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José Bertin, Jean-Pierre Demailly,
Luc Illusie, and Chris Peters

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Introduction to Hodge Theory

Jose Bertin, Jean-Pierre Demailly,
Luc Illusie, and Chris Peters

Translated by
James Lewis
Chris Peters

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Société Mathématique de France

Introduction à la Théorie de Hodge
(Introduction to Hodge Theory)

by José Bertin, Jean-Pierre Demailly, Luc Illusie, and Chris Peters

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L^2 Hodge theory and vanishing theorems by Jean-Pierre Demailly and *Frobenius and Hodge degeneration* by Luc Illusie were translated from the French by James Lewis.

Variations of Hodge structure. Calabi- Yau manifolds and, mirror symmetry by José Bertin and Chris Peters was translated from the French by Chris Peters.

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ABSTRACT. Hodge theory is a powerful tool in analytic and algebraic geometry. This book consists of expositions of aspects of modern Hodge theory, with the purpose of providing the nonexpert reader with a clear idea of the current state of the subject. The three main topics are: L^2 Hodge theory and vanishing theorems; Hodge theory in characteristic *p;* and variations of Hodge structures and mirror symmetry. Each section has a detailed introduction and numerous references. Many open problems are also included. The reader should have some familiarity with differential and algebraic geometry, with other prerequisites varying by chapter. The book is suitable as an accompaniment to a second course in algebraic geometry. This is the English translation of a volume previously published as volume 3 in the Panoramas et Synthèses series.

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Foreword

Each of the three chapters collected in this book is concerned with various aspects – important ones in several respects – of Hodge theory. The text is an expanded version, including substantial additions, of lectures presented on the occasion of the meeting "l'Etat de la Recherche" devoted to Hodge theory, that has been held at Université Joseph Fourier in Grenoble from Friday November 25, 1994 till Sunday November 27, under the auspices of the SMF (Société Mathématique de France). The authors wishes would be fulfilled if, in accordance with the general goals of sessions "l'Etat de la Recherche", this book could help the nonexpert reader to get a precise idea of the current status of Hodge theory.

The three main subjects developed here (L^2 Hodge theory and vanishing theorems, Frobenius and Hodge degeneration, Variations of Hodge structures and mirror symmetry) cover a wide range of techniques: elliptic PDE theory, complex differential geometry, algebraic geometry in characteristic p, cohomological and sheaf-theoretic methods, deformation theory of complex varieties, Calabi-Yau manifolds, a few aspects of singularity theory ... This accumulation of tools arising from various fields probably makes the access to the theory rather uneasy for newcomers. We hope that the present book will greatly facilitate this access: a special effort has been made to approach various themes by their most natural starting point, each of the three chapters being supplemented with a detailed introduction and numerous references. The reader will find precise statements of quite a number of open problems which have been the subject of active research in the last years.

The authors are grateful to SMF and MESR (Ministère de l'Enseignement Supérieur et de la Recherche) for their decisive action – both psychological and financial – without which the Grenoble session "Hodge theory" would probably never have taken place. They address special thanks to the Scientific Committee of Sessions l'Etat de la Recherche, in behalf of its two successive directors Pierre Schapira and Colette Mœglin, as well as to Michèle Audin, Editor in Chief of the Journal "Panoramas et Synthèses", for her strong encouragement to publish the present manuscript. Finally, they express their gratitude to the referee for his careful reading of the manuscript and a large number of invaluable suggestions.

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