| \le \left(\int_{r_0}^r f^*(\beta) \wedge f^*(\overline{\beta}) \right)^{1/2} \cdot \left(\int_{r_0}^r dg \wedge d^c g \right)^{1/2}. \tag{3.7}$$

An easy computation gives $0 < \int_{r_0}^r dg \wedge d^c g < C_1 \log(r)$. Since X is compact, we can find a constant C_2 such that $\beta \wedge \overline{\beta} \le C_2 c_1(L)$. The conclusion follows. □

3.2. **The Ahlfors current associated to an entire curve.** With this in mind, we can associate to an entire map a closed positive current which will play the role of intersection theory.

Let X be smooth projective and L an ample line bundle on it. Suppose that the metric on L is positive.

Let $f : \mathbf{C} \to X$ be an entire map. Consider the family of positive currents on X

$$\begin{aligned} T_r : A^{1,1} \longrightarrow & \quad \mathbf{R} \\ \alpha \longrightarrow & \quad \frac{T_f(r,\alpha)}{T_f(r, c_1(L))}. \end{aligned}$$

It is a family of currents which is bounded for the standard norm on $A^{1,1}$ (but also for the L^∞ norm); consequently we can extract from it a sequence converging, in the weak topology, to a current T_f.

We now list some properties of the current T_f:

a) Even if the current T_f in principle, depends on the choice of the sequence, we will call it *the closed positive Ahlfors current associated to* f.

b) The current T_f is not zero because $T_f(c_1(L)) = 1$.

c) By Lemma 3.3, the current T_f is closed.

d) Let M be a line bundle on X. Suppose that we fixed a metric on it. since T_f is closed, then $T_f(c_1(M)$ is independent to the chosen metric on M. For instance, given a divisor D on X we can speak about the real number $T_f(D)$.

e) if D is an *effective divisor* on X such that $f(\mathbf{C}) \not\subset D$, then $T_f(D) \geq 0$.

Indeed, the number $T_f(D)$ is independent on the metric chosen on $\mathscr{O}(D)$, we can choose a metric on it such that $\sup \|D\| \leq 1$. On a donc $m(r, D) \leq C$ where C is independent on r. By Nevanlinna First Main Theorem 3.2, we have $T_f(r, c_1(\mathscr{O}(D)) \geq C$. Thus $T_f(D) \geq 0$.

f) Point (f) implies that, if M is a line bundle on X whose base locus is empty, we have $T_f(M) \geq 0$ and points (b) and (f) imply that if M Is ample then $T_f(M) > 0$.

g) More generally, $T_f(M) \geq 0$ as soon as $f(\mathbf{C})$ is not contained in the stable base locus of M.

h) If the image of f is Zariski dense, in order to construct the Ahlfors current associated to it, we may suppose that L is just big: indeed, in this case, we may suppose that $L = A + D$ with A ample line bundle and D effective divisor. Thus, by the First Main Theorem 3.2, we can find a constant t independent on r, such that $T_f(r, c_1(L)) \geq T_f(r, c_1(A)) + C$. As a consequence we find:

Proposition 3.4. *Let $p : X_1 \to X$ be a birational morphism and $f : \mathbf{C} \to X$ be an entire map with Zariski dense image. Let $f_1 : \mathbf{C} \to X_1$ the lift of f to X_1. Then we can construct the Ahlfors currents T_f and T_{f_1} associated to f and f_1 respectively, in such a way that*

$$p_*(T_{f_1}) = T_f. \tag{3.8}$$

Since T_f is a closed current, we may consider its class in cohomology. Properties (e), (f) and (g) above show that this class is very similar to a the class of an algebraic curve in X. This is why we see T_f as a substitute of intersection theory in this contest.

3.3. **The tautological inequality.** Let X be a smooth projective variety, $p : \mathbf{P}(\Omega^1_X) \to X$ be the projective bundle associated to the sheaf of differentials and $\mathscr{O}(1)$ be the tautological line bundle on it.

If C is a smooth projective curve of genus g and $g : C \to X$ is a non constant morphism. Then, as we saw before, we can lift f to a morphism $f' : C \to \mathbf{P}(\Omega^1_X)$ such that $p \circ f' = f$ and the following inequality holds

$$\deg(f'^*(\mathscr{O}(1)) \leq 2g - 2. \tag{3.9}$$

If $f : \mathbf{C} \to X$ is an entire map, we also have a canonical lift $f' : \mathbf{C} \to \mathbf{P}(\Omega^1_X)$ such that $p \circ f' = f$. Consequently we can associate to f the current T_f on X and the current $T_{f'}$ on $\mathbf{P}(\Omega^1_X)$. We will now show that we can choose $T_{f'}$ in such a way that $p_*(T_{f'}) = T_f$ and $T_{f'}(\mathscr{O}(1)) \leq 0$. This second inequality should be interpreted as an analytic analogue of the inequality 3.9.

We suppose that the line bundle L is equipped with a positive metric. Thus $c_1(L)$ will define a metric on Ω^1_X and from exact sequence 2.7 we obtain a metric on $\mathscr{O}(1)$.

Theorem 3.5. *In the situation above, we can find a set $E \subset \mathbf{R}$ of finite Lebesgue measure and a positive constant C such that, for every $r \notin E$ we have*

$$T_{f'}(r, c_1(\mathscr{O}(1)) \leq C \log(T_f(r, c_1(L)). \tag{3.10}$$

Proof. We begin by fixing some notation: over $\mathbf{C}$ we fix a coordinate z, thus a trivialization of $\Omega^1_{\mathbf{C}} = \mathscr{O}dz$; it will be equipped with the metric such that $\|dz\| = 1$. We denote by $\mathbf{P}_1$ the projective bundle $p_1 : \mathbf{P}_1 := \mathbf{P}(\mathscr{O}_X \oplus \Omega^1_X) \to X$. The tautological line bundle on $\mathbf{P}_1$ will be denote by $\mathbb{M}$. By the standard Euler exact sequence on $\mathbf{P}_1$, the trivial metric on $\mathscr{O}_X$ and the metric $c_1(L)$ on Ω^1_X induce a metric on $\mathbb{M}$.

The natural surjection $\mathscr{O}_X \oplus \Omega^1_X \to \Omega^1_X$ induces an inclusion $\mathbf{P}(\Omega^1_X) \hookrightarrow \mathbf{P}_1$ as a divisor. For the time being, to avoid confusions, we will denote this divisor by D. We have $\mathscr{O}_{\mathbf{P}_1}(D) = \mathbb{M}$.

The natural surjection $\mathscr{O}_X \oplus \Omega^1_X \to \mathscr{O}_X$ induces a section $P_1 : X \to \mathbf{P}_1$.

The natural inclusion $\Omega^1_X \to \mathscr{O}_X \oplus \Omega^1_X$ induces a rational map $h : \mathbf{P}_1 \dashrightarrow \mathbf{P}(\Omega^1_X)$. Let $q : \widetilde{\mathbf{P}_1} \to \mathbf{P}_1$ be the blow up along the section P_0 and denote by E the exceptional divisor. The map rational map h extends to a morpism $\widetilde{h} : \widetilde{\mathbf{P}_1} \to \mathbf{P}(\Omega^1_X)$. If (just for this proof) we denote by $\mathbb{L}$ the tautological line bundle on $\mathbf{P}(\Omega^1_X)$, we have then

$$\widetilde{h}^*(\mathbb{L}) = q^*(\mathbb{M})(-E). \tag{3.11}$$

Moreover, the isomorphism above induces a metric on the line bundle $\mathscr{O}_{\widetilde{\mathbf{P}_1}}(E)$.

The morphism $f : \mathbf{C} \to X$ induces the following surjective morphism:

$$\begin{aligned} f^*(\mathscr{O}_X \oplus \Omega^1_X) &\longrightarrow \quad \Omega^1_{\mathbf{C}} \\ (a, \alpha) &\longrightarrow \quad f^*(a) \cdot dz + f^*(\alpha). \end{aligned}$$

Consequently it induces morphisms $f_1 : \mathbf{C} \to \mathbf{P}_1$ and $\widetilde{f} : \mathbf{C} \to \widetilde{\mathbf{P}_1}$. By construction we have $\widetilde{h} \circ \widetilde{f} = f'$ and $q \circ \widetilde{f} = f_1$. Remark that, by construction, the image of f_1 never intersects D.

By Nevanlinna First Main Theorem 3.2 applied to the divisors $\widetilde{f}^*(E)$ and $f_1^*(D)$, we can find constants C_E and C_D, independent on r, such that

$$T_{\widetilde{f}}(r, c_1(E)) = N(f^*(E), r) - m(\widetilde{f}^*(E), r) + C_E \tag{3.12}$$

and

$$T_{f_1}(r, c_1(\mathbb{M})) = -m(f_1^*(D), r) + C_D. \tag{3.13}$$

Remark that we used that the image of f_1 do not intersect D. Consequently, from relation 3.11 and the fact that $N(f^*(D), r) \geq 0$ we deduce that

$$T_{f'}(r, c_1(\mathbb{L})) \leq m(f^*(E), r) - m(f^*(D), r) + C_0 \tag{3.14}$$

Where C_0 is a constant independent on r.

There exists a positive real function $F(z)$ such that $f^*(c_1(L)) = iF(z)dz \wedge d\overline{z}$.

A local expression of f, f_1 etc, implies that $\|\widetilde{f}^*(E)\|^2 = \frac{F(z)}{F(z)+1}$ and $\|f_1^*(D)\|^2 = \frac{1}{F(z)+1}$. Consequently, in order to conclude, we need to find an upper bound to

$$M_f(r) := \int_0^{2\pi} \log F(re^{i\theta}) \frac{d\theta}{2\pi r}. \tag{3.15}$$

We have that

$$\frac{d}{dr} T_f(r, c_1(L)) = \frac{1}{r} \int_{|z| \leq r} iF dz \wedge d\overline{z}. \tag{3.16}$$

Define $S(r) := r\frac{d}{dr} T_f(r, c_1(L))$. An easy computation in polar coordinates shows that

$$\frac{d}{dr} S(r) = \int_0^{2\pi} F(re^{i\theta}) r d\theta. \tag{3.17}$$

As soon as $r \geq 1$, the concavity of the log imply

$$\log(\frac{d}{dr} S(r)) \geq M_f(r). \tag{3.18}$$

The conclusion follows then from a double application of the Lemma below. □

Lemma 3.6. *Let $H(x)$ be a derivable positive increasing function on $\mathbf{R}$. For every positive ϵ, there exists a subset $E \subset \mathbf{R}$ whose measure is bounded above by $\int_{1+\epsilon}^{\infty} \frac{1}{t \log^{1+\epsilon}(t)} dt < \infty$ such that, for every $x \notin E$ we have*

$$H'(x) \leq H(x) \log^{1+\epsilon}(H(x)). \tag{3.19}$$

Proof. consider the set $F := \{x \in \mathbf{R} \ / \ H'(x) \geq H(x) \log^{1+\epsilon} H(x)\}$. Denote by $\mathbb{1}_F(x)$ the characteristic function of F. By construction we have

$$\mathbb{1}_F(x) \leq \frac{H'(x)}{H(x) \log^{1+\epsilon} H(x))}. \tag{3.20}$$

The conclusion follows by integrating the inequality above and taking the change of variable $H(x) = t$. □

Let $p : \mathbf{P}(\Omega^1_X) \to X$ be the projective bundle of differentials. Since we may suppose that $\mathscr{O}(1) \otimes p^*(L)$ is ample on it, Theorem 3.5 implies the following important theorem, which is called *Tautological inequality*:

Theorem 3.7. *Let $f : \mathbf{C} \to X$ be an entire curve and $f' : \mathbf{C} \to \mathbf{P}(\Omega^1_X)$ the canonical lift of it. Then we can choose the Ahlfors currents T_f and $T_{f'}$ associated to f and f' respectively in such a way that*

$$p_*(T_{f'}) = T_f \tag{3.21}$$

and

$$T_{f'}(\mathscr{O}(1)) = 0 \tag{3.22}$$

Proof. It suffices to remark that, by Theorem 3.5, we have that $T_{f'}(r, c_1(\mathscr{O}(1)) + p^*(c_1(L)) \leq CT_f(r, c_1(L))$ for a suitable constant C. Thus we can extract from the set of currents $\frac{T_{f'}(r,\cdot)}{T_f(r,c_1(L))}$ a sequence converging to a closed positive current $T_{f'}$ having the properties (a) – (h) of section 3.1 □

3.4. **The tautological inequality on the bundle of logarithmic differentials.** We recall that an effective divisor H on a smooth projective variety X is said to be *simple normal crossing* if, locally, for the euclidean topology, we can find coordinates $z_, \dots, z_n$ on X such that H is given by $z_1 \cdot z_2 \cdots z_r = 0$ (with $r \leq n$).

If H is a simple normal crossing divisor, we can associate to it, the *bundle of differentials with logarithmic poles* on it. This is the sheaf $\Omega^1_X(\log(H))$ of meromorphic differentials with simple poles around H. More explicitly, if on a open set where H is given by $z_1 \cdot z_2 \cdots z_r = 0$, then a section of $\Omega^1_X(\log(H))$ is a differential of the form $f_1 \frac{dz_1}{z_1} + \dots f_r \frac{dz_r}{z_r} + f_{r+1} dz_{r+1} + \dots f_n dz_n$ with f_i holomorphic functions.

The sheaf $\Omega^1_X(\log(H))$ is locally free of rank $n = \dim(X)$. Its dual will be denoted by $T_X(-\log(H))$, it is the bundle of derivations which locally can be written as $\partial = g_1 z_1 \partial_{z_1} + \cdots + g_r z_r \partial_{z_r} + g_{r+1} \partial_{z_{r+1}} + \cdots + g_n \partial_{z_n}$.

If $H = \sum_i H_i$, there is a natural exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log(H) \longrightarrow \bigoplus_i \mathscr{O}_{H_i} \longrightarrow 0 \tag{3.23}$$

Where the first inclusion is just the natural inclusion of holomorphic differential forms inside the space of differential forms with logarithmic poles.

In particular, as a consequence, we find that

$$c_1(\Omega^1_X(\log(H))) = K_X(H). \tag{3.24}$$

We will denote by $h_H : \mathbf{P}_H \to X$ the projective bundle $\mathbf{P}(\Omega^1_X(\log(H))$ and by $\mathscr{O}_H(1)$ its tautological line bundle.

Suppose that L is an ample hermitian line bundle on X with positive first Chern form $c_1(L)$. Suppose that $H = \sum H_i$ and that each $\mathscr{O}_X(H_i)$ is equipped with a smooth metric.

For a suitable positive constant A, the singular $(1, 1)$ form

$$\omega^{sm} := Ac_1(L) + \sum \frac{d\|H_i\| \wedge d^c\|H_i\|}{\|H_i\|^2} \tag{3.25}$$

induces a *smooth* metric on $T_X(-\log(H))$: If $\partial = g_1 z_1 \partial_{z_1} + \cdots + g_r z_r \partial_{z_r} + g_{r+1} \partial_{z_{r+1}} + \cdots + g_n \partial_{z_n}$ is a local element of $T_X(-\log(H)$, Then $\omega^{sm}(\partial, \partial)$ is locally comparable to $\sum_{i=1}^r |g_i|^2(|z_i|^2 + 1) + \sum_{j=r+1}^n |g_j|^2$. Thus ω^{sm} induces a smooth metric on $\Omega^1_X(\log(H)$ and eventually a metric on $\mathscr{O}_H(1)$.

If M is a Riemann surface and S is a discrete set of points of M with no accumulation points with at most countable cardinality (remark that if M is compact, then this implies that S is a finite set), we can define $\Omega^1_M(\log(S))$ in the same way. It is a line bundle on M.

If $f : M \to X$ is an analytic map such that $f(M) \not\subset H$ we may consider the set $f^*(H)_{red} := h^{-1}(H)$ (set wise pre image). A local computation shows that there is a natural map (analogous to the map of differentials)

$$h^*(\Omega^1_X(\log(H))) \longrightarrow \Omega^1_M(\log(f^*(H)_{red}). \tag{3.26}$$

Thus a functorial lifting $h'_H : M \to \mathbf{P}_H$.

Suppose now that $f : \mathbf{C} \to X$ is an analytic map such that $f(\mathbf{C}) \cap H = \emptyset$.

The analogous of Theorem 3.5 in this situation is the following *refined tautological inequality*:

Theorem 3.8. *In the situation above, we may find a set $E \subset \mathbf{R}$ and a constant C_H such that, for every $r \notin E$ we have*

$$T_{f'_H}(r, c_1(\mathscr{O}_X(1)) \leq C_H \log(T_f(r, c_1(H)). \tag{3.27}$$

Proof. The proof follows the same path then the proof of 3.5. We introduce $\mathbf{P}_{1,H} = \mathbf{P}(\mathscr{O} \oplus \Omega^1_X(\log H))$ and its tautological line bundle $\mathbb{M}_H$. Again, we will denote by $\mathbb{L}_H$ the tautological line bundle of $\mathbf{P}_H$. We will denote by $\|\cdot\|^{sm}_{\mathbb{M}}$ and by $\|\cdot\|^{sm}_{\mathbb{L}}$ the smooth metrics induced by ω_{sm} on $\mathbb{M}_H$ and $\mathbb{L}_H$ respectively. Denote by $c_1(\mathbb{M}_H)^{sm}$ and $c_1(\mathbb{L})^{sm}$ their first Chern forms.

We introduce the singular $(1,1)$ form

$$\widetilde{\omega} := Ac_1(L) + \sum \frac{d\|H_i\| \wedge d^c\|H_i\|}{\|H_i\|^2(\log\|H_i\|)^2}. \tag{3.28}$$

If ∂ is a local section of $T_X(-\log(H))$ as above, then $\widetilde{\omega}(\partial, \partial)$ is locally comparable with $\sum_{i=1}^r |g_i|^2(|z_i|^2 + \frac{1}{(\log|z_i|)^2}) + \sum_{j=r+1}^n |g_j|^2$.

Again, $\widetilde{\omega}$ induces a *singular* metrics $\|\cdot\|^{sing}_{\mathbb{M}}$ and $\|\cdot\|^{sing}_{\mathbb{L}}$ on $\mathbb{M}_H$ and $\mathbb{L}_H$ respectively. Denote by $c_1(\mathbb{M})^{sing}$ and $c_1(\mathbb{L})^{sing}$ their singular first Chern forms.

Remark that the restriction of the metrics $\|\cdot\|^{sing}_{\mathbb{M}}$ and $\|\cdot\|^{sing}_{\mathbb{L}}$ to $f'_H(\mathbf{C})$ are smooth because $f(\mathbf{C})$ do not intersect H.

As in the proof of Theorem 3.5, we can find a positive smooth function F on $\mathbf{C}$ such that $f^*(\widetilde{\omega}) = iF dz \wedge d\overline{z}$.

And, again, following the same strategy of Theorem 3.5, we find that

$$T_{f'_H}(r, c_1(\mathbb{L})^{sing}) \leq \int_0^{2\pi} \log F(re^{i\theta})\frac{d\theta}{2\pi r} + C_H \tag{3.29}$$

for a suitable constant C_H.

Consider now the function

$$T_f(r, \widetilde{\omega}) = \int_0^r \frac{dt}{t}\int_{|z|<t} f^*(\widetilde{\omega}) = \int_0^r \frac{dt}{t}\int_{|z|<t} iF dz \wedge d\overline{z}. \tag{3.30}$$

Observe again that $f^*(\widetilde{\omega})$ is a smooth form on $\mathbf{C}$. As in the proof of Theorem 3.5, if we define $S(r) := r\frac{d}{dr}T_f(r, \widetilde{\omega})$, we obtain

$$\log(\frac{d}{dr}S(r)) \geq \int_0^{2\pi} \log F(re^{i\theta})\frac{d\theta}{2\pi r}. \tag{3.31}$$

The conclusion will follow from Lemma 3.6 and the following two claims:

– *Claim 1* There exists a consent C_1 such that

$$T_{f'_H}(r, c_1(\mathbb{L})^{sm}) \leq T_{f'_H}(r, c_1(\mathbb{L})^{sing}) + C_1 \log(T_f(r, c_1(L)). \tag{3.32}$$

Proof. Outside the pre image of H, we can find a smooth positive function h such that $\| \cdot \|_{\mathbb{L}}^{sing} = h \cdot \| \cdot \|_{\mathbb{L}}^{sm}$. Moreover, a local computation and compactness of X implies that we can find a constant C_2 such that

$$h \leq C_2 \prod_i (\log \|H_i\|)^2. \tag{3.33}$$

Consequently $c_1(\mathbb{L})^{sing} = c_1(\mathbb{L})^{sm} - dd^c \log h$ and

$$T_{f'_H}(r, c_1(\mathbb{L})^{sm}) \leq T_{f'_H}(r, c_1(\mathbb{L})^{sm}) + \left| \int_0^r \frac{dt}{t} \int_{|z|=t} dd^c((f'_H)^*(h) \right|. \tag{3.34}$$

A double application of Stokes theorem gives

$$\left| \int_0^r \frac{dt}{t} \int_{|z|=t} dd^c((f'_H)^*(h) \right| \leq \int_0^{2\pi} \log(h)(re^{i\theta}) \frac{d\theta}{2\pi r} + C_3. \tag{3.35}$$

Where C_3 is independent on r.

Now, the bound 3.33 implies

$$\int_0^{2\pi} \log(h)(re^{i\theta}) \frac{d\theta}{2\pi r} \leq \sum_i \int_0^{2\pi} \log(\log \|H_i\|))^2 (re^{i\theta}) \frac{d\theta}{2\pi r} + C_4$$

Concavity of the log implies

$$\sum_i \int_0^{2\pi} \log(\log \|H_i\|))^2 (re^{i\theta}) \frac{d\theta}{2\pi r} \leq \sum_i 2 \log(\int_0^{2\pi} |\log \|H_i\|)|(re^{i\theta}) \frac{d\theta}{2\pi r} \tag{3.36}$$

and, since L is equipped with a positive metric, we can find a constant C_5 such that

$$\int_0^{2\pi} |\log \|H_i\|)|(re^{i\theta}) \frac{d\theta}{2\pi r} \leq T_f(r, c_1(\mathscr{O}(H_i)) \leq C_5 T_f(r, c_1(L)). \tag{3.37}$$

The claim follows. □

– *Claim 2* There exists a constant C_2 such that

$$T_{f_H}(r, \widetilde{\omega}) \leq C_2 T_f(r, c_1(L)). \tag{3.38}$$

Proof. The following equality holds:

$$-dd^c \log(\log \|H_i\|)^2 = \frac{d\|H_i\| \wedge d^c \|H_i\|}{\|H_i\|^2 \log^2(\|H_i\|)} + \frac{1}{|\log \|H_i\||} c_1(\mathscr{O}_X(H_i)). \tag{3.39}$$

And, since X is compact, we can uniformly bound the last term on the right. Consequently we can find a constant A such that

$$\widetilde{\omega} \leq Ac_1(L) - \sum_i dd^c \log(\log \|H_i\|)^2 \tag{3.40}$$

Again, an application of Stokes theorem gives

$$T_f(r, \widetilde{\omega}) \leq AT_f(r, c_1(L)) + \sum_i \left| \int_0^{2\pi} \log(\log^2 \|H_i\|(re^{i\theta}) \frac{d\theta}{2\pi r} \right| \tag{3.41}$$

and again positivity of $c_1(L)$ implies

$$\sum_i \int_0^{2\pi} |\log \|H_i\|(re^{i\theta})| \frac{d\theta}{2\pi r} \leq C_6 T_f(r, c_1(L)) \tag{3.42}$$

and concavity of the log implies the conclusion of the claim. □

The conclusion of the theorem follows. □

As a consequence we find the analogous of Theorem 3.7 on the logarithmic case:

Theorem 3.9. *Let X be a smooth projective variety and H a simple normal crossing divisor on it. Let $f : \mathbf{C} \to X$ be an entire curve such that $f(\mathbf{C}) \cap H = \emptyset$. Let $f'_H : \mathbf{C} \to \mathbf{P}(\Omega^1_X(\log(H))$ the canonical lift of f. Then we can choose the Ahlfors currents T_f and $T_{f'_H}$ associated to f and f'_H respectively in such a way that*

$$(h_H)_*(T_{f'_H}) = T_f \tag{3.43}$$

and

$$T_{f'_H}(\mathscr{O}(1)) = 0. \tag{3.44}$$

3.5. **Varieties with ample cotangent bundle are hyperbolic.** We show now how tautological inequality can be applied to give a proof of the fact that there is no, non constant, map $f : \mathbf{C} \to X$ when X is a variety with ample cotangent bundle. The reader should remark how the proof is, formally identical to the proof of the analogous algebraic statement Theorem 2.2.

Theorem 3.10. *Let X be a smooth projective variety with ample cotangent bundle. Then every entire map $f : \mathbf{C} \to X$ is constant.*

Proof. Fix an ample line bundle L on X. Consider the projective bundle $p : \mathbf{P}(\Omega^1_X) \to X$. Since, by hypothesis, $\mathscr{O}(1)$ is ample on it, we may suppose that $\mathscr{O}(N) \otimes p^*(L^{-1})$ is ample as soon as N is sufficiently big.

Let $f : \mathbf{C} \to X$ an entire map. Consider the Ahlfors currents T_f on X and $T_{f'}$ on $\mathbf{P}(\Omega^1_X)$. Since L is ample we have $T_f(L) > 0$. Since $\mathscr{O}(N) \otimes p^*(L^{-1})$ is ample we have $T_{f'}(\mathscr{O}(N) \otimes p^*(L^{-1})) > 0$. But by the tautological inequality we have

$$T_{f'}(\mathscr{O}(N) \otimes p^*(L^{-1})) < 0 \tag{3.45}$$

Thus we have a contradiction if f is not constant. □

3.6. **Green–Griffiths conjecture for surfaces with big cotangent bundle.** We will now show how to prove the Green–Griffiths conjecture for surfaces of general type with big cotangent bundle. The first part of the proof will follow the strategy of Bogomolov theorem 2.3 in the algebraic case. The second part of the proof will be more involved and will require a deeper analysis of properties of the foliations on surfaces.

Theorem 3.11. *(McQuillan) Let X be a surface with big cotangent bundle. Then there exists a proper Zariski closed subset $Z \subset X$ with the following property: every non constant entire map $f : \mathbf{C} \to X$ factorizes through Z.*

Proof. Let $\mathbf{P} := \mathbf{P}(\Omega^1_X) \xrightarrow{p} X$ be the projective cotangent bundle and $\mathscr{O}(1)$ the tautological line bundle on it. We also fix an ample line bundle L on X. As in the proof of Theorem 3.10, we associate to the map f the Ahlfors currents T_f on X and $T_{f'}$ on $\mathbf{P}$. We may suppose that f is Zariski dense.

Using the tautological inequality and the same strategy of the first part of Theorem 2.3 we see that f' *factorizes through a two dimensional component of the stable base locus of* $\mathscr{O}(1)$.

By the same proof of Lemma 2.5 we obtain the analogous lemma:

Lemma 3.12. *There exists a smooth surface of general type Y equipped with a foliation $\mathscr{F}$ and a dominant morphism $h : Y \to X$ with the following property: f lifts to an entire map $f_Y : \mathbf{C} \to Y$ and the image of f_Y is leaf of the foliation $\mathscr{F}$.*

We remark that:

– By hypothesis, the image of f_Y is Zariski dense.

– The surface Y is of general type because it dominates a surface of general type.

Thus Theorem 3.11 is consequence of the following theorem:

Theorem 3.13. *Let $(X, \mathscr{F})$ be a smooth projective surface equipped with a foliation. Then every map $f : \mathbf{C} \to X$ which is leaf of the foliation is algebraically degenerate: the Zariski closure of it is one dimensional.*

□

3.7. **Some heuristics about the proof of McQuillan's Theorem.** We make here some remarks which, we hope, will clarify a bit the ideas behind the proof of Theorem 3.13

Suppose *for the moment* that the foliation is smooth. This means that the exact sequence

$$0 \longrightarrow N^*_{\mathscr{F}} \longrightarrow \Omega^1_X \longrightarrow K_{\mathscr{F}} \longrightarrow 0 \tag{3.46}$$

is exact. More precisely $N^*_{\mathscr{F}}$ is a *sub bundle* of Ω^1_X.

The first Chern class of $N^*_{\mathscr{F}}$ is a class in $H^1(X, \Omega^1_X)$.

Lemma 3.14. *We have that*

$$c_1(N^*_{\mathscr{F}})) \in Im\{H^1(X, N^*_{\mathscr{F}}) \to H^1(X, \Omega^1_X)\}. \tag{3.47}$$

Proof. The inclusion $N^*_{\mathscr{F}} \to \Omega^1_X$ give rise to a global section $\omega \in H^0(X, \Omega^1_X \otimes N_{\mathscr{F}})$.

Choose a covering by polydisks $\{U_\alpha\}_{\alpha\in A}$ of X which trivializes everything: the restriction of N, Ω^1_X, the foliation etc. are trivial on every open set.

Since the foliation is smooth. Over every open set U_α we can choose coordinates z_α and w_α such that $\omega|_{U_\alpha} := \omega_\alpha = f_\alpha dz_\alpha$ for a suitable non vanishing holomorphic function f_α.

Moreover

$$\omega_\alpha = g_{\alpha,\beta}\omega_\beta \tag{3.48}$$

on $U_{\alpha\beta} := U_\alpha \cap U_\beta$, where $g_{\alpha,\beta} \in \mathscr{O}^*_{U_{\alpha\beta}}$ is a cocycle representing $N_{\mathscr{F}}$. Recall that, with this notation, the cocycle $\{\frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}}\}$ represents $c_1(N_{\mathscr{F}})$ in $H^1(X, \Omega^1_X)$.

By construction we can find holomorphic forms γ_α on U_α such that

$$d\omega_\alpha = \gamma_\alpha \wedge \omega_\alpha \tag{3.49}$$

(they will be $\gamma_\alpha = \frac{df_\alpha}{f_\alpha}$). If we differentiate equation 3.48 and we use 3.49 we obtain

$$\left(\gamma_\alpha - \gamma_\beta - \frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}}\right) \wedge \omega_\beta = 0. \tag{3.50}$$

Which means that, up to a coboundary, $\frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}}$ is proportional to ω_β. This is the conclusion of the lemma because $c_1(N_{\mathscr{F}})$ is represented by the class of $\frac{dg_{\alpha,\beta}}{g_{\alpha,\beta}}$. $\square$

We would show now how the Lemma below may explain the ideas behind the proof of McQuillan Theorem 3.11.

Suppose for the moment that the foliation in Theorem 3.13 is without singularities.

In this case, the foliation give rise to a section $p_{\mathscr{F}} : X \to \mathbf{P}$ such that $p^*_{\mathscr{F}}(\mathscr{O}(1)) = K_{\mathscr{F}}$.

Let $\iota : Y \hookrightarrow X$ be a smooth projective curve which is leaf of the foliation. By construction we will have that $\iota' : Y \to \mathbf{P}$ factorises through the image of $p_{\mathscr{F}}$ and we have then that

$$\deg(\iota^*(K_{\mathscr{F}})) = 2g(Y) - 2. \tag{3.51}$$

On the other side, since Y is a leaf of the foliation, the composite map

$$H^1(X, N^*_{\mathscr{F}}) \longrightarrow H^1(X, \Omega^1_X) \longrightarrow H^1(Y, \Omega^1_Y) \tag{3.52}$$

is the zero map. Thus Lemma 3.14 tels us that

$$\deg(\iota^*(N^*_{\mathscr{F}})) = 0 \tag{3.53}$$

, Consequently, since $K_X = N^*_{\mathscr{F}} \otimes K_{\mathscr{F}}$, we obtain that, if Y is a compact leaf of the foliation, then

$$\deg(K_X|_Y) \leq 2g(Y) - 2. \tag{3.54}$$

Suppose that $(X, \mathscr{F})$ is again a smooth surface of general type equipped with a smooth foliation. Suppose that $\iota : \mathbf{C} \to X$ is a leaf of the foliation.

Since we still have the section $p_{\mathscr{F}} : X \to \mathbf{P}$, the Tautological inequality 3.7 implies that

$$T_\iota(K_{\mathscr{F}}) \leq 0 \tag{3.55}$$

Indeed, $p^*_{\mathscr{F}}(\mathscr{O}(1)) = K_{\mathscr{F}}$ and we may suppose that $(p_{\mathscr{F}})_*(T_f) = T_{f'}$.

Lemma 3.14 tells us that, up to $\overline{\partial}$ of smooth forms, the first Chern form of $N^*_{\mathscr{F}}$ is represented by a class which is in $A^{1,1}(N^*_{\mathscr{F}})$ thus it will vanish when restricted to $\iota(\mathbf{C})$. Thus Lemma 3.3 implies that

$$T_\iota(N^*_{\mathscr{F}}) = 0 \tag{3.56}$$

Consequently, we obtain that $T_\iota(K_X) \leq 0$. This is impossible if X is of general type and $\iota(\mathbf{C})$ is Zariski dense.

In general the foliation in Theorem 3.13 cannot supposed to be regular everywhere and singularities of it prevent the argument above to work directly. Never the less, the proof in the general case will follow the ideas above.

In order to prove Theorem 3.13 we will then compute the value of the Ahfors current on the canonical line bundle and on the conormal line bundle of the foliation.

The presence of singularities of the foliation makes the proof much more involved. Since the general case is, technically, quite complicate, we will make some restrictive hypothesis on the possible singularities.

Before we start the main part of the proof, we need to recall some advances properties of foliations on surfaces.

3.8. **Some advanced facts about foliations on complex surfaces.** Let Δ be a two dimensional disk with coordinates (z, w). A foliation $\mathscr{F}$ on Δ is given by a derivation $\partial = a_1(z, w)\partial_z + a_2(z, w)\partial_w$, where $a_i(z, w)$ are analytic functions such that $Sing(\partial) := \{a_1(z, w) = a_2(z, w) = 0\}$ is zero dimensional. We will suppose that $\mathscr{F}$ is singular. Shrinking Δ if necessary we may suppose that $Sing(\partial) = \{(0, 0)\}$.

Let $\mathfrak{m}$ be the maximal ideal of $(0, 0)$ in $\mathscr{O}_\Delta$. Since $\mathscr{F}$ is singular in $(0, 0)$, the derivation ∂ defines a $\mathbf{C}$–linear map $\mathfrak{m} \to \mathfrak{m}$ verifying Liebnitz rule. Consequently it defines a linear map $L_{\mathscr{F}} : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$. Denote by λ_1 and λ_2 the eigenvalues of $L_{\mathscr{F}}$.

Definition 3.15. *The singularity* $(0, 0)$ *of the foliation* $\mathscr{F}$ *is said to be* reduced*, if at least one of the* λ_i*, for instance* λ_2*, is different from zero, and* $\lambda := \frac{\lambda_1}{\lambda_2} \notin \mathbf{Q}_{>0}$.

Of course, if X is a smooth surface and $\mathscr{F}$ is a foliation on it, we may give the definition of *reduced singularity* for every $p \in X$ which is singular for the foliation.

One of the main theorems of the theory of foliations is the following "resolution of singularities of foliations":

Theorem 3.16. *(Seidenberg) Let* X *be a smooth surface and* $\mathscr{F}$ *be a foliation on it. There exists a birational morphism* $p : \widetilde{X} \to X$ *such that the foliation* $p^*(\mathscr{F})$ *is with only reduced singularities. The birational morphism* p *is obtained by successive blow up of points which are singular for the foliation on the corresponding varieties.*

Consequently, in the proof of Theorem 3.13 we may suppose that the foliation has reduced singularities.

We will now list some properties of reduced singularities:

We suppose that $(X, \mathscr{F})$ is a surface with a foliation with only only reduced singularities.

a) If $p \in X$ is a singular point for the foliation, then there are at least one and at most two leaves passing through p. These leaves intersect properly (distinct tangents) in p and they are smooth.

b) If $p \in X$ and $h_p : X_p \to X$ is the blow up of X at p, then $h_p^*(\mathscr{F})$ is a foliation with only reduced singularities.

Observe that if $p \in X$ is a smooth point for the foliation, then $h_p^*(\mathscr{F})$ acquires a singular point which lies over the exceptional divisor of X_p. This singular point is the intersection between the strict transform of the leaf through p and the exceptional divisor. Moreover, the exceptional divisor is a leaf of the the foliation.

c) If $p \in X$ is a singular point of the foliation and $h_p : X_p \to X$ is the blow up of X at p then $h_p^*(K_{\mathscr{F}}) = K_{h_p^*(\mathscr{F})}$. Thus, the canonical sheaf of the foliation is essentially an invariant by blow up of foliations with reduced singularities.

The canonical sheaf $K_{\mathscr{F}}$ is not a true invariant by blow up of foliations with reduced singularities because if $p \in X$ is a smooth point of the foliation and E_p is the exceptional divisor of X_p, then $K_{h_p^*(\mathscr{F})} = h_p^*(K_{\mathscr{F}})(E_p)$.

A special kind of reduced foliations are *linear reduced foliations*: these are foliations which, near each point, are either regular or that we may choose local coordinates centered in the point in such a way the foliation is generated by a derivation of the form

$$\partial = \lambda_1 z \partial_z + \lambda_2 w \partial_w \tag{3.57}$$

with $\lambda_1 \cdot \lambda_2 \neq 0$ and $\frac{\lambda_1}{\lambda_2} \notin \mathbf{Q}_{>0}$.

To avoid many technical complications, from now on, we will suppose the following **restrictive hypothesis**:

The foliation involved in Theorem 3.13 has only linear reduced singularities.

Observe that, the fact that the singularities are reduced is not restrictive because of Seidenberg theorem 3.16. On the other side, the hypothesis that the singularities are linear is a true restriction with respect to the general case. The main steps and ideas of the proof are well described under this restrictive hypothesis and the proof in the general case requires only some more technical considerations.

We would like to check here that properties (a) and (b) above are verified for a surface with only linear reduced singularities.

Remark that the problem is local so we may suppose that we are on a two dimensional disk.

Let $\partial = \lambda_1 z \partial_z + \lambda_2 w \partial_w$ be a linear reduced foliation on a disk. Thus $\lambda_i \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbf{Q}_{>0}$.

(a_L) The only leaves passing through the origin $(0, 0)$ are the axes $z = 0$ and $w = 0$.

Proof. We begin by remarking that the axes are indeed leaves of the foliation. The conormal sheaf of the foliation is locally generated by the differential form $\omega_{\mathscr{F}} = \lambda_2 w dz - \lambda_1 z dw$. Thus the leaves (up to the leaf $w = 0$) will be locally solutions of the differential equation

$$z'(w) = \frac{\lambda_1}{\lambda_2} \cdot \frac{z(w)}{w}. \tag{3.58}$$

The fact that $\frac{\lambda_1}{\lambda_2} \notin \mathbf{Q}$ implies that the only solution through $(0, 0)$ is the the curve $z = 0$. □

b_L) If p is a singular point of a linear reduced foliation $\mathscr{F}$ and $h_p : X_p \to X$ is the blow up of X at p then $K_{h_p^*(\mathscr{F})} = h_p^*(K_{\mathscr{F}})$.

Proof. We recall that, if E_p is the exceptional divisor of X_p, then $K_{X_p} = h_p^*(K_X)(E_p)$.

The equation of the blow up is locally given by $w = zu$. Thus the derivation ∂ acts on the variable u by $\partial(u) = \partial\frac{w}{z} = \frac{z\cdot\partial(w) - w\cdot\partial(z)}{z^2} = (\lambda_2 - \lambda_1)u$. Which means that the foliation $h_p^*(\mathscr{F})$ is locally generated by $\lambda_1 z\partial_z + (\lambda_2 - \lambda_1)u\partial_u$. This shows that the foliation $h_p^*(\mathscr{F})$ is again linear reduced.

The conormal sheaf $N^*_{h_p^*(\mathscr{F})}$ is (locally) generated by $(\lambda_2 - \lambda_1)udz - \lambda_1 zdw$. On the other side, $h_p^*(N^*_{\mathscr{F}})$ is generated by $h_p^*(\lambda_2 wdz - \lambda_1 zdw) = \lambda_2 zudz - \lambda_1 z(zdu + udz) = z((\lambda_2 - \lambda_1)udz - \lambda_1 zdw)$. Since the equation of the exceptional divisor E_p is locally given by $z = 0$, the equation above means that $h_p^*(N^*_{\mathscr{F}})(E_p) = N^*_{h_p^*(\mathscr{F})}$.

Since $K_{X_p} = N^*_{h_p^*(\mathscr{F})} \otimes K_{h_p^*(\mathscr{F})}$ and $K_X = N^*_{\mathscr{F}} \otimes K_{\mathscr{F}}$, the conclusion follows. □

We also have the following property:

c_L) Suppose that $(X, \mathscr{F})$ is a foliated variety, $h_p : X_p \to X$ is the blow up of X at a singular point of $\mathscr{F}$ and E_p is the exceptional divisor of X_p. Then:

1) the foliation $h_p^*(\mathscr{F})$ has two singular points on E_p: they correspond to the intersection of E_p with the strict transforms of the leaves of $\mathscr{F}$ through p.

2) E_p is a leaf of $h_p^*(\mathscr{F})$.

3.9. **Foliations and logarithmic differentials.** Let $(X; \mathscr{F})$ be a surface with a foliation with only linear reduced singularities. Blowing up X we may suppose that the following property is verified:

Every algebraic curve $C \subset X$ which is a leaf of $\mathscr{F}$ is smooth.

By Jouanoulou Theorem 2.7, there are only finitely many algebraic leaves of $\mathscr{F}$ (otherwise, the foliation is a fibration and it cannot contain a dense leaf).

Consequently, let $F = \sum F_i$ be the divisor of algebraic leaves of $\mathscr{F}$. Each of the F_i is smooth and the divisor F is a simple normal crossing divisor.

Let $P := \{p_1, \ldots, p_n\}$ be the set of the singular points of the foliation and let $F_1 = \sum F_j \leq F$ be an effective divisor. Denote by $P_{F_1} \subset P$ the subset of points of P which are contained in F_1 and by P^{F_1} the set $P \setminus P_{F_1}$.

The exact sequence 3.23 induces an inclusion $N^*_{\mathscr{F}} \hookrightarrow \Omega^1_X(\log(F_1))$. Let $N^*_{\mathscr{F}}(\log(F_1))$ be the saturation of it inside $\Omega^1_X(\log(F_1))$. It can be considered as a "foliation with logarithmic poles".

Here we show:

Proposition 3.17. *The sheaf* $\Omega^1_X(\log(F_1))/N^*_{\mathscr{F}}(\log(F_1))$ *is locally free on* $X \setminus P^{F_1}$.

In particular, $\Omega^1_X(\log(F_1))/N^*_{\mathscr{F}}(\log(F_1))$ is locally free in the intersections $F_i \cap F_j$ with F_i and F_j appearing in F_1.

Proof. The problem is local, thus we may suppose that X is a two dimensional disk with coordinates z ad w centered at a point $(0,0) \in X \setminus P^{F_1}$.

If $p \notin F_1$ then in a neighborhood U_p of p, we have that $\Omega^1_X|_{U_p} \simeq \Omega^1_X(\log(F_1))|_{U_p}$, moreover by hypothesis, $\mathscr{F}$ is regular at p thus there is nothing to prove.

If $p \in F_i \setminus \bigcup_{j\neq i} F_j$ and p is regular for the foliation, then we may suppose that, in a neighborhood U_p of p, the equation of F_i is $w = 0$, $\Omega^1_X(\log(F_1))|_{U_p} = \mathscr{O}dz \oplus \mathscr{O}\frac{dw}{w}$ and the foliation $\mathscr{F}|_{U_p}$ is generated by ∂_z. Thus $N^*_{\mathscr{F}}|_{U_p} = \mathscr{O}dw$. Consequently, we have that $N^*_{\mathscr{F}}(\log(F_1)|_{U_p} = \mathscr{O}\frac{dw}{w}$ and $\Omega^1_X(\log(F_1))/N^*_{\mathscr{F}}(\log(F_1))|_{U_p} = \mathscr{O}dz$, thus it is free of rank one.

If $p \in F_i \setminus \bigcup_{j\neq i} F_j$ is not regular for the foliation, then again we may suppose that, in a neighborhood U_p of p, the equation of F_i is $w = 0$, $\Omega^1_X(\log(F_1))|_{U_p} = \mathscr{O}dz \oplus \mathscr{O}\frac{dw}{w}$ and, since the foliation is linear reduced, the foliation $\mathscr{F}|_{U_p}$ is generated by $\lambda_1 z\partial_z + \lambda_2 w\partial_w$. In this case $N^*_{\mathscr{F}}|_{U_p} = \mathscr{O}(\lambda_2 wdz - \lambda_1 zdw)$ and $N^*_{\mathscr{F}}(\log(F_1)|_{U_p} = \mathscr{O}(\lambda_2 dz - \lambda_1 z\frac{dw}{w})$. Thus $\Omega^1_X(\log(F_1))/N^*_{\mathscr{F}}(\log(F_1))|_{U_p}$ is free of rank one.

If $p \in F_i \cap F_j$ then we may suppose that in a neighborhood U_p of p, the equation of the divisor F_1 is $zw = 0$, $\Omega^1_X(\log(F_1))|_{U_p} = \mathscr{O}\frac{dz}{z} \oplus \mathscr{O}\frac{dw}{w}$ and again the foliation $\mathscr{F}|_{U_p}$ is generated by $\partial = \lambda_1 z\partial_z + \lambda_2 w\partial_w$. Thus $N^*_{\mathscr{F}}|_{U_p} = \mathscr{O}(\lambda_2 wdz - \lambda_1 zdw)$. In this case, $N^*_{\mathscr{F}}(\log(F_1)|_{U_p} = (\lambda_2\frac{dz}{z} + \lambda_1\frac{dw}{w})\mathscr{O}$ and thus $\Omega^1_X(\log(F_1))/N^*_{\mathscr{F}}(\log(F_1))|_{U_p}$ is free of rank one. □

We observe that, as corollary of the proof we find:

Corollary 3.18. *In the hypothesis above, we have*

$$N^*_{\mathscr{F}}(\log(F_1)) = N^*_{\mathscr{F}}(F_1). \tag{3.59}$$

Moreover, as a consequence of the corollary and formula 3.24 we find

Corollary 3.19. *In the hypothesis above, we have an exact sequence*

$$0 \longrightarrow N^*_{\mathscr{F}}(F_1) \longrightarrow \Omega^1_X(\log(F_1) \longrightarrow I_Z K_{\mathscr{F}} \longrightarrow 0 \tag{3.60}$$

where I_Z *is the ideal sheaf of the set of points* P^{F_1} *with reduced structure.*

3.10. **Intersection with the canonical bundle of the foliation.** We suppose that $(X, \mathscr{F})$ is a foliated surface of general type and that all the singular points are reduced linear. Let $f : \mathbf{C} \to X$ be a leaf of the foliation which is "parametrized by $\mathbf{C}$" and which is Zariski dense.

In this section we would like to prove

Theorem 3.20. *If T_f is the Ahlfors current associated to f then*

$$T_f(K_{\mathscr{F}}) \leq 0. \tag{3.61}$$

Proof. Let P be the set of singular points for the foliation. By hypothesis, through each $p \in P$ there are two leaves of the foliation which are smooth at p and meet transversally. By property (c_L), taking a blow up of X if necessary, we may suppose that one of the leaves through p is an algebraic curve.

We will say that $p \in P$ is *a small singular point for $\mathscr{F}$* is both leaves through it are algebraic. Otherwise we say that p is a *a big singular point.* Let $P_B := \{p_1 \ldots, p_N\}$ be the set of big singular points of $\mathscr{F}$.

Taking a blow up of X if necessary, we may suppose that each algebraic leaf of $\mathscr{F}$ is smooth and contains at most two singular points of the foliation. Moreover, by Jouanoulou theorem 2.7, there are only finitely many algebraic leaves.

Let $h : \widetilde{X} \to X$ be the blow up of X at all the big singular points of P. For each big singular point p_i let $E_i \subset \widetilde{X}$ be the exceptional divisor such that $h(E_i) = p_i$. Denote by $\widetilde{\mathscr{F}}$ the foliation $h^*(\mathscr{F})$.

The following properties hold:

– Property (c_L) implies that the cardinality of the set of big singular points of $h^*(\mathscr{F})$ is the same then the cardinality of the set of big singular points of $\mathscr{F}$.

– Since $\mathscr{F}$ is with only reduced singularities, we have $K_{\widetilde{\mathscr{F}}} = h^*(K_{\mathscr{F}})$.

– The map f lifts to a map $\widetilde{f} : \mathbf{C} \to \widetilde{X}$ and we denote by $T_{\widetilde{f}}$ the Ahlfors current associated to $\widetilde{f}$ and such that $h_*(T_{\widetilde{f}}) = T_f$.

– Since T_f and $T_{\widetilde{f}}$ are closed currents of type $(1,1)$ we may consider their class in $H^{1,1}(X)$ and $H^{1,1}(\widetilde{X})$ respectively. We can find real numbers ρ_i such that $T_{\widetilde{f}} = h^*(T_f) + \sum_i \rho_i E_i$.

As classes in cohomology, we may consider the self intersection of the Ahlfors classes and we find the relation

$$T^2_{\widetilde{f}} = T^2_f - \sum_i^n \rho_i^2 \tag{3.62}$$

Since $T_{\widetilde{f}}$ is a class which intersects every effective divisor positively, we have that $T^2_{\widetilde{f}} \geq 0$, consequently

$$\sum_i^n \rho_i^2 \leq T^2_f. \tag{3.63}$$

Moreover, remark that $T^2_{\widetilde{f}} \leq T^2_f$.

The couple $(\widetilde{X}, \widetilde{\mathscr{F}})$ will be again a foliated surface with only reduced linear singularities equipped with a leaf $\widetilde{f} : \mathbf{C} \to \widetilde{X}$. Observe that the number of singular points of the foliation $\widetilde{\mathscr{F}}$ will be the double of the number of the singular points of $\mathscr{F}$ but the number of big singular points of $\widetilde{\mathscr{F}}$ will be N.

We can iterate the process and obtain then a sequence of foliated surfaces

$$\cdots \longrightarrow (X_k;\mathscr{F}_k) \xrightarrow{h_{k-1}} (X_{k-1};\mathscr{F}_{k-1}) \xrightarrow{h_{k-2}} \cdots \xrightarrow{h_1} (X_1;\mathscr{F}_1) = (\widetilde{X};\mathscr{F}_1) \xrightarrow{h_0=h} (X_0;\mathscr{F}_0) = (X;\mathscr{F}) \tag{3.64}$$

Such that:

1) Each of the $h_k : X_{k+1} \to X_k$ is the blow up of X_k in the big singular points of the foliation $\mathscr{F}_k$.

2) The foliation $\mathscr{F}_k$ is $h^*_{k-1}(\mathscr{F}_{k-1})$. Consequently each $\mathscr{F}_k$ has only linear reduced singularities and $K_{\mathscr{F}_k} = h^*_{k-1}(K_{\mathscr{F}_{k-1}})$.

3) The map $f : \mathbf{C} \to X$ lifts to maps $f_k : \mathbf{C} \to X_k$. Thus we have *positive* Ahlfors currents T_{f_k} on X_k such that $(h_{k-1})_*(T_{f_k}) = T_{f_{k-1}}$.

Observe that property (3) uses in an essential way the fact that the image of the f_k are Zariski dense. If one make the analogous construction with an algebraic curve, the class associated to the strict transform of it in X_k will not be positive for k big enough.

4) The number of big singular points of the foliation $\mathscr{F}_k$ is N. Denote the set of big singular points of X_k by $\{p^k_1, \dots, p^k_N\}$. For each $j_k = 1, \dots, N$ denote by $E^k_{j_k}$ the exceptional divisor of X_k such that $h_{k-1}(E^k_{j_k}) = p^{k-1}_{j_k}$.

5) We can find real numbers $\rho^k_{j_k}$ such that, in $H^{1,1}(X_k)$, we have $T_{f_k} = h^*_{k-1}(T_{f_{k-1}}) + \sum_{j_k=1}^N \rho^k_{j_k} E^k_{j_k}$.

The same computation as before gives

$$0 \le T^2_{f_k} = T^2_{f_{k-1}} - \sum_{j_k=1}^{N} (\rho^k_{j_k})^2 \tag{3.65}$$

which, by induction gives

$$\sum_{k=1}^{\infty} \sum_{j_k=1}^{N} (\rho^k_{j_k})^2 \le T^2_f. \tag{3.66}$$

Which, in particular, implies that

$$\lim_{k\to\infty} \sum_{j_k=1}^{N} \rho^k_{j_k} = 0 \tag{3.67}$$

Definition of small and big singular points can be given for each of the foliation $\mathscr{F}_k$ on X_k. Let $S_{\mathscr{F}_k}$ be the set of algebraic leaves of $\mathscr{F}_k$ which contain only small singular points of $\mathscr{F}_k$. and $F_{\mathscr{F}_k} = \sum_{F\in S_{\mathscr{F}_k}} F$.

By construction, the divisor $F_{\mathscr{F}_k}$ is simple normal crossing and every small singular point of $\mathscr{F}_k$ is contained in it.

Denote by $g_k : \mathbf{P}_k := \mathbf{P}(\Omega^1_{X_k}(\log(F_{\mathscr{F}_k})) \to X_k$ the projective bundle of differentials on X_k with logarithmc poles on $(F_{\mathscr{F}})_k$ and by $\mathscr{O}_k(1)$ the tautological line bundle on it.

Moreover we have that, since $f_k(\mathbf{C})$ is a leaf of $\mathscr{F}_k$ which do not pass through the singular points contained in $F_{\mathscr{F}_k}$, we have $f_k(\mathbf{C}) \cap F_{\mathscr{F}_k} = \emptyset$.

By Corollary 3.19, the morphism $h_k : X_{k+1} \to X_k$ factorise through a morphism $h'_k : X_{k+1} \to \mathbf{P}_k$ and we have $(h'_k)^*(\mathscr{O}_k(1)) = h_k^*(K_{\mathscr{F}_k})(-\sum_{j_k} E^k_{j_k})$.

Since $f_k(\mathbf{C})$ is a leaf of the foliation $\mathscr{F}_k$, the morphism f lifts to a morphism $f'_{F_{\mathscr{F}_k}} : \mathbf{C} \to \mathbf{P}_k$ such that $(h'_k) \circ f_{k+1} = f'_{F_{\mathscr{F}_k}}$. Moreover, we can choose Ahlfors currents for f_{k+1} and $f'_{F_{\mathscr{F}_k}}$ (on X_{k+1} and $\mathbf{P}_k$ respectively) in such a way that $(h'_k)_*(T_{f'_{F_{\mathscr{F}_k}}}) = T_{f_k}$.

Since $h^*_{k-1}(K_{\mathscr{F}_{k-1}}) = K_{\mathscr{F}_k}$, by induction we obtain that

$$T_{f_{k+1}}(h^*_k(K_{\mathscr{F}_k})) = T_f(K_{\mathscr{F}}). \tag{3.68}$$

And on the other side we have

$$T_{f_{k+1}}(h^*_k(K_{\mathscr{F}_k})) = T_{f_{k+1}}((h'_k)^*(\mathscr{O}_k(1)) - \sum_{j_k} E^k_{j_k}) \tag{3.69}$$

The tautological inequality in the logarithmic case 3.9 and $f_k(\mathbf{C}) \cap F_{\mathscr{F}_k} = \emptyset$ imply that

$$T_{f_{k+1}}(h^*_k(K_{\mathscr{F}_k})) \leq -\sum_{j_k} T_{f_{k+1}}(E^k_{j_k}). \tag{3.70}$$

In $H^{1,1}(X_{k+1})$ we have $T_{f_{k+1}} = h^*_k(T_{f_k}) + \sum_{j_k} \rho^k_{j_k} E^k_{j_k}$. Consequently

$$-\sum_{j_k} T_{f_{k+1}}(E^k_{j_k}) = (h^*_k(T_{f_k}) + \sum_{j_k} \rho^k_{j_k} E^k_{j_k}; -\sum_{j_k} E^k_{j_k}) = \sum_{j_k=1}^{N} \rho^k_{j_k}. \tag{3.71}$$

Since the last sum converges to zero, the conclusion of the theorem follows. □

3.11. **Intersection with the conormal bundle of the foliation.** Let $N^a_{\mathscr{F}} st$ be the conormal sheaf of the foliation. We want now give an upper bound for the real number $T_f(c_1(N^*_{\mathscr{F}})$.

The aim of this section is to prove

Theorem 3.21. *Under the hypotheses above we have that*

$$T_f(c_1(N^*_{\mathscr{F}})) \leq 0. \tag{3.72}$$

This is definitely the hardest part of the proof. Here the restrictive hypothesis simplify the proof. In order to remove the restrictive hypothesis, it would be necessary to analyze more in details the dynamic of the foliation. In any case, the main ideas of the proof are already visible under the restrictive hypothesis which we assume here.

Before we start the proof, we recall the following facts:

Fact 1: *(Siu decomposition of currents)*: If T is a closed positive current of type $(1,1)$ on a smooth projective surface X, then there exist (at most countable many) projective curves $C_i \subset X$, positive numbers λ_i and a closed positive current R such that

$$T = \sum_i \lambda_i \delta_{C_i} + R \tag{3.73}$$

where:

a) δ_{C_i} is the current of integration along C_i;

b) R is a current such that $\dim\{z \in X \ / \ \nu(R, z) \geq 0\} = 0$; where $\nu(R, z)$ is the Lelong number of R at z.

Fact 2: *(Intersection of the conormal bundle with algebraic leaves)*: Let C be an algebraic leaf of the foliation. Let Z_C be the number of singular points of the foliation which are on C. Under the restrictive hypothesis for the foliation, the following equality holds:

$$(N^*_{\mathscr{F}}; C) = -C^2 - Z_C. \tag{3.74}$$

Definition 3.22. *A current T is said to be invariant under the foliation, if for every (local) section $\alpha \in A^{1,1}(N^*_{\mathscr{F}})$ we have $T(\alpha) = 0$.*

Let $T_f = \sum_i \lambda_i \delta_{C_i} + R_f$ be the Siu decomposition of the Ahlfors current T_f. Since the image of f is invariant under the foliation, one easily sees that the curves C_i must be leaves of the foliation and the current R also must be invariant under the foliation.

We recall that, by Jouanoulou theorem 2.7, there are only finitely many leaves C_j of the foliation which are algebraic curves and by Seidenberg theorem 3.16 the divisor $D := \sum_j C_j$ of the invariant curves may be supposed simple normal crossing. Remark that we can also suppose that each algebraic leaf contains at most 2 singular points for the foliation.

The first step toward the proof of Theorem 3.21 is the following which tells us that we can concentrate only on the "non algebraic part" of the Ahlfors current T_f:

Proposition 3.23. *Let D be the simple normal crossing divisor of algebraic leaves of the foliation $\mathscr{F}$. The following inequality holds:*

$$T_f(N^*_{\mathscr{F}}) \leq R_f(N^*_{\mathscr{F}}(D)). \tag{3.75}$$

Proof. Since, the divisors C_j intersect properly and only on singular points of the foliation, we have, by Fact 2,

$$(N^*_{\mathscr{F}}; C_j) \leq -C_j^2 - \sum_{i \neq j} (C_i; C_j). \tag{3.76}$$

From this we obtain that

$$(N^*_{\mathscr{F}}(D); \sum_j \lambda_j C_j) \leq 0. \tag{3.77}$$

But since, T_f is a nef class we have that $T_f(D) = \sum_j \lambda_j (C_j; D) + R_f(D) \geq 0$. Thus $\sum_j \lambda_j (C_j; D) \geq -R_f(D)$. Consequently

$$\begin{aligned} T_f(N^*_{\mathscr{F}}) = & \ \sum_j \lambda_j (C_j; N^*_{\mathscr{F}}) + R_f(N^*_{\mathscr{F}}) \\ \leq & \ -\sum_j \lambda_j C_j, D) + R_f(N^*_{\mathscr{F}}) \\ \leq & \ \qquad R_f(N^*_{\mathscr{F}}(D)). \end{aligned}$$

$\square$

The foliation $\mathscr{F}$ is defined by a global section $\alpha \in H0(X;\Omega(N_{\mathscr{F}}))$ and it induces a logarithmic foliation

$$0 \longrightarrow N^*_{\mathscr{F}}(D) \longrightarrow \Omega^1_X(\log(D)) \longrightarrow I_{Z'}K_{\mathscr{F}} \longrightarrow 0 \tag{3.78}$$

Where $I_{Z'}$ is the ideal of the Ideal sheaf of the *big singular points* of the foliation (cf. the proof of Theorem 3.20). Remark that small singular points are smooth for the logarithmic foliation.

Similarly, the logarithmic foliation 3.78 induces a global section

$$\omega \in H^0(X;\Omega^1_X(\log(D))(N_{\mathscr{F}}(-D)). \tag{3.79}$$

Let's analyze the structure of the differential forms α and ω:

Cover X by open sets U_i boholomorphic to two dimensional disks with coordinates (z_1, w_i) such that:

a) Each singular point is contained in only one of these disks which is centered on it.

b) If the divisor D intersects U_i and the restriction of the foliation to U_i is smooth, then the equation of $D|_{U_i}$ is $z_i = 0$.

c) If the divisor D intersects U_i, and the restriction of the foliation to U_i is singular (thus in this case, the only singular point of the foliation is $(0,0)$) and $(0,0)$ is a *big singular point* for the foliation, then the equation of $D|_{U_i}$ is $z_i = 0$.

d) If the divisor D intersects U_i, and the restriction of the foliation to U_i is singular (thus in this case, the only singular point of the foliation is $(0,0)$) and $(0,0)$ is a *small singular point* for the foliation, then the equation of $D|_{U_i}$ is $z_iw_i = 0$.

The local description of the form α is the following:

Denote by α_i the restriction of α to U_i. On $U_{ij} := U_i \cap U_j$ we have

$$\alpha_i = h_{ij}\alpha_j \tag{3.80}$$

where $h_{ij} \in \mathscr{O}^*_{U_{ij}}$ and $\{h_{ij}\}$ is a cocycle representing $N_{\mathscr{F}}$.

Moreover:

a.1) When U_i do not contain a singular point of the foliation, there is an holomorphic function $k_i \in \mathscr{O}^*_{U_i}$ such that $\alpha_i = k_i dz_i$.

b.1) When U_i contains a singular point of the foliation, then there is a holomorphic function $k_i \in \mathscr{O}^*_{U_i}$ such that $\alpha_i = k_i(\lambda_i w_i dz_i - z_i dw_i)$.

The local description of the form ω is the following:

Denote by ω_i the restriction of ω to U_i. On $U_{ij} := U_i \cap U_j$ we have

$$\omega_i = g_{ij}\omega_j \tag{3.81}$$

where $g_{ij} \in \mathscr{O}^*_{U_{ij}}$ and $\{g_{ij}\}$ is a cocycle representing $N_{\mathscr{F}}(-D)$.

Moreover:

a.2) When $D \cap U_i = \emptyset$, there is an holomorphic function $f_i \in \mathscr{O}^*_{U_i}$ such that $\omega_i = f_i dz_i$.

b.2) In the case (b) there is an holomorphic function $f_i \in \mathscr{O}^*_{U_i}$ such that $\omega_i = f_i \frac{dz_i}{z_i}$.

c.2) In the case (c) there is an holomorphic function $f_i \in \mathscr{O}^*_{U_i}$ such that $\omega_i = f_i(\lambda_i w_i \frac{dz_i}{z_i} - dw_i)$.

d.2) In the case (d) there is an holomorphic function $f_i \in \mathscr{O}^*_{U_i}$ such that $\omega_i = f_i(\lambda_i \frac{dz_i}{z_i} - \frac{dw_i}{w_i})$.

The second step is about the structure of the current R_f.

Proposition 3.24. *We can find a positive measure τ such that*

$$R_f = \tau_f \cdot \alpha \wedge \overline{\alpha} \tag{3.82}$$

Proof. Since the current R_f is positive of type $(1,1)$ it suffices to prove the proposition when X is replaced by the open set where the foliation is smooth (the singularities of $\mathscr{F}$ are just points, thus in codimension two). When the foliation is smooth, the statement is local on X so we may suppose that the foliation is generated by dz.

The current R_f can be written as $\tau_1 \cdot dz \wedge d\overline{z} + \tau_2 \cdot dz \wedge d\overline{w} + \tau_3 \cdot dw \wedge d\overline{z} + \tau_4 \cdot dw \wedge d\overline{w}$, where τ_i are positive measures. Since R_f is invariant under the foliation we must have $\tau_i = 0$ for $i \neq 1$. □

We now begin the computation of the "intersection" between the class R_f and the normal bundle of the foliation. The reader should compare this with the "heuristic section" 3.6.

Theorem 3.25. *Under the hypotheses above, we have*

$$R_f(N^*_{\mathscr{F}}(D)) \leq 0. \tag{3.83}$$

Proof. We fix a covering $\mathscr{U} = \{U_i\}$ of X as before and a partition of unity $\{\rho_i\}$ submitted to it. If $p \in X$ is a singular point of the foliation, we denote by U_p the only open set of $\mathscr{U}$ containing it and by $(z_p.w_p)$ the corresponding coordinates on it.

We fix a smooth non decreasing function $\chi(t)$ on $\mathbb{R}$ such that $\chi(t) = 1$ if $t \geq 1/\sqrt{2}$ and $\chi(t) = 0$ if $t \leq 1/2$.

Using the notation as before, when U_i is a open set of type (a) and (b) denote by β_i the holomorphic differential form $\frac{df_i}{f_i}$. When U_i is of type (d) (thus U_i contains a small singular point), then we pose $\beta_i = 0$ (as holomorphic differential form).

When U_i is of type (c) (thus U_i contains a big singular point), we pose

$$\beta_i = \frac{df_i}{f_i} + \gamma_i \tag{3.84}$$

where

$$\gamma_i := \frac{1}{|z_i|^2 + |w_i|^2}(\lambda \overline{z}_i dz_i + \overline{w}_i dw_i). \tag{3.85}$$

Observe that, in this case, the form β_i *are not holomorphic*, but only C^∞ in the complementary of the big singular points.

We eventually pose

$$\tau_i = \sum_k \rho_k(\beta_i - \beta_k - \frac{dg_{ik}}{g_{ik}}). \tag{3.86}$$

It is easy to verify that, for every i, we have

$$d\omega_i = \beta_i \wedge \omega_i \tag{3.87}$$

and that, over $U_{ij} := U_i \cap U_j$,

$$(\beta_i - \tau_i) - (\beta_j - \tau_j) = \frac{dg_{ij}}{g_{ij}}. \tag{3.88}$$

The equation 3.88 above tells us that the $(1,1)$ form (which a priory is just smooth)

$$\frac{1}{2\pi i} \cdot \overline{\partial}(\beta_i - \tau_i) \tag{3.89}$$

represents the first Chern class of $N^*_{\mathscr{F}}(D)$.

An explicit computation gives $\tau_i \wedge \omega_i = 0$, thus we obtain $\overline{\partial}\tau_i \wedge \omega_i = 0$ and from 3.87 we obtain $\overline{\partial}\beta_i \wedge \omega_i = 0$.

Denote by V the union of closed small disks of positive radius and centeder on big singular points, by U the open surface $X \setminus V$ and by $\mathbf{1_U}$ the characteristic function of U.

The considerations above imply that $\mathbf{1}_U R_f(c_1(N^*_{\mathscr{F}}(D)) = 0$. Indeed, we just proved that (as in the "heuristic case") over U, the first Chern Class of $N^*_{\mathscr{F}}(D)$ is represented, modulo exact forms, by a class in $A^{1,1}(N^*_{\mathscr{F}}(D))$. Thus, its intersection with R_f vanishes.

Consequently, in order to estimate $R_f(c_1(N^*_{\mathscr{F}}(D))$ we may restrict our attention to small disks around big singular points.

Fix a big singular point p. Denote by $\|z\|^2$ the norm $(|z_p|^2 + |w_p|^2)^2$, by α, β and τ the corresponding forms on the open set U_p.

In order to conclude, it suffices to prove that

$$\lim_{r \to 0} \int_{U_p} \tau_f \cdot \frac{1}{2\pi i} \cdot (\overline{\partial}\left(\chi(\|z\|^{2r})(\beta - \tau)\right) \wedge \alpha \wedge \overline{\alpha} = 0. \tag{3.90}$$

Since $\tau \wedge \alpha = 0$, $\overline{\partial}\tau \wedge \alpha = 0$ and $\overline{\partial}\beta \wedge \alpha = 0$, this yields to estimating the limit

$$\lim_{r \to 0} r \int_{U_p} \frac{\chi'(\|z\|^{2r})\|z\|^{2r}}{\|z\|^2} \cdot \overline{\partial}(\|z\|^2) \wedge \beta \wedge \alpha \wedge \overline{\alpha}. \tag{3.91}$$

Relation 3.87 gives $\beta \wedge \alpha = z_p d\omega = \lambda dw_p \wedge dz_p$. Consequently we get

$$\|\overline{\partial}(\|z\|^2) \wedge \beta \wedge \alpha \wedge \overline{\alpha}\|^2 = |\lambda|^2 \|z\|^2 \|dz_p \wedge d\overline{z}_p \wedge dw_p \wedge d\overline{w}_p\|^2 \tag{3.92}$$

Thus the conclusion follows. □

3.12. **Conclusion of the proof of Theorem 3.11.** We can now conclude the proof of the Green Griffiths conjecture for surfaces with big cotangent bundle. In order to prove it, it suffices to prove Theorem 3.13.

Suppose that $f : \mathbf{C} \to X$ is an entire curve satisfying the hypotheses of Theorem 3.13, and whose image is Zariski dense. We may suppose that the involved foliation $\mathscr{F}$ has only reduced singularities and we will suppose that they are reduced and linear. Denote by $K_{\mathscr{F}}$ the canonical line bundle of the foliation and by $N^*_{\mathscr{F}}$ its conormal sheaf.

Let T_f be the Ahlfors current associated to f.

By Theorem 3.20, we have $T_f(K_{\mathscr{F}}) \leq 0$.

By Theorem 3.21, we have $T_f(N^*_{\mathscr{F}}) \leq 0$.

Thus, since $K_X = K_{\mathscr{F}} + N^*_{\mathscr{F}}$, we deduce that $T_f(K_X) \leq 0$.

But, by hypothesis, X is of general type, which means that K_X is big. Thus, there is an ample line bundle L and an effective divisor D such that, for a positive integer m we have $mK_X = L + D$. By construction we may suppose that $T_f(L) > 0$ and, since T_f is nef, $T_f(D) \geq 0$. Thus $T_f(K_X) > 0$ and a contradiction follows.

We would like to remark that the restrictive hypothesis do not play an important role in the proof of 3.20. Only properties of reduced foliations are used and this hypothesis is not restrictive because of Seidenberg Theorem 3.16.

On the other side, the restrictive hypothesis plays a crucial role when we estimate the intersection of the Ahlfors current with the conormal bundle. Even if the main ideas of the proof are already described here, the general case, when we suppose only that the singularities are reduced, requires a deeper analysis of the Ahlfors current (and of its Lelong numbers) and of the local structure of reduced singularities of foliations.

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