L^2 Hodge Theory and Vanishing Theorems

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0. Introduction

The aim of these notes is to describe two fundamental applications of L^2 Hilbert space techniques to analytic or algebraic geometry: Hodge theory, and the theory of L^2 estimates for the $\overline{\partial}$ operator. The point of view adopted here is essentially analytic.

The first part is focussed on Hodge theory and it is intended to be rather introductory. Thus the reader will find here only the most elementary topics, mostly those due to W.V.D. Hodge himself [**Hod41**] or to A. Weil [**Wei57**]. Hodge theory, as first conceived by its creator, consists of the study of the cohomology of Riemannian or Kählerian manifolds, by means of a description of harmonic forms and their properties. We refer to the treatment of J. Bertin-Ch. Peters [**BePe95**] and L. Illusie [**Ill95**] for a presentation of more advanced topics and applications (variation of Hodge structure, application of periods, Hodge theory in characteristic $p > 0$ ...). We consider a Riemannian manifold X and a Euclidean or Hermitian bundle E over X. We assume that E is equipped with a connection D compatible with the metric: A connection is by definition a differential operator analogous to exterior differentiation, acting on forms of arbitrary degree with values in E, and satisfies Leibniz rule for the exterior product. The *Laplace-Beltrami operator* is the self-adjoint differential operator of second order $\Delta_E = D_E D_E^* + D_E^* D_E$, where D_E^* is the Hilbert space adjoint of D_E. One easily shows that Δ_E is an *elliptic operator*. The finiteness theorem for elliptic operators shows then that the space $\mathcal{H}^q(X,E)$ of harmonic q-forms with values in E is finite dimensional if X is compact (we say that a form u is harmonic if $\Delta_E u = 0$). If we assume in addition that the connection satisfies $D_E^2 = 0$, the operator D_E acting on forms of all degrees defines a complex called the *de Rham complex* with values in the local system of coefficients defined by E. The corresponding cohomology groups will be denoted by $H^q_{\mathrm{DR}}(X,E)$. The fundamental observation of Hodge theory is that any cohomology class contains a unique harmonic representative, since X is compact. It leads then to an isomorphism, called the *Hodge isomorphism*

$$H^q_{\mathrm{DR}}(X,E) \simeq \mathcal{H}^q_{\mathrm{DR}}(X,E). \tag{0.1}$$

When the manifold X and the bundle E are holomorphic, there exists a unique connection D_E called the *Chern connection*, compatible with the Hermitian metric on E and has the following properties: D_E splits into a sum $D_E = D'_E + D''_E$ of a connection D'_E of type $(1,0)$ and a connection D''_E of type $(0,1)$, such that $D'^2_E = D''^2_E = 0$ and $D'_E D''_E + D''_E D'_E = \Theta(E)$ (Chern curvature tensor of the bundle). The operator D''_E acting on the forms of bidegree (p,q) defines then for fixed p, a complex called the *Dolbeault complex.* When X is compact, the Dolbeault cohomology groups $H^{p,q}(X,E)$ satisfy a Hodge isomorphism analogous to (0.1), namely

$$H^{p,q}(X,E) \simeq \mathcal{H}^{p,q}(X,E), \tag{0.2}$$

where $\mathcal{H}^{p,q}(X,E)$ denotes the space of harmonic (p,q)-forms with values in E, relative to the anti-holomorphic Laplacian $\Delta''_E = D''_E D''^*_E + D''^*_E D''_E$. By utilizing this latter result, one easily proves the *Serre duality theorem*

$$H^{p,q}(X,E)^* \simeq H^{n-p,n-q}(X,E^*), \quad n = \dim_{\mathbb{C}} X, \tag{0.3}$$

which is the complex version of the Poincaré duality theorem. The central theorem of Hodge theory concerns compact Kähler manifolds: A Hermitian manifold (X,ω) is called *Kählerian* if the Hermitian $(1,1)$-form $\omega = \mathrm{i}\sum_{j,k}\omega_{jk}dz_j\wedge d\overline{z}_k$ satisfies $d\omega = 0$. A fundamental example of a compact Kählerian manifold is given by the projective algebraic manifolds. If X is compact Kählerian and if E is a local system of coefficients on X, the *Hodge decomposition theorem* asserts that

$$H^k_{\mathrm{DR}}(X,E) = \bigoplus_{p+q=k} H^{p,q}(X,E) \quad \text{(Hodge decomposition)} \tag{0.4}$$

$$\overline{H^{p,q}(X,E)} \simeq H^{q,p}(X,E^*). \qquad \text{(Hodge symmetry)} \tag{0.5}$$

The intrinsic character of the decomposition will be shown here in a somewhat original way, via the utilization of the Bott-Chern cohomology groups ($\partial\overline{\partial}$-cohomology groups). It follows from these results that the Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C})$ satisfy the symmetry property $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$, and that they are connected to the Betti numbers $b_k = \dim_{\mathbb{C}} H^k_{\mathrm{DR}}(X,\mathbb{C})$ by the relation $b_k = \sum_{p+q=k} h^{p,q}$. A certain number of other remarkable cohomological properties of compact Kähler manifolds are obtained by means of the primitive decomposition and the hard Lefschetz theorems (which in turn is a result of the existence of an $\mathfrak{sl}(2,\mathbb{C})$ action on harmonic forms). These results allow us to describe in a precise way the structure of the Picard group $\mathrm{Pic}(X) = H^1(X,\mathcal{O}^*)$ in the Kählerian case. In a more general setting, we discuss the Hodge-Frölicher spectral sequence (the spectral sequence connecting Dolbeault to de Rham cohomology), and we show how one can utilize this spectral sequence to obtain some general results on the Hodge numbers $h^{p,q}$ of compact complex manifolds. Finally, we establish the semi-continuity of the dimension of the cohomology groups $H^q(X_t,E_t)$ of bundles arising from a proper and smooth holomorphic fibration $\mathfrak{X}\to S$ (result due to Kodaira-Spencer), and we deduce from it that the Hodge numbers $h^{p,q}(X_t)$ are constant if the fibers X_t are Kählerian (invariance of the $h^{p,q}$ under deformations); the holomorphic nature of the Hodge filtration $F^pH^k(X_t,\mathbb{C}) = \oplus_{r\geq p}H^{r,k-r}(X_t,\mathbb{C})$ relative to the Gauss-Manin connection is proven by means of the theorem on the coherence of direct images, applied to the relative de Rham complex $\Omega^\bullet_{\mathfrak{X}/S}$ of $\mathfrak{X}\to S$.

In the second part, after recalling some of the relevant concepts of positivity and pseudoconvexity, we establish the *Bochner-Kodaira-Nakano identity* connecting the Laplacians Δ'_E and Δ''_E. The identity in question furnishes an explicit expression of the difference $\Delta''_E - \Delta'_E$ in terms of the curvature $\Theta(E)$ of the bundle. Under adequate hypothesis (weak pseudoconvexity of X, positivity of the curvature of E), one arrives a priori at the estimate

$$||D''_E u||^2 + ||D''^*_E u||^2 \geq \int_X \lambda(z)|u|^2 dV(z)$$

where λ is a positive function depending on the eigenvalues of curvature. The inequality is valid here for any form u of bidegree (n,q), $n = \dim X$, $q\geq 1$, with values in E, u belonging to the Hilbert space domains of D''_E and D''^*_E. By an argument of Hilbert space duality one deduces from this the following fundamental theorem, essentially due to Hörmander [**Hör65**] and Andreotti-Vesentini [**AV65**]:

0.6. Theorem. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let E be a Hermitian line bundle and suppose that the* *eigenvalues of the curvature form* $\mathrm{i}\Theta(E)$ *with respect to the metric* ω *at each point* $x \in X$, *satisfy*

$$\gamma_1(x) \le \cdots \le \gamma_n(x).$$

Further, suppose that the curvature is semi-positive, i.e. $\gamma_1 \ge 0$ *everywhere. Then for any form* $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ *such that*

$$D''_E g = 0 \quad \textit{and} \quad \int_X (\gamma_1 + \cdots + \gamma_q)^{-1} |g|^2 dV_\omega < +\infty,$$

there exists $f \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ *such that*

$$D''_E f = g \quad \textit{and} \quad \int_X |f|^2 dV_\omega \le \int_X (\gamma_1 + \cdots + \gamma_q)^{-1} |g|^2 dV_\omega.$$

An important observation is that the above theorem still remains valid when the metric h of E acquires singularities. The metric h is then given in each chart by a weight $\mathrm{e}^{-2\varphi}$ associated to a plurisubharmonic function φ (by definition φ is psh if the matrix of second derivatives $(\partial^2\varphi/\partial z_j \partial \overline{z}_k)$, calculated in the sense of distributions, is semi-positive at each point). Taking into account Theorem (0.6), it is natural to introduce the *multiplier ideal sheaf* $\mathcal{J}(h) = \mathcal{J}(\varphi)$, made up of the germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $\int_V |f|^2 \mathrm{e}^{-2\varphi}$ converges in a sufficiently small neighbourhood V of x. A recent result of A. Nadel [**Nad89**] guarantees that $\mathcal{J}(\varphi)$ is always a coherent analytic sheaf, whatever the singularities of φ. In this context, one deduces from (0.6) the following qualitative version, concerning the cohomology with values in the coherent sheaf $\mathcal{O}(K_X \otimes E) \otimes \mathcal{J}(h)$ ($K_X = \Lambda^n T_X^*$ being the *canonical bundle* of X).

0.7. Nadel Vanishing Theorem ([**Nad89**], [**Dem93b**]). *Let* (X, ω) *be a weakly pseudoconvex Kähler manifold, and let* E *be a holomorphic line bundle over* X *equipped with a singular Hermitian metric* h *of weight* φ. *Suppose that there exists a continuous positive function* ϵ *on* X *such that the curvature satisfies the inequality* $\mathrm{i}\Theta_h(E) \ge \epsilon\omega$ *in the sense of currents. Then*

$$H^q(X, \mathcal{O}(K_X \otimes E) \otimes \mathcal{J}(h)) = 0 \quad \textit{for all } q \ge 1.$$

In spite of the relative simplicity of the techniques involved, it is an extremely powerful theorem, which by itself contains many of the most fundamental results of analytic or algebraic geometry. Theorem (0.7) also contains the solution of the Levi problem (equivalence of holomorphic convexity and pseudoconvexity), the vanishing theorems of Kodaira-Serre, Kodaira-Akizuki-Nakano and Kawamata-Viehweg for projective algebraic manifolds, as well as the Kodaira embedding theorem characterizing these manifolds among the compact complex manifolds. By its intrinsic character, the "analytic" statement of Nadel's theorem appears useful even for purely algebraic applications. (The algebraic version of the theorem, known as the Kawamata-Viehweg vanishing theorem, utilizes the resolution of singularities and does not give such a clear description of the multiplier sheaf $\mathcal{J}(h)$.) In a recent work [**Siu96**], Y.T. Siu has shown the following remarkable result, by utilizing only the Riemann-Roch formula and an inductive Noetherian argument for the multiplier sheaves. The technique is described in §16 (with some improvements developed in [**Dem96**]).

0.8. Theorem [**Siu96**], [**Dem96**]). *Let X be a projective manifold and L an ample line bundle (i.e. has positive curvature) on X. Then the bundle $K_X^{\otimes 2} \otimes L^{\otimes m}$ is very ample for $m \geq m_0(n) = 2 + \binom{3n+1}{n}$, where $n = \dim X$.*

The importance of having an effective bound for the integer $m_0(n)$ is that one can also obtain embeddings of manifolds X in projective space, with a precise control of the degree of the embedding. As a consequence of this, one has a rather simple proof of a significant finiteness theorem, namely "*Matsusaka's big theorem*" (cf. [**Mat72**], [**KoM83**], [**Siu93**], [**Dem96**]):

0.9. Matsusaka's Big Theorem. *Let X be a projective manifold and L an ample line bundle over X. There exists an explicit bound $m_1 = m_1(n, L^n, K_X \cdot L^{n-1})$ depending only on the dimension $n = \dim X$ and on the first two coefficients of the Hilbert polynomial of L, such that mL is very ample for $m \geq m_1$.*

From this theorem, one easily deduces numerous finiteness results, in particular the fact that there exist only a finite number of families of deformations of polarized projective manifolds (X, L), where L is an ample line bundle with given intersection numbers L^n and $K_X \cdot L^{n-1}$.

Part I: L^2 Hodge Theory

1. Vector bundles, connections and curvature

The goal of this section is to recall some basic definitions of Hermitian differential geometry with regard to the concepts of connection, curvature and the first Chern class of line bundles.

1.A. Dolbeault cohomology and the cohomology of sheaves. Assume given X a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T_X^*$ the bundle of differential forms of bidegree (p,q) on X, i.e. differential forms which can be written

$$u = \sum_{|I|=p,\ |J|=q} u_{I,J} dz \wedge \overline{z}_J, \quad dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J := d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q},$$

where $(z_1, \ldots, z_n)$ are local holomorphic coordinates, and where $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multi-indices (increasing sequences of integers in the interval $[1, \ldots, n]$, with lengths $|I| = p,\ |J| = q$). Let $\mathcal{A}^{p,q}$ be the sheaf of germs of differential forms of bidegree (p,q) with complex valued C^∞ coefficients. We recall that the exterior derivative d decomposes into $d = d' + d''$ where

$$\begin{aligned} d'u &= \sum_{|I|=p,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J, \\ d''u &= \sum_{|I|=p,|J|=q,1\le k\le n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J \end{aligned}$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well known Dolbeault-Grothendieck Lemma asserts that all d''-closed forms of type (p,q) with $q > 0$ are locally d''-exact (this is the analogue for d'' of the usual Poincaré Lemma for d, see for example [**Hor66**]). In other words, the complex of sheaves $(\mathcal{A}^{p,\bullet}, d'')$ is exact in degree $q > 0$: and in degree $q = 0$, $\operatorname{Ker} d''$ is the sheaf Ω_X^p of germs of holomorphic forms of degree p on X.

More generally, if E is a holomorphic vector bundle of rank r over X, there exists a natural operator d'' acting on the space $C^\infty(X, \Lambda^{p,q}T_X^* \otimes E)$ of C^∞ (p,q)-forms with values in E. Indeed, if $s = \sum_{1\le\lambda\le r} s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of E, we can define $d''s := \sum (d''s_\lambda) \otimes e_\lambda$; by first observing that the transition matrices corresponding to a change of holomorphic frame are holomorphic, and which commute with the operation of d''. It then follows that the Dolbeault-Grothendieck Lemma still holds for forms with values in E. For every integer $p = 0,\ 1, \ldots, n$, the Dolbeault cohomology groups $H^{p,q}(X,E)$ are defined as being the cohomology of the complex of global forms of type (p,q) (indexed by q):

$$H^{p,q}(X,E) = H^q(C^\infty(X, \Lambda^{p,\bullet}T_X^* \otimes E)). \tag{1.1}$$

There is the following fundamental result of sheaf theory (de Rham-Weil Isomorphism Theorem): Let $(\mathcal{L}^\bullet, \delta)$ be a resolution of a sheaf $\mathcal{F}$ by acyclic sheaves, i.e. a complex $(\mathcal{L}^\bullet, \delta)$ given by an exact sequence of sheaves

$$0 \to \mathcal{F} \xrightarrow{j} \mathcal{L}^0 \xrightarrow{\delta^0} \mathcal{L}^1 \to \cdots \to \mathcal{L}^q \xrightarrow{\delta^q} \mathcal{L}^{q+1} \to \cdots,$$

where $H^s(X, \mathcal{L}^q) = 0$ for all $q \geq 0$ and $s \geq 1$. (To arrive at this latter condition of acyclicity, it is enough for example that the $\mathcal{L}^q$ are flasque or soft, for example a sheaf of modules over the sheaf of rings $\mathcal{C}^\infty$.) Then there is a functorial isomorphism

$$H^q(\Gamma(X, \mathcal{L}^\bullet)) \to H^q(X, \mathcal{F}). \tag{1.2}$$

We apply this in the following situation. Let $\mathcal{A}^{p,q}(E)$ be the sheaf of germs of C^∞ sections of $\Lambda^{p,q}T_X^* \otimes E$. Then $(\mathcal{A}^{p,\bullet}(E), d'')$ is a resolution of the locally free $\mathcal{O}_X$-module $\Omega_X^p \otimes \mathcal{O}(E)$ (Dolbeault-Grothendieck Lemma), and the sheaves $\mathcal{A}^{p,q}(E)$ are acyclic as $\mathcal{C}^\infty$-modules. According to (1.2), we obtain

1.3. Dolbeault Isomorphism Theorem (1953). *For all holomorphic vector bundles E on X, there exists a canonical isomorphism*

$$H^{p,q}(X, E) \simeq H^p(X, \Omega_X^p \otimes \mathcal{O}(E)).$$

If X is projective algebraic and if E is an algebraic vector bundle, the theorem of Serre (GAGA) [**Ser56**] shows that the algebraic cohomology groups $H^q(X, \Omega_X^p \otimes \mathcal{O}(E))$ computed via the corresponding algebraic sheaf in the Zariski topology are isomorphic to the corresponding analytic cohomology groups. Since our point of view here is exclusively analytic, we will no longer need to refer to this comparison theorem.

1.B. Connections on differentiable manifolds. Assume given a real or complex C^∞ vector bundle E of rank r on a differentiable manifold M of class C^∞. A *connection* D on E is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T_M^* \otimes E) \to C^\infty(M, \Lambda^{q+1} T_M^* \otimes E)$$

such that D satisfies Leibnitz rule:

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{1.4}$$

for all forms $f \in C^\infty(M, \Lambda^p T_M^*)$, $u \in C^\infty(X, \Lambda^q T_M^* \otimes E)$. On an open set $\Omega \subset M$ where E admits a trivialization $\tau : E_{|\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\tau \; du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T_M^* \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is a given matrix of 1-forms and where d acts componentwise on $u \simeq_\tau \; (u_\lambda)_{1 \leq \lambda \leq r}$. It is then easy to verify that

$$D^2 u \simeq_\tau \; (d\Gamma + \Gamma \wedge \Gamma) \wedge u \text{ on } \Omega.$$

Since D^2 is a globally defined operator, there exists a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T_M^* \otimes \mathrm{Hom}(E, E)) \tag{1.5}$$

such that $D^2 u = \Theta(D) \wedge u$ for any form u with values in E. This 2-form with values in $\mathrm{Hom}(E, E)$ is called the curvature tensor of the connection D.

Now suppose that E is equipped with a Euclidean metric (resp. Hermitian) of class C^∞ and that the isomorphism $E_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ).

We then have a canonical bilinear pairing, (resp. sesquilinear).

(1.6)
$$\begin{aligned} C^\infty(M, \Lambda^p T_M^* \otimes E) \times C^\infty(M, \Lambda^q T_M^* \otimes E) &\to C^\infty(M, \Lambda^{p+q} T_M^* \otimes \mathbb{C}) \\ (u, v) &\mapsto \{u, v\} \end{aligned}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \quad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is called *Hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u}\{u, Dv\}.$$

By assuming that (e_λ) is orthonormal, one easily verifies that D is Hermitian if and only if $\Gamma^* = -\Gamma$. In this case $\Theta(D)^* = -\Theta(D)$, therefore

$$\mathrm{i}\Theta(D) \in C^\infty(M, \Lambda^2 T_M^* \otimes \mathrm{Herm}(E, E)).$$

1.7. A particular case. For a complex line bundle L (a complex vector bundle of rank 1), the connection form Γ of a Hermitian connection D can be taken to be a 1-form with purely imaginary coefficients $\Gamma = \mathrm{i}A$ (A real). We then have $\Theta(D) = d\Gamma = \mathrm{i}dA$. In particular $\mathrm{i}\Theta(L)$ is a closed 2-form. The *first Chern class* of L is defined to be the cohomology class

$$c_1(L)_\mathbb{R} = \left\{ \frac{\mathrm{i}}{2\pi} \Theta(D) \right\} \in H^2_{\mathrm{DR}}(M, \mathbb{R}).$$

This cohomology class is independent of the choice of connection, since any other connection D_1 differs by a global 1-form, $D_1 u = Du + B \wedge u$, so that $\Theta(D_1) = \Theta(D) + dB$. It is well-known that $c_1(L)_\mathbb{R}$ is the image in $H^2(M, \mathbb{R})$ of an integral class $c_1(L) \in H^2(M, \mathbb{Z})$. Indeed if $\mathcal{A} = \mathcal{C}^\infty$ is the sheaf of C^∞ functions on M, then via the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{A} \xrightarrow{e^{2\pi \mathrm{i} \bullet}} \mathcal{A}^* \to 0,$$

$c_1(L)$ can be defined in Čech cohomology as the image of the cocycle $\{g_{jk}\} \in H^1(M, \mathcal{A}^*)$ defining L by the coedge map $H^1(M, \mathcal{A}^*) \to H^2(M, \mathbb{Z})$. See for example [**GH78**] for more details.

1.C. Connections on complex manifolds. We now study those properties of connections governed by the existence of a complex structure on the base manifold. If $M = X$ is a complex manifold, any connection D on a complex C^∞ vector bundle E can be split in a unique manner as a sum of a $(1,0)$-connection and a $(0,1)$-connection, $D = D' + D''$. In a local trivialization τ given by a C^∞ frame, one can write

$$D'u \simeq_\tau d'u + \Gamma' \wedge u, \tag{1.8$'$}$$
$$D''u \simeq_\tau d''u + \Gamma'' \wedge u, \tag{1.8$''$}$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is Hermitian if and only if $\Gamma' = -(\Gamma'')^*$ relative to any orthonormal frame. As a consequence, there exists a unique Hermitian connection D associated to a $(0,1)$-connection prescribed by D''.

Now suppose that the bundle E is endowed with a *holomorphic* structure. The unique Hermitian connection whose component D'' is the operator d'' defined in §1.A is called the *Chern connection* of E. With respect to a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by the Hermitian matrix $H = (h_{\lambda\mu})$ where $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. We have

$$\{u, v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu \ = \ u^\dagger \wedge H\overline{v},$$

where $u^\dagger$ is the transpose matrix of u, and an easy calculation gives

$$\begin{aligned} d\{u, v\} &= (du)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge (dH \wedge \overline{v} + H\overline{dv}) \\ &= (du + \overline{H}^{-1} d'\overline{H} \wedge u)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge \overline{(dv + \overline{H}^{-1} d'\overline{H} \wedge v)}, \end{aligned}$$

by using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Consequently the Chern connection D coincides with the Hermitian connection defined by

$$\left\{ \begin{array}{ll} Du & \simeq_\tau du + \overline{H}^{-1} d'\overline{H} \wedge u, \\ D' & \simeq_\tau d' + \overline{H}^{-1} d'\overline{H} \wedge \bullet = \overline{H}^{-1} d'(\overline{H}\bullet), \quad D'' = d''. \end{array} \right. \tag{1.9}$$

These relations show that $D'^2 = D''^2 = 0$. Consequently $D^2 = D'D'' + D''D'$, and the curvature tensor $\Theta(D)$ is of type $(1, 1)$. Since $d'd'' + d''d' = 0$, we obtain

$$(D'D'' + D''D')u \simeq_\tau \overline{H}^{-1} d'\overline{H} \wedge d''u + d''(\overline{H}^{-1} d'\overline{H} \wedge u) = d''(\overline{H}^{-1} d'\overline{H}) \wedge u.$$

1.10. Proposition. *The Chern curvature tensor* $\Theta(E) := \Theta(D)$ *satisfies*

$$\mathrm{i}\Theta(E) \in C^\infty(X, \Lambda^{1,1} T^*_X \otimes \mathrm{Herm}(E, E)).$$

If $\tau : E_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ *is a holomorphic trivialization and if* H *is the Hermitian matrix representative of the metric along the fibers of* $E_{\restriction\Omega}$*, then*

$$\mathrm{i}\Theta(E) \simeq_\tau \mathrm{i}d''(\overline{H}^{-1} d'\overline{H}) \quad on \quad \Omega. \qquad \square$$

If $(z_1, \ldots, z_n)$ are holomorphic coordinates on X and if $(e_\lambda)_{1\le\lambda\le r}$ is an orthogonal frame of E, one can write

$$\mathrm{i}\Theta(E) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu, \tag{1.11}$$

where $(c_{jk\lambda\mu}(x))$ are the coefficients of the curvature tensor of E at any point $x \in X$.

2. Differential operators on vector bundles

We first describe some basic concepts concerning differential operators (symbol, composition, ellipticity, adjoint), in the general context of vector bundles. Assume given M a manifold of differentiable class C^∞, $\dim_{\mathbb{R}} M = m$, and E, F given $\mathbb{K}$ vector bundles on M, over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ such that rank $E = r$, rank $F = r'$.

2.1. Definition. A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : C^\infty(M,E) \to C^\infty(M,F),\ u \mapsto Pu$ of the form

$$Pu(x) = \sum_{|\alpha|\le\delta} a_\alpha(x) D^\alpha u(x),$$

where $E_{\restriction\Omega} \simeq \Omega\times\mathbb{K}^r$, $F_{\restriction\Omega} \simeq \Omega\times\mathbb{K}^{r'}$ are local trivializations on an open chart $\Omega \subset M$ with local coordinates $(x_1,\ldots,x_m)$, and the coefficients $a_\alpha(x)$ are $r'\times r$ matrices $(a_{\alpha\lambda\mu}(x))_{1\le\lambda\le r',1\le\mu\le r}$ with C^∞ coefficients on Ω. One writes here $D^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_m)^{\alpha_m}$ as usual, and the matrices $u=(u_\mu)_{1\le\mu\le r}$, $D^\alpha u = (D^\alpha u_\mu)_{1\le\mu\le r}$ are viewed as column vectors.

If $t\in\mathbb{K}$ is a parameter and $f\in C^\infty(M,\mathbb{K})$, $u\in C^\infty(M,E)$, an easy calculation shows that $e^{-tf(x)}P(e^{tf(x)}u(x))$ is a polynomial of degree δ in t, of the form

$$e^{-tf(x)}P(e^{tf(x)}u(x)) = t^\delta\sigma_P(x,df(x))\cdot u(x)\ +\ \text{terms } c_j(x)t^j \text{ of degree } j<\delta,$$

where σ_P is a homogeneous polynomial map $T^*_M \to \mathrm{Hom}(E,F)$ defined by

$$(2.2)\qquad T^*_{M,x}\ni\xi\mapsto\sigma_P(x,\xi)\in\mathrm{Hom}(E_x,F_x),\quad \sigma_P(x,\xi)=\sum_{|\alpha|=\delta}a_\alpha(x)\xi^\alpha.$$

Then $\sigma_P(x,\xi)$ is a C^∞ function of the variables $(x,\xi)\in T^*_M$, and this function is independent of the choice of coordinates or trivialization used for E, F. σ_P is called the principal symbol of P. The principal symbol of a composition $Q\circ P$ of differential operators is simply the product.

$$(2.3)\qquad \sigma_{Q\circ P}(x,\xi)=\sigma_Q(x,\xi)\sigma_P(x,\xi),$$

calculated as a product of matrices. The differential operators for which the symbols are injective play a very important role:

2.4. Definition. A differential operator P is said to be elliptic if $\sigma_P(x,\xi)\in\mathrm{Hom}(E_x,F_x)$ is injective for all $x\in M$ and $\xi\in T^*_{M,x}\setminus\{0\}$.

Let us now assume that M is oriented and assume given a C^∞ volume form $dV(x)=\gamma(x)dx_1\wedge\cdots\wedge dx_m$, where $\gamma(x)>0$ is a C^∞ density. If E is a Euclidean or Hermitian vector bundle, we can define a Hilbert space $L^2(M,E)$ of global sections with values in E, being the space of forms u with measurable coefficients which are square summable sections with respect to the scalar product

$$(2.5)\qquad ||u||^2=\int_M |u(x)|^2dV(x),$$

$$(2.5')\qquad \langle\langle u,v\rangle\rangle=\int_M\langle u(x),v(x)\rangle dV(x),\quad u,v\in L^2(M,E).$$

2.6. Definition. If $P:C^\infty(M,E)\to C^\infty(M,F)$ is a differential operator and if the bundles E, F are Euclidean or Hermitian, there exists a unique differential operator

$$P^*:C^\infty(M,F)\to C^\infty(M,E),$$

called the formal adjoint of P, such that for all sections $u\in C^\infty(M,E)$ and $v\in C^\infty(M,F)$ one has an identity

$$\langle\langle Pu,v\rangle\rangle=\langle\langle u,P^*v\rangle\rangle,\quad \textit{whenever}\ \mathrm{Supp}\,u\cap\mathrm{Supp}\,v\ \subset\subset\ M.$$

PROOF. The uniqueness is easy to verify, being a consequence of the density of C^∞ forms with compact support in $L^2(M,E)$. By a partition of unity argument, we reduce the verification of the existence of P^* to the proof of its local existence. Now let $Pu(x) = \Sigma_{|\alpha|\le\delta} a_\alpha(x) D^\alpha u(x)$ be the description of P relative to the trivializations of E, F associated to an orthonormal frame and to the system of local coordinates on an open set $\Omega \subset M$. By assuming $\mathrm{Supp}\, u \cap \mathrm{Supp}\, v \subset\subset \Omega$, integration by parts gives

$$\begin{aligned}\langle\langle Pu, v\rangle\rangle &= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} a_{\alpha\lambda\mu} D^\alpha u_\mu(x) \overline{v}_\lambda(x)\gamma(x) dx_1,\ldots,dx_m \\ &= \int_\Omega \sum_{|\alpha|\le\delta,\lambda,\mu} (-1)^{|\alpha|} u_\mu(x) \overline{D^\alpha(\gamma(x)\overline{a}_{\alpha\lambda\mu} v_\lambda(x)} dx_1,\ldots,dx_m \\ &= \int_\Omega \langle u, \sum_{|\alpha|\le\delta} (-1)^{|\alpha|}\gamma(x)^{-1} D^\alpha\big(\gamma(x)\overline{a}_\alpha^\dagger v(x)\big)\rangle dV(x).\end{aligned}$$

We thus see that P^* exists, and is defined in a unique way by

$$P^* v(x) = \sum_{|\alpha|\le\delta} (-1)^{|\alpha|}\gamma(x)^{-1} D^\alpha\big(\gamma(x)\overline{a}_\alpha^\dagger v(x)\big). \square \tag{2.7}$$

Formula (2.7) shows immediately that the principal symbol of P^* is given by

$$\sigma_{P^*}(x,\xi) = (-1)^\delta \sum_{|\alpha|=\delta} \overline{a}_\alpha^\dagger \xi^\alpha = (-1)^\delta \sigma_P(x,\xi)^*. \tag{2.8}$$

If rank E = rank F, the operator P is elliptic if and only if $\sigma_P(x,\xi)$ is invertible for $\xi \neq 0$, therefore the ellipticity of P is equivalent to that of P^*.

3. Fundamental results on elliptic operators

We assume throughout this section that M is a compact oriented C^∞ manifold of dimension m, with volume form dV. Let $E \to M$ be a C^∞ Hermitian vector bundle of rank r on M.

3.A. Sobolev spaces. For any real number s, we define the Sobolev space $W^s(\mathbb{R}^m)$ to be the Hilbert space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^m)$ such that the Fourier transform $\hat{u}$ is a L^2_{loc} function satisfying the estimate

$$||u||_s^2 = \int_{\mathbb{R}^m} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\lambda(\xi) \ < \ +\infty. \tag{3.1}$$

If $s \in \mathbb{N}$, we have

$$||u|_s^2 \sim \int_{\mathbb{R}^m} \sum_{|\alpha|\le s} |D^\alpha u(x)|^2 d\lambda(x),$$

therefore $W^s(\mathbb{R}^m)$ is the Hilbert space of functions u such that all the derivatives $D^\alpha u$ of order $|\alpha| \le s$ are in $L^2(\mathbb{R}^m)$.

More generally, we denote by $W^s(M,E)$ the Sobolev space of sections $u : M \to E$ whose components are locally in $W^s(\mathbb{R}^m)$ on all open charts. More precisely, choose a finite subcovering (Ω_j) of M by open coordinate charts $\Omega_j \simeq \mathbb{R}^m$ on which E is trivial.

Consider an orthonormal frame $(e_{j,\lambda})_{1\le\lambda\le r}$ of $E_{\restriction\Omega_j}$ and write u in terms of its components, i.e. $u=\sum u_{j,\lambda}e_{j,\lambda}$. We then set

$$||u||_s^2=\sum_{j,\lambda}||\psi_j u_{j,\lambda}||_s^2$$

where (ψ_j) is a "partition of unity" subordinate to (Ω_j), such that $\sum\psi_j^2=1$. The equivalence of norms $||\ \ ||_s$ is independent of choices made. We will need the following fundamental facts, that the reader will be able to find in many of the specialized works devoted to the theory of partial differential equations.

3.2. Sobolev lemma. *For an integer $k\in\mathbb{N}$ and any real numbers $s\ge k+\frac{m}{2}$, we have $W^s(M,E)\subset C^k(M,E)$ and the inclusion is continuous.* □

It follows immediately from the Sololev lemma that

$$\bigcap_{s\ge 0}W^s(M,E)=C^\infty(M,E),$$

$$\bigcup_{s\le 0}W^s(M,E)=\mathcal{D}'(M,E).$$

3.3. Rellich lemma. *For all $t>s$, the inclusion*

$$W^t(M,E)\hookrightarrow W^s(M,E)$$

is a compact linear operator. □

3.B. Pseudodifferential operators. If $P=\sum_{|\alpha|\le\delta}a_\alpha(x)D^\alpha$ is a differential operator on $\mathbb{R}^m$, the Fourier inversion formula gives

$$Pu(x)=\int_{\mathbb{R}^m}\sum_{|\alpha|\le\delta}a_\alpha(x)(2\pi\mathrm{i}\xi)^\alpha\hat{u}(\xi)e^{2\pi\mathrm{i}x\cdot\xi}d\lambda(\xi),\quad\forall u\in\mathcal{D}(\mathbb{R}^m),$$

where $\hat{u}(\xi)=\int_{\mathbb{R}^m}u(x)e^{-2\pi\mathrm{i}x\cdot\xi}d\lambda(x)$ is the Fourier transform of u. We call

$$\sigma(x,\xi)=\sum_{|\alpha|\le\delta}a_\alpha(x)(2\pi\mathrm{i}\xi)^\alpha,$$

the symbol (or total symbol) of P.

A *pseudodifferential operator* is an operator Op_σ defined by a formula of the type

$$\mathrm{Op}_\sigma(u)(x)=\int_{\mathbb{R}^m}\sigma(x,\xi)\hat{u}(\xi)e^{2\pi\mathrm{i}x\cdot\xi}d\lambda(\xi),\quad u\in\mathcal{D}(\mathbb{R}^m),\tag{3.4}$$

where σ belongs to a suitable class of functions on $T^*_{\mathbb{R}^m}$. The standard class of symbols $S^\delta(\mathbb{R}^m)$ is defined as follows: Assume given $\delta\in\mathbb{R}$, $S^\delta(\mathbb{R}^m)$ is the class of C^∞ functions $\sigma(x,\xi)$ on $T^*_{\mathbb{R}^m}$ such that for any α, $\beta\in\mathbb{N}^m$ and any compact subset $K\subset\mathbb{R}^m$ one has an estimate

$$|D_x^\alpha D_\xi^\beta\sigma(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|)^{\delta-|\beta|},\quad\forall(x,\xi)\in K\times\mathbb{R}^m,\tag{3.5}$$

where $\delta\in\mathbb{R}$ is regarded as the "degree" of σ. Then $\mathrm{Op}_\sigma(u)$ is a well defined C^∞ function on $\mathbb{R}^m$, since $\hat{u}$ belongs to the class $\mathcal{S}(\mathbb{R}^m)$ of functions having rapid decay. In the more general situation of operators acting on a bundle E and having values in a bundle F over a compact manifold M, we introduce the analogous space of symbols $S^\delta(M;E,F)$. The elements of $S^\delta(M;E,F)$ are the functions

$$T_M^* \ni (x,\xi) \mapsto \sigma(x,\xi) \in \mathrm{Hom}(E_x,F_x)$$

satisfying condition (3.5) in all coordinate systems. Finally, we take a finite trivializing cover (Ω_j) of M and a "partition of unity" (ψ_j) subordinate to Ω_j such that $\sum \psi_j^2 = 1$, and we define

$$\mathrm{Op}_\sigma(u) = \sum \psi_j \mathrm{Op}_\sigma(\psi_j u), \quad u \in C^\infty(M,E),$$

in a way which reduces the calculations to the situation of $\mathbb{R}^m$. The basic results pertaining to the theory of pseudodifferential operators are summarized below.

3.6. Existence of extensions to the spaces W^s. If $\sigma \in S^\delta(M;E,F)$, then Op_σ extends uniquely to a continuous linear operator

$$\mathrm{Op}_\sigma : W^s(M,E) \to W^{s-\delta}(M,F). \qquad \square$$

In particular if $\sigma \in S^{-\infty}(M;E,F) := \bigcap S^\delta(M;E,F)$, then Op_σ is a continuous operator sending an arbitrary distributional section of $\mathcal{D}'(M,E)$ into $C^\infty(M,F)$. Such an operator is called a *regular operator*. It is a standard result in the theory of distributions that the class $\mathcal{R}$ of regular operators coincides with the class of operators defined by means of a C^∞ kernel $K(x,y) \in \mathrm{Hom}(E_y,F_x)$. That is, the operators of the form

$$R : \mathcal{D}'(M,E) \to C^\infty(M,F), \quad u \mapsto Ru, \quad Ru(x) = \int_M K(x,y)\cdot u(y)dV(y).$$

Conversely, if $dV(y) = \gamma(y)dy_1\cdots dy_m$ on Ω_j and if we write $Ru = \sum R(\theta_j u)$, where (θ_j) is a partition of unity, the operator $R(\theta_j\bullet)$ is the pseudodifferential operator associated to the symbol σ defined by the partial Fourier transform

$$\sigma(x,\xi) = \big(\gamma(y)\theta_j(y)K(x,y)\big)^\wedge_y(x,\xi), \quad \sigma \in S^{-\infty}(M;E,F).$$

When one works with pseudodifferential operators, it is customary to work modulo the regular operators and to allow operators more generally of the form $\mathrm{Op}_\sigma + R$ where $R \in \mathcal{R}$ is an arbitrary regular operator.

3.7. Composition. *If $\sigma \in S^\delta(M;E,F)$ and $\sigma' \in S^{\delta'}(M;F,G)$, δ, $\delta' \in \mathbb{R}$, there exists a symbol $\sigma'\Diamond\sigma \in S^{\delta+\delta'}(M;E,G)$ such that $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_\sigma = \mathrm{Op}_{\sigma'\Diamond\sigma}$ mod $\mathcal{R}$. Moreover*

$$\sigma'\Diamond\sigma - \sigma'\cdot\sigma \in S^{\delta+\delta'-1}(M;E,G).$$

3.8. Definition. A pseudodifferential operator Op_σ of degree δ is called elliptic if it can be defined by a symbol $\sigma \in S^\delta(M,E,F)$ such that

$$|\sigma(x,\xi)\cdot u| \geq c|\xi|^\delta|u|, \quad \forall(x,\xi)\in T_M^*, \quad \forall u \in E_x$$

for $|\xi|$ large enough, the estimate being uniform for $x \in M$.

If E and F have the same rank, the ellipticity condition implies that $\sigma(x,\xi)$ is invertible for large ξ. By taking a suitable truncating function $\theta(\xi)$ equal to 1 for large ξ, one sees that the function $\sigma'(x,\xi) = \theta(\xi)\sigma(x,\xi)^{-1}$ defines a symbol in the space $S^{-\delta}(M;F,E)$, and according to (3.8) we have $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_\sigma = \mathrm{Id} + \mathrm{Op}_\rho$, $\rho \in S^{-1}(M;E,E)$. Choose a symbol τ asymptomatically equivalent (at infinity) to the expansion $\mathrm{Id} - \rho + \rho^{\diamondsuit 2} + \cdots + (-1)^j \rho^{\diamondsuit j} + \cdots$. It is clear then that one obtains an inverse $\mathrm{Op}_{\tau\diamondsuit\sigma'}$ of Op_σ modulo $\mathcal{R}$. An easy consequence of this observation is the following:

3.9. Gårding inequality. Assume given $P : C^\infty(M,E) \to C^\infty(M,F)$ an elliptic differential operator of degree δ, where rank E = rank $F = r$, and let $\tilde{P}$ be an extension of P with distributional coefficient sections. For all $u \in W^0(M,E)$ such that $\tilde{P}u \in W^s(M,F)$, one then has $u \in W^{s+\delta}(M,E)$ and

$$||u||_{s+\delta} \leq C_s(||\tilde{P}u||_s + ||u||_0),$$

where C_s is a positive constant depending only on s.

PROOF. Since P is elliptic, there exists a symbol $\sigma \in S^{-\delta}(M;F,E)$ such that $\mathrm{Op}_\sigma \circ \tilde{P} = \mathrm{Id} + R$, $R \in \mathcal{R}$. Then $||\mathrm{Op}_\sigma(v)||_{s+\delta} \leq C||v||_s$ by applying (3.6). Consequently, in setting $v = \tilde{P}u$, we see that $u = \mathrm{Op}_\sigma(\tilde{P}u) - Ru$ satisfies the desired estimate. □

3.C. Finiteness theorem. We conclude this section with the proof of the following fundamental finiteness theorem, which is the starting point of L^2 Hodge theory.

3.10. FINITENESS THEOREM. *Assume given E, F Hermitian vector bundles on a compact manifold M, such that rank E = rank $F = r$; and given $P : C^\infty(M,E) \to C^\infty(M,F)$ an elliptic differential operator of degree δ. Then:*

i) $\mathrm{Ker}\, P$ *is finite dimensional.*

ii) $P(C^\infty(M,E))$ is closed and of finite codimension in $C^\infty(M,F)$; moreover, if P^* is the formal adjoint of P, there exists a decomposition.

$$C^\infty(M,F) = P(C^\infty(M,E)) \oplus \mathrm{Ker}\, P^*$$

as an orthogonal direct sum in $W^0(M,F) = L^2(M,F)$.

PROOF. (i) The Gårding inequality shows that $||u||_{s+\delta} \leq C_s||u||_0$ for all $u \in \mathrm{Ker}\, P$. By the Sobolev Lemma, this implies that $\mathrm{Ker}\, P$ is closed in $W^0(M,E)$. Moreover, the $||\ \ ||_0$-closed unit ball of $\mathrm{Ker}\, P$ is contained in the $||\ ||_\delta$-ball of radius C_0, therefore it is compact according to the Rellich Lemma. Riesz Theorem implies that $\dim \mathrm{Ker}\, P < +\infty$.

(ii) We first show that the extension

$$\tilde{P} : W^{s+\delta}(M,E) \to W^s(M,F)$$

has closed image for all s. For any $\epsilon > 0$, there exists a finite number of elements $v_1, \ldots, v_N \in W^{s+\delta}(M,F)$, $N = N(\epsilon)$, such that

$$||u||_0 \leq \epsilon||u||_{s+\delta} + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0|. \tag{3.11}$$

Indeed the set:

$$K_{(v_j)} = \left\{u \in W^{s+\delta}(M,F)\ ;\ \epsilon||u||_{s+\delta} + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0| \leq 1\right\},$$

is relatively compact in $W^0(M,F)$ and $\bigcap_{(v_j)} \overline{K}_{(v_j)} = \{0\}$. It follows that there are elements (v_j) such that $\overline{K}_{(v_j)}$ are contained in the unit ball of $W^0(M,E)$, as required. Substituting the main term $||u||_0$ given by (3.11) in the Gårding inequality; we obtain

$$(1 - C_s\epsilon)||u||_{s+\delta} \leq C_s\bigg(||\tilde{P}u||_s + \sum_{j=1}^{N} |\langle\langle u, v_j\rangle\rangle_0|\bigg).$$

Define $T = \{u \in W^{s+\delta}(M,E)\ ;\ u \perp v_j,\ 1 \leq j \leq n\}$ and put $\epsilon = 1/2C_s$. It follows that

$$||u||_{s+\delta} \leq 2C_s||\tilde{P}u||_s, \quad \forall u \in T.$$

This implies that $\tilde{P}(T)$ is closed. As a consequence

$$\tilde{P}(W^{s+\delta}(M,E)) = \tilde{P}(T) \ + \ \mathrm{Vect}\big(\tilde{P}(v_1), \ldots, \tilde{P}(v_N)\big)$$

is closed in $W^s(M,E)$. Consider now the case $s = 0$. Since $C^\infty(M,E)$ is dense in $W^\delta(M,E)$, we see that in $W^0(M,E) = L^2(M,E)$, one has

$$\Big(\tilde{P}\big(W^\delta(M,E)\big)\Big)^\perp = \Big(P\big(C^\infty(M,E)\big)\Big)^\perp = \mathrm{Ker}\,\tilde{P}^*.$$

We have thus proven that

$$W^0(M,E) = \tilde{P}\big(W^\delta(M,E)\big) \oplus \mathrm{Ker}\,\tilde{P}^*. \tag{3.12}$$

Since P^* is also elliptic, it follows that $\mathrm{Ker}\,\tilde{P^*}$ is finite dimensional and that $\mathrm{Ker}\,\tilde{P^*} = \mathrm{Ker}\,P^*$ is contained in $C^\infty(M,F)$. By applying the Gårding inequality, the decomposition formula (3.12) gives

$$W^s(M,E) = \tilde{P}\big(W^{s+\delta}(M,E)\big) \oplus \mathrm{Ker}\,P^*, \tag{3.13}$$

$$C^\infty(M,E) = P\big(C^\infty(M,E)\big) \oplus \mathrm{Ker}\,P^*. \tag{3.14}$$

We finish this section by the construction of the Green's operator associated to a self-adjoint elliptic operator.

3.15. Theorem. *Assume given E a Hermitian vector bundle of rank r on a compact manifold M, and $P : C^\infty(M,E) \to C^\infty(M,E)$ a self-adjoint elliptic differential operator of degree δ. Then if H denotes the orthogonal projection operator $H : C^\infty(M,E) \to \mathrm{Ker}\,P$, there exists a unique operator G on $C^\infty(M,E)$ such that*

$$PG \ + \ H = GP \ + \ H \ = \ \mathrm{Id},$$

moreover G is a pseudo-differential operator of degree $-\delta$, called the Green's operator associated to P.

Proof. According to Theorem 3.10, $\mathrm{Ker}\,P = \mathrm{Ker}\,P^*$ is finite dimensional and $\mathrm{Im}\,P = (\mathrm{Ker}\,P)^\perp$. It then follows that the restriction of P to $(\mathrm{Ker}\,P)^\perp$ is a bijective operator. One defines G to be $0 \oplus P^{-1}$ relative to the orthogonal decomposition $C^\infty(M,E) = \mathrm{Ker}\,P \oplus (\mathrm{Ker}\,P)^\perp$. The relations $PG + H = GP + H = \mathrm{Id}$ are then obvious, as well as the uniqueness of G. Moreover, G is continuous in the Fréchet space topology of $C^\infty(M,E)$, according to the Banach theorem. One also uses the fact that there exists a pseudo-differential operator Q of order $-\delta$ which is an inverse of P modulo $\mathcal{R}$, i.e. $PQ = \mathrm{Id} + R$, $R \in \mathcal{R}$. It then follows that

$$Q = (GP + H)Q = G(\mathrm{Id} + R) + HQ = G + GR + HQ,$$

where GR and HG are regular. (H is a regular operator of finite rank defined by the kernel $\sum \varphi_s(x) \otimes \varphi_s^*(y)$, if (φ_s) is a basis of eigenfunctions of $\mathrm{Ker}\, P \subset C^\infty(M, E)$.) Consequently $G = Q \bmod \mathcal{R}$ and G is a pseudodifferential operator of order $-\delta$. □

3.16. Corollary. *Under the hypotheses of 3.15, the eigenvalues of P form a real sequence λ_k such that $\lim_{k\to+\infty} |\lambda_k| = +\infty$, the eigenspaces V_{λ_k} of P are finite dimensional, and one has a Hilbert space direct sum*

$$L^2(M, E) = \widehat{\bigoplus_k} V_{\lambda_k}.$$

For any integer $m \in \mathbb{N}$, an element $u = \sum_k u_k \in L^2(M, E)$ is in $W^{m\delta}(X, E)$ if and only if $\sum |\lambda_k|^{2m} ||u_k||^2 < +\infty$.

Proof. The Green's operator extends to a self-adjoint operator

$$\tilde{G} : L^2(M, E) \to L^2(M, E)$$

which factors through $W^\delta(M, E)$, and is therefore compact. This operator defines an inverse to $\tilde{P} : W^\delta(M, E) \to L^2(M, E)$ on $(\mathrm{Ker}\, P)^\perp$. The spectral theory of compact self-adjoint operators shows that the eigenvalues μ_k of $\tilde{G}$ form a real sequence tending to 0 and that $L^2(M, E)$ is a direct sum of Hilbert eigenspaces. The corresponding eigenvalues of $\tilde{P}$ are $\lambda_k = \mu_k^{-1}$ if $\mu_k \neq 0$ and according to the ellipticity of $P - \lambda_k \mathrm{Id}$, the eigenspaces $V_{\lambda_k} = \mathrm{Ker}(P - \lambda_k \mathrm{Id})$ are finite dimensional and contained in $C^\infty(M, E)$. Finally, if $u = \sum_k u_k \in L^2(M, E)$, the Gårding inequality shows that $u \in W^{m\delta}(M, E)$ if and only if $\tilde{P}^m u \in L^2(M, E) = W^0(M, E)$, which easily gives the condition $\sum |\lambda_k|^{2m} ||u_k||^2 < +\infty$.

4. Hodge theory of compact Riemannian manifolds

The establishment of Hodge theory as a well developed subject, was carried out by W.V.D Hodge during the decade 1930-1940 (see [**Hod41**], [**DR55**]). The principal goal of the theory is to describe the de Rham cohomology algebra of a Riemannian manifold in terms of its harmonic forms. The principal result is that any cohomology class has a unique harmonic representative.

4.A. Euclidean structure of the exterior algebra. Let (M, g) be an oriented Riemannian C^∞ manifold of dimension m, and let $E \to M$ be a Hermitian vector bundle of rank r on M. We denote respectively by $(\xi_1, \ldots, \xi_m)$ and $(e_1, \ldots, e_r)$ orthonormal frames of T_M and of E on a coordinate chart $\Omega \subset M$, and let $(\xi_1^*, \ldots, \xi_m^*)$, $(e_1^*, \ldots, e_r^*)$ be the corresponding dual coframes of T_M^*, E^* respectively. Further, let dV be the Riemannian volume element on M. The exterior algebra $\Lambda^\bullet T_M^*$ is endowed with a natural inner product $\langle \bullet, \bullet \rangle$, given by

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle = \det(\langle u_j, v_k \rangle)_{1 \le j,k \le p}, \quad u_j, v_k \in T_M^* \tag{4.1}$$

for all p, with $\Lambda^\bullet T_M^* = \bigoplus \Lambda^p T_M^*$ an orthogonal direct sum. Thus the family of covectors $\xi_I^* = \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^*$, $i_1 < i_2 < \cdots < i_p$, defines an orthonormal basis of $\Lambda^\bullet T_M^*$. One denotes by $\langle \bullet, \bullet \rangle$ the corresponding inner product on $\Lambda^\bullet T_M^* \otimes E$.

4.2. Hodge star operator. The Hodge-Poincaré-de Rham $\star$ operator is the endomorphism of $\Lambda^\bullet T^*_M$ defined by a collection of linear maps such that

$$\star : \Lambda^p T^*_M \to \Lambda^{m-p} T^*_M, \quad u \wedge \star v = \langle u, v\rangle dV, \quad \forall u, v \in \Lambda^p T^*_M.$$

The existence and uniqueness of this operator follows easily from the duality pairing

$$\Lambda^p T^*_M \times \Lambda^{m-p} T^*_M \to \mathbb{R}$$

$$(u, v) \mapsto u \wedge v/dV = \sum \epsilon(I, \complement I) u_I v_{\complement I}, \tag{4.3}$$

where $u = \sum_{|I|=p} u_I \xi^*_I$, $v = \sum_{|J|=m-p} v_J \xi^*_J$, and where $\epsilon(I, \complement I)$ is the sign of the permutation $(1, 2, \ldots, m) \mapsto (I, \complement I)$ defined by I followed by the complementary (ordered) multi-indices $\complement I$. From this, we deduce

$$\star v = \sum_{|I|=p} \epsilon(I, \complement I) v_I \xi^*_{\complement I}. \tag{4.4}$$

More generally, the sesquilinear pairing $\{\bullet, \bullet\}$ defined by (1.6) induces an operator $\star$ on the vector-valued forms, such that

$$\star : \Lambda^p T^*_M \otimes E \to \Lambda^{m-p} T^*_M \otimes E, \quad \{s, \star t\} = \langle s, t\rangle dV, \tag{4.5}$$

$$\star t = \sum_{|I|=p,\lambda} \epsilon(I, \complement I) t_{I,\lambda} \xi^*_{\complement I} \otimes e_\lambda, \quad \forall s, t \in \Lambda^p T^*_M \otimes E, \tag{4.6}$$

for $t = \sum t_{I,\lambda} \xi^*_I \otimes e_\lambda$. Since $\epsilon(I, \complement I)\epsilon(\complement I, I) = (-1)^{p(m-p)} = (-1)^{p(m-1)}$, we immediately obtain

$$\star \star t = (-1)^{p(m-1)} t \text{ on } \Lambda^p T^*_M \otimes E. \tag{4.7}$$

It is clear that $\star$ is an isometry of $\Lambda^\bullet T^*_M \otimes E$. We will also need a variant of the $\star$ operator, namely the antilinear operator

$$\# : \Lambda^p T^*_M \otimes E \to \Lambda^{m-p} T^*_M \otimes E^*$$

defined by $s \wedge \# t = \langle s, t\rangle dV$, where the exterior product $\wedge$ is combined with the canonical pairing $E \times E^* \to \mathbb{C}$. We have

$$\# t = \sum_{|I|=p,\lambda} \epsilon(I, \complement I) \overline{t}_{I,\lambda} \xi^*_{\complement I} \otimes e^*_\lambda. \tag{4.8}$$

4.9. Contraction by a vector field. Assume given a tangent vector $\theta \in T_M$ and a form $u \in \Lambda^p T^*_M$. The contraction $\theta \lrcorner u \in \Lambda^{p-1} T^*_M$ is defined by

$$\theta \lrcorner u(\eta_1, \ldots, \eta_{p-1}) = u(\theta, \eta_1, \ldots, \eta_{p-1}), \quad \eta_j \in T_M.$$

In terms of the basis (ξ_j), $\bullet \lrcorner \bullet$ is the bilinear operator characterized by

$$\xi_l \lrcorner (\xi^*_{i_1} \wedge \cdots \wedge \xi^*_{i_p}) = \begin{cases} 0 & \text{if } l \notin \{i_1, \ldots, i_p\}, \\ (-1)^{k-1} \xi^*_{i_1} \wedge \cdots \widehat{\xi^*_{i_k}} \cdots \xi^*_{i_p} & \text{if } l = i_k. \end{cases}$$

This same formula is also valid when (ξ_j) is not orthonormal. An easy calculation shows that $\theta \lrcorner \bullet$ is a *derivation* of the exterior algebra, i.e. that

$$\theta \lrcorner (u \wedge v) = (\theta \lrcorner u) \wedge v + (-1)^{\deg u} u \wedge (\theta \lrcorner v).$$

Moreover, if $\tilde\theta = \langle \bullet, \theta\rangle \in T^*_M$, the operator $\theta \lrcorner \bullet$ is the adjoint of $\tilde\theta \wedge \bullet$, i.e.,

$$\langle \theta \lrcorner u, v\rangle = \langle u, \tilde\theta \wedge v\rangle, \quad \forall u, v \in \Lambda^\bullet T^*_M. \tag{4.10}$$

Indeed, this property is immediate when $\theta = \xi_l,\ u = \xi^*_I,\ v = \xi^*_J$.

4.B. Laplace-Beltrami operator. Let E be a Hermitian vector bundle on M, and let D_E be a Hermitian connection on E. We consider the Hilbert space $L^2(M, \Lambda^p T^*_M \otimes E)$ of p-forms on M with values in E, with the given L^2 scalar product

$$\langle\langle s, t\rangle\rangle = \int_M \langle s, t\rangle dV$$

already introduced in (2.5). Here $\langle s, t\rangle$ is the specific scalar product on $\Lambda^p T^*_M \otimes E$ associated to the Riemannian scalar product on $\Lambda^p T^*_M$ and the Hermitian pairing on E.

4.11. Theorem. *The formal adjoint of D_E acting on $C^\infty(M, \Lambda^p T^*_M \otimes E)$ is given by*

$$D^*_E = (-1)^{mp+1} \star D_E \star .$$

Proof. If $s \in C^\infty(M, \Lambda^p T^*_M \otimes E)$ and $t \in C^\infty(M, \Lambda^{p+1} T^*_M \otimes E)$ have compact support, we have

$$\begin{aligned}\langle\langle D_E s, t\rangle\rangle &= \int_M \langle D_E s, t\rangle dV = \int_M \{D_E s, \star t\} \\ &= \int_M d\{s, \star t\} - (-1)^p \{s, D_E \star t\} = (-1)^{p+1} \int_M \{s, D_E \star t\}\end{aligned}$$

by an application of Stokes theorem. As a consequence, (4.5) and (4.7) imply

$$\langle\langle D_E s, t\rangle\rangle = (-1)^{p+1}(-1)^{p(m-1)} \int_M \{s, \star\star D_E \star t\} = (-1)^{mp+1} \langle\langle s, \star D_E \star t\rangle\rangle.$$

The desired formula follows. □

4.12. Remark. In the case of the trivial connection d on $E = M \times \mathbb{C}$, the formula becomes $d^* = (-1)^{m+1} \star d \star$. If m is even, these formulas reduce to

$$d^* = -\star d \star, \quad D^*_E = -\star D_E \star .$$

4.13. Definition. The Laplace-Beltrami operator is the second order differential operator acting on the bundle, $\Lambda^p T^*_M \otimes E$, such that

$$\Delta_E = D_E D^*_E + D^*_E D_E.$$

In particular, the Laplace-Beltrami operator acting on $\Lambda^p T^*_M$ is $\Delta = dd^* + d^* d$. This latter operator does not depend on the Riemannian structure (M, g).

It is clear that the Laplacian Δ is formally self-adjoint i.e. $\langle\langle \Delta_E s, t\rangle\rangle = \langle\langle s, \Delta_E t\rangle\rangle$ whenever the forms s, t are C^∞ and that one of them has compact support.

4.14. Calculation of the symbol. For every C^∞ function f, Leibnitz rule gives $e^{-tf}D_E(e^{tf}s) = tdf \wedge s + D_E s$. By definition of the symbol, we therefore find

$$\sigma_{D_E}(x,\xi)\cdot s = \xi \wedge s, \quad \forall \xi \in T^*_{M,x}, \quad \forall s \in \Lambda^p T^*_M \otimes E.$$

From formula (2.8), we obtain $\sigma_{D_E^*} = -(\sigma_{D_E})^*$, therefore

$$\sigma_{D_E^*}(x,\xi)\cdot s = -\tilde{\xi} \lrcorner s$$

where $\tilde{\xi} \in T_M$ is the adjoint tangent vector of ξ. The equality $\sigma_{\Delta_E} = \sigma_{D_E}\sigma_{D_E^*} + \sigma_{D_E^*}\sigma_{D_E}$ implies that

$$\begin{aligned}
&\sigma_{\Delta_E}(x,\xi)\cdot s = -\xi \wedge (\tilde{\xi}\lrcorner s) - \tilde{\xi}\lrcorner(\xi \wedge s) = -(\tilde{\xi}\lrcorner\xi)s,\\
&\sigma_{\Delta_E}(x,\xi)\cdot s = -|\xi|^2 s.
\end{aligned}$$

In particular, Δ_E is always an elliptic operator. In the special case where M is an open subset of $\mathbb{R}^m$ with the constant metric $g = \sum_{i=1}^m dx_i^2$, all these operators $d,\ d^*,\ \Delta$ have constant coefficients. They are completely determined by their principal symbol (no term of lower order can appear). One easily computes:

$$\begin{aligned}
s &= \sum_{|I|=p} s_I dx_I, \quad ds = \sum_{|I|=p,j} \frac{\partial s_I}{\partial x_j} dx_j \wedge dx_I,\\
d^*s &= -\sum_{I,j} \frac{\partial s_I}{\partial x_j}\frac{\partial}{\partial x_j}\lrcorner dx_I,\\
\Delta s &= -\sum_I \Big(\sum_j \frac{\partial^2 s_I}{\partial x_j^2}\Big) dx_I.
\end{aligned}$$

Consequently Δ has the same expression as the elementary Laplacian operator, up to a minus sign.

4.C. Harmonic forms and the Hodge isomorphism. Let E be a Hermitian vector bundle on a compact Riemannian manifold (M,g). We assume that E is given a Hermitian connection D_E such that $\Theta(D_E) = D_E^2 = 0$. Such a connection is said to be integrable or flat. It is known that this is equivalent to such an E given by a representation $\pi_1(M) \to U(r)$. Such a bundle is called a *flat bundle* or a *local system of coefficients*. A standard example is the trivial bundle $E = M \times \mathbb{C}$ with its obvious connection $D_E = d$. Our assumption implies that D_E defines a generalized de Rham complex

$$C^\infty(M,E) \xrightarrow{D_E} C^\infty(M,\Lambda^1 T^*_M \otimes E) \to \cdots \to C^\infty(M,\Lambda^p T^*_M \otimes E) \xrightarrow{D_E} \cdots.$$

The cohomology groups of this complex are denoted by $H^p_{DR}(M,E)$.

The space of harmonic forms of degree p relative to the Laplace-Beltrami operator $\Delta_E = D_E D_E^* + D_E^* D_E$ is defined by

$$\mathcal{H}^p(M,E) = \big\{ s \in C^\infty(M,\Lambda^p T^*_M \otimes E)\ ;\ \Delta_E s = 0 \big\}.$$

Since $\langle\langle \Delta_E s, s\rangle\rangle = ||D_e s||^2 + ||D_E^* s||^2$, we see that $s \in \mathcal{H}^p(M,E)$ if and only if $D_E s = D_E^* s = 0$.

4.16. THEOREM. *For all p, there exists an orthogonal decomposition*

$$C^\infty(M, \Lambda^p T^*_M \otimes E) = \mathcal{H}^p(M,E) \oplus \mathrm{Im} D_E \oplus \mathrm{Im} D^*_E, \quad \textit{where}$$

$$\mathrm{Im} D_E = D_E\big(C^\infty(M, \Lambda^{p-1} T^*_M \otimes E)\big),$$
$$\mathrm{Im} D^*_E = D^*_E\big(C^\infty(M, \Lambda^{p+1} T^*_M \otimes E)\big).$$

PROOF. It is immediate that $\mathcal{H}^p(M,E)$ is orthogonal to the two subspaces $\mathrm{Im} D_E$ and $\mathrm{Im} D^*_E$. The orthogonality of these two subspaces is also obvious, as a result of the hypothesis $D^2_E = 0$, namely:

$$\langle\langle D_E s, D^*_E t\rangle\rangle = \langle\langle D^2_E s, t\rangle\rangle = 0.$$

We now apply th. 3.10 to the elliptic operator $\Delta_E = \Delta^*_E$ acting on the p-forms, i.e. the operator $\Delta_E : C^\infty(M,F) \to C^\infty(M,F)$ acting on the bundle $F = \Lambda^p T^*_M \otimes E$. We obtain

$$C^\infty(M, \Lambda^p T^*_M \otimes E) = \mathcal{H}^p(M,E) \oplus \Delta_E(C^\infty(M, \Lambda^p T^*_M \otimes E)),$$
$$\mathrm{Im}\Delta_E = \mathrm{Im}(D_E D^*_E + D^*_E D_E) \subset \mathrm{Im} D_E + \mathrm{Im} D^*_E.$$

Further, since $\mathrm{Im} D_E$ and $\mathrm{Im} D^*_E$ are orthogonal to $\mathcal{H}^p(M,E)$, these spaces are contained in $\mathrm{Im}\Delta_E$. □

4.17. HODGE ISOMORPHISM THEOREM. *The de Rham cohomology groups $H^p_{\mathrm{DR}}(M,E)$ are finite dimensional; moreover $H^p_{\mathrm{DR}}(M,E) \simeq \mathcal{H}^p(M,E)$.*

PROOF. From the decomposition in (4.16), we obtain

$$B^p_{\mathrm{DR}}(M,E) = D_E(C^\infty(M, \Lambda^{p-1} T^*_M \otimes E)),$$
$$Z^p_{\mathrm{DR}}(M,E) = \mathrm{Ker}\, D_E = (\mathrm{Im} D^*_E)^\perp = \mathcal{H}^p(M,E) \oplus \mathrm{Im} D_E.$$

This shows that any de Rham cohomology class contains a unique harmonic representative. □

4.18. Poincaré duality. The pairing

$$H^p_{\mathrm{DR}}(M,E) \times H^{m-p}_{\mathrm{DR}}(M,E^*) \to \mathbb{C}, \quad (s,t) \mapsto \int_M s \wedge t$$

is a non-degenerate bilinear form, and thus defines a duality between $H^p_{\mathrm{DR}}(M,E)$ and $H^{m-p}_{\mathrm{DR}}(M,E^*)$.

PROOF. First observe that there is a naturally defined flat connection D_{E^*} such that for all $s \in C^\infty(M, \Lambda^\bullet T^*_M \otimes E)$, $t \in C^\infty(M, \Lambda^\bullet T^*_M \otimes E^*)$, one has

$$d(s \wedge t) = (D_E s) \wedge t + (-1)^{\deg s} s \wedge D_{E^*} t. \tag{4.19}$$

It then follows from Stokes theorem that the bilinear map $(s,t) \mapsto \int_M s \wedge t$ factors through the cohomology groups. For $s \in C^\infty(M, \Lambda^p T^*_M \otimes E)$, the reader can easily verify the following formulas (use (4.19) in a similar way to that which was done for the proof of th. 4.11):

(4.20)

$$D_{E^*}(\#s) = (-1)^p \# D^*_E s, \quad (D_{E^*})^*(\#s) = (-1)^{p+1} \# D_E s, \quad \Delta_{E^*}(\#s) = \#\Delta^s_E.$$

Consequently $\#s \in \mathcal{H}^{m-p}(M,E^*)$ if and only if $s \in \mathcal{H}^p(M,E)$. Since

$$\int_M s \wedge \#s = \int_M |s|^2 dV = ||s||^2,$$

it follows that the Poincaré duality pairing has trivial kernel in the left factor $\mathcal{H}^p(M,E) \simeq H^p_{\mathrm{DR}}(M,E)$. By symmetry, it also has trivial kernel in the right. This completes the proof. □

5. Hermitian and Kähler manifolds

Let X be a complex manifold of dimension n. A Hermitian metric on X is a positive definite Hermitian C^∞ form on T_X. In terms of local coordinates $(z_1,\ldots,z_n)$, such a form can be written

$$h(z) = \sum_{1\le j,k\le n} h_{jk}(z) dz_j \otimes d\overline{z}_k,$$

where (h_{jk}) is a positive Hermitian matrix with C^∞ coefficients. The *fundamental* $(1,1)$*-form* associated to h is

$$\omega = -\mathrm{Im} h = \frac{\mathrm{i}}{2} \sum h_{jk} dz_j \wedge d\overline{z}_k, \quad 1 \le j,k \le n.$$

5.1. DEFINITION.

a) A Hermitian manifold is a pair (X,ω) where ω is a positive definite C^∞ $(1,1)$-form on X.

b) The metric ω is said to be Kähler if $d\omega = 0$.

c) X is called a Kähler manifold if X has at least one Kähler metric.

Since ω is real, the conditions $d\omega = 0$, $d'\omega = 0$, $d''\omega = 0$ are all equivalent. In local coordinates, we see that $d'\omega = 0$ if and only if

$$\frac{\partial h_{jk}}{\partial z_l} = \frac{\partial h_{lk}}{\partial z_j}, \quad 1 \le j,k,l \le n.$$

A simple calculation gives

$$\frac{\omega^n}{n!} = \det(h_{jk}) \bigwedge_{1\le j\le n} \left(\frac{\mathrm{i}}{2} dz_j \wedge d\overline{z}_j\right) = \det(h_{jk}) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_n = x_n + \mathrm{i}y_n$. Consequently the (n,n) form

$$dV = \frac{1}{n!}\omega^n \tag{5.2}$$

is positive and coincides with the Hermitian volume element of X. If X is compact, then $\int_X \omega^n = n!\mathrm{Vol}_\omega(X) > 0$. This simple observation already implies that a compact Kähler manifold must satisfy certain restrictive topological conditions:

5.3. CONSEQUENCE.

a) *If (X,ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X,\mathbb{R})$, then $\{\omega\}^n \ne 0$.*

b) *If X is compact Kähler, then $H^{2k}(X,\mathbb{R}) \ne 0$ for $0 \le k \le n$. Indeed, $\{\omega\}^k$ is a non-zero class of $H^{2k}(X,\mathbb{R})$.*

5.4. EXAMPLE. Complex projective space $\mathbb{P}^n$ is endowed with a natural Kähler metric ω, called the *Fubini-Study metric,* defined by

$$p^*\omega = \frac{\mathrm{i}}{2\pi} d'd'' \log(|\zeta_o|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2)$$

where $\zeta_0, \zeta_1, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be the non-homogeneous coordinates of the chart $\mathbb{C}^n \subset \mathbb{P}^n$. A calculation shows that

$$\omega = \frac{\mathrm{i}}{2\pi} d'd'' \log(1 + |z|^2) = \frac{\mathrm{i}}{2\pi} \Theta(\mathcal{O}(1)), \qquad \int_{\mathbb{P}^n} \omega^n = 1.$$

Since the only non-zero integral cohomology groups of $\mathbb{P}^n$ are $H^{2p}(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ for $0 \le p \le n$, we see that $h = \{\omega\} \in H^2(\mathbb{P}^n, \mathbb{Z})$ is a generator of the cohomology ring $H^\bullet(\mathbb{P}^n, \mathbb{Z})$. In other words, $H^\bullet(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[h]/(h^{n+1})$ as rings.

5.5. EXAMPLE. A *complex torus* is a quotient $X = \mathbb{C}^n/\Gamma$ of $\mathbb{C}^n$ by a lattice Γ of rank $2n$. This gives a compact complex manifold. Any positive definite Hermitian form $\omega = \mathrm{i} \sum h_{jk} dz_j \wedge d\overline{z}_k$ with constant coefficients on $\mathbb{C}^n$ defines a Kähler metric on X.

5.6. EXAMPLE. Any complex submanifold X of a Kähler manifold (Y, ω') is Kähler with the induced metric $\omega = \omega'_{\restriction X}$. In particular, any projective manifold is Kähler (by definition, a projective manifold is a closed submanifold $X \subset \mathbb{P}^n$ of projective space). In this case, if ω' denotes the Fubini-Study metric on $\mathbb{P}^n$, we have the additional property that the class $\{\omega\} := \{\omega'\}_{\restriction X} \in H^2_{\mathrm{DR}}(X, \mathbb{R})$ is integral, i.e. is the image of an integral class of $H^2(X, \mathbb{Z})$. A Kähler metric ω with integral cohomology class is called a *Hodge metric.*

5.7. EXAMPLE. Consider the complex surface

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where $\Gamma = \{\lambda^n \; ; \; n \in \mathbb{Z}\}$, $\lambda \in \,]0, 1[$, is viewed as a group of dilations. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}^*_+ \times S^3$, we have $X \simeq S^1 \times S^3$. As a consequence, $H^2(X, \mathbb{R}) = 0$ by an application of the Künneth formula, and property 5.3 b) shows that X is not Kähler. More generally, one can take for Γ an infinite cyclic group generated by the holomorphic contractions of $\mathbb{C}^2$, of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda z_1 \\ \lambda z_2 + z_1^p \end{pmatrix},$$

where $\lambda, \lambda_1, \lambda_2$ are complex numbers such that $0 < |\lambda_1| \le |\lambda_2| < 1$, $0 < |\lambda| < 1$, and p a positive integer. These non-Kähler surfaces are called *Hopf surfaces.* □

The following theorem shows that a Hermitian metric ω on X is Kähler if and only if the metric ω is tangent to order 2 to a Hermitian metric with constant coefficients at any point of X.

5.8. THEOREM. *Let ω be a positive definite C^∞ $(1,1)$-form on X. For ω to be Kähler, it is necessary and sufficient to show that at any point $x_0 \in X$, there exists a holomorphic coordinate system $(z_1, \ldots, z_n)$ centered at x_0 such that*

$$\omega = \mathrm{i} \sum_{1 \le l, m \le n} \omega_{lm} dz_l \wedge d\overline{z}_m, \quad \omega_{lm} = \delta_{lm} + O(|z|^2). \tag{5.9}$$

If ω is Kähler, the coordinates $(z_j)_{1\le j\le n}$ can be chosen so that

$$\omega_{lm} = \langle \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \rangle = \delta_{lm} - \sum_{1\le j,k\le n} c_{jklm} z_j \overline{z}_k + O(|z|^3), \tag{5.10}$$

where (c_{jklm}) are the coefficients of the Chern curvature tensor

$$\Theta(T_X)_{x_0} = \sum_{j,k,l,m} c_{jklm} dz_j \wedge d\overline{z}_k \otimes \left(\frac{\partial}{\partial z_l}\right)^* \otimes \frac{\partial}{\partial z_m} \tag{5.11}$$

associated to (T_X, ω) at x_0. Such a system (z_j) is called a geodesic coordinate system at x_0.

PROOF. It is clear that (5.9) implies $d_{x_0}\omega = 0$, consequently the condition is sufficient. Assume now that ω is Kähler. Then one can choose local coordinates $(\zeta_1, \ldots, \zeta_n)$ such that $(d\zeta_1, \ldots, d\zeta_n)$ are a ω-orthonormal basis of $T^*_{x_0}X$. As a consequence

$$\begin{aligned} &\omega = \mathrm{i} \sum_{1\le l,m\le n} \tilde{\omega}_{lm} d\zeta_l \wedge d\overline{\zeta}_m, \quad \text{where} \\ &\tilde{\omega}_{lm} = \delta_{lm} + O(|\zeta|) = \delta_{lm} + \sum_{1\le j\le n} (a_{jlm}\zeta_j + a'_{jlm}\overline{\zeta}_j) + O(|\zeta|^2). \end{aligned} \tag{5.12}$$

Since ω is real, we have $a'_{jlm} = \overline{a}_{jml}$. Furthermore, the Kähler condition $\partial\omega_{lm}/\partial\zeta_j = \partial\omega_{jm}/\partial\zeta_l$ at x_0 implies that $a_{jlm} = a_{ljm}$. Now put

$$z_m = \zeta_m + \frac{1}{2}\sum_{j,l} a_{jlm}\zeta_j\zeta_l, \quad 1 \le m \le n.$$

Then (z_m) is a local coordinate system at x_0, and

$$\begin{aligned} dz_m &= d\zeta_m + \sum_{j,l} a_{jlm}\zeta_j d\zeta_l, \\ \mathrm{i}\sum_m dz_m \wedge d\overline{z}_m &= \mathrm{i}\sum_m d\zeta_m \wedge d\overline{\zeta}_m + \mathrm{i}\sum_{j,l,m} a_{jlm}\zeta_j d\zeta_l \wedge d\overline{\zeta}_m \\ &\qquad + \mathrm{i}\sum_{j,l,m} \overline{a}_{jlm}\overline{\zeta}_j d\zeta_m \wedge d\overline{\zeta}_l + O(|\zeta|^2) \\ &= \mathrm{i}\sum_{l,m} \tilde{\omega}_{lm} d\zeta_l \wedge d\overline{\zeta}_m + O(|\zeta|^2) = \omega + O(|z|^2). \end{aligned}$$

Thus we have shown condition (5.9). Now let us assume the coordinates (ζ_m) were chosen initially so that (5.9) is satisfied for (ζ_m). By continuing the Taylor expansion (5.12) to order two, we arrive at

$$\begin{aligned} \tilde{\omega}_{lm} &= \delta_{lm} + O(|\zeta|^2)) \\ &= \delta_{lm} + \sum_{jk} (a_{jklm}\zeta_j\overline{\zeta}_k + a'_{jklm}\zeta_j\zeta_k + a''_{jklm}\overline{\zeta}_j\overline{\zeta}_k) + O(|\zeta|^3). \end{aligned} \tag{5.13}$$

The new coefficients introduced satisfy the relation

$$a'_{jklm} = a'_{kjlm}, \quad a''_{jklm} = \overline{a}'_{jkml}, \quad \overline{a}_{jklm} = a_{kjml}.$$

The Kähler condition $\partial\omega_{lm}/\partial\zeta_j = \partial\omega_{jm}/\partial\zeta_l$ at $\zeta = 0$ furnishes the equality $a'_{jklm} = a'_{lkjm}$; in particular a'_{jklm} is invariant under all permutations of j, k, l. If one puts

$$z_m = \zeta_m + \frac{1}{3}\sum_{j,k,l} a'_{jklm}\zeta_j\zeta_k\zeta_l, \quad 1 \le m \le n,$$

then from (5.13) one finds

$$\begin{aligned} dz_m &= d\zeta_m + \sum_{j,k,l} a'_{jklm}\zeta_j\zeta_k d\zeta_l, \quad 1 \le m \le n, \\ \omega &= \mathrm{i}\sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i}\sum_{j,k,l,m} a_{jklm}\zeta_j\overline{\zeta}_k d\zeta_l \wedge d\overline{\zeta}_m + O(|\zeta|^3), \\ \omega &= \mathrm{i}\sum_{1\le m\le n} dz_m \wedge d\overline{z}_m + \mathrm{i}\sum_{j,k,l,m} a_{jklm} z_j \overline{z}_k dz_l \wedge d\overline{z}_m + O(|z|^3). \end{aligned} \tag{5.14}$$

It is now easy to calculate the Chern curvature tensor $\Theta(T_X)_{x_0}$ in terms of the coefficients a_{jklm} and to verify that $c_{jklm} = -a_{jklm}$. We leave this as an exercise for the reader.

6. Fundamental identities of Kählerian geometry

6.A. Hermitian geometric operators. Assume given (X, ω) a Hermitian manifold and let $z_j = x_j + \mathrm{i}y_j$, $1 \le j \le n$, be $\mathbb{C}$-analytic coordinates about a point $a \in X$, such that $\omega(a) = \mathrm{i}\sum dz_j \wedge d\overline{z}_j$ is diagonalized at this point. The associated Hermitian form is $h(a) = 2\sum dz_j \otimes d\overline{z}_j$ and its real part is the Euclidean metric $2\sum (dx_j)^2 + (dy_j)^2$. It follows that $|dx_j| = |dy_j| = 1/\sqrt{2}$, $|dz_j| = |d\overline{z}_j| = 1$, and that $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ is an orthonormal basis of $(T^*_a X, \omega)$. Formula (4.1) for u_j, v_k in the orthogonal sum $(\mathbb{C} \otimes T_X)^* = T^*_X \oplus \overline{T^*_X}$ defines a natural inner product on the exterior algebra $\Lambda^\bullet(\mathbb{C} \otimes T_X)^*$. The norm of a form

$$u = \sum_{I,J} u_{I,J} dz_I \wedge d\overline{z}_J \in \Lambda^\bullet(\mathbb{C} \otimes T_X)^*.$$

at a point a is then given by

$$|u(a)|^2 = \sum_{I,J} |u_{I,J}(a)|^2. \tag{6.1}$$

The Hodge $\star$ operator (4.2) can be extended to the complex-valued forms by the formula

$$u \wedge \overline{\star v} = \langle u, v\rangle dV. \tag{6.2}$$

It follows that $\star$ is a $\mathbb{C}$-linear isometry

$$\star : \Lambda^{p,q} T^*_X \to \Lambda^{n-q,n-p} T^*_X.$$

The standard Hermitian geometric operators are the operators d, $\delta = - \star d \star$, the Laplacian $\Delta = d\delta + \delta d$ already defined, and their complex analogues

$$\left\{\begin{array}{l} d = d' + d'', \\ \delta = d'^* + d''^*, \quad d'^* = (d')^* = - \star d'' \star, \quad d''^* = (d'')^* = - \star d' \star, \\ \Delta' = d'd'^* + d'^*d', \quad \Delta'' = d''d''^* + d''^*d''. \end{array}\right. \tag{6.3}$$

We say that an operator is of pure degree r if it transforms a form of degree k to a form of degree $k + r$, and similarly an operator of pure bidegree (s,t) is an operator which transforms the (p,q)-forms to forms of bidegree, $(p+s, q+t)$. (Its total degree is then of course $r = s + t$.) Thus d', d'', d'^*, d''^*, Δ', Δ'' are of bidegree $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, $(0,0)$, $(0,0)$ respectively. Another important operator is the operator L of bidegree $(1,1)$ defined by

$$Lu = \omega \wedge u, \tag{6.4}$$

and its adjoint $\Lambda = L^* = \star^{-1} L \star$ of bidegree $(-1,-1)$:

$$\langle u, \Lambda v\rangle = \langle Lu, v\rangle. \tag{6.5}$$

We observe that the unitary group $\mathrm{U}(T_X) \simeq \mathrm{U}(n)$ has a natural action on the space of (p,q)-forms, given by

$$\mathrm{U}(n) \times \Lambda^{p,q} T_X^* \ni (g, v) \mapsto (g^{-1})^* v.$$

This action makes $\Lambda^{p,q}T_X^*$ a unitary representation of $\mathrm{U}(n)$. Since the metric ω is invariant, it is clear that L and Λ commute with the action of $\mathrm{U}(n)$.

6.B. Commutivity identities. If A, B are endomorphisms (of pure degree) of the graded module $M^\bullet = C^\infty(X, \Lambda^{\bullet,\bullet}T_X^*)$, their *graded commutator* (or *graded Lie bracket*) is defined by

$$[A, B] = AB - (-1)^{ab} BA \tag{6.6}$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, one has the following formal *Jacobi identity.*

$$(-1)^{ca}\big[A, [B, C]\big] + (-1)^{ab}\big[B, [C, A]\big] + (-1)^{bc}\big[C, [A, B]\big] = 0. \tag{6.7}$$

For all $\alpha \in \Lambda^{p,q}T_X^*$, we will still denote by α the associated endomorphism of type (p,q), operating on $\Lambda^{\bullet,\bullet}T_X^*$ by the formula $u \mapsto \alpha \wedge u$.

Let $\gamma \in \Lambda^{1,1}T_X^*$ be a real $(1,1)$-form. There exists a ω-orthogonal basis $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ of T_X which diagonalizes the two forms ω and γ simultaneously:

$$\omega = \mathrm{i} \sum_{1\le j\le n} \zeta_j^* \wedge \overline{\zeta}_j^*, \quad \gamma = \mathrm{i} \sum_{1\le j\le n} \gamma_j \zeta_j^* \wedge \overline{\zeta}_j^*, \ \gamma_j \in \mathbb{R}.$$

6.8. Proposition. *For any form $u = \sum u_{j,k} \zeta_J^* \wedge \overline{\zeta}_K^*$, one has*

$$[\gamma, \Lambda] u = \sum_{J,K} \Big(\sum_{j\in J} \gamma_j + \sum_{j\in K} \gamma_j - \sum_{1\le j\le n} \gamma_j \Big) u_{J,K} \zeta_J^* \wedge \overline{\zeta}_K^*.$$

PROOF. If u is of type (p,q), a brutal calculation gives

$$\begin{aligned}
\Lambda u &= \mathrm{i}(-1)^p \sum_{J,K,l} u_{J,K}(\zeta_l \lrcorner \zeta_J^*) \wedge (\overline{\zeta}_l \lrcorner \overline{\zeta}_K^*), \quad 1 \le l \le n,\\
\gamma \wedge u &= \mathrm{i}(-1)^p \sum_{J,K,m} \gamma_m u_{J,K} \zeta_m^* \wedge \zeta_J^* \wedge \overline{\zeta}_m^* \wedge \overline{\zeta}_K^*, \quad 1 \le m \le n,\\
[\gamma,\Lambda]u &= \sum_{J,K,l,m} \gamma_m u_{J,K} \Big(\big(\zeta_l^* \wedge (\zeta_m \lrcorner \zeta_J^*)\big) \wedge \big(\overline{\zeta}_l^* \wedge (\overline{\zeta}_m \lrcorner \overline{\zeta}_K^*)\big)\\
&\qquad\qquad - \big(\zeta_m \lrcorner (\zeta_l^* \wedge \zeta_J^*)\big) \wedge \big(\overline{\zeta}_m \lrcorner (\overline{\zeta}_l^* \wedge \overline{\zeta}_K^*)\big)\Big)\\
&= \sum_{J,K,m} \gamma_m u_{J,K} \Big(\zeta_m^* \wedge (\zeta_m \lrcorner \zeta_J^*) \wedge \overline{\zeta}_K^*\\
&\qquad\qquad + \zeta_J^* \wedge \overline{\zeta}_m^* \wedge (\overline{\zeta}_m \lrcorner \overline{\zeta}_K^*) - \zeta_J^* \wedge \overline{\zeta}_K^* \Big)\\
&= \sum_{J,K} \Big(\sum_{m\in J} \gamma_m + \sum_{m \in K} \gamma_m - \sum_{1 \le m \le n} \gamma_m \Big) u_{J,K} \zeta_J^* \wedge \overline{\zeta}_K^*. \qquad \square
\end{aligned}$$

6.9. COROLLARY. *For all $u \in \Lambda^{p,q}T_X^*$, one has $[L,\Lambda]u = (p+q-n)u$.*

PROOF. Indeed, if $\gamma = \omega$, the eigenvalues of γ are $\gamma_1 = \cdots = \gamma_n = 1$. $\square$

We introduce the operator $B = [L,\Lambda]$ which satisfies $Bu = (p+q-n)u$ for u of bidegree (p,q). Since L has degree 2, one immediately obtains $[B,L] = 2L$, and similarly $[B,\Lambda] = -2\Lambda$. This suggests introducing the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (matrices with zero trace, with the usual commutator bracket $[\alpha,\beta] = \alpha\beta - \beta\alpha$ of matrices), for which the basis of 3 matrices

$$\ell = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.10}$$

satisfies the commutivity relations

$$[\ell,\lambda] = b, \quad [b,\ell] = 2\ell, \quad [b,\lambda] = -2\lambda.$$

6.11. COROLLARY. *There is a natural action of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on the vector space $\Lambda^{\bullet,\bullet}T_X^*$, i.e. a morphism of Lie algebras $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\Lambda^{\bullet,\bullet}T_X^*)$, given by $\rho(\ell) = L$, $\rho(\lambda) = \Lambda$, $\rho(b) = B$.*

We now mention the other very important commutivity identities. Let us first assume that $X = \Omega \subset \mathbb{C}^n$ is open in $\mathbb{C}^n$ and that ω is the standard Kähler metric,

$$\omega = \mathrm{i} \sum_{1 \le j \le n} dz_j \wedge d\overline{z}_j.$$

For any form $u \in C^\infty(\Omega, \Lambda^{p,q}T_X^*)$ one has

$$d'u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J, \tag{6.12$'$}$$

$$d''u = \sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J. \tag{6.12$''$}$$

Since the global L^2 scalar product is given by

$$\langle\langle u, v\rangle\rangle = \int_\Omega \sum_{I,J} u_{I,J}\overline{v}_{I,J} dV,$$

some elementary calculations similar to those of the example in 4.12 show that

$$d'^* u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial \overline{z}_k} \frac{\partial}{\partial z_k} \lrcorner (dz_I \wedge d\overline{z}_J), \tag{6.13$'$}$$

$$d''^* = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k} \frac{\partial}{\partial \overline{z}_k} \lrcorner (dz_i \wedge d\overline{z}_J). \tag{6.13$''$}$$

We first state a lemma due to Akizuki and Nakano [**AN54**].

6.14. LEMMA. *In* $\mathbb{C}^n$, *one has* $[d''^*, L] = \mathrm{i}d'$.

PROOF. Formula (6.13$''$) can more succinctly be written

$$d''^* u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\frac{\partial u}{\partial z_k}\Big).$$

We then obtain

$$[d''^*, L]u = -\sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\frac{\partial}{\partial z_k}(\omega \wedge u)\Big) + \omega \wedge \sum_k \frac{\partial}{\partial \overline{z}_k} \lrcorner \Big(\frac{\partial u}{\partial z_k}\Big).$$

Since ω has constant coefficients, one has $\frac{\partial}{\partial z_k}(\omega \wedge u) = \omega \wedge \frac{\partial u}{\partial z_k}$ and consequently

$$\begin{aligned}[d''^*, L]u &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner (\omega \wedge \frac{\partial u}{\partial z_k}) - \omega \wedge (\frac{\partial}{\partial \overline{z}_k} \lrcorner \frac{\partial u}{\partial z_k})\Big)\\ &= -\sum_k \Big(\frac{\partial}{\partial \overline{z}_k} \lrcorner\, \omega\Big) \wedge \frac{\partial u}{\partial \overline{z}_k}.\end{aligned}$$

However, it is clear that $\frac{\partial}{\partial \overline{z}_k} \lrcorner \omega = \mathrm{i}dz_k$, therefore

$$[d''^*, L]u = \mathrm{i}\sum_k dz_k \wedge \frac{\partial u}{\partial z_k} = \mathrm{i}d'u. \qquad \square$$

We are now ready to establish the basic commutivity relations in the situation of an arbitrary Kähler manifold (X, ω).

6.15. THEOREM. *If* (X, ω) *is Kähler, then*

$$\begin{array}{rclcrcl} [d''^*, L] &=& \mathrm{i}d', && [d'^*, L] &=& -\mathrm{i}d'', \\ [\Lambda, d''] &=& -\mathrm{i}d'^*, && [\Lambda, d'] &=& \mathrm{i}d''^*. \end{array}$$

PROOF. It suffices to establish the first relation, since the second is the conjugate of the first, and the relations in the second line are the adjoint of the relations in the first line. If (z_j) is a geodesic coordinate system at a point $x_0 \in X$, then for all (p,q)-forms u, v with compact support in a neighbourhood of x_0, (5.9) implies that

$$\langle\langle u,v\rangle\rangle = \int_M \Big(\sum_{IJ} u_{IJ}\overline{v}_{IJ} + \sum_{I,J,K,L} a_{IJKL}u_{IJ}\overline{v}_{KL}\Big)dV,$$

with $a_{IJKL}(z) = O(|z|^2)$ at x_0. An integration by parts analogous to that used to obtain (4.12) and (6.13$''$) gives

$$d''^{*}u = -\sum_{I,J,k} \frac{\partial u_{I,J}}{\partial z_k}\frac{\partial}{\partial \overline{z}_k} \lrcorner (dz_I \wedge d\overline{z}_J) + \sum_{I,J,K,L} b_{IJKL}u_{IJ}dz_k \wedge d\overline{z}_L,$$

where the coefficients b_{IJKL} are obtained by differentiation of a_{IJKL}. Consequently we have $b_{IJKL} = O(|z|)$. Since $\partial\omega/\partial z_k = O(|z|)$, the proof of lemma 6.14 above implies $[d''^{*}, L]u = \mathrm{i}d'u + O(|z|)$. In particular the two terms coincide at the given point $x_0 \in X$. □

6.16. COROLLARY. *If (X,ω) is Kähler, the complex Laplace-Beltrami operators satisfy*

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

PROOF. We first show that $\Delta'' = \Delta'$. One has

$$\Delta'' = [d'', d''^{*}] = -\mathrm{i}\big[d'', [\Lambda, d']\big].$$

Since $[d', d''] = 0$, the Jacobi identity (6.7) implies that

$$-\big[d'', [\Lambda, d']\big] + \big[d', [d'', \Lambda]\big] = 0,$$

hence $\Delta'' = \big[d', -\mathrm{i}[d'', \Lambda]\big] = [d', d'^{*}] = \Delta'$. Furthermore,

$$\Delta = [d' + d'', d'^{*} + d''^{*}] = \Delta' + \Delta'' + [d', d''^{*}] + [d'', d'^{*}].$$

It therefore suffices to prove:

6.17. LEMMA. $[d', d''^{*}] = 0$, $[d'', d'^{*}] = 0$.

PROOF. We have $[d', d''^{*}] = -\mathrm{i}\big[d', [\Lambda, d']\big]$ and (6.7) implies that

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^{*}] = 0$. The second relation $[d'', d'^{*}] = 0$ is the adjoint of the first. □

6.18. THEOREM. *If (X,ω) is Kähler, Δ commutes with all the operators $\star$, d', d'', d'^{*}, d''^{*}, L, Λ.*

PROOF. The identities $[d', \Delta'] = [d'^{*}, \Delta'] = 0$, $[d'', \Delta''] = [d''^{*}, \Delta''] = 0$ and $[\Delta, \star] = 0$ are immediate. Moreover, the equality $[d', L] = d'\omega = 0$, combined with the Jacobi identity, implies that

$$[L, \Delta'] = \big[L, [d', d'^{*}]\big] = -\big[d', [d'^{*}, L]\big] = \mathrm{i}[d', d''] = 0.$$

Taking adjoints, we obtain $[\Delta', \Lambda] = 0$. □

6.C. Primitive elements and the Lefschetz isomorphism theorem. To establish the Lefschetz Theorem, it is convenient to use the representation of $\mathfrak{sl}(2,\mathbb{C})$ exhibited in Cor. 6.11. We first recall that if $\mathfrak{g}$ is a Lie sub-algebra (real or complex) of the Lie algebra $\mathfrak{sl}(r,\mathbb{C}) = \mathrm{End}(\mathbb{C}^r)$ of complex matrices and if $G = \exp(\mathfrak{g}) \subset \mathrm{GL}(r,\mathbb{C})$ is the associated Lie group, a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of the Lie algebra in a complex vector space V induces by exponentiation a representation $\tilde{\rho} : G \to \mathrm{GL}(V)$ of the group G. Conversely, a representation $\tilde{\rho} : G \to \mathrm{GL}(V)$ induces by differentiation a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of Lie algebras; there is therefore an identification between these two notions. If G is compact, a classical lemma of H. Weyl shows that all representations of $\mathfrak{g}$ are broken down into a direct sum of irreducible representations (one says that $\mathfrak{g}$ is *reductive*): the Haar measure of G indeed allows the construction of an invariant Hermitian metric on V, and one exploits the fact that the orthogonal complement of a sub-representation is a sub-representation. In particular the Lie algebra $\mathfrak{su}(r)$ of the compact group $\mathrm{SU}(r)$ is reductive. It is the same as for $\mathfrak{sl}(r,\mathbb{C})$, which is the complexification of $\mathfrak{su}(r)$. We will need the following well-known lemma from representation theory.

6.19. LEMMA. *Let $\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(V)$ be a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on a finite dimensional complex vector space V, and let*

$$L = \rho(\ell),\ \ \Lambda = \rho(\lambda),\ \ B = \rho(b) \in \mathrm{End}(V)$$

be the endomorphisms of V associated to the basis elements of $\mathfrak{sl}(2,\mathbb{C})$. Then:

a) *$V = \oplus_{\mu\in\mathbb{Z}} V_\mu$ is a (finite) direct sum of eigenspaces of B, whose eigenvalues μ are integers. An element $v \in V_\mu$ is said to be an element of pure weight μ.*

b) *L and Λ are nilpotent, satisfying $L(V_\mu) \subset V_{\mu+2}$, $\Lambda(V_\mu) \subset V_{\mu-2}$ for all $\mu \in \mathbb{Z}$.*

c) *We denote by $P = \mathrm{Ker}\,\Lambda = \{v \in V\ ;\ \Lambda v = 0\}$, the set of primitive elements. One then has a direct sum decomposition*

$$V = \bigoplus_{r\in\mathbb{N}} L^r(P).$$

d) *V is isomorphic to a finite direct sum $\oplus_{m\in\mathbb{N}} S(m)^{\oplus\alpha_m}$ of irreducible representations, where $S(m) \simeq S^m(\mathbb{C}^2)$ is the representation of $\mathfrak{sl}(2,\mathbb{C})$ induced by the m-th symmetric product of the natural representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$, and $\alpha_m = \dim P_m$ is the multiplicity of the isotypic component $S(m)$.*

e) *If $P_\mu = P \cap V_\mu$, then $P_\mu = 0$ for $\mu > 0$ and $P = \oplus_{\mu\in\mathbb{Z},\mu\le 0} P_\mu$. The endomorphism $L^r : P_{-m} \to V_{m+2r}$ is injective for $r \le m$ and zero for $r > m$.*

f) *$V_\mu = \bigoplus_{r\in\mathbb{N},r\ge\mu} L^r(P_{\mu-2r})$, where $L^r : P_{\mu-2r} \to L^r(P_{\mu-2r})$ is bijective.*

g) *For any $r \in \mathbb{N}$, the endomorphism $L^r : V_{-r} \to V_r$ is bijective.*

PROOF. We first observe the following fact: If $v \in V_\mu$, then Lv has pure weight $\mu + 2$ and Λv has pure weight $\mu - 2$. Indeed, one has

$$\begin{aligned} BLv &= LBv + [B, L]v = L(\mu v) + 2Lv = (\mu + 2)Lv,\\ B\Lambda v &= \Lambda Bv + [B, \Lambda]v = \Lambda(\mu v) - 2\Lambda v = (\mu - 2)\Lambda v. \end{aligned}$$

Now suppose $V \ne 0$ and let $v \in V_\mu$ be a non-zero eigenvector. If the vectors $(\Lambda^k v)_{k\in\mathbb{N}}$ were all non-zero, one would have an infinite number of eigenvectors of B with $\mu - 2k$ distinct eigenvalues, which is impossible. Therefore there exists an integer $r \ge 0$ such that $\Lambda^r v \ne 0$ and $\Lambda^k v = 0$ for $k > r$. Consequently $\Lambda^r v$ is a non-zero primitive element of pure weight $\mu' = \mu - 2r$. Thus we conclude that for some $\mu \in \mathbb{C}$, there exists $w \in P$, a non-zero element of pure weight μ. The same reasoning as above applied to the powers $L^k w$ shows that there exists an integer $m > 0$ such that $L^m w \neq 0$ and $L^{m+1} w = 0$. The vector space W of dimension $m+1$ generated by $w_k = L^k w,\ 0 \leq k \leq m$ is stable under the action of $\mathfrak{sl}(2,\mathbb{C})$. Indeed one has $Bw_k = (\mu + 2k) w_k,\ Lw_k = w_{k+1}$ by definition, while

$$\begin{aligned}\Lambda w_k = \Lambda L^k w &= L^k \Lambda w - \sum_{0\leq j\leq k-1} L^{k-j-1}[L,\Lambda]L^j w\\ &= 0 - \sum L^{k-j-1} B L^j w = - \sum_{0\leq j\leq k-1} (\mu+2j) L^{k-1} w\\ &= k(-\mu - k + 1) w_{k-1}.\end{aligned}$$

By applying this relation to the indice $k = m+1$ for which $w_{m+1} = 0$, it follows that one must necessarily have $\mu = -m \leq 0$. We remark that $B_{\restriction W}$ is diagonalizable (the eigenvectors of W being the vectors w_k of integral weight $2k - m$), and that the primitive elements of W are reduced to the line $\mathbb{C}w$, such that $W = \oplus L^r(\mathbb{C}w)$. Properties (a,b,c,d) mentioned above are then easily obtained by induction on $\dim V$. By considering the quotient representation V/W one can argue by induction that the eigenvalues of B are integers and that $L,\ \Lambda$ are nilpotent. It is easy to verify that $W \simeq S^m(\mathbb{C}^2)$ as a representation of $\mathrm{SL}(2,\mathbb{C})$. (If $e_1,\ e_2$ are two basis vectors of $\mathbb{C}^2$, the isomorphism sends $w = w_0$ to e_1^m and w_k to $L^k e_1^m = m(m-1)\cdots(m-k+1) e_1^k e_2^{m-k}$.) The fact that one has a direct sum of representations $V = V' \oplus W$ (with $V' = W^\perp \subset V$ for a certain $\mathrm{SU}(2,\mathbb{C})$-invariant metric) involves the diagonalizability of B, by induction on $\dim V$, as well as the formula $V = \oplus L^r(P)$ and the decomposition in d).

e) The relation $[B,\Lambda] = -2\Lambda$ shows that $P = \operatorname{Ker}\Lambda$ is stable under B, consequently

$$P = \bigoplus (P \cap V_\mu) = \bigoplus P_\mu.$$

The above calculations show that the non-zero primitive elements w are of weight $-m \leq 0$, so that $P_\mu = 0$ if $\mu > 0$. The latter assertion of e) follows from the fact that for $0 \neq w \in P_{-m}$, one has $L^r w \neq 0$ if and only if $r \leq m$.

f) An immediate consequence of e) and the decomposition $V = \oplus_{r\in\mathbb{N}} L^r(P)$, if one restricts only to elements of pure weight μ. One can only have $L^r(P_{\mu-2r}) \neq 0$ if either $r \leq m = -(\mu - 2r)$, or $r \geq \mu$.

g) It suffices to verify the assertion in the case of an irreducible representation $V \simeq S^m(\mathbb{C}^2)$. In this case, the result is clear, since the weights $2k - m,\ 0 \leq k \leq m$ are distributed symmetrically in the interval $[-m, m]$ and that V is generated by $(L^k w)_{0\leq k\leq m}$ for any non-zero vector w of V_{-m}. □

We now interpret these results in the case of a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $V = \Lambda^{\bullet,\bullet} T_X^*$. The component $\Lambda^k(\mathbb{C}\otimes T_X)^* = \oplus_{p+q=k} \Lambda^{p,q} T_X^*$ can then be identified with the eigenspace V_μ of B of weight $\mu = k - n = p + q - n$ (by definition of B, see (6.9)).

6.20. Definition. A homogeneous form $u \in \Lambda^k(\mathbb{C}^k \otimes T_X)^*$ is called primitive if $\Lambda u = 0$. The space of primitive forms of total degree k is denoted by

$$\mathrm{Prim}^k T_X^* = \bigoplus_{p+q=k} \mathrm{Prim}^{p,q} T_X^*.$$

Since the operator Λ commutes with the action of $\mathrm{U}(T_X) \simeq \mathrm{U}(n)$ on the exterior algebra, it is clear that $\mathrm{Prim}^{p,q}T_X^* \subset \Lambda^{p,q}T_X^*$ is a $\mathrm{U}(n)$-invariant subspace. One further sees (prop. 6.24) that $\mathrm{Prim}^{p,q}T_X^*$ is in fact an irreducible representation of $\mathrm{U}(n)$. Properties (6.19 e, f, g) successively imply

6.21. PROPOSITION. *We have* $\mathrm{Prim}^k T_X^* = 0$ *for* $k > n$. *Moreover, if* $u \in \mathrm{Prim}^k T_X^*$, $k \le n$, *then* $L^r u = 0$ *for* $r > n - k$.

6.22. Primitive decomposition formula. For any $u \in \Lambda^k(\mathbb{C} \otimes T_X)^*$, there exists a unique decomposition

$$u = \sum_{r \ge (k-n)_+} L^r u_{k-2r}, \quad u_{k-2r} \in \mathrm{Prim}^{k-2r}T_X^*.$$

Consequently, one obtains a decomposition into a direct sum of representations of $\mathrm{U}(n)$

$$\Lambda^k(\mathbb{C} \otimes T_X)^* = \bigoplus_{r \ge (k-n)_+} L^r \mathrm{Prim}^{k-2r}T_X^*,$$

$$\Lambda^{p,q}(\mathbb{C} \otimes T_X)^* = \bigoplus_{r \ge (p+q-n)_+} L^r \mathrm{Prim}^{p-r,q-r}T_X^*.$$

6.23. LEFSCHETZ ISOMORPHISM THEOREM. *The linear operators*

$$L^{n-k} : \Lambda^k(\mathbb{C} \otimes T_X)^* \to \Lambda^{2n-k}(\mathbb{C} \otimes T_X)^*,$$
$$L^{n-p-q} : \Lambda^{p,q}T_X^* \to \Lambda^{n-q,n-p}T_X^*,$$

are isomorphisms for all integers $k \le n$ *and* (p, q) *satisfying* $p + q \le n$.

6.24. PROPOSITION. *For any* $(p, q) \in \mathbb{N}^2$ *satisfying* $p+q \le n$, $\mathrm{Prim}^{p,q}T_X^*$ *is an irreducible representation of* $\mathrm{U}(n)$*; more precisely, it is the irreducible representation associated to the highest weight* $\epsilon_1 + \cdots + \epsilon_q - (\epsilon_{n-p+1} + \cdots + \epsilon_n)$*, where* (ϵ_j) *is the canonical basis of characters of the maximal commutative subgroup* $\mathrm{U}(1)^n \subset \mathrm{U}(n)$*. The primitive decomposition of* $\Lambda^{p,q}T_X^*$ *or of* $\Lambda^k(\mathbb{C} \otimes T_X)^*$ *is the same as the decomposition into irreducible components under the action of* $\mathrm{U}(n)$.

PROOF. First observe that $\mathrm{Prim}^{p,q}T_X^* \neq 0$, since for example

$$dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z}_{p+1} \wedge \cdots \wedge d\overline{z}_{p+q} \in \mathrm{Prim}^{p,q}T_X^*.$$

Further, the primitive decomposition gives

$$\Lambda^{p,q}T_X^* = \bigoplus_{0 \le r \le m} L^r \mathrm{Prim}^{p-r,q-r}T_X^*$$

with $m = \min(p, q)$, which shows that the $\mathrm{U}(n)$-module $\Lambda^{p,q}T_X^*$ has at least $m + 1$ non-trivial irreducible components, to account for each of the terms $\mathrm{Prim}^{p-r,q-r}T_X^*$, $0 \le r \le m$. To see that these are irreducible , it suffices to show that the $\mathrm{U}(n)$-module $\Lambda^{p,q}T_X^*$ has no more than $(m + 1)$ irreducible components. However, by complexification of the representation of $\mathrm{U}(n)$, one obtains a representation isomorphic to that of $\mathrm{GL}(n, \mathbb{C})$ on $\Lambda^p T_X^* \otimes \Lambda^q T_X$ given by $g \cdot (u \otimes \xi) = (g^{-1})^* u \otimes g_* \xi$. The representation theory of linear groups shows that the irreducible components of a representation are in bijective correspondence with the eigenvectors associated to the action of the Borel subgroup B_n of upper triangular matrices. We leave it to the reader to show that these eigenvectors correspond precisely to the (p,q)-forms

$$L^r(dz_{n-p+r+1}\wedge\cdots\wedge dz_n\wedge d\overline{z}_1\wedge\cdots\wedge d\overline{z}_{q-r}),\quad 0\le r\le m,$$

for which the weight under the action of $\mathrm{U}(1)^n$ is $\epsilon_1+\cdots+\epsilon_{q-r}-(\epsilon_{n-p+r+1}+\cdots+\epsilon_n)$. □

7. The groups $\mathcal{H}^{p,q}(X,E)$ and Serre duality

We now arrive at some holomorphic consequences of Hodge theory. A large part of this theory was developed by K. Kodaira, S. Lefschetz and A. Weil. The reader can profitably consult the Completed Works of Kodaira [**Kod75**] and the book by A. Weil [**Wei57**]; see also [**Wel80**] for a more recent account.

Let (X,ω) be a *compact Hermitian* manifold and E a Hermitian holomorphic vector bundle of rank r over X. We will denote by D_E the Chern connection of E, $D_E^*=-\star D_E\star$ the formal adjoint of D_E, and $D_E'^*$, $D_E''^*$ the components of D_E^* of type $(-1,0)$ and $(0,-1)$. A similar calculation to that done in 4.14 shows that

$$\sigma_{D''_E}(x,\xi)\cdot s=\xi^{0,1}\wedge s,\quad \xi\in{}^{\mathbb{R}}T_X^*=\mathrm{Hom}_{\mathbb{R}}(T_X,\mathbb{R}),\ s\in E_x,$$

where $\xi^{(0,1)}$ is the type $(0,1)$ part of the real 1-form ξ. Consequently, we see that the principal part of the operator $\Delta''_E=D''_ED''^*_E+D''^*_ED''_E$ is given by

$$\sigma_{\Delta''_E}(x,\xi)\cdot s=-|\xi^{0,1}|^2s=-\frac{1}{2}|\xi|^2s,$$

and there is a similar result for Δ'_E. In particular $\sigma_{\Delta'_E}=\sigma_{\Delta''_E}=\frac{1}{2}\sigma_{\Delta_E}$ and Δ''_E is a self-adjoint elliptic operator on each of the spaces $C^\infty(X,\Lambda^{p,q}T_X^*\otimes E)$. Using $D''^2_E=0$, one arrives at the following result, in the same way as obtained in §4.C.

7.1. Theorem. *For any bidegree (p,q), there exists an orthogonal decomposition*

$$C^\infty(X,\Lambda^{p,q}T_X^*\otimes E)=\mathcal{H}^{p,q}(X,E)\oplus\mathrm{Im}\ D''_E\oplus\mathrm{Im}\ D''^*_E$$

where $\mathcal{H}^{p,q}(X,E)$ is the space of Δ''_E-harmonic forms in $C^\infty(X,\Lambda^{p,q}T_X^*\otimes E)$.

The above decomposition shows that the subspace of q-cocycles of the complex $(C^\infty(X,\Lambda^{p,\bullet}T_X^*\otimes E),d'')$ is $\mathcal{H}^{p,q}(X,E)\oplus\mathrm{Im}\ D''_E$. From here, we deduce the

7.2. Theorem (Hodge isomorphism). *The Dolbeault cohomology groups $H^{p,q}(X,E)$ are finite dimensional, and there is an isomorphism*

$$H^{p,q}(X,E)\simeq\mathcal{H}^{p,q}(X,E).$$

Another interesting consequence is a proof of the Serre duality theorem for compact complex manifolds. See Serre [**Ser55**] for a proof in a somewhat more general context.

7.3. Theorem (Serre duality). *The bilinear pairing*

$$H^{p,q}(X,E)\times H^{n-p,n-q}(X,E^*)\to\mathbb{C},\quad (s,t)\mapsto\int_M s\wedge t$$

is a non-degenerate duality.

PROOF. Let $s_1 \in C^\infty(X, \Lambda^{p,q}T_X^* \otimes E)$, $s_2 \in C^\infty(X, \Lambda^{n-p,n-q-1}T_X^* \otimes E)$. Since $s_1 \wedge s_2$ is of bidegree $(n, n-1)$, we have

$$d(s_1 \wedge s_2) = d''(s_1 \wedge s_2) = d''s_1 \wedge s_2 + (-1)^{p+q} s_1 \wedge d''s_2. \tag{7.4}$$

Stokes theorem implies that the bilinear pairing above can be factored through the Dolbeault cohomology groups. The operator $\#$ defined is §4.A satisfies

$$\# : C^\infty(X, \Lambda^{p,q}T_X^* \otimes E) \to C^\infty(X, \Lambda^{n-p,n-q}T_X^* \otimes E^*).$$

Moreover, (4.20) implies

$$\begin{aligned} &D''_{E^*}(\#s) = (-1)^{\deg s}\#(D''_E)^* s, \quad (D''_{E^*})^*(\#s) = (-1)^{\deg s+1}\#D''^*_E s, \\ &\Delta''_{E^*}(\#s) = \#\Delta''_E s, \end{aligned}$$

where D_{E^*} is the Chern connection of E^*. Consequently, $s \in \mathcal{H}^{p,q}(X, E)$ if and only if $\#s \in \mathcal{H}^{n-p,n-q}(X, E^*)$. Theorem 7.3 is then a consequence of the fact that the integral $||s||^2 = \int_X s \wedge \#s$ is non-vanishing if $s \neq 0$.

8. Comohology of compact Kähler manifolds

8.A. Bott-Chern cohomology groups. Let X be a complex manifold, for the moment not necessarily compact. The following "cohomology groups" are useful for describing certain aspects of the Hodge theory of compact complex manifolds, which are not necessarily Kähler.

8.1. DEFINITION. The Bott-Chern cohomology groups of X are given by

$$H^{p,q}_{\mathrm{BC}}(X, \mathbb{C}) = (C^\infty(X, \Lambda^{p,q}T_X^*) \cap \operatorname{Ker} d)/d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^*).$$

The cohomology $H^{\bullet,\bullet}_{\mathrm{BC}}(X, \mathbb{C})$ has a bigraded algebra structure, which we call the Bott-Chern cohomology algebra of X.

Since the group $d'd''C^\infty(X, \Lambda^{p-1,q-1}T_X^*)$ is also contained in the group of coboundaries $d''C^\infty(X, \Lambda^{p,q-1}T_X^*)$ of the Dolbeault complex as well as that in coboundaries of the de Rham complex $dC^\infty(X, \Lambda^{p+q-1}(\mathbb{C} \otimes T_X)^*)$, there are canonical morphisms

$$H^{p,q}_{\mathrm{BC}}(X, \mathbb{C}) \to H^{p,q}(X, \mathbb{C}), \tag{8.2}$$

$$H^{p,q}_{\mathrm{BC}}(X, \mathbb{C}) \to H^{p+q}(X, \mathbb{C}), \tag{8.3}$$

of the Bott-Chern cohomology to the Dolbeault or de Rham cohomology. These morphisms are $\mathbb{C}$-algebra homomorphisms. It is also clear from the definition that we have the symmetry property $H^{q,p}_{\mathrm{BC}}(X, \mathbb{C}) = \overline{H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})}$. One can show from the Hodge-Frölicher spectral sequence (see §10) that $H^{p,q}_{\mathrm{BC}}(X, \mathbb{C})$ is always finite dimensional if X is compact.

8.B. Hodge decomposition theorem. We assume from now on that (X, ω) is a *compact Kähler manifold*. The equality $\Delta = 2\Delta''$ shows that Δ is homogeneous with respect to bidegree and that there is an orthogonal decomposition

$$\mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, \mathbb{C}). \tag{8.4}$$

Since $\overline{\Delta''} = \Delta' = \Delta''$, one has the equality $\mathcal{H}^{q,p}(X,\mathbb{C}) = \overline{\mathcal{H}^{p,q}(X,\mathbb{C})}$. By applying the Hodge isomorphism theorem for de Rham cohomology and for Dolbeault cohomology, one obtains:

8.5. THEOREM (Hodge Decomposition). *On a compact Kähler manifold, there are canonical isomorphisms*

$$H^k_{\mathrm{DR}}(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}) \qquad (\textit{Hodge decomposition}),$$
$$H^{q,p}(X,\mathbb{C}) \simeq \overline{H^{p,q}(X,\mathbb{C})} \qquad (\textit{Hodge symmetry}).$$

The only point that is not *a priori* obvious is that isomorphisms are independent of the choice of Kähler metric. To show that this is indeed the case, one can use the following lemma, which will allow us to compare the three types of cohomology groups considered in §8.A.

8.6. LEMMA. *Let u be a d-closed (p,q)-form. The following properties are equivalent:*

a) *u is d-exact;*
b$'$) *u is d'-exact;*
b$''$) *u is d''-exact;*
c) *u is $d'd''$-exact, i.e. u can be written $u = d'd''v$.*
d) *u is orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$.*

PROOF. It is evident that c) implies a), b$'$), b$''$), and that a) or b$'$) or b'') implies d). It suffices therefore to prove that d) implies c). Since $du = 0$, we have $d'u = d''u = 0$, and since u is assumed orthogonal to $\mathcal{H}^{p,q}(X,\mathbb{C})$, th. 7.1 implies that $u = d''s$, $s \in C^\infty(X, \Lambda^{p,q-1}T^\star_X)$. The analogous theorem to th. 7.1 for d' (which can be deduced by complex conjugation) shows that one can write $s = h + d'v + d'^\star w$, where $h \in \mathcal{H}^{p,q-1}(X,\mathbb{C})$, $v \in C^\infty(X,\Lambda^{p-1,q-1}T^\star_X)$ and $w \in C^\infty(X,\Lambda^{p+1,q-1}T^\star_X)$. Consequently

$$u = d''d'v + d''d'^\star w = -d'd''v - d'^\star d''w$$

by an application of Lemma 6.16. Since $d'u = 0$, the component $d'^\star d''w$ orthogonal to $\operatorname{Ker} d'$ must be zero. □

From Lemma 8.6 we deduce the following corollary, which in turn implies that the Hodge decomposition does not depend on the choice of Kähler metric.

8.7. COROLLARY. *Let X be a compact Kähler manifold. Then the natural morphisms*

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C}), \qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^k_{\mathrm{DR}}(X,\mathbb{C})$$

are isomorphisms.

PROOF. The surjectivity of $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C})$ follows from the fact that any class in $H^{p,q}(X,\mathbb{C})$ can be represented by a harmonic (p,q)-form, therefore by a d-closed (p,q)-form; the injectivity property is nothing more than the equivalence (8.5b$''$) $\Leftrightarrow$ (8.5c). Therefore $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C}) \simeq \mathcal{H}^{p,q}(X,\mathbb{C})$, and the isomorphism

$$\bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^k_{\mathrm{DR}}(X,\mathbb{C})$$

is a consequence of (8.4). $\square$

We now mention two simple consequences of Hodge theory. The first concerns the calculation of the Dolbeault cohomology of $\mathbb{P}^n$. Since $H^{p,p}(\mathbb{P}^n,\mathbb{C})$ contains the non-zero class $\{\omega^p\}$ and since $H^{2p}_{\mathrm{DR}}(\mathbb{P}^n,\mathbb{C})=\mathbb{C}$, the Hodge decomposition formula implies:

8.8. Consequence. The Dolbeault cohomology groups of $\mathbb{P}^n$ are

$$H^{p,p}(\mathbb{P}^n,\mathbb{C})=\mathbb{C}\ \text{ for } 0\le p\le n,\quad H^{p,q}(\mathbb{P}^n,\mathbb{C})=0\ \text{ for } p\ne q. \qquad \square$$

8.9. Proposition. *Any holomorphic p-form on a compact Kähler manifold X is d-closed.*

Proof. If u is a holomorphic form of type $(p,0)$ then $d''u=0$. Moreover d''^*u is of type $(p,-1)$, hence $d''^*u=0$. Consequently $\Delta u=2\Delta''u=0$, which implies that $du=0$. $\square$

8.10. Example. Consider the *Heisenberg group* $G\subset \mathrm{Gl}_3(\mathbb{C})$, defined by the subgroup of matrices

$$M=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\quad,\quad (x,y,z)\in\mathbb{C}^3.$$

Let Γ be a discrete subgroup of matrices with the property that the coefficients x,y,z belong to the ring $\mathbb{Z}[\mathrm{i}]$ (or more generally in the ring of imaginary quadratic integers). Then $X=G/\Gamma$ is a compact complex manifold of dimension 3, called a *Iwasawa manifold*. The equality

$$M^{-1}dM=\begin{pmatrix}0 & dx & dz-xdy\\ 0 & 0 & dy\\ 0 & 0 & 0\end{pmatrix},$$

shows that $dx,\ dy,\ dz-xdy$ are left invariant 1-forms on G. These forms induce holomorphic 1-forms on the quotient $X=G/\Gamma$. Since $dz-xdy$ is not d-closed, one concludes that X cannot be Kähler.

8.11. Remark. For simplicity of notation we work here with constant coefficients, but the reader can easily verify that one has analogous results for cohomology with values in a local system of coefficients (flat Hermitian bundle), as in §4.C. It is enough to replace everywhere in the proof the operator $d=d'+d''$ by $D_E=D'_E+D''_E$, and to observe that one still has $\Delta'_E=\Delta''_E=\frac{1}{2}\Delta_E$ (proof identical to that of Cor. (6.16)). One can then deduce the existence of isomorphisms

$$H^{p,q}_{\mathrm{BC}}(X,E)\to H^{p,q}(X,E),\qquad \bigoplus_{p+q=k}H^{p,q}_{\mathrm{BC}}(X,E)\to H^k_{\mathrm{DR}}(X,E)$$

and a canonical decomposition

$$H^k_{\mathrm{DR}}(X,E)=\bigoplus_{p+q=k}H^{p,q}(X,E).$$

In this context, the symmetry property of Hodge becomes

$$\overline{H^{p,q}(X,E)}\simeq H^{q,p}(X,E^*),$$

via the antilinear operator # considered in §4 and §7. These observations are useful for the study of variations of Hodge structure.

8.C. Primitive decomposition and hard Lefschetz theorems. We first introduce some standard notation. The Betti numbers and the Hodge numbers of X are by definition

$$b_k = \dim_{\mathbb{C}} H^k(X,\mathbb{C}), \qquad h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C}). \tag{8.12}$$

According to the Hodge decomposition, the numbers satisfy the relations

$$b_k = \sum_{p+q+k} h^{p,q}, \qquad h^{q,p} = h^{p,q}. \tag{8.13}$$

Consequently, the Betti numbers b_{2k+1} of a compact Kähler manifold are even. Note that the Serre duality theorem gives the additional relation $h^{p,q} = h^{n-p,n-q}$, provided that X is compact. As we will see, the existence of the primitive decomposition implies many other interesting characteristic properties of the cohomology algebra of a compact Kähler manifold.

8.14. Lemma. *If $u = \sum_{r\ge(k-n)_+} L^r u_r$ is the primitive decomposition of a harmonic k-form u, then all the components u_r are harmonic.*

Proof. Since $[\Delta, L] = 0$, one obtains $0 = \Delta u = \sum_r L^r \Delta u_r$, therefore $\Delta u_r = 0$ according to the uniqueness of the decomposition. □

Denote by $\mathcal{H}^p_{\mathrm{prim}}(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p+q}_{\mathrm{prim}}(X,\mathbb{C})$ the space of primitive harmonic k-forms and let $h^{p,q}_{\mathrm{prim}}$ be the dimension of the component of bidegree (p,q). Lemma (8.14) gives

$$\mathcal{H}^{p,q}(X,\mathbb{C}) = \bigoplus_{r\ge(p+q-n)_+} L^r \mathcal{H}^{p-r,q-r}_{\mathrm{prim}}(X,\mathbb{C}), \tag{8.15}$$

$$h^{p,q} = \sum_{r\ge(p+q-n)_+} h^{p-r,q-r}_{\mathrm{prim}}. \tag{8.16}$$

Formula (8.16) can be written as

$$\begin{cases} \text{If } p+q\le n,\ h^{p,q} = h^{p,q}_{\mathrm{prim}} + h^{p-1,q-1}_{\mathrm{prim}} + \cdots \\[2ex] \text{If } p+q\ge n,\ h^{p,q} = h^{n-q,n-p}_{\mathrm{prim}} + h^{n-q-1,n-p-1}_{\mathrm{prim}} + \cdots. \end{cases} \tag{8.16$'$}$$

8.17. Corollary. *The Hodge and Betti numbers satisfy the following inequalities.*

a) *If $k = p+q \le n$, then $h^{p,q} \ge h^{p-1,q-1}$, $b_k \ge b_{k-2}$,*
b) *If $k = p+q \ge n$, then $h^{p,q} \ge h^{p+1,q+1}$, $b_k \ge b_{k+2}$.* □

Another important result of Hodge theory (that is in fact a direct consequence of Cor. 6.23) is the

8.18. Hard Lefschetz Theorem. *The cup product morphisms*

$$L^{n-k} : H^k(X,\mathbb{C}) \to H^{2n-k}(X,\mathbb{C}), \quad k\le n,$$
$$L^{n-p-q} : H^{p,q}(X,\mathbb{C}) \to H^{n-q,n-p}(X,\mathbb{C}), \quad p+q\le n,$$

are isomorphisms. □

Another way of stating the hard Lefschetz Theorem is to introduce the *Hodge-Riemann bilinear form* on $H^k_{\rm DR}(X,\mathbb{C})$, defined by

$$Q(u,v)=(-1)^{k(k-1)/2}\int_X u\wedge v\wedge\omega^{n-k}. \tag{8.19}$$

The hard Lefschetz Theorem combined with Poincaré duality says that Q is non-degenerate. Moreover Q is of parity $(-1)^k$ (symmetric if k is even, alternating if k is odd). When ω is a Hodge metric, that is a Kähler metric such that $\{\omega\}\in H^2(X,\mathbb{Z})$, it is clear that Q takes integer values when restricted to $H^k(X,\mathbb{Z})/(\text{torsion})$. The Hodge-Riemann bilinear form satisfies the following additional properties: For $p+q=k$,

$$Q(H^{p,q},H^{p',q'})=0\quad\text{if } (p',q')\neq(q,p), \tag{8.20′}$$

$$\text{If } 0\neq u\in H^{p,q}_{\rm prim}(X,\mathbb{C}),\quad\text{then } \mathrm{i}^{p-q}Q(u,\overline{u})=||u||^2>0. \tag{8.20″}$$

In fact (8.20′) is clear and (8.20″) will be shown if we can check that any (p,q)-primitive form u satisfies

$$(-1)^{k(k-1)/2}\mathrm{i}^{p-q}\omega^{n-k}\wedge\overline{u}=\star\overline{u}.$$

Since $\mathrm{Prim}^{p,q}T^*_X$ is an irreducible representation of $\mathrm{U}(n)$, it suffices to verify the formula for a conveniently chosen (p,q)-form u. One can take for example $u=dz_1\wedge\cdots\wedge dz_p\wedge d\overline{z}_{p+1}\wedge\cdots\wedge d\overline{z}_{p+q}$ from an orthonormal basis for ω. The necessary verification is easy for the reader to work out as an exercise.

8.D. A description of the Picard group. Another important application of Hodge theory is a description of the *Picard group* $H^1(X,\mathcal{O}^*)$ of a compact Kähler manifold. We assume here that X is connected. The exponential exact sequence $0\to\mathbb{Z}\to\mathcal{O}\to\mathcal{O}^*\to 1$ gives

$$0\to H^1(X,\mathbb{Z})\to H^1(X,\mathcal{O})\to H^1(X,\mathcal{O}^*)\xrightarrow{c_1}H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O}), \tag{8.21}$$

taking into account the fact that the map $\exp(2\pi\mathrm{i}\bullet):H^0(X,\mathcal{O})=\mathbb{C}\to H^0(X,\mathcal{O}^*)=\mathbb{C}^*$ is surjective. One has $H^1(X,\mathcal{O})\simeq H^{0,1}(X,\mathbb{C})$ by the Dolbeault isomorphism theorem. The dimension of this group is called the *irregularity of* X and it is usually denoted by

$$q=q(X)=h^{0,1}=h^{1,0}. \tag{8.22}$$

Consequently we have $b_1=2q$ and

$$H^1(X,\mathcal{O})\simeq\mathbb{C}^q,\quad H^0(X,\Omega^1_X)=H^{1,0}(X,\mathbb{C})\simeq\mathbb{C}^q. \tag{8.23}$$

8.24. Lemma. *The image of $H^1(X,\mathbb{Z})$ in $H^1(X,\mathcal{O})$ is a lattice.*

Proof. Consider the morphism

$$H^1(X,\mathbb{Z})\to H^1(X,\mathbb{R})\to H^1(X,\mathbb{C})\to H^1(X,\mathcal{O})$$

induced by the inclusions $\mathbb{Z}\subset\mathbb{R}\subset\mathbb{C}\subset\mathcal{O}$. Since the Čech cohomology groups with values in $\mathbb{Z}$ or $\mathbb{R}$ can be calculated by a finite covering of open sets for which each is diffeomorphic to an open convex set, and the same for all their mutual intersections, it is clear that $H^1(X,\mathbb{Z})$ is a $\mathbb{Z}$-module of finite type and that the image of the $H^1(X,\mathbb{Z})$ in $H^1(X,\mathbb{R})$ is a lattice. It suffices therefore to show that the map $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$ is an isomorphism. However, the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{C} & \to & \mathcal{A}^0 & \xrightarrow{d} & \mathcal{A}^1 & \xrightarrow{d} & \mathcal{A}^2 & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{O} & \to & \mathcal{A}^{0,0} & \xrightarrow{d''} & \mathcal{A}^{0,1} & \xrightarrow{d''} & \mathcal{A}^{0,2} & \to \cdots
\end{array}$$

shows that the map $H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O})$ corresponds, for de Rham and Dolbeault cohomology, to the composite map

$$H^1_{\mathrm{DR}}(X,\mathbb{R}) \subset H^1_{\mathrm{DR}}(X,\mathbb{C}) \to H^{0,1}(X,\mathbb{C}).$$

Since $H^{1,0}(X,\mathbb{C})$ and $H^{0,1}(X,\mathbb{C})$ are complex conjugate subspaces in the complexification $H^1_{\mathrm{DR}}(X,\mathbb{C})$ of $H^1_{\mathrm{DR}}(X,\mathbb{R})$, we can easily deduce that $H^1_{\mathrm{DR}}(X,\mathbb{R}) \to H^{0,1}(X,\mathbb{C})$ is an isomorphism. $\square$

As a consequence of this lemma, $H^1(X,\mathbb{Z})$ is of rank $2q$, i.e. $H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2q}$. The complex torus of dimension q

$$\mathrm{Jac}(X) = H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) \tag{8.25}$$

is called the *Jacobian variety of* X. It is isomorphic to the subgroup of $H^1(X,\mathcal{O}^*)$ corresponding to the line bundles with zero first Chern class. In other words, the kernel of the arrow

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}) = H^{0,2}(X,\mathbb{C}),$$

which defines the integral cohomology classes of type $(1,1)$, is equal to the image of the morphism $c_1(\bullet)$ in $H^2(X,\mathbb{Z})$. This subgroup is called the *Néron-Severi group* of X, and is denoted by $NS(X)$. Its rank $\rho(X)$ is called the *Picard number* of X. The exact sequence (8.21) then gives

$$0 \to \mathrm{Jac}(X) \to H^1(X,\mathcal{O}^*) \xrightarrow{c_1} NS(X) \to 0. \tag{8.26}$$

The Picard group $H^1(X,\mathcal{O}^*)$ is therefore an extension of the complex torus $\mathrm{Jac}(X)$ by the $\mathbb{Z}$-module of finite type $NS(X)$.

8.27. Corollary. *The Picard group of $\mathbb{P}^n$ is $H^1(\mathbb{P}^n,\mathcal{O}^*) \simeq \mathbb{Z}$ with $\mathcal{O}(1)$ as generator, i.e. any line bundle over $\mathbb{P}^n$ is isomorphic to one of the line bundles $\mathcal{O}(k)$, $k \in \mathbb{Z}$.*

Proof. We have $H^k(\mathbb{P}^n,\mathcal{O}) = H^{0,k}(\mathbb{P}^n,\mathbb{C}) = 0$ for $k \geq 1$ by applying conseq. 8.8, therefore $\mathrm{Jac}(\mathbb{P}^n) = 0$ and $NS(\mathbb{P}^n) = H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z}$. Moreover, $c_1(\mathcal{O}(1))$ is a generator of $H^2(\mathbb{P}^n,\mathbb{Z})$. $\square$

9. The Hodge-Frölicher spectral sequence

Assume given X a complex manifold (i.e. not necessarily compact) of dimension n. We consider the double complex $K^{p,q} = C^\infty(X,\Lambda^{p,q}T^*_X)$ with its total differential $d = d' + d''$. The *Hodge-Frölicher spectral sequence* (or *Hodge to de Rham spectral sequence*) is by definition the spectral sequence associated to this double complex.

We first recall the algebraic machinery of spectral sequences, which applies to an arbitrary double complex $(K^{p,q}, d' + d'')$ of modules over a ring. We assume here for simplicity that $K^{p,q} = 0$ if $p < 0$ or $q < 0$. One first associates to $K^{\bullet,\bullet}$ the total complex $(K^\bullet, d)$ such that $K^l = \oplus_{p+q=l} K^{p,q}$, equipped with the total differential $d = d' + d''$. Then $K^\bullet$ admits a decreasing filtration formed from the subcomplexes $F^p K^\bullet$ where

$$F^p K^l = \bigoplus_{p \le j \le l} K^{j,l-j}. \tag{9.1}$$

One obtains an induced filtration on the cohomology groups $H^l(K^\bullet)$ of the total complex by setting

$$F^p H^l(K^\bullet) := \mathrm{Im}\big(H^l(F^p K^\bullet) \to H^l(K^\bullet)\big), \tag{9.2}$$

and one denotes by $G^p H^l(K^\bullet) = F^p H^l(K^\bullet)/F^{p+1} H^l(K^\bullet)$ the associated graded module. The theory of spectral sequences (see for example [**God57**]) says that there exists a sequence of double complexes $E_r^{\bullet,\bullet}$, $r \ge 1$, equipped with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ of bidegree $(r, -r+1)$ such that $E_{r+1} = H^\bullet(E_r)$ is calculated recursively as the cohomology of the complex $(E_r^{\bullet,\bullet}, d_r)$, and where the limit $E_\infty^{p,q} = \lim_{r\to+\infty} E_r^{p,q}$ is identified with the graded module $G^\bullet H^\bullet(K^\bullet)$, more precisely $E_\infty^{p,q} = G^p H^{p+q}(K^\bullet)$. The E_1 terms are defined as the cohomology groups of the partial complex $d'' : K^{p,q} \to K^{p,q+1}$ by passing to the second differential, that is

$$E_1^{p,q} = H^q\big((K^{p,\bullet}, d'')\big), \tag{9.4}$$

and the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is induced by the first differential d':

$$d' : H^q\big((K^{p,\bullet}, d'')\big) \to H^q\big((K^{p+1,\bullet}, d'')\big). \tag{9.5}$$

In fact, one has $E_r^{p,q} = 0$ unless $p, q \ge 0$, and the limit $E_\infty = \lim E_r$ is stationary, more precisely

$$E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q} \quad \text{when } r \ge \max(p+1, q+2),$$

as one sees by considering the indices in which d_r can be non-zero. One says that the spectral sequence converges to the graded filtered module $H^\bullet(K^\bullet)$, and it is customary to represent this situation by the notation

$$E_1^{p,q} \Rightarrow G^p H^{p+q}(K^\bullet).$$

A careful examination of the terms of small degree leads to the exact sequence

$$0 \to E_2^{1,0} \to H^1(K^\bullet) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(K^\bullet). \tag{9.6}$$

One says that the spectral sequence *degenerates* at E_{r_0} if $d_r = 0$ for all $r \ge r_0$ and for all bidegree (p, q). In this case one has $E_{r_0}^{\bullet,\bullet} = E_{r_0+1}^{\bullet,\bullet} = \cdots = E_\infty^{\bullet,\bullet}$.

In the case of the Hodge-Frölicher spectral sequence, the E_1 terms are the Dolbeault cohomology groups $E_1^{p,q} = H^{p,q}(X, \mathbb{C})$, and the cohomology of the total complex is precisely the de Rham cohomology $H^\bullet_{\mathrm{DR}}(X, \mathbb{C})$. One therefore obtains a spectral sequence

$$E_1^{p,q} = H^{p,q}(X, \mathbb{C}) \Rightarrow G^p H^{p+q}_{\mathrm{DR}}(X, \mathbb{C}) \tag{9.7}$$

of the Dolbeault cohomology to the de Rham cohomology. The corresponding filtration $F^pH^k_{\mathrm{DR}}(X,\mathbb{C})$ of cohomology groups is called the *Hodge(-Frölicher) filtration.*

Now assume that X is *compact.* All the terms $E_r^{p,q}$ are then finite dimensional vector spaces. Since $E_{r+1} = H^\bullet(E_r)$, the dimensions $\dim E_r^{p,q}$ are decreasing (or stationary) with r, therefore $\dim E_\infty^{p,q} \le \dim E_r^{p,q}$, and equality takes place if and only if the spectral sequence degenerates at E_r. In particular, the Betti numbers $b_l = \dim H^l(X,\mathbb{C})$ and the Hodge numbers $h^{p,q} = \dim E_1^{p,q}$ satisfy the inequality

$$b_l = \sum_{p+q=l} \dim E_\infty^{p,q} \le \sum_{p+q=l} \dim E_1^{p,q} = \sum_{p+q=l} h^{p,q}, \tag{9.8}$$

and equality is equivalent to the degeneration of the spectral sequence at $E_1^\bullet$. As a consequence, we have the

9.9. Theorem. *If X is a compact Kähler manifold, the following properties are equivalent:*

a) *The Hodge-Frölicher spectral sequence degenerates at $E_1^\bullet$.*

b) *One has the equality $b_l = \sum_{p+q=l} h^{p,q}$ for all l.*

c) *There exists an isomorphism $G^pH^{p+q}_{\mathrm{DR}}(X,\mathbb{C}) \simeq H^{p,q}(X,\mathbb{C})$ for all p,q.*

If one of these conditions is satisfied, the isomorphism c) *is given in a canonical way.*

We can now again interpret the results of §8.B as follows.

9.10. Theorem. *If X is a compact Kähler manifold, the Hodge-Frölicher spectral sequence degenerates at E_1 and there is a canonical decomposition*

$$H^l_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{p+q=l} H^{p,q}(X,\mathbb{C}), \quad H^{q,p}(X,\mathbb{C}) = \overline{H^{p,q}(X,\mathbb{C})}.$$

In terms of this decomposition, the filtration $F^pH^l_{\mathrm{DR}}(X,\mathbb{C})$ is given by

$$F^pH^l_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{j\ge p} H^{j,l-j}(X,\mathbb{C}).$$

In particular, the conjugate filtration $\overline{F^\bullet H^l_{\mathrm{DR}}}$ is opposed to the filtration $F^\bullet H^l_{\mathrm{DR}}$, i.e.

$$H^l_{\mathrm{DR}}(X,\mathbb{C}) = F^pH^l_{\mathrm{DR}}(X,\mathbb{C}) \oplus \overline{F^{l-p+1}H^l_{\mathrm{DR}}(X,\mathbb{C})}.$$

9.11. Definition. If X is a compact complex manifold, we say that X admits a Hodge decomposition if the Hodge-Frölicher spectral sequence degenerates at E_1 and if the conjugate filtration $\overline{F^\bullet H^l_{\mathrm{DR}}}$ is opposed to $F^\bullet H^l_{\mathrm{DR}}$, i.e. $H^l_{\mathrm{DR}} = F^pH^l_{\mathrm{DR}} \oplus \overline{F^{l-p+1}H^l_{\mathrm{DR}}}$ for all p.

If X admits a Hodge decomposition in the sense of def. 9.11 and if $p+q=l$, then it is immediate from the equality $H^l_{\mathrm{DR}} = F^{p+1}H^l_{\mathrm{DR}} \oplus \overline{F^qH^l_{\mathrm{DR}}}$ that

$$F^pH^l_{\mathrm{DR}} = F^{p+1}H^l_{\mathrm{DR}} \oplus (F^pH^l_{\mathrm{DR}} \cap \overline{F^qH^l_{\mathrm{DR}}}).$$

Therefore one obtains a canonical isomorphism

$$H^{p,q}(X,\mathbb{C}) \simeq F^pH^l_{\mathrm{DR}}/F^{p+1}H^l_{\mathrm{DR}} \simeq F^pH^l_{\mathrm{DR}} \cap \overline{F^qH^l_{\mathrm{DR}}} \subset H^l_{\mathrm{DR}}. \tag{9.12}$$

We deduce from this that there are canonical isomorphisms

$$H^l_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{p+q=l} H^{p,q}(X,\mathbb{C}), \quad H^{q,p}(X,\mathbb{C}) = \overline{H^{p,q}(X,\mathbb{C})},$$

as expected. Note that (9.12) furnishes another proof of the fact that the Hodge decomposition of a compact Kähler manifold does not depend on the choice of Kähler metric (all the groups and morphisms concerned in (9.12) are intrinsic). In fact, we have shown that a compact Kähler manifold satisfies a still stronger property, that will be convenient to call a *strong Hodge decomposition,* since this one trivially implies the existence of a Hodge decomposition in the sense of Definition 9.11.

9.13. DEFINITION. If X is a compact complex manifold, we say that X admits a strong Hodge decomposition if the morphisms

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C}), \qquad \bigoplus_{p+q=l} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^l_{\mathrm{DR}}(X,\mathbb{C})$$

are isomorphisms.

9.14. REMARK. Deligne [**Del68, 72**] has given an algebraic criterion for the degeneration of the Hodge spectral sequence, including the case of the relative situation. More recently, Deligne and Illusie [**DeI87**] have given a proof of the degeneration of the Hodge spectral sequence which does not use analytic methods (their idea is to work in characteristic p and to relate the result in characteristic 0). It is necessary to observe that the degeneration of the Hodge-Frölicher spectral sequence *does not automatically imply* the Hodge symmetry property $H^{q,p}(X,\mathbb{C}) = \overline{H^{p,q}(X,\mathbb{C})}$ nor the existence of a canonical decomposition of de Rham groups. In fact, it is not difficult to show that the Hodge-Frölicher spectral sequence of a compact complex surface always degenerates at E_1; however if X is not Kähler, then b_1 is odd, and one can show using the index theorem of Hirzebruch that $h^{0,1} = h^{1,0} + 1$ and $b_1 = 2h^{1,0} + 1$ (see [**BPV84**]). One can show that the existence of a Hodge decomposition (resp. strong Hodge) is preserved by contraction morphisms (replacement of X by X', if $\mu : X \to X'$ is a modification); this is an easy consequence of the existence of a direct image functor μ_* acting on all the cohomology groups concerned, such that $\mu_*\mu^* = \mathrm{Id}$. In the analytic context, μ_* is easily constructed by calculating cohomology with the aid of currents, since one has on those a natural direct image functor. As any Moishezon manifold admits a projective algebraic modification, we deduce that Moishezon manifolds also admit a strong Hodge decomposition. It would be interesting to know if there exists examples of compact complex manifolds possessing a Hodge decomposition without having a strong Hodge decomposition (there are indeed immediate examples of abstract double complexes having this property). □

In general, when X is not Kähler, a certain amount of interesting information can be deduced from the spectral sequence. For example, (9.6) implies

$$b_1 \geq \dim E_2^{1,0} + (\dim E_2^{0,1} - \dim E_2^{2,0})_+. \tag{9.15}$$

In addition, $E_2^{1,0}$ is the cohomology group defined by the sequence

$$d_1 = d' : E_1^{0,0} \to E_1^{1,0} \to E_1^{2,0},$$

and since $E_1^{0,0}$ is the space of global holomorphic functions on X, the first arrow d_1 is zero (by the maximum principal, the holomorphic functions are constant on each connected component of X). Therefore $\dim E_2^{1,0} \geq h^{1,0} - h^{2,0}$. Similarly, $E_2^{0,1}$ is the kernel of the map $E_1^{0,1} \to E_1^{1,1}$, therefore $\dim E_2^{0,1} \geq h^{0,1} - h^{1,1}$. From (9.15) we deduce

$$b_1 \geq (h^{1,0} - h^{2,0})_+ + (h^{0,1} - h^{1,1} - h^{2,0})_+. \tag{9.16}$$

Another interesting relation concerns the topological Euler-Poincaré characteristic

$$\chi_{\mathrm{top}}(X) = b_0 - b_1 + \cdots - b_{2n-1} + b_{2n}.$$

We utilize the following simple lemma.

9.17. Lemma. *Let $(C^\bullet, d)$ be a bounded complex of finite dimensional vector spaces over a field. Then the Euler characteristic*

$$\chi(C^\bullet) = \sum (-1)^q \dim C^q$$

is equal to the Euler characteristic $\chi\big(H^\bullet(C^\bullet)\big)$ of the cohomology module.

Proof. Set

$$c_q = \dim C^q, \quad z_q = \dim Z^q(C^\bullet), \quad b_q = \dim B^q(C^\bullet), \quad h_q = \dim H^q(C^\bullet).$$

Then

$$c_q = z_q + b_{q+1}, \quad h_q = z_q - b_q.$$

Consequently we find

$$\sum (-1)^q c_q = \sum (-1)^q z_q - \sum (-1)^q b_q = \sum (-1)^q h_q. \qquad \square$$

In particular, if the term $E_r^\bullet$ of the spectral sequence of a filtered complex $K^\bullet$ is a bounded complex of finite dimension, one has

$$\chi(E_r^\bullet) = \chi(E_{r+1}^\bullet) = \cdots = \chi(E_\infty^\bullet) = \chi(H^\bullet(K^\bullet))$$

because $E_{r+1}^\bullet = H^\bullet(E_r^\bullet)$ and $\dim E_\infty^l = \dim H^l(K^\bullet)$. In the Hodge-Frölicher spectral sequence one additionally has $\dim E_1^l = \sum_{p+q=l} h^{p,q}$, therefore:

9.18. Theorem. *For any compact complex manifold X, the topological Euler characteristic can be written*

$$\chi_{\mathrm{top}}(X) = \sum_{0 \leq l \leq 2n} (-1)^l b_l = \sum_{0 \leq p,q \leq n} (-1)^{p+q} h^{p,q}.$$

We now translate the Hodge-Frölicher spectral sequence in terms of the spectral sequence of hypercohomology associated to the holomorphic de Rham complex. First let us briefly explain what this spectral sequence consists of. Assume given a bounded complex of sheaves of abelian groups $\mathcal{A}^\bullet$ over a topological space X. Then the *hypercohomology groups* of $\mathcal{A}^\bullet$ are defined as the groups

$$\mathbb{H}^k(X, \mathcal{A}^\bullet) := H^k(\Gamma(X, \mathcal{L}^\bullet)),$$

where $\mathcal{L}^\bullet$ is a complex of acyclic sheaves (flasque sheaves or sheaves of $\mathcal{C}^\infty$ modules for example) chosen so that one has a quasi-isomorphism $\mathcal{A}^\bullet \to \mathcal{L}^\bullet$ (a morphism of complexes of sheaves inducing an isomorphism $\mathcal{H}^k(\mathcal{A}^\bullet) \to \mathcal{H}^k(\mathcal{L}^\bullet)$ on the cohomology of sheaves). It is easy to see that hypercohomology does not depend up to isomorphism on the complex of acyclic sheaves $\mathcal{L}^\bullet$ chosen. Hypercohomology is a functor from the category of complexes of sheaves of abelian groups to the category of graded groups. By definition, if $\mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism, then $\mathbb{H}^k(X, \mathcal{A}^\bullet) \to \mathbb{H}^k(X, \mathcal{B}^\bullet)$ is an isomorphism; moreover hypercohomology reduces to the usual cohomology $H^k(X, \mathcal{E})$ of the sheaf $\mathcal{E}$ for a complex $\mathcal{A}^\bullet$ reduced to a single term $\mathcal{A}^0 = \mathcal{E}$. Suppose that one has for each term $\mathcal{A}^p$ of the complex $\mathcal{A}^\bullet$ a resolution $\mathcal{A}^p \to \mathcal{L}^{p,\bullet}$ by acyclic sheaves $\mathcal{L}^{p,q}$, giving rise to a double complex of sheaves $(\mathcal{L}^{p,q}, d' + d'')$. Then the associated total complex $(\mathcal{L}^\bullet, d)$ is an acyclic complex quasi-isomorphic to $\mathcal{A}^\bullet$, and one therefore has

$$\mathbb{H}^k(X, \mathcal{A}^\bullet) = H^k(\Gamma(X, \mathcal{L}^\bullet)).$$

Further, the double complex $K^{p,q} = \Gamma(X, \mathcal{L}^{p,q})$ defines a spectral sequence such that

$$E_1^{p,q} = H^q(K^{p,\bullet}, d'') = H^q(X, \mathcal{A}^p),$$

converges to the associated graded cohomology of the total complex $H^k(K^\bullet) = \mathbb{H}^k(X, \mathcal{A}^\bullet)$. One therefore obtains a spectral sequence called the *hypercohomology spectral sequence*

$$E_1^{p,q} = H^q(X, \mathcal{A}^p) \Rightarrow G^p\mathbb{H}^{p+q}(X, \mathcal{A}^\bullet). \tag{9.19}$$

The filtration F^p of hypercohomology groups is by definition obtained by taking the image of the morphism

$$\mathbb{H}^k(X, F^p\mathcal{A}^\bullet) \to \mathbb{H}^k(X, \mathcal{A}^\bullet),$$

where $F^p\mathcal{A}^\bullet$ denotes the complex truncated to the left

$$\cdots \to 0 \to 0 \to \mathcal{A}^p \to \mathcal{A}^{p+1} \to \cdots \to \mathcal{A}^N \cdots.$$

Consider now the case where X is any given complex manifold and where $\mathcal{A}^\bullet = \Omega_X^\bullet$ is the holomorphic de Rham complex (with the usual exterior differential). The holomorphic Poincaré Lemma shows that $\Omega_X^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_X$, i.e., one has a quasi-isomorphism of complexes of sheaves $\mathbb{C}_X \to \Omega_X^\bullet$, where $\mathbb{C}_X$ denotes the complex reduced to a single term in degree 0. By definition of hypercohomology, one therefore has

$$H^k(X, \mathbb{C}_X) = \mathbb{H}^k(X, \Omega_X^\bullet), \tag{9.20}$$

and the exact sequence of hypercohomology of the complex $\Omega_X^\bullet$ furnishes a spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow G^pH^{p+q}(X, \mathbb{C}_X). \tag{9.21}$$

Because the groups $\mathbb{H}^k(X, \Omega_X^\bullet)$ can be calculated by using the resolution of $\Omega_X^\bullet$ by the Dolbeault complex $\mathcal{L}^{p,q} = \mathcal{C}^\infty(\Lambda^{p,q}T_X^*)$ (these sheaves are certainly acyclic!), one then sees that the hypercohomology spectral sequence (9.21) is precisely the Hodge-Frölicher spectral sequence previously defined.

10. Deformations and the semi-continuity theorem

The purpose of this section is to study the dependence of the groups $H^{p,q}(X_t, \mathbb{C})$ or more generally the cohomology groups $H^q(X_t, E_t)$, when the pair (X_t, E_t) depends holomorphically on a parameter t in a certain complex space S. Our approach is to adopt the point of view of Kodaira-Spencer, such as is developed in their original work on the theory of deformations (see for example the complete works of Kodaira [**Kod75**]). The method of Kodaira-Spencer exploits the continuity properties or semi-continuity of proper spaces of Laplacians as a function of the parameter t. Another approach furnishing more precise results consists of utilizing the theorem of direct images of Grauert [**Gra60**].

10.1 Definition. A deformation of compact complex manifolds is given by a proper analytic morphism $\sigma : \mathfrak{X} \to S$ of connected complex spaces, for which all the fibers $X_t = \sigma^{-1}(t)$ are smooth manifolds of the same dimension n, and satisfy the following local condition:

(H) Any point $\zeta \in \mathfrak{X}$ admits a neighbourhood $\mathcal{U}$ such that there exists a biholomorphism $\psi : U \times V \to \mathcal{U}$ where U is open in $\mathbb{C}^n$ and V is a neighbourhood of $t = \sigma(\zeta)$, satisfying $\sigma \circ \psi = \mathrm{pr}_2 : U \times V \to V$ (second projection).

We say that $(X_t)_{t \in S}$ is a holomorphic family of deformations of any given fiber X_{t_0}, and that S is the base of the deformation. A holomorphic family of vector bundles (resp. sheaves) $E_t \to X_t$ is given by a family of bundles (resp. sheaves) obtained from a global bundle (resp. global sheaf) $\mathcal{E} \to \mathfrak{X}$, by restriction to the fibers X_t.

If S is smooth, the hypothesis (H) is equivalent to assuming that σ is a holomorphic submersion, as a consequence of the theorem of constant rank. There are nevertheless situations where one must necessarily consider also the case of a singular base S (for example when one seeks to construct the "universal deformation" of a manifold). In a topological setting (differentiable or smooth), we have the following lemma, known as Ehresmann's Lemma.

10.2. Ehresmann's Lemma. *Let $\sigma : \mathfrak{X} \to S$ be a smooth and proper differentiable submersion.*

a) *If S is contractible, then for any $t_0 \in S$, there exists a commutative diagram*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\Phi} & X_{t_0} \times S \\ \mathrm{pr}_1 \searrow & & \swarrow \sigma \\ & S & \end{array}$$

where Φ is a diffeomorphism.

b) *For any given base S, $\mathfrak{X} \to S$ is a locally trivial bundle (differentiable). In particular, if S is connected, the fibers are all diffeomorphic.*

Proof. a) Let $H : S \times [0,1] \to S$ be a differentiable homotopy between $H(\bullet, 0) = \mathrm{Id}_S$ and $H(\bullet, 1) =$ constant map $S \to \{t_0\}$. The fiber product

$$\tilde{\mathfrak{X}} = \{(x, s, t) \in \mathfrak{X} \times S \times [0,1] \; ; \; \sigma(x) = H(s,t)\}$$

with projection $\tilde{\sigma} = \mathrm{pr}_2 \times \mathrm{pr}_3 : \tilde{\mathfrak{X}} \to S \times [0,1]$ is still a differentiable submersion, as one can easily verify. One deduces that there exists a vector field ξ on $\tilde{\mathfrak{X}}$ which lifts the vector field $\frac{\partial}{\partial t}$ on $S \times [0,1]$, i.e. $\sigma_* \xi = \frac{\partial}{\partial t}$. (There exists a local lifting by the submersive property, and one glues together these liftings by means of a partition of unity.) Let φ_t be a flow of this lifting: Then, if $(x,s,0) \in \tilde{\mathfrak{X}}_{\restriction S\times\{0\}} \simeq \mathfrak{X}$, one has by construction $\varphi_t(x,s,0) = (?,s,t)$, therefore $\Phi = \varphi_1$ defines a diffeomorphism of $\tilde{\mathfrak{X}}_{\restriction S\times\{0\}} \simeq \mathfrak{X}$ on $\tilde{\mathfrak{X}}_{\restriction S\times\{1\}} \simeq X_{t_0} \times S$, commuting with the projection on S.

b) is deduced immediately from a). □

It follows from b) that the bundle $t \mapsto H^k(X_t,\mathbb{C})$ is a locally trivial bundle of $\mathbb{C}$-vector spaces of finite dimension. Furthermore, in each fiber we have a free abelian subgroup $\mathrm{Im}H^k(X_t,\mathbb{Z}) \subset H^k(X_t,\mathbb{C})$ of rank b_k which generates $H^k(X_t,\mathbb{C})$ as a $\mathbb{C}$-vector space. The transition matrices of this locally constant system are in $\mathrm{SL}_{b_k}(\mathbb{Z})$. Since the transition matrices are locally constant, the bundle $t \mapsto H^k(X_t,\mathbb{C})$ is equipped with a connection D such that $D^2 = 0$: This connection is called the *Gauss-Manin connection.* The following lemma is useful.

10.3. LEMMA. *Let $\sigma : \mathfrak{X} \to S$ be a smooth and proper differentiable submersion and $\mathcal{E}$ a C^∞ vector bundle over $\mathfrak{X}$. Consider a family of elliptic operators*

$$P_t : C^\infty(X_t,E_t) \to C^\infty(X_t,E_t)$$

of degree δ. We assume that P_t is self-adjoint semipositive relative to a metric h_t on E_t and a volume form dV_t on X_t, and that the coefficients of P_t, h_t and dV_t are C^∞ on $\mathfrak{X}$. Then the eigenvalues of P_t, computed with multiplicity, can be arranged in a sequence

$$\lambda_0(t) \le \lambda_1(t) \le \cdots \le \lambda_k(t) \to +\infty,$$

where the k-th eigenvalue $\lambda_k(t)$ is a continuous function of t. Moreover, if λ is not in the spectrum $\{\lambda_k(t_0)\}_{k\in\mathbb{N}}$ of P_{t_0}, the direct sum $W_{\lambda,t} \subset C^\infty(X_t,E_t)$ of eigenspaces of P_t with eigenvalues $\lambda_k(t) \le \lambda$ defines a C^∞ vector bundle, $t \mapsto W_{t,\lambda}$, in a neighbourhood of t_0.

PROOF. Since the results are local over S, one can assume that $\mathfrak{X} = X_{t_0} \times S$ and $\mathcal{E} = \mathrm{pr}_1^* E_{t_0}$, that is, their fibers X_t and E_t are independent of t (but the forms dV_t on X_t and the metrics h_t on E_t are in general dependent on t). Let $\Pi_{\lambda,t}$ be the orthogonal projection operator on $W_{\lambda,t}$ in $L^2(X_t,E_t) \simeq L^2(X_{t_0},E_{t_0})$. If $\Gamma(0,\lambda)$ denotes the circle with center 0 and with radius λ in the complex plane, Cauchy's formula gives

$$\Pi_{\lambda,t} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma(0,\lambda)} (z\mathrm{Id} - P_t)^{-1} dz,$$

where the integral is viewed as an integral with vector values in the space of bounded operators on $L^2(M_{t_0},E_{t_0})$. (It suffices to verify the formula on the eigenvectors of P_t, which is elementary.) The arguments made in §3 show that there exists a family of pseudodifferential operators Q_t of order $-\delta$, for which the symbol depends in a C^∞ manner with t (and with uniform estimates by differentiation in t), such that $P_tQ_t = \mathrm{Id} + R_t$ for regular operators R_t, for which the kernel also depends in a C^∞ manner in t. Since Q_t is a family of compact operators on $L^2(X_t,E_t)$ which depend in a C^∞ manner in t, the eigenvalues of Q_t depend continuously in t. Up to changing Q_{t_0} on a subspace of finite dimension, one can assume that Q_{t_0} is an isomorphism of $L^2(X_{t_0},E_{t_0})$ onto $W^\delta(X_{t_0},E_{t_0})$. It will be the same for Q_t in a neighbourhood of t_0, and consequently $z\mathrm{Id} - P_t$ is invertible if and only if $(z\mathrm{Id} - P_t)Q_t = \mathrm{Id} + R_t + zQ_t$ is invertible. If λ is not in the spectrum of P_{t_0}, it follows that for all $z \in \Gamma(0,\lambda)$, the inverse $(z\mathrm{Id} - P_t)^{-1} = Q_t(\mathrm{Id} + R_t + zQ_t)^{-1}$ depends in a C^∞ way in t. This implies that $t \mapsto W_{t,\lambda}$ is a locally trivial C^∞ fibration in a neighbourhood of t_0. The continuity of the eigenvalue $\lambda_k(t)$ of P_t follows from the constant rank of $W_{\lambda,t}$ in a neighbourhood of t_0, for $\lambda = \lambda_k(t_0) \pm \epsilon$. □

10.4. Semi-continuity Theorem (Kodaira-Spencer). *If $\mathfrak{X} \to S$ is a smooth, proper $\mathbb{C}$-analytic morphism and if $\mathcal{E}$ is a locally free sheaf on $\mathfrak{X}$, the dimensions $h^q(t) = h^q(X_t, \mathcal{E}_t)$ are upper semi-continuous functions. More precisely, the alternating sums*

$$h^q(t) - h^{q-1}(t) + \cdots + (-1)^q h^0(t), \quad 0 \le q \le n = \dim X_t$$

are upper semi-continuous functions.

Proof. Let E_t the holomorphic vector bundle associated to $\mathcal{E}_t$. Equip $\mathcal{E}$ and $\mathfrak{X}$ with arbitrary Hermitian metrics. According to the Hodge isomorphism for the d''-cohomology, one can interpret $H^q(X_t, \mathcal{E}_t)$ as the space of harmonic forms for the Laplacian Δ''^q_t acting on $C^\infty(X_t, \Lambda^{0,q} T^*_{X_t} \otimes E_t)$. Fix a point $t_0 \in S$ and a real $\lambda > 0$ which does not belong in the spectrum of the operators $\Delta''^q_{t_0}, 0 \le q \le n = \dim X_t$. Then

$$W^q_t = W^q_{\lambda,t} = \text{direct sum of eigenspaces of } \Delta''^q_t \text{ with eigenvalues} \le \lambda$$

defines a C^∞ bundle W^q in a neighbourhood of t_0. Moreover the differential d''_t commutes with Δ''_t and thus sends the eigenspaces of Δ''^q_t into the eigenspaces of Δ''^{q+1}_t associated to the same eigenvalues. This shows that $(W^\bullet_t, d''_t)$ is a subcomplex of finite dimension of the Dolbeault complex $\big(C^\infty(X_t, \Lambda^{0,q} T^*_{X_t} \otimes E_t), d''_t\big)$. The cohomology of this subcomplex coincides with $H^q(X_t, E_t)$ since the relation $d''_t d''^*_t + d''^*_t d''_t = \Delta''_t$ shows that $\frac{1}{\lambda_k} d''^*_t$ is a homotopy operator on the subcomplex formed from the eigenspaces with eigenvalue λ_k when $\lambda_k \ne 0$. If Z^q_t denotes the kernel of the morphism $d''^q_t : W^q_t \to W^{q+1}_t$, then $z^q(t) := \dim Z^q_t$ is an upper semi-continuous function in the Zariski topology, as one can easily see by considering the rank of the minors of the matrix defining the morphism $d''^q : W^q \to W^{q+1}$. From the truncated complex

$$0 \to W^0_t \to W^1_t \to \cdots \to W^{q-1}_t \to Z^q_t \to 0$$

having for the cohomology the groups $H^j(X_t, E_t)$ with indices $0 \le j \le q$, one obtains

$$h^q(t) - h^{q-1}(t) + \cdots + (-1)^q h^0(t) = z^q(t) - w^{q-1} + w^{q-2} + \cdots + (-1)^q w^0,$$

where w^q denotes the rank of W^q. The upper semi-continuity of the term on the left follows, and that of $h^q(t)$ is then immediate by induction on q. □

10.5. Invariance of the Hodge numbers. Let $\mathfrak{X} \to S$ be a smooth and proper $\mathbb{C}$-analytic morphism. We assume that the fibers X_t are Kähler manifolds. Then the Hodge numbers $h^{p,q}(X_t)$ are constant. Moreover, in the decomposition

$$H^k(X_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t, \mathbb{C}),$$

the bundles $t \mapsto H^{p,q}(X_t, \mathbb{C})$ define C^∞ subbundles (in general, not holomorphic subbundles) of the bundle $t \mapsto H^k(X_t, \mathbb{C})$.

PROOF. Lemma 10.2 implies that the Betti numbers $b_k = \dim H^k(X_t, \mathbb{C})$ are constant. Since, according to th. 10.4, $h^{p,q}(X_t, \mathbb{C}) = h^q(X_t, \Omega^p_{X_t})$ is upper semi-continuous, and

$$h^{p,q}(X_t) = b_k - \sum_{r+s=k,(r,s)\neq(p,q)} h^{r,s}(X_t),$$

these functions are likewise lower semi-continuous. Consequently they are continuous and therefore constant. A theorem of Kodaira [**Kod75**] shows that if a fiber X_{t_0} is Kähler, then the neighbouring fibers X_t are Kähler and the Kähler metrics ω_t can be chosen so that they depend in a C^∞ way with t. The spaces of harmonic (p,q)-forms therefore depend in a C^∞ way with t according to th. 10.4, and one deduces that $t \mapsto H^{p,q}(X_t, \mathbb{C})$ is a C^∞ subbundle of $H^k(X_t, \mathbb{C})$. □

It is possible to obtain more precise and general results by means of the theorem of direct images of Grauert [**Gra60**]. Recall that if we are given a continuous map $f : X \to Y$ between topological spaces and a sheaf $\mathcal{E}$ of abelian groups on X, then one can define the direct image sheaf $R^k f_* \mathcal{E}$ on Y, as being the sheaf associated to the presheaf $U \mapsto H^k(f^{-1}(U), \mathcal{E})$, for all open U in Y. More generally, being given a complex of sheaves $\mathcal{A}^\bullet$, we have the direct image sheaves $\mathbb{R}^q f_* \mathcal{A}^\bullet$, obtained from the hypercohomology presheaves

$$U \mapsto \mathbb{H}^k(f^{-1}(U), \mathcal{A}^\bullet).$$

The proof of the theorem of direct images as given by [**FoK71**] and [**KiV71**] (also see [**DoV72**]) furnishes the following fundamental result.

10.6. THEOREM OF DIRECT IMAGES. *Let $\sigma : \mathfrak{X} \to S$ be a proper morphism of complex analytic spaces and $\mathcal{A}^\bullet$ a bounded complex of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules. Then*

a) *The direct image sheaves $\mathbb{R}^k \sigma_* \mathcal{A}^*$ are coherent sheaves on S.*
b) *Any point of S admits a neighbourhood $U \subset S$ on which there exists a bounded complex $\mathcal{W}^\bullet$ of sheaves of locally free $\mathcal{O}_S$-modules in which the cohomology sheaves $\mathcal{H}^k(\mathcal{W}^\bullet)$ are isomorphic to the sheaf $\mathbb{R}^k \sigma_* \mathcal{A}^\bullet$.*
c) *If the fibers of σ are equidimensional ("geometrically flat morphism"), the hypercohomology of the fiber $X_t = \sigma^{-1}(t)$ with values in $\mathcal{A}^\bullet_t = \mathcal{A}^\bullet \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_t}$ (where $\mathcal{O}_{X_t} = \mathcal{O}_{\mathfrak{X}}/\sigma^* \mathfrak{m}_{S,t}$) is given by*

$$H^k(X_t, \mathcal{A}^\bullet_t) = H^k(W^\bullet_t),$$

where $(W^\bullet_t)$ is the complex of finite dimensional spaces $W^k_t = \mathcal{W}^k \otimes_{\mathcal{O}_{S,t}} \mathcal{O}_{S,t}/\mathfrak{m}_{S,t}$.
d) *Under the hypothesis of* c), *if the hypercohomology spaces $\mathbb{H}^k(X_t, \mathcal{A}^\bullet_t)$ of the fibers are of constant dimension, the sheaves $\mathbb{R}^k \sigma_* \mathcal{A}^\bullet$ are locally free on S.*

The same results are true in particular for the direct images $R^k \sigma_* \mathcal{E}$ of a coherent sheaf $\mathcal{E}$ on $\mathfrak{X}$, and the cohomology groups $H^k(X_t, \mathcal{E}_t)$ of the fibers.

One notes that property d) is in fact a formal consequence of c), because the hypothesis guarantees that the holomorphic matrices defining morphisms $\mathcal{W}^k \to \mathcal{W}^{k+1}$ are of constant rank at each point $t \in S$. From (10.6b) one then deduces the following result due to [**Fle81**] with an identical argument to that in th. 10.4.

10.7 SEMI-CONTINUITY THEOREM. *If $\mathfrak{X} \to S$ is a proper analytic morphism with equidimensional fibers and if $\mathcal{E}$ is a coherent sheaf on $\mathfrak{X}$, then the alternating* *sums*

$$h^q(t) - h^{q-1}(t) + \cdots + (-1)^q h^0(t),$$

with dimensions $h^k(t) = h^k(X_t, \mathcal{E}_t)$, are upper semi-continuous functions of t in the analytic Zariski topology (topology of whose closed are the analytic sets).

Let $\sigma : \mathfrak{X} \to S$ be a $\mathbb{C}$-analytic proper and smooth submersion. One assumes that the Hodge spectral sequence of the fibers X_t degenerates at E_1 for all $t \in S$ (according to (10.7) this is in fact an open property for the analytic Zariski topology on S). If $U \subset S$ is open and contractible, then $\sigma^{-1}(U) \simeq X_t \times U$ for any fiber over $t \in U$. If $\mathbb{Z}_{\mathfrak{X}}, \mathbb{C}_{\mathfrak{X}}$, denotes the locally constant sheaves with base $\mathfrak{X}$ and with fibers $\mathbb{Z}$, $\mathbb{C}$, one obtains

$$\Gamma(U, R^k\sigma_*\mathbb{Z}_{\mathfrak{X}}) = H^k(\sigma^{-1}(U), \mathbb{Z}) = H^k(X_t, \mathbb{Z}),$$

$$\Gamma(U, R^k\sigma_*\mathbb{C}_{\mathfrak{X}}) = H^k(\sigma^{-1}(U), \mathbb{C}) = H^k(X_t, \mathbb{C}),$$

so that $R^k\sigma_*\mathbb{Z}_{\mathfrak{X}}$ and $R^k\sigma_*\mathbb{C}_{\mathfrak{X}}$ are locally constant sheaves on S, with fibers $H^k(X_t, \mathbb{Z})$ and $H^k(X_t, \mathbb{C})$. The bundle $t \mapsto H^k(X_t, \mathbb{C})$, equipped with the flat connection D (Gauss-Manin connection), possesses a canonical holomorphic structure induced by the component $D^{0,1}$ of the Gauss-Manin connection. The flat bundle $\oplus_k H^k(X_t, \mathbb{C})$ is called the Hodge bundle of the fibration $\mathfrak{X} \to S$.

Now consider the relative de Rham complex $(\Omega^\bullet_{\mathfrak{X}/S}, d_{\mathfrak{X}/S})$ of the fibration $\mathfrak{X} \to S$. This complex furnishes a resolution of the sheaf $\sigma^{-1}\mathcal{O}_S$ ("purely sheafified" inverse image of $\mathcal{O}_S$), consequently

$$\mathbb{R}^k\sigma_*\Omega^\bullet_{\mathfrak{X}/S} = R^k\sigma_*(\sigma^{-1}\mathcal{O}_S) = (R^k\sigma_*\mathbb{C}_{\mathfrak{X}}) \otimes_{\mathbb{C}} \mathcal{O}_S. \tag{10.8}$$

The latter equality is obtained immediately by an argument using $\mathcal{O}_S(U)$ linearity for the cohomology calculated on the open set $\sigma^{-1}(U)$ (the complex structure of $\sigma^{-1}(U)$ does not intervene here). In other words, $\mathbb{R}^k\sigma_*\Omega^\bullet_{\mathfrak{X}/S}$ is the locally free $\mathcal{O}_S$-module associated to the flat bundle $t \mapsto H^q(X_t, \mathbb{C})$. One has a relative hypercohomology spectral sequence

$$E_1^{p,q} = R^q\sigma_*\Omega^p_{\mathfrak{X}/S} \Rightarrow G^p\mathbb{R}^{p+q}\sigma_*\Omega^\bullet_{\mathfrak{X}/S} = G^pR^{p+q}\sigma_*\mathbb{C}_{\mathfrak{X}}$$

(the relative spectral sequence is obtained simply by a "sheafification" of the absolute hypercohomology spectral sequence (9.19) of the complex $\Omega^\bullet_{\mathfrak{X}/S}$ over the open set $\sigma^{-1}(U)$). Since the cohomology of $\Omega^p_{\mathfrak{X}/S}$ on the fiber X_t is precisely the space $H^q(X_t, \Omega^p_{X_t})$ of constant rank, th. 10.6d) shows that the direct image sheaves $R^p\sigma_*\Omega^p_{\mathfrak{X}/S}$ are locally free. In addition, the filtration $F^pH^k(X_t, \mathbb{C}) \subset H^k(X_t, \mathbb{C})$ is obtained on the level of locally free $\mathcal{O}_S$-modules associated with taking the image of the $\mathcal{O}_S$-linear morphism

$$\mathbb{R}^k\sigma_*F^p\Omega^\bullet_{\mathfrak{X}/S} \to \mathbb{R}^k\sigma_*\Omega^\bullet_{\mathfrak{X}/S},$$

which is therefore a coherent subsheaf (and likewise a locally free subsheaf, according to the property of constant rank on the fibers X_t). From (10.8) one deduces the

10.9. THEOREM (holomorphic Hodge filtration). *The Hodge filtration $F^pH^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C})$ defines a holomorphic subbundle relative to the holomorphic structure defined by the Gauss-Manin connection.*

One sees that in general there is no reason for $H^{p,q}(X_t,\mathbb{C}) = F^pH^k(X_t,\mathbb{C}) \cap \overline{F^qH^k(X_t,\mathbb{C})}$ to be a holomorphic subbundle of $H^k(X_t,\mathbb{C})$ for any $p+q=k$, although $H^{p,q}(X_t,\mathbb{C})$ possesses a natural holomorphic bundle structure (obtained from the coherent sheaf $R^q\sigma_*\Omega^p_{\mathfrak{X}/S}$, or as a quotient of $F^pH^k(X_t,\mathbb{C})$). In other words, this is the Hodge *decomposition* which is not holomorphic.

10.10 EXAMPLE. Let $S = \{\tau \in \mathbb{C};\ \mathrm{Im}\ \tau > 0\}$ be the upper half plane and $\mathfrak{X} \to S$ the "universal" family of elliptic curves over S, defined by $X_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. The two basis elements of the Hodge fiber $H^1(X_\tau,\mathbb{C})$, dual to the basis $(1,\tau)$ of the lattice of periods, are $\alpha = dx - \mathrm{Re}\ \tau/\mathrm{Im}\ \tau dy$ and $\beta = (\mathrm{Im}\ \tau)^{-1}dy$ ($z = x + \mathrm{i}y \in \mathbb{C}$ denotes the coordinates on X_τ). These elements therefore satisfy $D\alpha = D\beta = 0$ and define the holomorphic structure of the Hodge bundle; the subbundle $H^{1,0}(X_\tau,\mathbb{C})$ generated by the 1-form $dz = \alpha + \tau\beta$ is clearly holomorphic (as it should be!), however one sees that the components $\beta^{1,0} = -\frac{\mathrm{i}}{2}(\mathrm{Im}\ \tau)^{-1}dz$ and $\beta^{0,1} = -\frac{\mathrm{i}}{2}(\mathrm{Im}\ \tau)^{-1}d\overline{z}$ are not holomorphic in τ.

Part II: L^2 Estimations and Vanishing Theorems

11. Concepts of pseudoconvexity and of positivity

The statements and proofs of the vanishing theorems brings into play many concepts of pseudoconvexity and positivity. We first present a summary, by bringing together the concepts that we deem necessary.

11.A. Plurisubharmonic functions. The plurisubharmonic functions were introduced independently by Lelong and Oka in 1942 in the study of holomorphic convexity. We refer to [**Lel67, 69**] for more details.

11.1. Definition. A function $u : \Omega \to [-\infty, +\infty[$ defined on an open set $\Omega \subset \mathbb{C}^n$ is called plurisubharmonic (abbreviated psh) if

a) u is upper semi-continuous;

b) for any complex line $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for any $a \in \Omega$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, \complement\Omega)$, the function u satisfies the mean inequality
$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + \mathrm{e}^{\mathrm{i}\Theta}\xi)d\theta.$$
The set of psh functions on Ω is denoted by $\mathrm{Psh}(\Omega)$.

We give below a list of some fundamental properties satisfied by the psh functions. All these properties come about easily from the definition.

11.2. Fundamental properties.

a) Any function $u \in \mathrm{Psh}(\Omega)$ is subharmonic in the $2n$ real variables, i.e. satisfies the mean value inequality on the Euclidean ball (or sphere):
$$u(a) \leq \frac{1}{\pi^n r^{2n}/n!} \int_{B(a,r)} u(z) d\lambda(z)$$
for all $a \in \Omega$ and all $r < d(a, \complement\Omega)$. In this case, one has either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}$ on every connected component of Ω.

b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

c) Assume given $u \in \mathrm{Psh}(\Omega)$ such that $u \not\equiv -\infty$ on all connected components of Ω. If (ρ_ϵ) is a family of regular kernels, then $u \star \rho_\epsilon$ is C^∞ and psh on
$$\Omega_\epsilon = \{x \in \Omega;\ d(x, \complement\Omega) > \epsilon\},$$
the family $(u \star \rho_\epsilon)$ is increasing in ϵ, and $\lim_{\epsilon \to 0} u \star \rho_\epsilon = u$.

d) Assume given $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \to \mathbb{R}$ a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each variable t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(\mathrm{e}^{u_1} + \cdots + \mathrm{e}^{u_p})$ are psh on Ω. □

11.3. Lemma. *A function $u \in C^2(\Omega, \mathbb{R})$ is psh on Ω if and only if the Hermitian form $Hu(a)(\xi) = \sum_{1 \leq j,k \leq n} \partial^2 u/\partial z_j \partial \overline{z}_k(a) \xi_j \overline{\xi}_k$ is semi-positive at every point $a \in \Omega$.*

PROOF. This is an easy consequence of the following standard formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(a+\mathrm{e}^{\mathrm{i}\Theta}\xi)d\theta - u(a) = \frac{2}{\pi}\int_0^1 \frac{dt}{t}\int_{|\zeta|<t} Hu(a+\zeta\xi)(\xi)d\lambda(\zeta),$$

where $d\lambda$ is the Lebesque measure on $\mathbb{C}$. Lemma 11.3 strongly suggests that plurisubharmonicity is the complex analog of the property of linear convexity in the real case. $\square$

For nonregular functions, one obtains an analogous characterization of plurisubharmonicity by means of a process of regularization.

11.4. THEOREM. *If $u \in \mathrm{Psh}(\Omega)$ with $u \not\equiv -\infty$ on every connected component of Ω, then for every $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1\le j,k\le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\xi_j\overline{\xi}_k \in \mathcal{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathcal{D}'(\Omega)$ is given such that $Hv(\xi)$ is a positive measure for all $\xi \in \mathbb{C}^n$, then there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u. $\square$

In order to obtain a better geometrical comprehension of the notion of plurisubharmonicity, we assume more generally that the function u lives on a complex manifold X of dimension n. If $\Phi : X \to Y$ is a holomorphic map and if $v \in C^2(Y,\mathbb{R})$, we have $d'd''(v\circ\Phi) = \Phi^* d'd''v$, therefore

$$H(v\circ\Phi)(a,\xi) = Hv(\Phi(a),\Phi'(a).\xi).$$

In particular Hu, viewed as a Hermitian form on T_X, is independent of the choice coordinates $(z_1,\ldots,z_n)$. Consequently, the notion of a psh function makes sense on any complex manifold. More generally, we have

11.5. PROPOSITION. *If $\Phi : X \to Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v\circ\Phi \in \mathrm{Psh}(X)$.* $\square$

11.6. EXAMPLE. It is well known that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Therefore $\log|f| \in \mathrm{Psh}(X)$ for any holomorphic function $f \in H^0(X,\mathcal{O}_X)$. More generally

$$\log(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}) \in \mathrm{Psh}(X)$$

for any choice of functions $f_j \in H^0(X,\mathcal{O}_X)$ and real $\alpha_j \ge 0$ (apply property 11.2d with $u_j = \alpha_j \log|f_j|$). We will be interested more particularly with singularities of this function along the variety of zeros $f_1 = \cdots = f_q = 0$, when the α_j are rational numbers. $\square$

11.7. DEFINITION. One says that a psh function $u \in \mathrm{Psh}(X)$ has analytic singularities (resp. algebraic) if u can be written locally in the form

$$u = \frac{\alpha}{2}\log(|f_1|^2 + \cdots + |f_N|^2) + v,$$

with holomorphic functions (resp. algebraic) f_j, $\alpha \in \mathbb{R}_+$, (resp. $\alpha \in \mathbb{Q}_+$), and where v is a bounded function.

We introduce then the ideal $\mathfrak{J} = \mathfrak{J}(u/\alpha)$ of germs of holomorphic functions h such that there exists a constant $C \geq 0$ for which $|h| \leq C e^{u/\alpha}$, i.e.

$$|h| \leq C(|f_1| + \cdots + |f_N|).$$

One therefore obtains a global sheaf of ideals defined on X, locally equal to the integral closure $\overline{\mathfrak{I}}$ of the sheaf of ideals $\mathfrak{I} = (f_1, \ldots, f_N)$; consequently $\mathfrak{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are the local generators of $\mathfrak{J}$, we still have

$$u = \frac{\alpha}{2} \log(|g_1|^2 + \cdots + |g_{N'}|^2) + O(1).$$

From an algebraic point of view, the singularities of u are in bijective correspondence with the "algebraic data" $(\mathfrak{J}, \alpha)$. We will later see another even more significant way to associate to a psh function, a sheaf of ideals.

11.B. Positive currents. The theory of currents was founded by G. de Rham [**DR55**]. We mention here only the most basic definitions. The reader can consult [**Fed69**] for a much more complete treatment of this theory. In the complex situation, the important characteristic concept of a positive current was studied and emanated by P. Lelong [**Lel57,69**].

A *current* of degree q on a differential manifold M, is nothing more than a differential q-form Θ with distribution coefficients. The space of currents of degree q on M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, one can consider the currents of degree q as the elements Θ of the dual $\mathcal{D}'_p(M) := \big(\mathcal{D}^p(M)\big)'$ of the space $\mathcal{D}^p(M)$ of C^∞ differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$\langle \Theta, \alpha \rangle = \int_M \Theta \wedge \alpha, \quad \alpha \in \mathcal{D}^p(M). \tag{11.8}$$

A fundamental example is the *current of integration* $[S]$ on a compact oriented submanifold S (possibly with boundary) of M:

$$\langle [S], \alpha \rangle = \int_S \alpha, \quad \deg \alpha = p = \dim_{\mathbb{R}} S. \tag{11.9}$$

Then $[S]$ is a current with measurable coefficients, and Stokes theorem shows that $d[S] = (-1)^{q-1}[\partial S]$. In particular $d[S] = 0$ if and only if S is a submanifold without boundary. Because of this example, the integer p is called the dimension of $\Theta \in \mathcal{D}'_p(M)$. One says that the current Θ is *closed* if $d\Theta = 0$.

On a complex manifold X, we have the analogous concept of bidegree and of bidimension. As in the real case, we denote by

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X), \quad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p, n-q)$ on X. Following [**Lel57**], a current Θ of bidimension (p,p) is called *(weakly) positive* if for any choice of C^∞ $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X, the distribution

$$\Theta \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p \quad \text{is a positive measure.} \tag{11.10}$$

11.11. Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that they are dominated up to a constant by the trace measure

$$\sigma_\Theta = \Theta \wedge \frac{1}{p!}\beta^p = 2^{-p}\sum \Theta_{I,I}, \quad \text{where } \beta = \frac{\mathrm{i}}{2}d'd''|z|^2 = \frac{\mathrm{i}}{2}\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j,$$

is a positive measure.

Indication. Observe that $\sum \Theta_{I,I}$ is invariant under a unitary change of coordinates, and that the (p,p)-forms $\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \cdots \wedge \mathrm{i}\alpha \wedge \overline{\alpha}_p$ generate $\Lambda^{p,p}T^*_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

One easily sees that a current $\Theta = \mathrm{i}\sum_{1\le j,k\le n}\Theta_{jk}dz_j \wedge d\overline{z}_k$ of bidegree $(1,1)$ is positive if and only if the complex measure $\sum \lambda_j\overline{\lambda}_k\Theta_{jk}$ is a positive measure for any n-tuple $(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$.

11.12. Example. If u is a psh function (not identically $-\infty$) on X, one can associate to u a closed positive current $\Theta = \mathrm{i}\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, any closed positive current of bidegree $(1,1)$ can be written in this form on any open subset $\Omega \subset X$ satisfying $H^2_{\mathrm{DR}}(\Omega,\mathbb{R}) = H^1(\Omega,\mathcal{O}) = 0$, for example on open coordinate charts biholomorphic to a ball (exercise for the reader). □

It is not difficult to show that a product $\Theta_1 \wedge \cdots \wedge \Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined. (This is certainly the case if all but one of the Θ_j are C^∞.) Other much finer conditions exist, but we will not pursue this subject here.

We now discuss another very important example of a closed positive current. For any closed analytic set A in X, of pure dimension p, one associates a current of integration

$$\langle [A], \alpha\rangle = \int_{A_{\mathrm{reg}}} \alpha, \quad \alpha \in \mathcal{D}^{p,p}(X), \tag{11.13}$$

obtained by integrating α on the set of regular points of A. To check that (11.13) gives a legitimate definition of a current on X, it should be shown that A_{reg} is locally of finite area in a neighbourhood of each point of A_{sing}. This result which due to [**Lel57**], can be shown as follows. Suppose (after a change of coordinates) that $0 \in A_{\mathrm{sing}}$. From the local parameterization theorem for analytic sets, one deduces that there exists a linear change of coordinates on $\mathbb{C}^n$ such that all the projections

$$\pi_I : (z_1,\ldots,z_n) \mapsto (z_{i_1},\ldots,z_{i_p})$$

define a finite ramified covering over the intersection $A \cap \Delta_I$ of A with a small polydisk $\Delta_I = \Delta'_I \times \Delta''_I$ of $\mathbb{C}^n = \mathbb{C}^p \times \mathbb{C}^{n-p}$, over the polydisk Δ'_I of $\mathbb{C}^p$. Let n_I be the number layers of each of these coverings. Then, if $\Delta = \cap\Delta_I$, the p-dimensional area of $A\cap\Delta$ is bounded above by the sum of the areas of its projections computed with multiplicities, i.e.

$$\text{Surface Area}(A\cap\Delta) \le \sum n_I \mathrm{Vol}(\Delta'_I).$$

The fact that $[A]$ is positive is easy. In fact, in terms of local coordinates $(w_1,\ldots,w_p)$ on A_{reg}, one has

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2 \mathrm{i}w_1 \wedge \overline{w}_1 \wedge \cdots \mathrm{i}w_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$. This shows that such a product of forms is ≥ 0 by comparison to the canonical orientation defined by $\mathrm{i}w_1 \wedge \overline{w}_1 \wedge \cdots \wedge \mathrm{i}w_p \wedge \overline{w}_p$. A deeper result, also proven by P. Lelong [**Lel57**], is that $[A]$ is a d-closed current on X, in other words, the set A_{sing} (which is of real dimension $\leq 2p-2$) does not contribute to the boundary current $d[A]$. Finally, in connection with example 11.12, we have the important

11.14. Lelong-Poincaré equation. Let $f \in H^0(X, \mathcal{O}_X)$ be a nonzero holomorphic function, $Z_f = \sum m_j Z_j$, $m_j \in \mathbb{N}$, the divisor of zeros of f, and $[Z_f] = \sum m_j [Z_j]$ the associated current of integration. Then

$$\frac{\mathrm{i}}{\pi} \partial\overline{\partial} \log |f| = [Z_f].$$

Proof (outline). It is clear that $\mathrm{i}d'd'' \log |f| = 0$ in a neighbourhood of each point $x \notin \mathrm{Supp}(Z_f) = \cup Z_j$, consequently it suffices to verify the equation in a neighbourhood of any point of $\mathrm{Supp}(Z_f)$. Let A be the set of singular points of $\mathrm{Supp}(Z_f)$, i.e. the union of the intersections $Z_j \cap Z_k$ and of their singularities $Z_{j,\mathrm{sing}}$; we then have $\dim A \leq n-2$. In a neighbourhood of any point $x \in \mathrm{Supp}(Z_f) \backslash A$ there exists local coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_1^{m_j}$, where m_j is the multiplicity of f along the component Z_j which contains x, and where $z_1 = 0$ is a local equation of Z_j near x. Since $\frac{\mathrm{i}}{\pi} d'd'' \log |z| =$ Dirac measure δ_0 in $\mathbb{C}$, we find $\frac{\mathrm{i}}{\pi} d'd'' \log |z_1| =$ [hyperplane $z_1 = 0$], therefore

$$\frac{\mathrm{i}}{\pi} d'd'' \log |f| = m_j \frac{\mathrm{i}}{\pi} d'd'' \log |z_1| = m_j [Z_j]$$

in a neighbourhood of x. This shows that the equation is valid on $X \backslash A$. Consequently, the difference $\frac{\mathrm{i}}{\pi} d'd'' \log |f| - [Z_f]$ is a closed current of degree 2 with measurable coefficients for which the support is contained in A. This current is necessarily zero because A is of too small a dimension for to be able to carry its support. (A is stratified into submanifolds of real codimension ≥ 4, whereas the current itself is of real codimension 2.) □

To conclude this section we now revisit the de Rham and Dolbeault cohomology in the context of the theory of currents. A basic observation is that the Poincaré and Dolbeault-Grothendieck Lemmas are still valid for currents. More precisely, if $(\mathcal{D}'^q, d)$ and $(\mathcal{D}'(F)^{p,q}, d'')$ denotes the complexes of sheaves of currents of degree q (resp. currents of bidegree (p,q) with values in a holomorphic vector bundle F), one still has resolutions of de Rham and of Dolbeault sheaves

$$0 \to \mathbb{R} \to \mathcal{D}'^{\bullet}, \quad 0 \to \Omega_X^p \otimes \mathcal{O}(F) \to \mathcal{D}'(F)^{p,\bullet}.$$

As a result, there are canonical isomorphisms

$$\begin{aligned} H^q_{\mathrm{DR}}(M, \mathbb{R}) &= H^q\big((\Gamma(M, \mathcal{D}'^{\bullet}), d)\big), \\ H^{p,q}(X, F) &= H^q\big((\Gamma(X, \mathcal{D}'(F)^{p,\bullet}), d'')\big). \end{aligned} \tag{11.15}$$

In other words, one can attach a cohomology class $\{\Theta\} \in H^q_{\mathrm{DR}}(M, \mathbb{R})$ to any closed current Θ of degree q, resp. a cohomology class $\{\Theta\} \in H^{p,q}(X, F)$ for any d''-closed current of bidegree (p,q). By replacing if necessary the respective currents by their

C^∞ representatives of the same cohomology class, one sees that there exists a well defined cup product pairing, given by the exterior product of differential forms

$$\begin{aligned} H^{q_1}(M,\mathbb{R}) \times \cdots \times H^{q_m}(M,\mathbb{R}) &\to H^{q_1+\cdots+q_m}(M,\mathbb{R}),\\ (\{\Theta_1\},\ldots,\{\Theta_1\}) &\mapsto \{\Theta_1\}\wedge\cdots\wedge\{\Theta_m\}.\end{aligned}$$

In particular, if M is a compact oriented manifold and if $q_1+\cdots+q_m = \dim M$, one obtains a well defined intersection number

$$\{\Theta_1\}\cdot\{\Theta_2\}\cdot\cdots\cdot\{\Theta_m\} = \int_M \{\Theta_1\}\wedge\cdots\wedge\{\Theta_m\}.$$

We note however that the specific product $\Theta_1\wedge\cdots\wedge\Theta_m$ does not exist in general.

11.C. Positive vector bundles. Let (E,h) be a Hermitian holomorphic vector bundle on a complex manifold X. Its Chern curvature tensor

$$\Theta(E) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^*\otimes e_\mu$$

can be identified with a Hermitian form on $T_X\otimes E$, viz.

$$\tilde\Theta(E)(\xi\otimes v) = \sum_{1\le j,k\le n, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda\overline{v}_\mu, \quad \overline{c}_{jk\lambda\mu} = c_{kj\mu\lambda}. \tag{11.16}$$

This leads us naturally to the concept of positivity, in the following definitions introduced by Kodaira [**Kod53**], Nakano [**Nak55**] and Griffiths [**Gri66**].

11.17. Definition. The Hermitian holomorphic vector bundle E is called

a) positive in the sense of Nakano if:
$\tilde\Theta(E)(\tau) > 0$ for any non-zero tensor $\tau = \sum \tau_{j\lambda}\partial/\partial z_j\otimes e_\lambda \in T_X\otimes E$.
b) positive in the sense of Griffiths if:
$\tilde\Theta(E)(\xi\otimes v) > 0$ for any non-zero decomposable tensor $\xi\otimes v\in T_X\otimes E$.

The corresponding concepts of semi-positivity are defined by replacing the strict inequalities by the broader inequalities.

11.18. The particular case of rank 1 bundles. Suppose that E is a line bundle. The Hermitian matrix $H=(h_{11})$ associated to a trivialization $\tau : E_{\restriction\Omega}\simeq \Omega\times\mathbb{C}$ is then simply a positive function, and it will be convenient to denote it by $\mathrm{e}^{-2\varphi}$, $\varphi\in C^\infty(\Omega,\mathbb{R})$. In this case, the curvature form $\Theta(E)$ can be identified with the $(1,1)$-form $2d'd''\varphi$, and

$$\frac{\mathrm{i}}{2\pi}\Theta(E) = \frac{\mathrm{i}}{\pi}d'd''\varphi = dd^c\varphi, \quad \text{where } d^c = \frac{\mathrm{i}}{2\pi}(d''-d')$$

is a real $(1,1)$-form. Therefore E is semipositive (in the sense of Nakano or in the sense of Griffiths) if and only if φ is psh, resp. positive if and only if φ is *strictly psh*. In this context, the Lelong-Poincaré equation can be generalized as follows: Let $\sigma\in H^0(X,E)$ be a non-zero holomorphic section. Then

$$dd^c\log||\sigma|| = [Z_\sigma] - \frac{\mathrm{i}}{2\pi}\Theta(E). \tag{11.19}$$

Formula (11.19) is immediate if one writes $||\sigma|| = |\tau(\sigma)|\mathrm{e}^{-\varphi}$ and if one applies the Lelong-Poincaré equation to the holomorphic function $f=\tau(\sigma)$. As we will see later, it is important for applications to consider the case of singular Hermitian metrics (cf. [**Dem90b**]).

11.20. Definition. A singular (Hermitian) metric on a line bundle E is a metric given in any trivialization $\tau : E_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by

$$||\xi|| = |\tau(\xi)|e^{-\varphi(x)}, \quad x \in \Omega,\ \xi \in E_x$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the weight of the metric with respect to the trivialization τ.

If $\tau' : E_{\restriction\Omega'} \to \Omega' \times \mathbb{C}$ is another trivialization, φ' the associated weight and $g \in \mathcal{O}^*(\Omega \cap \Omega')$ the transition function, then $\tau'(\xi) = g(x)\tau(\xi)$ for all $\xi \in E_X$, and therefore $\varphi' = \varphi + \log|g|$ on $\Omega \cap \Omega'$. The curvature form of E is then formally given by the current of degree $(1,1)$, $\frac{i}{\pi}\Theta(E) = dd^c\varphi$ on Ω; moreover the hypothesis $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ guarantees that $\Theta(E)$ exists in the sense of distributions. As in the C^∞ case, the form $\frac{i}{\pi}\Theta(E)$ is globally defined on X and independent of the choice of trivializations, and its de Rham cohomology class is the image of the first Chern class $c_1(E) \in H^2(X,\mathbb{Z})$ in $H^2_{\mathrm{DR}}(X,\mathbb{R})$. Before going further, we discuss two fundamental examples.

11.21. Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $E = \mathcal{O}(D)$ be the associated invertible sheaf, defined as the sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \geq 0$. The corresponding line bundle can be given a singular metric defined by $||u|| = |u|$ (modulus of the meromorphic function u). If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$, then $\tau(u) = u \prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ on Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log|g_j|$. The Lelong-Poincaré equation implies that

$$\frac{i}{\pi}\Theta(\mathcal{O}(D)) = dd^c\varphi = [D],$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration on D. □

11.22. Example. Suppose that $\sigma_1, \ldots, \sigma_N$ are non-zero holomorphic sections of E. One can then define a natural (possibly singular) Hermitian metric on E^*, by setting

$$||\xi^*||^2 = \sum_{1\leq j\leq n} |\xi^*.\sigma_j(x)|^2 \quad \text{for } \xi^* \in E^*_x.$$

The dual metric of E is given by

$$||\xi^*||^2 = \frac{|\tau(\xi)|^2}{|\tau(\sigma_1(x))|^2 + \cdots + |\tau(\sigma_N(x))|^2}$$

with respect to any local trivialization τ. The associated weight function is therefore given by $\varphi(x) = \log(\sum_{1\leq j\leq N} |\tau(\sigma_j(x))|^2)^{1/2}$. In this case φ is a psh function, therefore $i\Theta(E)$ is a closed positive current. Denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and $B_\Sigma = \cap \sigma_j^{-1}(0)$ its base locus. One has a meromorphic map

$$\Phi_\Sigma : X \backslash B_\Sigma \to \mathbb{P}^{N-1}, \quad x \mapsto [\sigma_1(x) : \sigma_2(x) : \cdots : \sigma_N(x)].$$

With this notation, the curvature $\frac{i}{2\pi}\Theta(E)$ restricted to $X\backslash B_\Sigma$ is identified with the inverse image by Φ_Σ of the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{i}{2\pi} d'd'' \log(|z_1|^2 + \cdots + |z_N|^2)$ on $\mathbb{P}^{N-1}$. It is therefore semi-positive. □

11.23. Ample and very ample line bundles. A holomorphic line bundle E on a compact complex manifold X is called

a) very ample if the map $\Phi_{|E|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|E| = \mathbb{P}(H^0(X, E))$ is a regular embedding. (This implies in particular that the base locus is empty, i.e. $B_{|E|} = \emptyset$.)

b) ample if there exists a multiple mE, $m > 0$, which is very ample.

We adopt here the additive notation for $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*)$, the symbol mE representing the line bundle $E^{\otimes m}$. By refering to example 11.22, it follows that any ample line bundle E has a C^∞ Hermitian metric, having a positive definite curvature form. Indeed, if the linear system $|mE|$ gives an embedding in projective space, then one obtains a C^∞ Hermitian metric on $E^{\otimes m}$, and the m-th root gives a metric on E such that $\frac{\mathrm{i}}{2\pi}\Theta(E) = \frac{1}{m}\Phi^*_{|mE|}\omega_{\mathrm{FS}}$. Conversely, Kodaira's embedding theorem [**Kod54**] says that any positive line bundle E is ample (see exercise 15.11 for a direct analytic proof of this fundamental theorem).

12. Hodge theory of complete Kähler manifolds

The goal of this section is primarily to extend to the case of complete Kähler manifolds the results of Hodge theory already proven in the compact case.

12.A. Complete Riemannian manifolds. Before treating the complex situation, we will need to discuss some general results on the Hodge theory of complete Riemannian manifolds. Recall that a Riemannian manifold (M, g) is said to be complete if the geodesic distance δ_g is complete, or what amounts to the same thing (Hopf-Rinow Lemma below), if the closed geodesic balls are all compact. We will need the following more precise characterization.

12.1. LEMMA (Hopf-Rinow). *The following properties are equivalent:*

a) (M, g) *is complete;*

b) *the closed geodesic balls* $\overline{B}_g(a, r)$ *are compact;*

c) *there exists an exhaustive function* $\psi \in C^\infty(M, \mathbb{R})$ *such that* $|d\psi|_g \le 1$*;*

d) *there exists in* M *an exhaustive sequence* $(K_\nu)_{\nu\in\mathbb{N}}$ *of compact sets and functions* $\theta_\nu \in \mathcal{C}^\infty(M, \mathbb{R})$ *such that*

$$\theta_\nu = 1 \quad \textit{on a neighbourhood of } K_\nu, \quad \mathrm{Supp}\,\theta_\nu \subset K^\circ_{\nu+1},$$
$$0 \le \theta_v \le 1 \ \textit{and} \ |d\theta_\nu|_g \le 2^{-\nu}.$$

PROOF. a) $\Longrightarrow$ b). The point x being fixed, one denotes by $r_0 = r_0(x)$, the supremum of the real numbers $r > 0$ such that $\overline{B}_g(a, r)$ is compact. Suppose $r_0 < +\infty$. Being given a sequence of points (x_ν) in $\overline{B}_g(a, r_0)$ and $\epsilon > 0$, one chooses a sequence of points $x_{\nu,\epsilon} \in \overline{B}(a, r_0 - \epsilon)$ such that $\delta_g(x_\nu, x_{\nu,\epsilon}) < 2\epsilon$. By compactness of $\overline{B}_g(a, r_0 - \epsilon)$, one can extract from $(x_{\nu,\epsilon})$ a convergent subsequence for each $\epsilon > 0$. By applying a diagonal process, one easily sees that one can extract from (x_ν) a Cauchy subsequence. Consequently this sequence converges and $\overline{B}_g(a, r_0)$ is compact. The local compactness of M implies that $\overline{B}_g(a, r_0 + \eta)$ is still compact for $\eta > 0$ small enough, which is a contradiction if $r_0 < +\infty$.
b) $\Longrightarrow$ c). Suppose M is connected. Choose a point $x_0 \in M$ and set $\psi_0(x) = \frac{1}{2}\delta(x_0, x)$. Then ψ_0 is exhaustive, and this is a Lipschitz function of order $\frac{1}{2}$, therefore ψ_0 is differentiable almost everywhere on M. One obtains the sought for function ψ by regularization.

c) $\Longrightarrow$ d). Let ψ be as in a) and let $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that $\rho = 1$ on $]-\infty, 1.1]$, $\rho = 0$ on $[1.99, +\infty[$ and $0 \le \rho' \le 2$ on $[1,2]$. Then

$$K_\nu = \{x \in M;\ \psi(x) \le 2^{\nu+1}\}, \quad \theta_\nu(x) = \rho\big(2^{-\nu-1}\psi(x)\big)$$

satisfies the desired properties.
d) $\Longrightarrow$ c). Set $\psi = \sum 2^{\nu-1}(1-\theta_\nu)$.
c) $\Longrightarrow$ b). The inequality $|d\psi|_g \le 1$ implies $|\psi(x) - \psi(y)| \le \delta_g(x,y)$ for any $x, y \in M$, therefore the geodesic ball $\overline{B}_g(a,r) \subset \{x \in M;\ \delta_g(x,a) \le \psi(a) + r\}$ is relatively compact.
b) $\Longrightarrow$ a). This is obvious! □

Let (M,g) be a Riemannian manifold, not necessarily complete for the moment, E a Hermitian vector bundle on M, with a given Hermitian connection D. One considers the unbounded operator between Hilbert spaces, still denoted by D

$$D : L^2(M, \Lambda^p T^*_M \otimes E) \to L^2(M, \Lambda^{p+1} T^*_M \otimes E),$$

for which the domain Dom D is defined as follows: A section $u \in L^2$ is said to be in Dom D if Du calculated in the sense of distributions is still in L^2. The domain thus defined is always dense in L^2, because Dom D contains the space $\mathcal{D}(M, \Lambda^p T^*_M E)$ of C^∞ sections with compact support, which is itself dense in L^2. Moreover, the operator D thus defined, albeit not bounded, is *closed,* that is to say its graph is closed; this follows at once from the fact that the differential operators are continuous in the weak distribution topology. In the same way, the formal adjoint D^* admits an extension to a closed operator

$$D^* : L^2(M, \Lambda^{p+1} T^*_M \otimes E) \to L^2(M, \Lambda^p, T^*_M \otimes E).$$

Some well-known elementary results of spectral theory due to Von Neumann guarantees, in addition, the existence of a closed operator $D^*_{\mathcal{H}}$ with dense domain, called the Hilbert space adjoint of D, defined as follows: An element $v \in L^2(M, \Lambda^{p+1} T^*_M \otimes E)$ is in Dom $D^*_{\mathcal{H}}$ if the linear form $L^2 \to \mathbb{C}$, $u \mapsto \langle\langle Du, v\rangle\rangle$ is continuous. It is thus written $u \mapsto \langle\langle u, w\rangle\rangle$ for a unique element $w \in L^2(M, \Lambda^p, T^*_M \otimes E)$. One sets $D^*_{\mathcal{H}} v = w$, so that $D^*_{\mathcal{H}}$ is defined by the usual adjoint relation

$$\langle\langle Du, v\rangle\rangle = \langle\langle u, D^*_{\mathcal{H}} v\rangle\rangle \quad \forall u \in \text{Dom } D.$$

(Note that the formal adjoint D^*, itself, is defined by requiring only the validity of their relation for $u \in \mathcal{D}(M, \Lambda^p, T^*_M \otimes E)$.) It is clear that one always has Dom $D^*_{\mathcal{H}} \subset$ Dom D^* and that $D^*_{\mathcal{H}} = D^*$ on Dom $D^*_{\mathcal{H}}$. In general, however, the domains are distinct (this is the case for example if $M =]0,1[$, $g = dx^2$, $D = d/dx$!). A fundamental observation is that this phenomenon cannot occur if the Riemannian metric is complete.

12.2. Proposition. *If the manifold (M,g) is complete, then:*

a) *The space $\mathcal{D}(M, \Lambda^\bullet T^*_M E)$ is dense in* Dom D, Dom D^* *and* Dom $D \cap$ Dom D^* *respectively, for the norms of the graphs*

$$u \mapsto ||u|| + ||Du||, \quad u \mapsto ||u|| + ||D^*u||, \quad u \mapsto ||u|| + ||Du|| + ||D^*u||.$$

b) $D^*_{\mathcal{H}} = D^*$ *(i.e. the two domains coincide), and $D^{**}_{\mathcal{H}} = D^{**} = D$.*

c) *Let $\Delta = DD^* + D^*D$ be the Laplacian calculated in the sense of distributions. For any $u \in \mathrm{Dom}\ \Delta \subset L^2(M, \Lambda^\bullet T_M^* \otimes E)$, one has $\langle u, \Delta u\rangle = ||Du||^2 + ||D^*u||^2$. In particular*

$$\mathrm{Dom}\ \Delta \subset \mathrm{Dom}\ D \cap \mathrm{Dom}\ D^*, \quad \mathrm{Ker}\,\Delta = \mathrm{Ker}\, D \cap \mathrm{Ker}\, D^*,$$

and Δ is self adjoint.

d) *If $D^2 = 0$, there is an orthogonal decomposition*

$$L^2(M, \Lambda^\bullet T_M^* \otimes E) = \mathcal{H}^\bullet_{L^2}(M, E) \oplus \overline{\mathrm{Im}\ D} \oplus \overline{\mathrm{Im}\ D^*},$$
$$\mathrm{Ker}\, D = \mathcal{H}^\bullet_{L^2}(M, E) \oplus \overline{\mathrm{Im}\ D},$$

where $\mathcal{H}^\bullet_{L^2}(M, E) = \{u \in L^2(M, \Lambda^\bullet T_M^* \otimes E);\ \Delta u = 0\}$ is the space of L^2 harmonic forms on M.

Proof. a) It is necessary to show for example that any element $u \in \mathrm{Dom}\ D$ can be approximated in the norm of the graph of D by C^∞ forms with compact support. By assumption, u and Du are in L^2. Let (θ_ν) be a sequence of truncating functions as in Lemma 12.1 d). Then $\theta_\nu u \to u$ in $L^2(M, \Lambda^\bullet T_M^* \otimes E)$ and $D(\theta_\nu u) = \theta_\nu Du + d\theta_\nu \wedge u$ where

$$|d\theta_\nu \wedge u| \le |d\theta_\nu||u| \le 2^{-\nu}|u|.$$

Consequently $d\theta_\nu \wedge u \to 0$ and $D(\theta_\nu u) \to Du$. By replacing u by $\theta_\nu u$, one can assume that u has compact support, and with the aid of a partition of unity, one is reduced to the case where Supp u is contained in a coordinate chart of M on which E is trivial. Let (ρ_ϵ) be a family of regular kernels. A classical lemma in the theory of PDE (Friedrich's Lemma), shows that for any differential operator P of order 1 with C^1 coefficients, one has $||P(\rho_\epsilon \star u) - \rho_\epsilon Pu||_{L^2} \to 0$, as ϵ tends to 0 (u being an L^2 section with compact support in the coordinate chart considered). By applying this lemma to $P = D,\ P = D^*$ respectively, one arrives at the desired properties of density.

b) is equivalent to the fact that

$$\langle\langle Du, v\rangle\rangle = \langle\langle u, D^*v\rangle\rangle, \quad \forall u \in \mathrm{Dom}\ D,\ \forall v \in \mathrm{Dom}\ D^*.$$

However, according to a), one can find $u_\nu,\ v_\nu \in \mathcal{D}(M, \Lambda^\bullet T_M^* \otimes E)$ such that

$$u_\nu \to u, \quad v_\nu \to v, \quad Du_\nu \to Du \quad \text{and} \quad D^*v_\nu \to D^*v \quad \text{in} \quad L^2(M, \Lambda^\bullet T_M^* \otimes E).$$

The desired equality is then the limit of the equality $\langle\langle Du_\nu, v_\nu\rangle\rangle = \langle\langle u_\nu, D^*v_\nu\rangle\rangle$.

c) Let $u \in \mathrm{Dom}\ \Delta$. Since $\Delta u \in L^2$ and that Δ is an elliptic operator of order 2, one obtains $u \in W^2_{\mathrm{loc}}$ by applying the local version of the Gårding inequality. In particular $Du,\ D^*u \in W^1_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}$, and we can apply integration by parts as needed, after multiplying the respective forms by C^∞ functions θ_ν with compact support. Some simple calculations then give

$$\begin{aligned}
&||\theta_\nu Du||^2 + ||\theta_\nu D^*u||^2 = \\
&= \langle\langle \theta_\nu^2 Du, Du\rangle\rangle + \langle\langle u, D(\theta_\nu^2 D^*u)\rangle\rangle \\
&= \langle\langle D(\theta_\nu^2, u), Du\rangle\rangle + \langle\langle u, \theta_\nu^2 DD^*u\rangle\rangle - 2\langle\langle \theta_\nu d\theta_\nu \wedge u, Du\rangle\rangle + 2\langle\langle u, \theta_\nu d\theta_\nu \wedge D^*u\rangle\rangle \\
&= \langle\langle \theta_\nu^2 u, \Delta u\rangle\rangle - 2\langle\langle d\theta_\nu \wedge u, \theta_\nu Du\rangle\rangle + 2\langle\langle u, d\theta_\nu \wedge (\theta_\nu D^*_u)\rangle\rangle \\
&\le \langle\langle \theta_\nu^2 u, \Delta u\rangle\rangle + 2^{-\nu}\big(2||\theta_\nu Du||||u|| + 2||\theta_\nu D^*u||||u||\big) \\
&\le \langle\langle \theta_\nu^2 u, \Delta u\rangle\rangle + 2^{-\nu}\big(||\theta_\nu Du||^2 + ||\theta_\nu D^*u||^2 + 2||u||^2\big).
\end{aligned}$$

Consequently

$$||\theta_\nu Du||^2 + ||\theta_\nu D^* u||^2 \le \frac{1}{1-2^{-\nu}}\big(\langle\langle \theta_\nu^2 u, \Delta u\rangle\rangle + 2^{1-\nu}||u||^2\big).$$

By letting ν tend to $+\infty$, one obtains $||Du||^2 + ||D^*u||^2 \le \langle\langle u, \Delta u\rangle\rangle$, in particular $Du,\ D^*u$ are in L^2. This implies

$$\langle\langle u, \Delta v\rangle\rangle = \langle\langle Du, Dv\rangle\rangle + \langle\langle D^*u, D^*v\rangle\rangle, \quad \forall u, v \in \mathrm{Dom}\ \Delta,$$

because the equality holds for $\theta_\nu u$ and v, and that $\theta_\nu u \to u,\ D(\theta_\nu u) \to Du$ and $D^*(\theta_\nu u) \to D^* u$ in L^2. It follows from this that Δ is self-adjoint.
d) If P is a closed operator with dense domain on a Hilbert space $\mathcal{H}$, then $\mathrm{Ker}\, P$ is closed and $\mathrm{Ker}\, P^* = (\mathrm{Im}\ P)^\perp$. Consequently $(\mathrm{Ker}\, P^*)^\perp = (\mathrm{Im}\ P)^{\perp\perp} = \overline{\mathrm{Im}\ P}$. Since $\mathrm{Ker}\, P^*$ itself is also closed, we have

$$\mathcal{H} = \mathrm{Ker}\, P^* \oplus (\mathrm{Ker}\, P^*)^\perp = \mathrm{Ker}\, P^* \otimes \overline{\mathrm{Im}\ P}.$$

This result applied to $P = \Delta$ gives

$$\mathcal{H}^\bullet_{L^2}(M, E) = \mathrm{Ker}\,\Delta \oplus \overline{\mathrm{Im}\ \Delta},$$

and it is clear according to (12.2 c) that $\mathrm{Im}\ \Delta \subset \mathrm{Im}\ D \oplus \mathrm{Im}\ D^*$. Furthermore, one easily sees that $\mathrm{Ker}\,\Delta$, $\overline{\mathrm{Im}\ D}$ and $\overline{\mathrm{Im}\ D^*}$ are pairwise orthogonal by using (12.2 a,c). Property d) follows as in the case where M is compact. $\square$

12.3. Definition. Assume given a Riemannian manifold (M, g) and a Hermitian bundle E with a flat Hermitian connection D. We denote by $H^p_{\mathrm{DR},L^2}(M, E)$, the L^2 de Rham cohomology groups, namely the cohomology groups of the complex $(K^\bullet, D)$ defined by

$$K^p = \big\{u \in L^2(M, \Lambda^p T^*_M \otimes E);\ Du \in L^2\big\}.$$

In other words, one has $H^p_{\mathrm{DR},L^2}(M, E) = \mathrm{Ker}\, D/\mathrm{Im}\ D$, where D is the L^2 extension of the connection calculated in the sense of distributions. Since $\mathcal{H}^p_{L^2}(M, E) = \mathrm{Ker}\, D/\overline{\mathrm{Im}\ D}$ according to (12.2 d), it follows that:

12.4. Proposition. *There is a canonical isomorphism*

$$\mathcal{H}^p_{L^2}(M, E) \simeq H^p_{\mathrm{DR},L^2}(M, E)_{\mathrm{sep}}$$

between $\mathcal{H}^p_{L^2}(M, E)$ and the separated space associated to the L^2 de Rham cohomology.

In general the space $H^p_{\mathrm{DR},L^2}(M, E)$ is not always separated, but it is in the important case where the L^2 cohomology is finite dimensional:

12.5. Corollary. *If (M, g) is complete and if $H^p_{\mathrm{DR},L^2}(M, E)$ is finite dimensional, then this space is separated and there is a canonical isomorphism*

$$\mathcal{H}^p_{L^2}(M, E) \simeq H^p_{\mathrm{DR},L^2}(M, E).$$

PROOF. The space K^p can be considered as the Hilbert space with norm $u \mapsto (||u||_{L^2} + ||Du||_{L^2})^{1/2}$. It is a question of seeing that $\operatorname{Im} D = D(K^{p-1})$ is closed in $\operatorname{Ker} D$, $\operatorname{Ker} D$ being itself closed in K^p. Now $D : K^{p-1} \to \operatorname{Ker} D$ is continuous and its image is of finite codimension by hypothesis. The fact that the image is closed is then a direct consequence of the Banach Theorem. □

12.6. Remark. For L^2 de Rham cohomology, observe that one obtains the identical cohomology groups when working with the subcomplex of global L^2 C^∞-forms, that is

$$\tilde{K}^p = \{u \in C^\infty(M, \Lambda^p T^*_M \otimes E);\ u \in L^2 \text{ and } Du \in L^2\} \subset K^p.$$

For that, it suffices to construct an operator $\tilde{K}^\bullet \to K^\bullet$ which is a homotopic inverse to the inclusion. This can be done by using a regularization process by flows of vector fields tending to 0 sufficiently quickly, near infinity.

12.B. Case of Hermitian and complete Kähler manifolds. The preceding results admit of course complex analogs, with almost identical proofs (the details will be therefore left to the reader). One says that a Hermitian or Kähler manifold (X, ω) is complete if the underlying Riemannian manifold is complete.

12.7. PROPOSITION. *Let (X, ω) be a complete Hermitian manifold and E a Hermitian holomorphic vector bundle over X. There is a canonical isomorphism*

$$\mathcal{H}^{p,q}_{L^2}(M, E) \simeq H^{p,q}_{L^2}(M, E)_{\mathrm{sep}}$$

between the space of L^2 harmonic forms and the separated L^2 Dolbeault cohomology group, this latter space being itself equal to $H^{p,q}_{L^2}(M, E)$ if the Dolbeault cohomology is finite dimensional.

12.8. COROLLARY. *Let (X, ω) be a Kähler manifold and E a flat Hermitian bundle over X.*

a) *Without further assumptions, there is, for any k, an orthogonal decomposition*

$$\mathcal{H}^k_{L^2}(M, E) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{L^2}(M, E), \quad \overline{\mathcal{H}^{p,q}_{L^2}(M, E)} = \mathcal{H}^{q,p}_{L^2}(M, E^*).$$

b) *If moreover (X, ω) is complete, there are canonical isomorphisms*

$$H^k_{L^2}(M, E)_{\mathrm{sep}} \simeq \bigoplus_{p+q=k} H^{p,q}_{L^2}(M, E)_{\mathrm{sep}}, \quad \overline{H^{p,q}_{L^2}(M, E)_{\mathrm{sep}}} \simeq H^{q,p}_{L^2}(M, E^*)_{\mathrm{sep}}.$$

c) *If (X, ω) is complete, and if the L^2 de Rham and Dolbeault cohomology groups are finite dimensional, there are canonical isomorphisms*

$$H^k_{L^2}(M, E) \simeq \bigoplus_{p+q=k} H^{p,q}_{L^2}(M, E), \quad \overline{H^{p,q}_{L^2}(M, E)} \simeq H^{q,p}_{L^2}(M, E^*).$$

12.C. Hodge theory of weakly pseudoconvex Kähler manifolds. The weakly pseudoconvex Kähler manifolds furnish an important example of complete Kähler manifolds.

12.9. DEFINITION. A complex manifold X is said to be weakly pseudoconvex if there exists a C^∞ psh exhaustion function ψ on X. (Recall that a function ψ is said to be exhaustive if for any $c > 0$ the level set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z tends towards infinity, according to the stratification of the complements of compact parts of X.)

In particular, the compact complete manifolds X are weakly pseudoconvex (take $\psi = 0$), as well as the Stein manifolds. For example the affine algebraic subvarieties of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$), the open balls $X = B(z_0, r)$ $\big($take $\psi(z) = 1/(r - |z - z_0|^2)\big)$, the open convex sets, and so on. A basic observation is the following:

12.10. PROPOSITION. *Any weakly pseudoconvex Kähler manifold (X, ω) has a complete Kähler metric $\hat{\omega}$.*

PROOF. For any increasing convex function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, we will consider the closed $(1,1)$-form

$$\omega_\chi = \omega + \mathrm{i}\ d'd''(\chi \circ \psi) = \omega + \chi'(\psi)\mathrm{i}\ d'd''\psi + \chi''(\psi)\mathrm{i}\ d'\psi \wedge d''\psi.$$

Since the three terms are positive or zero, this is a Kähler metric. The presence of the third term implies that the norm of $\chi''(\psi)^{1/2} d\psi$ by comparison to ω_χ is less than or equal to 1, therefore if ρ is a choice of $(\chi'')^{1/2}$ we have $|d(\rho \circ \psi)|_{\omega_\chi} \le 1$. According to (12.1 c), ω_χ will be complete as long as $\rho \circ \psi$ is exhaustive, that is, as long as $\lim_{+\infty} \rho(t) = +\infty$. We therefore obtain the sufficient condition

$$\int_{t_0}^{+\infty} \chi''(t)^{1/2} dt = +\infty,$$

which is realized, for example, for the choice $\chi(t) = t^2$ or $\chi(t) = t - \log t,\ t \ge 1$. □

We have now established a Hodge decomposition theorem for weakly pseudoconvex Kähler manifolds having "sufficiently many strictly pseudoconvex directions". Following Andreotti-Grauert [**AG62**], we introduce the:

12.11. DEFINITION. A complex manifold X is said to be ℓ-convex (resp. absolutely ℓ-convex) if X has an exhaustion function (resp. a psh exhaustion function) ψ, which is strongly ℓ-convex on the complement $X \setminus K$ of a compact part, i.e. such that $\mathrm{i}\ d'd''\psi$ has at least $n - \ell + 1$ positive eigenvalues at any point of $X \setminus K$, where $n = \dim_{\mathbb{C}} X$.

12.12. EXAMPLE. Let $\overline{X}$ be a smooth projective variety such that there exists a surjective morphism $F : \overline{X} \to \overline{Y}$ onto another smooth projective variety $\overline{Y}$. Let D be a divisor of $\overline{Y}$ and let $X = \overline{X} \setminus F^{-1}(D)$, $Y = \overline{Y} \setminus D$. We assume that F induces a submersion $\overline{X} \setminus F^{-1}(D) \to \overline{Y} \setminus D$ and that $\mathcal{O}(D)_{\restriction D}$ is ample. Then X is absolutely ℓ-convex for $\ell = \dim X - \dim Y + 1$. Indeed, the hypothesis of ampleness of $\mathcal{O}(D)_{\restriction D}$ implies that there exists a Hermitian metric on $\mathcal{O}(D)$ for which the curvature is positive definite in a neighbourhood of D, that is on an open set of the form $\overline{Y} \setminus K'$ where K' is a compact part of $\overline{Y} \setminus D$. Let $\sigma \in H^0(\overline{Y}, \mathcal{O}(D))$ be the canonical section of the divisor D. Then $-\log|\sigma|^2$ is strongly psh on $Y \setminus K'$, consequently $\psi = -\log|\sigma \circ F|^2$ is psh and strongly ℓ-convex on $X \setminus K$, where $K = F^{-1}(K')$. In addition, ψ clearly defines an exhaustion on X. Nothing is known of ψ on K, but it is enough to truncate ψ by taking a maximal regularized $\psi_C = \max_\epsilon(\psi, C)$ with a constant $C > \sup_K \psi$ to obtain an everywhere psh function ψ_C on X.

We now can state the Hodge decomposition theorem for absolutely ℓ-convex manifolds. This result is due to T. Ohsawa [**Ohs81, 87**]; we present here a simplified description of a proof of it in [**Dem90a**]. A purely algebraic approach of these results was obtained by Bauer-Kosarew [**BaKo89,91**] and [**Kos91**].

12.13. Theorem (Ohsawa [**Ohs81,87**], [**OT88**]). *Let (X, ω) be a Kähler manifold and $n = \dim_{\mathbb{C}} X$, and assume that X is absolutely ℓ-convex. Then, in suitable degrees, there is a Hodge decomposition and symmetry:*

$$H^k_{\mathrm{DR}}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}), \quad \overline{H^{p,q}(X, \mathbb{C})} \simeq H^{q,p}(X, \mathbb{C}), \quad k \geq n + \ell,$$

$$H^k_{\mathrm{DR},c}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}_c(X, \mathbb{C}), \quad \overline{H^{p,q}_c(X, \mathbb{C})} \simeq H^{q,p}_c(X, \mathbb{C}), \quad k \leq n - \ell,$$

all these groups being finite dimensional ($H^k_{\mathrm{DR},c}(X, \mathbb{C})$ and $H^{p,q}_c(X, \mathbb{C})$ denotes here the cohomology groups with compact support). Moreover, there is a Lefschetz isomorphism

$$\omega^{n-p-q} \wedge \bullet : H^{p,q}_c(X, \mathbb{C}) \to H^{n-q,n-p}(X, \mathbb{C}), \quad p + q \leq n - \ell.$$

Proof. The finiteness of the de Rham cohomology groups concerned is easily obtained by means of Morse theory. Recall briefly the argument: a suitably small perturbation of a strongly ℓ-convex exhaustion function gives a Morse function ψ which is still strongly ℓ-convex on the complement $X \backslash K$ of a compact set. The real Hessian $D^2\psi$ of ψ at a critical point induces a Hermitian form on the complexified tangent space $\mathbb{C} \otimes T_X$, and its restriction to $T^{1,0}_X$ is identified with the complex Hessian i $d'd''\psi$. Since the complex Hessian has by assumption at least $n - \ell + 1$ positive eigenvalues on $X \backslash K$, it follows from this that $D^2\psi$ has at most $2n - (n - \ell + 1) = n + \ell - 1$ negative eigenvalues on $X \backslash K$, without which the positive and negative eigenvalues of $D^2\psi$ would have a non-trivial intersection. Consequently all the critical points of index $\geq n + \ell$ are located in K and their number is finite. This implies that the groups $H^k_{\mathrm{DR}}(X, \mathbb{C})$ of degree $k \geq n + \ell$ are finite dimensional. The finiteness of the Dolbeault cohomology groups $H^{p,q}(X, \mathbb{C}) = H^q(X, \Omega^p_X)$ is a result of the theorem of Andreotti-Grauert [**AG62**] (*all the cohomology groups of higher degree than ℓ with values in a given coherent sheaf are separated and finite dimensional if the manifold is ℓ-convex*). It is noted however, that the ℓ-convexity, although sufficient to ensure the finiteness of the various groups involved, is not sufficient to guarantee the existence of a Hodge decomposition, nor even the Hodge symmetry. The reader will find a simple counterexample in Grauert-Riemenschneider [**GR70**].

Now let ω be a Kähler metric on X and ψ a strongly ℓ-convex psh exhaustion function on $X \backslash K$. As one can see, the existence of a Hodge decomposition follows directly from the fact that one has such a decomposition for the L^2 harmonic forms. The key point resides in the observation that any L^2_{loc} form of degree $k \geq n + \ell$ becomes globally L^2 for a suitable choice of metric $\omega_\chi = \omega + \mathrm{i}\, d'd''(\chi \circ \psi)$. The groups $H^k_{\mathrm{DR}}(X, \mathbb{C})$ and $H^{p,q}(X, \mathbb{C})$ could then be considered as the inductive limit of L^2 cohomology groups. In the sequel, we will use notation such as

$L^2_{\omega_\chi}(X,\Lambda^{p,q}T^*_X)$, $\mathcal{H}^{p,q}_{L^2,\omega_\chi}(X,\mathbb{C})$, to denote the spaces of L^2-forms (resp. harmonic forms) relative to ω_χ. Since ω_χ is Kähler, one has

$$(12.14)\qquad \mathcal{H}^k_{L^2,\omega_\chi}(M,\mathbb{C})=\bigoplus_{p+q=k}\mathcal{H}^{p,q}_{L^2,\omega_\chi}(M,\mathbb{C}),\ \overline{\mathcal{H}^{p,q}_{L^2,\omega_\chi}(M,\mathbb{C})}=\mathcal{H}^{q,p}_{L^2,\omega_\chi}(M,\mathbb{C}),$$

with an isomorphism $\mathcal{H}^k_{L^2,\omega_\chi}(M,\mathbb{C})\simeq H^k_{L^2,\omega_\chi}(M,\mathbb{C})_{\rm sep}$ as long as ω_χ is complete. In the sequel, we always assume that ω_χ is complete. It is enough, for example, to impose $\chi''(t)\ge 1$ on $[0,+\infty[$.

12.15. Lemma. *Let u be a form of bidegree (p,q) with $L^2_{\rm loc}$ coefficients on X. If $p+q\ge n+\ell$, then $u\in L^2_{\omega_\chi}(X,\Lambda^{p,q}T^*_X)$ as long as χ grows sufficiently quickly near infinity.*

Proof. At a fixed point $x\in X$, there exists an orthogonal basis $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ of $T_{X,x}$ for which

$$\omega(x)=\mathrm{i}\sum_{1\le j\le n}dz_j\wedge d\overline{z}_j,\quad \omega_\chi(x)=\mathrm{i}\sum_{1\le j\le n}\lambda_j(x)dz_j\wedge d\overline{z}_j,$$

where $\lambda_1\le\cdots\le\lambda_n$ are the eigenvalues of ω_χ relative to ω. Then the volume elements $dV_\omega=\omega^n/2^nn!$ and $dV_{\omega_\chi}=\omega_\chi^n/2^nn!$ are bound by the relation

$$dV_{\omega_\chi}=\lambda_1\cdots\lambda_n dV_\omega,$$

and for a (p,q)-form $u=\sum_{I,J}u_{I,J}dz_I\wedge d\overline{z}_J$ we find that

$$|u|^2_{\omega_\chi}=\sum_{|I|=p,|J|=q}\Big(\prod_{k\in I}\lambda_k\prod_{k\in J}\lambda_k\Big)^{-1}|u_{I,J}|^2.$$

In particular, it follows that

$$|u|^2_{\omega_\chi}dV_{\omega_\chi}\le\frac{\lambda_1\cdots\lambda_n}{\lambda_1\cdots\lambda_p\lambda_1\cdots\lambda_q}|u|^2_\omega dV_\omega=\frac{\lambda_{p+1}\cdots\lambda_n}{\lambda_1\cdots\lambda_q}|u|^2_\omega dV_\omega.$$

In addition, one has upper bounds

$$\lambda_j\le 1+C_1\chi'(\psi),\quad 1\le j\le n-1,\quad \lambda_n\le 1+C_1\chi'(\psi)+C_2\chi''(\psi)$$

where $C_1(x)$ is the largest eigenvalue of $\mathrm{i}\,d'd''\psi(x)$ and $C_2(x)=|\partial\psi(x)|^2$. For to obtain the first $n-1$ inequalities, one need only apply the minimum principal on the kernel of $\partial\psi$. Since $\mathrm{i}\,d'd''\psi$ has at most $\ell-1$ zero eigenvalues on $X\backslash K$, the minimum principal also gives lower bounds

$$\lambda_j\ge 1,\quad 1\le j\le\ell-1,\quad \lambda_j\ge 1+c\chi'(\psi),\quad \ell\le j\le n,$$

where $c(x)\ge 0$ is the ℓ-th eigenvalue of $\mathrm{i}\,d'd''\psi(x)$ and $c(x)>0$ on $X\backslash K$. If we assume $\chi'\ge 1$, then we can easily deduce

$$\begin{aligned}\frac{|u|^2_{\omega_\chi}dV_{\omega_\chi}}{|u|^2_\omega dV_\omega}&\le\frac{\big(1+C_1\chi'(\psi)\big)^{n-p-1}\big(1+C_1\chi'(\psi)+C_2\chi''(\psi)\big)}{\big(1+c\chi'(\psi)\big)^{q-\ell+1}}\\&\le C_3\big(\chi'(\psi)^{n+\ell-p-q-1}+\chi''(\psi)\chi'(\psi)^{-2}\big)\quad\text{on}\quad X\backslash K.\end{aligned}$$

For $p+q\ge n+\ell$, this is smaller or equal to

$$C_3\big(\chi'(\psi)^{-1}+\chi''(\psi)\chi'(\psi)^{-2}\big),$$

and it is easy to show that this quantity can be made arbitrarily small towards infinity on X as χ grows sufficiently quickly to infinity on $\mathbb{R}$. □

PROOF OF THE THEOREM (12.13), CONCLUSION. A well-known result of the Andreotti-Grauert [**AG62**] guarantees that the natural topology of the cohomology groups $H^q(X,\mathcal{F})$ of any given coherent sheaf $\mathcal{F}$ on a ℓ-convex manifold is separated for $q \geq \ell$. If $\mathcal{F} = \mathcal{O}(E)$ is the sheaf of sections of a holomorphic vector bundle, the groups $H^q(X,\mathcal{O}(E))$ are algebraically and topologically isomorphic to the cohomology groups of the Dolbeault complex of forms of type $(0,q)$ with $L^2_{\rm loc}$ coefficients for which the d''-differential has L^2 coefficients in terms of the Fréchet topology defined by the semi-norms $u \mapsto ||u||_{L^2(K)} + ||d''u||_{L^2(K)}$. To see this, one can begin again word for word the proof of Theorem (1.3), by observing that the $L^2_{\rm loc}$ complex still furnishes a resolution of $\mathcal{O}(E)$ by the (acyclic) sheaves of $\mathcal{C}^\infty$-modules. It follows from what proceeds this that the morphism

$$L^2_{\omega_\chi}(X,\Lambda^{p,q}T^*_X) \supset \operatorname{Ker} D''_{\omega_\chi} \to H^{p,q}(X,\mathbb{C}) = H^q(X,\Omega^p_X)$$

is continuous and with closed kernel. Consequently this kernel contains the image $\overline{\operatorname{Im} D''_{\omega_\chi}}$, and we obtain a factorization

$$\mathcal{H}^{p,q}_{\omega_\chi}(X,\mathbb{C}) \simeq \operatorname{Ker} D''_{\omega_\chi}/\overline{\operatorname{Im} D''_{\omega_\chi}} \to H^{p,q}(X,\mathbb{C}).$$

The proof of proposition (12.2) further shows that $\overline{\operatorname{Im} D''_{\omega_\chi}}$ coincides with the image of $D''\big(\mathcal{D}(X,\Lambda^{p,q}T^*_X)\big)$ in $L^2_{\omega_\chi}(X,\Lambda^{p,q}T^*_X)$. Consider the limit morphism

$$\lim_{\overrightarrow{\chi}} \mathcal{H}^{p,q}_{\omega_\chi}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C}), \tag{12.16}$$

where the inductive limit is extended to the set of increasing C^∞ convex functions χ, such that $\chi''(t) \geq 1$ on $[0,+\infty[$, with the order relation

$$\chi_1 \preceq \chi_2 \iff \chi_1 \leq \chi_2 \text{ and } L^2_{\omega_{\chi_1}}(X,\Lambda^{p,q}T^*_X) \subset L^2_{\omega_{\chi_2}}(X,\Lambda^{p,q}T^*_X) \text{ for } k = p+q.$$

It is easy to see that this order is filtered by again taking the arguments used for Lemma (12.15). Furthermore, it is well-known that the de Rham cohomology groups are always separated in the induced topology from the Fréchet topology on the space of forms, consequently one has a limit morphism

$$\lim_{\overrightarrow{\chi}} \mathcal{H}^{k}_{\omega_\chi}(X,\mathbb{C}) \to H^{k}_{\rm DR}(X,\mathbb{C}) \tag{12.16$_{\rm DR}$}$$

analogous to (12.16). The decomposition formula of Theorem (12.13) follows now from (12.14), and from the following elementary lemma.

12.17. LEMMA. *The limit morphisms* (12.16), $(12.16)_{\rm DR}$ *are bijective for* $k = p+q \geq n+\ell$.

PROOF. Let us treat for example the case of the morphism (12.16), and let u be a $L^2_{\rm loc}$ d''-closed form of bidegree (p,q), $p+q \geq n+\ell$. Then there exists a choice of χ for which $u \in L^2_{\omega_\chi}$, therefore $u \in \operatorname{Ker} D''_{\omega_\chi}$ and (12.16) is surjective. If a class $\{u\} \in \mathcal{H}^{p,q}_{\omega_{\chi_0}}(X,\mathbb{C})$ is sent to zero in $H^{p,q}(X,\mathbb{C})$, one can write $u = d''v$ for a certain form v with $L^2_{\rm loc}$ coefficients and of bidegree $(p,q-1)$. In the case $p+q > n+\ell$, we will have $v \in L^2_{\omega_\chi}$ for $\chi \succeq \chi_0$ large enough, therefore the class of $u = D''_{\omega_\chi}v$ in $H^{p,q}_{\omega_\chi}(X,\mathbb{C})$ is zero and (12.16) is injective. When $p+q = n+\ell$, the form v does not necessarily belong anymore to one of the spaces $L^2_{\omega_\chi}$, but it suffices to show that $u = d''v$ is in the image of Im D''_{ω_χ} for χ large enough. Let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a truncating function such that $\theta(t) = 1$ for $t \le 1/2$, $\theta(t) = 0$ for $t \ge 1$ and $|\theta'| \le 3$. Then

$$d''(\theta(\epsilon\psi)v) = \theta(\epsilon\psi)d''v + \epsilon\theta'(\epsilon\psi)d''\psi \wedge v.$$

According to the proof of lemma (12.15), there exists a continuous function $C(x) > 0$ such that $|v|^2_{\omega_\chi} dV_{\omega_\chi} \le C\big(1 + \chi''(\psi)/\chi'(\psi)\big)|v|^2_\omega dV_\omega$, whereas $|d''\psi|^2_{\omega_\chi} \le 1/\chi''(\psi)$ according to the same definition of ω_χ. We see therefore that the integral

$$\int_X |\theta'(\epsilon\psi)d''\psi \wedge v|^2_{\omega_\chi} dV_{\omega_\chi} \le \int_X C\big(1/\chi''(\psi) + 1/\chi'(\psi)\big)|v|^2 dV$$

is finite for χ large enough, and by dominated convergence $d''(\theta(\epsilon\psi)v)$ converges to $d''v = u$ in $L^2_{\omega_\chi}(X, \Lambda^{p,q}T^*_X)$. □

Poincaré and Serre duality show that the spaces $H^k_{\mathrm{DR},c}(X, \mathbb{C})$ and $H^{p,q}_c(X, \mathbb{C})$ with compact support are dual to the spaces $H^{2n-k}_{\mathrm{DR}}(X, \mathbb{C})$ and $H^{n-p,n-q}(X, \mathbb{C})$ since the latter are separated and of finite dimension, which is very much the case if $k = p + q \le n - \ell$. We therefore obtain a dual Hodge decomposition

$$(12.18) \quad H^k_c(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}_c(X, \mathbb{C}), \quad \overline{H^{p,q}_c(X, \mathbb{C})} \simeq H^{q,p}_c(X, \mathbb{C}), \quad k \le n - \ell.$$

In addition, it is easy to prove that the Lefschetz isomorphism

$$(12.19) \qquad \omega_\chi^{n-p-q} \wedge \bullet : \mathcal{H}^{p,q}_{\omega_\chi}(X, \mathbb{C}) \to \mathcal{H}^{n-q,n-p}_{\omega_\chi}(X, \mathbb{C})$$

given in the limit is an isomorphism between the cohomology with compact support and cohomology without supports (this result is due to Ohsawa [**Ohs81**]). Indeed, if $p + q \le n - \ell$, the natural morphism

$$(12.20) \qquad H^{p,q}_c(X, \mathbb{C}) = \operatorname{Ker} D''_{\mathcal{D}}/\mathrm{Im}\ D''_{\mathcal{D}} \to \operatorname{Ker} D''_{\omega_\chi}/\overline{\mathrm{Im}\ D''_{\omega_\chi}} \simeq \mathcal{H}^{p,q}_{\omega_\chi}(X)$$

is dual to the morphism $\mathcal{H}^{n-p,n-q}_{\omega_\chi}(X, \mathbb{C}) \to H^{n-p,n-q}(X, \mathbb{C})$, which is surjective for χ large enough according to Lemma (12.17) and the finiteness of the group $H^{n-p,n-q}(X, \mathbb{C})$. Therefore (12.20) is injective for χ large, and after composition with the Lefschetz isomorphism (12.19), we obtain an injection

$$\omega^{n-p-q} \wedge \bullet = \omega_\chi^{n-p-q} \wedge \bullet : H^{p,q}_c(X, \mathbb{C}) \to H^{n-q,n-p}_{L^2,\omega_\chi}(X, \mathbb{C})_{\mathrm{sep}} \simeq \mathcal{H}^{n-q,n-p}_{\omega_\chi}(X, \mathbb{C}).$$

(The equality $\omega^{n-p-q} \wedge \bullet = \omega_\chi^{n-p-q} \wedge \bullet$ follows from the fact that ω_χ has the same cohomology class as ω.) By taking the inductive limit on χ and in combination with the limit isomorphism (12.16), we obtain an injective map

$$(12.21) \qquad \omega^{n-p-q} \wedge \bullet \quad : \quad H^{p,q}_c(X, \mathbb{C}) \to H^{n-q,n-p}(X, \mathbb{C}), \quad p + q \le n - \ell.$$

Since the two groups have the same dimension by the Serre duality theorem and Hodge symmetry, the map is necessarily an isomorphism. □

12.22. REMARK. Since the Lefschetz isomorphism (12.21) can be factored through $H^{p,q}(X,\mathbb{C})$ or through $H_c^{n-q,n-p}(X,\mathbb{C})$, we deduce from this that the natural morphisms

$$H_c^{p,q}(X,\mathbb{C}) \to H^{p,q}(X,\mathbb{C})$$

are injective for $p+q \le n-\ell$ and surjective for $p+q \ge n+\ell$. Of course, there are entirely analogous properties for the de Rham cohomology groups.

13. Bochner techniques and vanishing theorems

Let X be a complex manifold with a given Kähler metric $\omega = \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Let (E,h) be a Hermitian holomorphic vector bundle over X. We denote by $D = D' + D''$ the Chern connection and $\Theta(E)$ the associated curvature tensor.

13.1. Basic commutivity relations. Let L be the operator $Lu = \omega \wedge u$ acting on the vector valued forms, and let $\Lambda = L^*$ be its adjoint. Then

$$[D''^*, L] = \mathrm{i}\ d', \qquad [D'^*, L] = -\mathrm{i}\ d'',$$
$$[\Lambda, D''] = -\mathrm{i}\ d'^*, \quad [\Lambda, D'] = \mathrm{i}\ d''^*.$$

PROOF (OUTLINE). This is a simple consequence of the commutivity relation (6.14) already shown for the trivial connection $d = d' + d''$ on $E = X \times \mathbb{C}$. Indeed, for any point $x_0 \in X$, there exists a local holomorphic frame $(e_\lambda)_{1 \le \lambda \le r}$ of E such that

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda\mu} + O(|z|^2).$$

(The proof is identical to that of Theorem 5.8.) For $s = \sum s_\lambda \otimes e_\lambda$ with $s_\lambda \in C^\infty(X, \Lambda^{p,q} T_X^*)$, we obtain

$$D''s = \sum d'' s_\lambda \otimes e_\lambda + O(|z|), \quad D''^* s = \sum d''^* s_\lambda \otimes e_\lambda + O(|z|).$$

The stated relations follow easily. □

13.2. The Bochner-Kodaira-Nakano identity. If (X,ω) is a Kähler manifold, the complex Laplacians Δ' and Δ'' acting on the forms with values in E satisfy the identity

$$\Delta'' = \Delta' + [\mathrm{i}\ \Theta(E), \Lambda].$$

PROOF. The latter equality (13.1) gives $D''^* = -\mathrm{i}[\Lambda, D']$, therefore

$$\Delta'' = [D'', D''^*] = -\mathrm{i}\big[D'', [\Lambda, D']\big].$$

The Jacobi identity implies

$$\big[D'', [\Lambda, D']\big] = \big[\Lambda, [D', D'']\big] + \big[D', [D'', \Lambda]\big] = [\Lambda, \Theta(E)] + \mathrm{i}[D', D'^*],$$

which is based on the fact that $[D', D''] = D^2 = \Theta(E)$. The stated identity follows. □

Assume that X is compact and let $u \in C^\infty(X, \Lambda^{p,q} T_X^* \otimes E)$ be an arbitrary (p,q)-form. Integration by parts gives

$$\langle \Delta' u, u \rangle = ||D'u||^2 + ||D'^* u||^2 \ge 0,$$

and one has an analogous equality for Δ''. From the Bochner-Kodaira-Nakano identity, one deduces a priori the inequality

$$||D''u||^2 + ||D''^*u||^2 \geq \int_X \langle [\mathrm{i}\ \Theta(E), \Lambda]u, u\rangle dV_\omega. \tag{13.3}$$

This inequality is the well-known Bochner-Kodaira-Nakano inequality (see [**Boc48**], [**Kod53**], [**Nak55**]). When u is Δ''-harmonic, we obtain

$$\int_X \langle [\mathrm{i}\ \Theta(E), \Lambda]u, u\rangle dV \leq 0.$$

If the Hermitian operator $[\mathrm{i}\ \Theta(E), \Lambda]$ is positive on each fiber of $\Lambda^{p,q}T_X^* \otimes E$, then one sees that u is necessarily zero, therefore

$$H^{p,q}(X, E) = \mathcal{H}^{p,q}(X, E) = 0$$

according to Hodge theory. In this approach, the essential point is to know how to calculate the curvature form $\Theta(E)$ and to find sufficient conditions for which the operator $[\mathrm{i}\ \Theta(E), \Lambda]$ is positive definite. Some elementary (albeit somewhat agonizing) calculations yields the following formula: If the curvature of E is written in the form (11.16) and if

$$u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda, \quad |J| = p,\ |K| = q,\ 1 \leq \lambda \leq r$$

is a (p, q)-form with values in E, then

$$\begin{aligned} \langle [\mathrm{i}\ \Theta(E), \Lambda]u, u\rangle = & \sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu} u_{J,jS,\lambda} \overline{u_{J,kS,\mu}} \\ & + \sum_{j,k,\lambda,\mu,R,K} c_{jk\lambda,\mu} u_{kR,K,\lambda} \overline{u_{jR,K\mu}} \\ & - \sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu} u_{J,K,\lambda} \overline{u_{J,K,\mu}}, \end{aligned} \tag{13.4}$$

where the summations are extended to all the indices $1 \leq j, k \leq n,\ 1 \leq \lambda, \mu \leq r$ and all the multi-indices $|J| = p,\ |K| = q,\ |R| = p-1,\ |S| = q-1$. (Here the notation $u_{JK\lambda}$ is applied to some not necessarily increasing multi-indices. Also, it is agreed that the sign of this coefficient is alternating, under the action of permutations.) Taking into account the complexity of the curvature term (13.4), the sign of this term is in general difficult to elucidate, except in some very particular case.

The simpler case is the case $p = n$. All of the terms of the extra second summation in (13.4) are then such that $j = k$ and $R = \{1, \ldots, n\} \setminus \{j\}$. Consequently the second and third summations are equal. It follows that

$$\langle [\mathrm{i}\ \Theta(E), \Lambda]u, u\rangle = \sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu} u_{J,jS,\lambda} \overline{u_{J,kS,\mu}}$$

is positive on the (n, q)-forms under the hypothesis that E is positive in the sense of Nakano. In this case, X is automatically Kähler since

$$\omega = \mathrm{Tr}_E(\mathrm{i}\ \Theta(E)) = \mathrm{i} \sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k = \mathrm{i}\ \Theta(\det E)$$

therefore defines a Kähler metric.

13.5. Nakano Vanishing Theorem (1955). *Let X be a compact complex manifold and let E be a positive vector bundle in the sense of Nakano on X. Then*

$$H^{n,q}(X,E) = H^q(X, K_X \otimes E) = 0 \quad \text{for all } q \geq 1. \qquad \square$$

Another approachable case is the case where E is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we can then choose a coordinate system, which simultaneously diagonalizes the Hermitian forms $\omega(x)$ and $\Theta(E)(x)$, in such a way that

$$\omega(x) = \mathrm{i} \sum_{1 \leq j \leq n} dz_j \wedge d\overline{z}_j, \quad \Theta(E)(x) = \mathrm{i} \sum_{1 \leq j \leq n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \leq \cdots \leq \gamma_n$. The eigenvalues of curvature $\gamma_j = \gamma_j(x)$ are then defined in a unique way and depend continuously in x. In the former notation, we have $\gamma_j = c_{jj11}$ and all the other coefficients $c_{jk\lambda\mu}$ are zero. For any (p,q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned} \langle [\mathrm{i}\ \Theta(E), \Lambda] u, u \rangle &= \sum_{|J|=p, |K|=q} \Big(\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j \Big) |u_{JK}|^2 \\ &\geq (\gamma_1 + \cdots + \gamma_q - \gamma_{n-p+1} - \cdots - \gamma_n) |u|^2. \end{aligned} \tag{13.6}$$

Assume that $\mathrm{i}\ \Theta(E)$ is positive. It is then natural to provide X with the particular Kähler metric $\omega = \mathrm{i}\ \Theta(E)$. Then $\gamma_j = 1$ for $j = 1, 2, \ldots, n$ and we obtain

$$\langle [\mathrm{i}\ \Theta(E), \Lambda] u, u \rangle = (p + q - n) |u|^2.$$

As a consequence:

13.7. Kodaira-Akizuki-Nakano Vanishing Theorem ([**AN54**]). *If E is a positive line bundle over a compact complex manifold X, then*

$$H^{p,q}(X,E) = H^q(X, \Omega_X^p \otimes E) = 0 \quad \text{for } p + q \geq n + 1. \qquad \square$$

More generally, if E is a positive vector bundle in the sense of Griffiths (or ample), of rank $r \geq 1$, Le Potier [**LP75**] has proven that $H^{p,q}(X,E) = 0$ for $p + q \geq n + r$. The proof is not a direct consequence of the Bochner technique. A simple enough proof has been obtained by M. Schneider [**Sch74**], by utilizing the Leray spectral sequence associated to the projection on X of the projective bundle $\mathbb{P}(E) \to X$.

13.8. Exercise. It is significant for various applications to formulate vanishing theorems which are also valid in the case of semi-positive line bundles. There is, for example, the following result due to J. Girbau [**Gir76**] : Let (X, ω) be a compact Kähler manifold, assume that E is a line bundle and that $\mathrm{i}\ \Theta(E) \geq 0$ has at least $n - k$ positive eigenvalues at each point, for a certain integer $k \geq 0$. Then $H^{p,q}(X,E) = 0$ for $p + q \geq n + k + 1$.

Indication. Use the Kähler metric $\omega_\epsilon = \mathrm{i}\ \Theta(E) + \epsilon\omega$ with small $\epsilon > 0$.

A more natural and powerful version of this result has been obtained by A. Sommese [**Som78, ShSo85**] : Following these authors, we say that E is k-ample if a certain multiple mE is such that the canonical map

$$\Phi_{|mE|} : X \backslash B_{|mE|} \to \mathbb{P}^{N-1}$$

has all its fibers of dimension $\le k$ and $\dim B_{|mE|} \le k$. If X is projective and if E is k-ample, then $H^{p,q}(X,E)=0$ for $p+q\ge n+k+1$.

INDICATION. Prove the dual result, that $H^{p,q}(X,E^{-1})=0$ for $p+q\le n-k-1$, by induction on k. First show that E is 0-ample if and only if E is positive. Then use some hyperplane sections $Y\subset X$ to prove the induction step, by considering the exact sequences

$$\begin{aligned}&0\to\Omega^p_X\otimes E^{-1}\otimes\mathcal{O}(-Y)\to\Omega^p_X\otimes E^{-1}\to(\Omega^p_X\otimes E^{-1})_{\restriction Y}\to 0,\\ &0\to\Omega^{p-1}_Y\otimes E^{-1}_{\restriction Y}\to(\Omega^p_X\otimes E^{-1})_{\restriction Y}\to\Omega^p_Y\otimes E^{-1}_{\restriction Y}\to 0.\end{aligned}\qquad\square$$

14. L^2 estimations and existence theorems

The starting point is the following L^2 existence theorem, which is essentially due to Hörmander [**Hör65, 66**], and Andreotti-Vesentini [**AV65**]. We only sketch the principal ideas, while referring for example to [**Dem82**] for a detailed exposition of the techniques considered in the situation here.

14.1. THEOREM. *Let (X,ω) be a complete Kähler manifold, and let E be a Hermitian vector bundle of rank r on X, such that the curvature operator $A=A^{p,q}_{E,\omega}=[\mathrm{i}\,\Theta(E),\Lambda_\omega]$ is semi-positive on all the fibers of $\Lambda^{p,q}T^*_X\otimes E$, $q\ge 1$. Let $g\in L^2(X,\Lambda^{p,q}T^*_X\otimes E)$ be a form satisfying*

$$D''g=0\quad\text{and}\quad\int_X\langle A^{-1}g,g\rangle dV_\omega<+\infty.$$

(At the points where A is not positive definite, we assume as a precondition that $A^{-1}g$ exists almost everywhere. We then choose the preconditional term $A^{-1}g$ of minimal norm, orthogonal to $\operatorname{Ker} A$*.) Then there exists $f\in L^2(X,\Lambda^{p,q-1}T^*_X\otimes E)$ such that*

$$D''f=g\quad\text{and}\quad\int_X|f|^2dV_\omega\le\int_X\langle A^{-1}g,g\rangle dV_\omega.$$

PROOF. Let $u\in L^2(X,\Lambda^{p,q}T^*_X\otimes E)$ be a form such that $D''u\in L^2$ and $D''^*u\in L^2$ in the sense of distributions. Lemma (12.2 a) shows (under the indispensable hypothesis that ω is complete) that u is the limit of a sequence of C^∞ forms u_ν with compact support in such a way that $u_\nu\to u$, $D''u_\nu\to D''u$ and $D''^*u_\nu\to D''^*u$ in L^2. It follows that a priori the inequality (13.3) extends to arbitrary forms u such that $u,\ D''u,\ D''^*u\in L^2$. Now, since $\operatorname{Ker} D''$ is weakly (and therefore strongly) closed, we obtain an orthogonal decomposition of the Hilbert space $L^2(X,\Lambda^{p,q}T^*_X\otimes E)$, namely

$$L^2(X,\Lambda^{p,q}T^*_X\otimes E)=\operatorname{Ker} D''\oplus(\operatorname{Ker} D'')^\perp.$$

Let $v=v_1+v_2$ be the corresponding decomposition of a C^∞ form $v\in\mathcal{D}^{p,q}(X,E)$ with compact support (in general, v_1, v_2 do not have compact support!). Since $(\operatorname{Ker} D'')^\perp=\overline{\operatorname{Im} D''^*}\subset\operatorname{Ker} D''^*$ by duality and $g,\ v_1\in\operatorname{Ker} D''$ by hypothesis, we obtain $D''^*v_2=0$ and

$$|\langle g,v\rangle|^2=|\langle g,v_1\rangle|^2\le\int_X\langle A^{-1}g,g\rangle dV_\omega\int_X\langle Av_1,v_1\rangle dV_\omega$$

by applying the Cauchy-Schwartz inequality. The inequality (13.3) a priori, applied to $u = v_1$ gives

$$\int_X \langle Av_1, v_1\rangle dV_\omega \leq ||D''v_1||^2 + ||D''^* v_1||^2 = ||D''^* v_1||^2 = ||D''^* v||^2.$$

Combining these two inequalities we find that

$$|\langle g, v\rangle|^2 \leq \left(\int_X \langle A^{-1} g, g\rangle dV_\omega\right) ||D''^* v||^2$$

for any C^∞ (p,q)-form v with compact support. This shows that there is a well-defined linear form

$$w = D''^* v \mapsto \langle v, g\rangle, \quad L^2(X, \Lambda^{p,q-1} T_X^* \otimes E) \supset D''^*(\mathcal{D}^{p,q}(E)) \to \mathbb{C}$$

on the image of D''^*. This linear form is continuous in the L^2 norm, and its norm is $\leq C$ with

$$C = \left(\int_X \langle A^{-1} g, g\rangle dV_\omega\right)^{1/2}.$$

According to the Hahn-Banach Theorem, there exists an element

$$f \in L^2(X, \Lambda^{p,q-1} T_X^* \otimes E)$$

such that $||f|| \leq C$ and $\langle v, g\rangle = \langle D''^* v, f\rangle$ for any v, consequently $D'' f = g$ in the sense of distributions. The inequality $||f|| \leq C$ is equivalent to the latter estimation in the theorem. □

The preceding L^2 existence theorem can be applied in the general context of *weakly pseudoconvex* Kähler manifolds (see definition (12.9)), and the same if the Kähler metric considered ω is not complete. Indeed, according to Proposition (12.10), we arrive at complete Kähler metrics by setting

$$\omega_\epsilon = \omega + \epsilon \mathrm{i}\; d'd''\psi^2 = \omega + 2\epsilon(2\mathrm{i}\;\psi d'd''\psi + \mathrm{i}\; d'\psi \wedge d''\psi)$$

with a C^∞ psh exhaustion function $\psi \geq 0$. As a consequence, the L^2 existence theorem (14.1) applies to each Kähler metric ω_ϵ. Indeed one can show (the calculations being left to the reader!) that the quantities $|g|^2_\omega dV_\omega$ and $\langle (A^{p,q}_{E,\omega})^{-1} g, g\rangle_\omega dV_\omega$ are decreasing functions of ω when $p = n = \dim_{\mathbb{C}} X$. For a D''-closed form g of bidegree (n,q), we therefore obtains solutions f_ϵ of the equation $D'' f_\epsilon = g$ satisfying

$$\int_X |f_\epsilon|^2_{\omega_\epsilon} dV_{\omega_\epsilon} \leq \int_X \langle (A^{p,q}_{E,\omega_\epsilon})^{-1} g, g\rangle_{\omega_\epsilon} dV_{\omega_\epsilon} \leq \int_X \langle (A^{p,q}_{E,\omega})^{-1} g, g\rangle_\omega dV_\omega.$$

These solutions f_ϵ can be uniformly bounded in the L^2 norm on any compact set. Thus we can extract a weakly convergent subsequence in L^2. The limit f is a solution of $D'' f = g$ and satisfies the required L^2 estimation relative to the metric ω initially given (which, to repeat, is not necessarily complete). A particularly important case is the following:

14.2. THEOREM. *Let (X,ω) be a Kähler manifold,* $\dim X = n$. *Assume that X is weakly pseudoconvex. Let E be a Hermitian line bundle and let*

$$\gamma_1(x) \leq \cdots \leq \gamma_n(x)$$

be the eigenvalues of curvature (i.e. the eigenvalues of $\mathrm{i}\,\Theta(E)$ *with respect to the metric ω) at any point x. Assume that the curvature is semi-positive, i.e. $\gamma_1 \geq 0$ everywhere. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ satisfying*

$$D''g = 0 \quad \text{and} \quad \int_X (\gamma_1 + \cdots + \gamma_q)^{-1}|g|^2 dV_\omega < +\infty,$$

(one assumes therefore $g(x) = 0$ almost everywhere at all points where $\gamma_1(x) + \cdots + \gamma_q(x) = 0$), there exists $f \in L^2(X, \Lambda^{n,q-1}T_X^* \otimes E)$ such that

$$D''f = g \quad \text{and} \quad 2\int_X |f|^2 dV_\omega \leq \int_X (\gamma_1 + \cdots + \gamma_q)^{-1}|g|^2 dV_\omega.$$

PROOF. Indeed, for $p = n$, formula (13.6) shows that

$$\langle Au, u\rangle \geq (\gamma_1 + \cdots + \gamma_q)|u|^2,$$

therefore $\langle A^{-1}u, u\rangle \geq (\gamma_1 + \cdots + \gamma_q)^{-1}|u|^2$. □

An important observation is that the above theorem still applies when the Hermitian metric of E is a singular metric with positive curvature in the sense of currents. Indeed, by a process of regularization (convolution of psh functions by regular kernels), the metric can made C^∞ and the solutions obtained by means of Theorems (14.1) or (14.2), since the regular metrics have limits satisfying the desired estimates. In particular, we obtain the following corollary.

14.3. COROLLARY. *Let (X,ω) be a Kähler manifold,* $\dim X = n$. *Assume that X is weakly pseudoconvex. Let E be a holomorphic bundle provided with a singular metric for which the local weight is denoted by $\varphi \in L^1_{\mathrm{loc}}$. Assume that*

$$\mathrm{i}\,\Theta(E) = 2\mathrm{i}\, d'd''\varphi \geq \epsilon\omega$$

for a certain $\epsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{n,q}T_X^* \otimes E)$ satisfying $D''g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T_X^* \otimes E)$ such that $D''f = g$ and

$$\int_X |f|^2 \mathrm{e}^{-2\varphi} dV_\omega \leq \frac{1}{q\epsilon}\int_X |g|^2 \mathrm{e}^{-2\varphi} dV_\omega. \qquad \square$$

We denoted here somewhat incorrectly the metric in the form $|f|^2\,\mathrm{e}^{-2\varphi}$, as if the weight φ were globally defined on X (certainly, this is not possible if E is globally trivial). By abuse of notation, we will nevertheless use this same notation because it clearly underlines the dependence of the L^2 norm on the psh function associated to the weight.

15. Vanishing theorems of Nadel and Kawamata-Viehweg

We begin by introducing the concept of *multiplier ideal sheaves,* following A. Nadel [**Nad89**]. The principal idea in fact goes back to the fundamental work of E. Bombieri [**Bom70**] and H. Skoda [**Sko72**].

15.1. Definition. Let φ be a psh function on an open set $\Omega \subset X$. We associate to φ, the sheaf of ideals $\mathcal{J}(\varphi) \subset \mathcal{O}_\Omega$ formed from the germs of holomorphic functions $f \in \mathcal{O}_{\Omega,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in the local coordinates x. This sheaf will be called the multiplier ideal sheaf associated to the weight φ.

The variety of zeros $V(\mathcal{J}(\varphi))$ is therefore the set of points in a neighbourhood for which $e^{-2\varphi}$ is non-integrable. Of course, such points cannot appear where φ has logarithmic poles. The precise formulation is the following.

15.2. Definition. We say that a psh function φ has a logarithmic pole with coefficient γ at a point $x \in X$ if the Lelong number

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log |z - x|}$$

is non-zero and if $\nu(\varphi, x) = \gamma$.

15.3. Lemma. *(Skoda* **[Sko72]***). Let φ be a psh function on an open set $\Omega \subset \mathbb{C}^n$ and let $x \in \Omega$.*

a) *If $\nu(\varphi, x) < 1$, then $e^{-2\varphi}$ is integrable in a neighbourhood of x, in particular $\mathcal{J}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*
b) *If $\nu(\varphi, x) \geq n + s$ for a certain integer $s \geq 0$, then $e^{-2\varphi} \geq C|z - x|^{-2n-2s}$ in a neighbourhood of x and $\mathcal{J}(\varphi)_x \subset \mathfrak{m}_{\Omega,x}^{s+1}$, where $\mathfrak{m}_{\Omega,x}$ denotes the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

Proof. The proof rests on some classical estimations of complex potential theory, see H. Skoda **[Sko72]**. $\square$

15.4. Proposition (**[Nad89]**). *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{J}(\varphi)$ is a coherent sheaf of ideals on Ω.*

Proof. Since the result is local we can assume that Ω is the unit ball in $\mathbb{C}^n$. Let $\mathcal{H}_\varphi(\Omega)$ be the set of the holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2\varphi} d\lambda < +\infty$. According to the strong Noetherian property of coherent sheaves, the set $\mathcal{H}_\varphi(\Omega)$ generates a coherent sheaf of ideals $\mathfrak{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathfrak{J} \subset \mathcal{J}(\varphi)$; for to show equality, it suffices to verify that $\mathfrak{J}_x + \mathcal{J}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{J}(\varphi)_x$ for any integer s, by virtue of Krull's lemma. Let $f \in \mathcal{J}(\varphi)_x$ be a germ defined on a neighbourhood V of x and let θ be a truncating function with support in V, such that $\theta = 1$ in a neighbourhood of x. We can solve the equation $d''u = g := d''(\theta f)$ by means of L^2 estimations of Hörmander (14.3), where E is the trivial line bundle $\Omega \times \mathbb{C}$ provided with the strictly psh weight

$$\tilde{\varphi}(z) = \varphi(z) + (n + s) \log |z - x| + |z|^2.$$

We obtain a solution u such that $\int_\Omega |u|^2 e^{-2\varphi} |z - x|^{-2(n+s)} d\lambda < \infty$, therefore $F = \theta f - u$ is holomorphic, $F \in \mathcal{H}_\varphi(\Omega)$ and $f_x - F_x = u_x \in \mathcal{J}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves our assertion. $\square$

The multiplier ideal sheaves satisfy the following essential functorial property, relative to the direct images of sheaves by modifications.

15.5. PROPOSITION. *Let $u : X' \to X$ be a modification of non-singular complex varieties (i.e. a proper holomorphic map that is generically $1:1$), and let φ be a psh function on X. Then*

$$\mu_*\big(\mathcal{O}(K_{X'}) \otimes \mathcal{J}(\varphi\circ\mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{J}(\varphi).$$

PROOF. Let $n = \dim X = \dim X'$ and let $S \subset X$ be an analytic subvariety such that $\mu : X'\backslash S' \to X\backslash S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X)\otimes\mathcal{J}(\varphi)$ is identified with the sheaf of holomorphic n-forms f on some open set $U \subset X$, satisfying $\mathrm{i}^{n^2} f\wedge\overline{f}\mathrm{e}^{-2\varphi} \in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded above, we can likewise consider the forms f which a priori are defined only on $U\backslash S$, because f is in $L^2_{\mathrm{loc}}(U)$, and thus automatically extends through S. The change of variables formula gives

$$\int_U \mathrm{i}^{n^2} f\wedge\overline{f}\mathrm{e}^{-2\varphi} = \int_{\mu^{-1}(U)} \mathrm{i}^{n^2}\mu^*f\wedge\overline{\mu^*f}\mathrm{e}^{-2\varphi\circ\mu},$$

therefore $f \in \Gamma(U, \mathcal{O}(K_X)\otimes\mathcal{J}(\varphi))$ if and only if $\mu^*f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_X)\otimes\mathcal{J}(\varphi\circ\mu))$. This proves Prop. 15.5. $\square$

15.6. REMARK. If φ has algebraic or analytic singularities (cf. definition 11.7), the calculation of $\mathcal{J}(\varphi)$ is reduced to a purely algebraic problem.

The first observation is that $\mathcal{J}(\varphi)$ is easily calculated if $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are smooth irreducible divisors with normal crossings. Then $\mathcal{J}(\varphi)$ is the sheaf of holomorphic functions h on the open set $U\subset X$, satisfying

$$\int_U |h|^2 \prod |g_j|^{-2\alpha_j} dV < +\infty.$$

Since the g_j can be taken as coordinate functions in suitable local coordinate systems $(z_1, \ldots, z_n)$, the integrability condition is that h is divisible by $\prod g_j^{m_j}$, where $m_j - \alpha_j > -1$ for each j, i.e. $m_j \ge \lfloor\alpha_j\rfloor$ (where $\lfloor\ \rfloor$ denotes the integral part). Consequently

$$\mathcal{J}(\varphi) = \mathcal{O}(-\lfloor D\rfloor) = \mathcal{O}(-\sum\lfloor\alpha_j\rfloor D_j)$$

where $\lfloor D\rfloor$ is the integral part of the $\mathbb{Q}$-divisor $D = \sum\alpha_j D_j$.

Now consider the general case of algebraic or analytic singularities and assume that

$$\varphi \sim \frac{\alpha}{2}\log\big(|f_1|^2 + \cdots + |f_N|^2\big)$$

in a neighbourhood of the poles. According to the remark stated after definition 11.7, we can assume that the (f_j) are generators of the sheaf of integrally closed ideals $\mathfrak{J} = \mathfrak{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h| \le C\exp(\varphi/\alpha)$. In this case, the calculation is done as follows.

Let us first choose a smooth modification $\mu : \tilde{X} \to X$ of X such that $\mu^*\mathfrak{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated to a divisor with normal crossings $D = \sum\lambda_j D_j$, where (D_j) are the components of the exceptional divisor of $\tilde{X}$. (Consider the blow-up X' of X along the ideal $\mathfrak{J}$, so that the inverse image of $\mathfrak{J}$ on X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow-up X' again so as to render X' smooth and D' with normal crossings, by invoking Hironaka [**Hi64**].) We then have $K_{\tilde{X}} =$

$\mu^*K_X + R$ where $R = \sum \rho_j D_j$ is the divisor of zeros of the jacobian J_μ of the blow-up map. From the direct image formula 15.5, we deduce

$$\mathcal{J}(\varphi) = \mu_*\big(\mathcal{O}(K_{\tilde X} - \mu^*K_X) \otimes \mathcal{J}(\varphi\circ\mu)\big) = \mu_*\big(\mathcal{O}(R) \otimes \mathcal{J}(\varphi\circ\mu)\big).$$

Now the $(f_j \circ \mu)$ are generators of the ideal $\mathcal{O}(-D)$, therefore

$$\varphi\circ\mu \sim \alpha \sum \lambda_j \log|g_j|$$

where the g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to calculating the multiplier ideal sheaf in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum \alpha\lambda_j D_j$. We obtain $\mathcal{J}(\varphi\circ\mu) = \mathcal{O}(-\sum\lfloor\alpha\lambda_j\rfloor D_j)$, therefore

$$\mathcal{J}(\varphi) = \mu_*\mathcal{O}_{\tilde X}\big(\textstyle\sum(\rho_j - \lfloor\alpha\lambda_j\rfloor)D_j\big). \ \square$$

15.7. EXERCISE. Calculate the multiplier ideal sheaf $\mathcal{J}(\varphi)$ associated to the psh function $\varphi = \log(|z_1|^{\alpha_1} + \cdots + |z_p|^{\alpha_p})$, for arbitrary real numbers $\alpha_j > 0$.

INDICATION. By using Parseval's formula and polar coordinates $z_j = r_j \mathrm{e}^{\mathrm{i}\Theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1, \ldots, \beta_p) \in \mathbb{N}^p$ the integral

$$\int_{[0,1]^p} \frac{r_1^{2\beta_1}\cdots r_p^{2\beta_p} r_1 dr_1 \cdots r_p dr_p}{r_1^{2\alpha_1} + \cdots + r_p^{2\alpha_p}} = \int_{[0,1]^p} \frac{t_1^{(\beta_1+1)/\alpha_1}\cdots t_p^{(\beta_p+1)/\alpha_p}}{t_1 + \cdots + t_p}\frac{dt_1}{t_1}\cdots\frac{dt_p}{t_p}$$

is convergent. Deduce from this that $\mathcal{J}(\varphi)$ is generated by the monomials $z_1^{\beta_1}\cdots z_p^{\beta_p}$ such that $\sum(\beta_p+1)/\alpha_p > 1$. (This exercise shows that the analytic definition of $\mathcal{J}(\varphi)$ is also sometimes very convenient for calculations). $\square$

Let E be a line bundle over X with a given singular metric h with curvature current $\Theta_h(E)$. If φ is the weight representing the metric h on an open set $\Omega \subset X$, the sheaf of ideals $\mathcal{J}(\varphi)$ is independent of the choice of the trivialization. It is therefore the restriction to Ω of a global coherent sheaf on X that we will denote by $\mathcal{J}(h) = \mathcal{J}(\varphi)$, by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results in algebraic or analytic geometry. (As we will see later, this theorem contains the Kawamata-Viehweg vanishing theorem as a special case.)

15.8. NADEL VANISHING THEOREM ([**Nad89**], [**Dem93b**]). *Let (X,ω) be a weakly pseudoconvex Kähler manifold, and let E be a holomorphic line bundle on X with a given singular Hermitian metric h of weight φ. Assume that there exists a positive continuous function ϵ on X such that* $\mathrm{i}\,\Theta_h(E) \geq \epsilon\omega$. *Then*

$$H^q\big(X, \mathcal{O}(K_X + E) \otimes \mathcal{J}(h)\big) = 0 \quad \text{for all } q \geq 1.$$

PROOF. Let $\mathcal{L}^q$ be the sheaf of germs of (n,q)-forms u with values in E and with measurable coefficients, for which $|u|^2\mathrm{e}^{-2\varphi}$ and $|d''u|^2\mathrm{e}^{-2\varphi}$ are simultaneously locally integrable. The operator d'' defines a complex of sheaves $(\mathcal{L}^\bullet, d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X + E) \otimes \mathcal{J}(\varphi)$: Indeed, the kernel of d'' in degree 0 consists of the germs of holomorphic n-forms with values in E which satisfy the integrability condition. Therefore the coefficient function belongs to $\mathcal{J}(\varphi)$, and the exactness at degree $q \geq 1$ arises from Corollary 14.3 applied to arbitrary small balls. Since each sheaf $\mathcal{L}^q$ is a C^∞-module, $\mathcal{L}^\bullet$ is a resolution by acyclic sheaves. Let ψ

be a C^∞ psh exhaustion function on X. We apply Corollary 14.3 globally on X, with the initial metric of E multiplied by the factor $\mathrm{e}^{-\chi\circ\psi}$, where χ is an increasing convex function of arbitrary growth at infinity. This factor can be used to ensure convergence of integrals at infinity. From Corollary 14.3, we then deduce that $H^q(\Gamma(X,\mathcal{L}^\bullet)) = 0$ for $q \geq 1$. The theorem follows by virtue of the de Rham-Weil Isomorphism Theorem (1.2). □

15.9. Corollary. *Let (X,ω), E and φ be given as in Theorem* 15.8*, and assume given $x_1,\ldots,x_N$ isolated points of the variety of zeros $V(\mathcal{J}(\varphi))$. Then there exists a surjective map*

$$H^0(X,\mathcal{O}(K_X+E)) \longrightarrow \bigoplus_{1\leq j\leq N} \mathcal{O}(K_X+E)_{x_j} \otimes \big(\mathcal{O}_X/\mathcal{J}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact cohomology sequence associated to the short exact sequence $0 \to \mathcal{J}(\varphi) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}(\varphi) \to 0$, twisted by $\mathcal{O}(K_X+E)$, and apply Theorem 15.8 to obtain the vanishing of the first group H^1. The stated surjective property follows. □

15.10. Corollary. *Let (X,ω), E and φ be given as in Theorem* 15.8*. Assume that the weight function φ satisfies $\nu(\varphi,x) \geq n+s$ at a given point $x \in X$ for which $\nu(\varphi,y) < 1$, for $y \neq x$ close enough to x. Then $H^0(X,K_X+E)$ generates all the s-jets of sections at the point x.*

Proof. Skoda's Lemma 15.3 b) shows that $\mathrm{e}^{-2\varphi}$ is integrable in a neighbourhood of any point $y \neq x$ sufficiently close to x, therefore $\mathcal{J}(\varphi)_y = \mathcal{O}_{X,y}$, whereas $\mathcal{J}(\varphi)_x \subset \mathfrak{m}_{X,x}^{s+1}$ according to 15.3 a). Corollary 15.10 is therefore a special case of 15.9. □

The philosophy of the results (which can be regarded as generalization of the Hörmander-Bombieri-Skoda Theorem [**Bom70**], [**Sko72,75**]), is that the problem of constructing holomorphic sections of K_X+E can be solved by constructing suitable Hermitian metrics on E such that the weight φ has isolated logarithmic points at the given points x_j.

15.11. Exercise. Assume that X is compact and that L is a positive line bundle on X. Let $\{x_1,\ldots,x_N\}$ be a finite set. Show that there exists constants $a,b\geq 0$ depending only on L and N such that for any $s\in\mathbb{N}$, the group $H^0(X,\mathcal{O}(mL))$ generates the jets of order s at any point x_j, for $m \geq as+b$.

Indication. Apply Corollary 15.9 to $E = -K_X + mL$, with a singular metric on L of the form $h = h_0\mathrm{e}^{-\epsilon\psi}$, where h_0 is C^∞ with positive curvature, $\epsilon>0$ small, and $\psi(z) \sim \log|z-x_j|$ in a neighbourhood of x_j. Deduce from this the Kodaira embedding theorem:

15.12. Kodaira Embedding Theorem. *If L is a line bundle on a compact complex manifold, then L is ample if and only if L is positive.* □

An equivalent way to state the Kodaira embedding theorem is the following:

15.13. Kodaira criterion for projectivity. A compact complex manifold X is projective algebraic if and only if X contains a Hodge metric. That is, a Kähler metric with integral cohomology class.

PROOF. If $X \subset \mathbb{P}^N$ is projective algebraic, then the restriction of the Fubini-Study metric to X is a Hodge metric. Conversely, if X has a Hodge metric ω, the cohomology class representative $\{\omega\}$ in $H^2(X, \mathbb{Z})$ defines a complex topological (i.e. C^∞) line bundle, say L. Since ω is of type $(1,1)$, the exponential exact sequence (8.20)

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) = H^{0,2}(X, \mathbb{C})$$

shows that the line bundle L can be represented by a cocycle in $H^1(X, \mathcal{O}_X^*)$. In other words, L is endowed with a complex structure. Moreover, there exists a Hermitian metric h on L such that $\frac{\mathrm{i}}{2\pi}\Theta_h(L) = \omega$. Consequently, L is ample and X is projective algebraic.

15.14. EXERCISE (Riemann conditions characterizing Abelian varieties). A complex torus $X = \mathbb{C}^n/\Gamma$ is called an *Abelian variety* if X is projective algebraic. Show by using (15.13) that a torus X is an Abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^n$ such that Im $H(\gamma_1, \gamma_2) \in \mathbb{Z}$ for all γ_1, γ_2 in the lattice Γ.

INDICATION. Use a process of averaging to reduce the proof to the case of Kähler metric invariant by translations. Observe that the real torus $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ defines a system of generators of the homology group $H_2(X, \mathbb{Z})$ and that $\int_{\mathbb{Z}\gamma_1+\mathbb{Z}\gamma_2} \omega = \omega(\gamma_1, \gamma_2)$.

15.15. EXERCISE (solution of the Levi problem). Show that the following two properties are equivalent.

a) X is strongly pseudoconvex, i.e. X admits a strongly psh exhaustion function.
b) X is a Stein, i.e. the global holomorphic functions separate points, furnishing a system of local coordinates at every point, and X is holomorphically convex. (By definition, this means that for any discrete sequence (z_ν) in X, there exists a function $f \in H^0(X, \mathcal{O}_X)$ such that $|f(z_\nu)| \to \infty$.) □

15.16. REMARK. As long as one is interested only in the case of forms of bidegree $(n, q), n = \dim X$, the L^2 estimates extend to the complex spaces acquiring arbitrary singularities. Indeed, if X is a complex space and φ a psh weight function on X, one can still define a sheaf $K_X(\varphi)$ on X, such that the sections of $K_X(\varphi)$ on an open set U are the holomorphic n-forms f on the regular part $U \cap X_{\mathrm{reg}}$, satisfying the integrability condition $\mathrm{i}^{n^2} f \wedge \overline{f} \mathrm{e}^{-2\varphi} \in L^1_{\mathrm{loc}}(U)$. In this context, the functorial property 15.5 can be written (or is written)

$$\mu_*(K_{X'}(\varphi \circ \mu)) = K_X(\varphi),$$

and it is valid for arbitrary complex spaces X, X', $\mu : X' \to X$ being a modification. If X is non-singular, one has $K_X(\varphi) = \mathcal{O}(K_X) \otimes \mathcal{J}(\varphi)$, however, if X is singular, the symbols K_X and $\mathcal{J}(\varphi)$ do not have to be dissociated. The statement of the Nadel vanishing theorem becomes $H^q(X, \mathcal{O}(E) \otimes K_X(\varphi)) = 0$ for $q \geq 1$, under the same hypothesis (X Kähler and weakly pseudoconvex, curvature of $E \geq \epsilon\omega$). The proof is obtained by restricting all the situations to X_{reg}. Although in general X_{reg} is not weakly pseudoconvex (a necessary condition being codim$X_{\mathrm{sing}} = 1$), X_{reg} is always Kählerian complete (the complement of an analytic subset in a weakly pseudoconvex Kähler space is Kählerian complete, see for example [**Dem82**]). As a consequence, the Nadel vanishing theorem is essentially insensitive to the presence of singularities. □

We now deduce an algebraic version of the Nadel vanishing theorem obtained independently by Kawamata [**Kaw82**] and Viehweg [**Vie82**]. (The original proof relies on a different method using cyclic coverings to reduce to the case situation of the ordinary Kodaira Theorem.) Before stating the theorem, we need a definition.

15.17. DEFINITION. A line bundle L on a compact complex manifold is called large if its Kodaira dimension is equal to $n = \dim X$, that is, if there exists a constant $c > 0$ such that

$$\dim H^0(X, \mathcal{O}(kL)) \geq ck^n, \quad k \geq k_0.$$

15.18. DEFINITION. A line bundle L on a projective algebraic manifold is called numerically effective (nef for short) if L satisfies one of the following three equivalent properties:

a) For any irreducible algebraic curve $C \subset X$, one has $L \cdot C = \int_C c_1(L) \geq 0$.
b) If A is an ample line bundle, then $kL + A$ is ample for all $k \geq 0$.
c) For any $\epsilon > 0$, there exists a C^∞ Hermitian metric h_ϵ on L such that $\Theta_{h_\epsilon}(L) \geq -\epsilon\omega$, where ω is a fixed Hermitian metric on X.

The equivalence of properties 15.18 a) and b) is well-known and we will omit it here (see for example Hartshorne [**Har70**] for the proof). It is clear in addition that 15.18 c) implies 15.18 a), while 15.18 b) implies 15.18 c). Indeed if $\omega = \frac{\mathrm{i}}{2\pi}\Theta(A)$ is the curvature of a metric of A with positive curvature, and if h_k is a metric on L inducing a metric with positive curvature on $kL + A$, it becomes $k\frac{\mathrm{i}}{2\pi}\Theta(L) + \frac{\mathrm{i}}{2\pi}\Theta(A) > 0$, where $\frac{\mathrm{i}}{2\pi}\Theta(L) \geq -\frac{1}{k}\omega$. Now, if $D = \sum \alpha_j D_j \geq 0$ is an effective $\mathbb{Q}$-divisor, we define the multiplier ideal sheaf $\mathcal{J}(D)$ to be the sheaf $\mathcal{J}(\varphi)$ associated to the psh function $\varphi = \sum \alpha_j \log|g_j|$ defined by the generators g_j of $\mathcal{O}(-D_j)$. According to remark 15.6, the calculation of $\mathcal{J}(D)$ can be done algebraically by making use of desingularizations $\mu : \tilde{X} \to X$ such that $\mu^* D$ becomes a divisor with normal crossings on $\tilde{X}$.

15.19. KAWAMATA-VIEHWEG VANISHING THEOREM. *Let X be a projective algebraic manifold, and let F be a line bundle on X such that a multiple mF of F can be written in the form $mF = L + D$, where L is a nef and large line bundle, and D an effective divisor. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{J}(m^{-1}D)\big) = 0 \quad \text{for} \quad q \geq 1.$$

15.20. COROLLARY. *If F is nef and large, then $H^q(X, \mathcal{O}(K_X + F)) = 0$ for $q \geq 1$.*

PROOF. Let A be a non-singular very ample divisor. There is an exact sequence

$$0 \to H^0(X, \mathcal{O}(kL - A)) \to H^0(X, \mathcal{O}(kL)) \to H^0(A, \mathcal{O}(kL)_{\restriction A}),$$

and dim $H^0(A, \mathcal{O}(kL)_{\restriction A}) \leq Ck^{n-1}$ for a certain constant $C \geq 0$. Since L is large, there exists an integer $k_0 \gg 0$ such that $\mathcal{O}(k_0 L - A)$ has a non-trivial section. If E is the divisor of this section, we have $\mathcal{O}(k_0 L - A) \simeq \mathcal{O}(E)$, therefore $\mathcal{O}(k_0 L) \simeq \mathcal{O}(A + E)$. Now, for $k \geq k_0$, we arrive at $\mathcal{O}(kL) = \mathcal{O}((k - k_0)L + A + E)$. According to 15.18 b), the line bundle $\mathcal{O}((k - k_0)L + A)$ is ample, therefore it comes with a C^∞ Hermitian metric $h_k = \mathrm{e}^{-\varphi_k}$, and with positive definite curvature form $\omega_k = \frac{\mathrm{i}}{2\pi}\Theta((k - k_0)L + A)$. Let $\varphi_D = \sum \alpha_j \log|g_j|$ be the weight of the singular metric on $\mathcal{O}(D)$ described in example 11.21, such that $\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(D)) = [D]$, and in a similar way, let φ_E be the weight such that $\frac{\mathrm{i}}{2\pi}\Theta(\mathcal{O}(E)) = [E]$. We define a singular metric on $\mathcal{O}(kL) = \mathcal{O}((k-k_0)L + A + E)$ by means of the weight $\varphi_k + \varphi_E$, and then we obtain a singular metric on $\mathcal{O}(mF) = \mathcal{O}(L+D)$, by considering the weight $\frac{1}{k}(\varphi_k + \varphi_E) + \varphi_D$. Finally, we obtain a metric on F of weight

$$\varphi_F = \frac{1}{km}(\varphi_k + \varphi_E) + \frac{1}{m}\varphi_D.$$

The corresponding curvature form is

$$\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{1}{km}(\omega_k + [E]) + \frac{1}{m}[D] \geq \frac{1}{km}\omega_k > 0.$$

Moreover φ_F has algebraic singularities, and by taking k sufficiently large we have

$$\mathcal{J}(\varphi_F) = \mathcal{J}\left(\frac{1}{km}E + \frac{1}{m}D\right) = \mathcal{J}\left(\frac{1}{m}D\right).$$

Indeed, $\mathcal{J}(\varphi_F)$ is calculated by taking the integral part of a $\mathbb{Q}$-divisor with normal crossings, obtained by the means of a suitable modification (as was explained in remark 15.6). The divisor $\frac{1}{km}E + \frac{1}{m}D$ furnishes therefore the same integral part as $\frac{1}{m}D$ when k is large. The Nadel Theorem then implies the desired vanishing result for all $q \geq 1$. □

16. On the conjecture of Fujita

Given an ample line bundle L, a fundamental question is of determining an effective integer m_0 such that mL is very ample for $m \geq m_0$. The example where X is a hyperelliptic curve of genus g and where $L = \mathcal{O}(p)$ is associated to one of the $2g+2$ Weierstrass points, shows that m_0 must be at least equal to $2g+1$ (additionally it is checked rather easily that $m_0 = 2g+1$ always answers the question for a curve). It follows from this that m_0 must necessarily depend on the geometry of X, and cannot depend only on the dimension of X. However, when mL is replaced by the "adjoint" line bundle $K_X + mL$, a simple universal answer seems likely to emerge.

16.1. FUJITA'S CONJECTURE ([**Fuj87**]). *If L is an ample line bundle on a projective manifold of dimension n, then*
i) *$K_X + (n+1)L$ is generated by its global sections;*
ii) *$K_X + (n+2)L$ is very ample.*

The bounds predicted by the conjecture are optimal for $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$, since in this case $K_X = \mathcal{O}(-n-1)$. The conjecture is easy to verify in the case of curves (exercise!), and I. Reider [**Rei88**] has solved the conjecture in the affirmative in the case $n = 2$. Ein-Lazarsfeld [**EL93**] and Fujita [**Fuj93**] arrived at establishing part i) in dimension 3, and a very thorough refinement of their technique allowed Kawamata [**Kaw95**] to also arrive at the case of dimension 4[1]. The other cases of the conjecture, namely i) for $n \geq 5$ and ii) for $n \geq 3$, remain for the time being unsolved. The first step in the direction of this conjecture for arbitrary dimension n has been realized in 1991 (work published 2 years later in [**Dem93**]), by means of an analytic method relying on a resolution of a Monge-Ampere equation. Similar results were obtained by Kollár [**Kol92**] employing entirely algebraic

[1]The technique of Fujita [**Fuj93**] and Kawamata [**Kaw95**] has just been simplified considerably and clarified by S. Helmke [**Hel96**].

methods. We refer to [**Laz93**] for an excellent article devoted to the synthesis of these developments, as well as [**Dem94**] for the analytic version of the theory.

This section is devoted to the proof of some results dependent on Kujita's conjecture in arbitrary dimension. The principal ideas of interest here are inspired by some recent work of Y.T. Siu [**Siu96**]. Siu's method, which is naturally algebraic and relatively elementary, consists of combining the Riemann-Roch formula with the Kawamata-Viehmeg vanishing theorem (however, it will be much more convenient to use this Nadel's formulation of the theorem, using the multiplier ideal sheaves). Subsequently, X will denote a projective algebraic manifold of dimension n. The first useful observation is the following classical consequence of the Riemann-Roch formula:

16.2. Particular case of the Riemann-Roch formula. Let $\mathfrak{J} \subset \mathcal{O}_X$ be a coherent sheaf of ideals on X such that the variety of zeros $V(\mathfrak{J})$ is of dimension d (with possibly some components of lower dimension). Let $Y = \sum \lambda_j Y_j$ be the effective algebraic cycle of dimension d associated to the components of dimension d of $V(\mathfrak{J})$ (the multiplicities λ_j taking into account the multiplicity of the length of the ideal $\mathfrak{J}$ along each component). Then, for any line bundle E, the Euler characteristic $\chi(X, \mathcal{O}(E+mL) \otimes \mathcal{O}_X/\mathcal{O}(\mathfrak{J}))$ is a polynomial $P(m)$ of degree d and with leading coefficient $L^d \cdot Y/d!$ $\square$

The second useful fact is an elementary lemma concerning the numerical polynomials (polynomials with rational coefficients, defining a map of $\mathbb{Z}$ into $\mathbb{Z}$).

16.3. Lemma. *Let $P(m)$ be a numerical polynomial of degree $d > 0$ and with leading coefficient $a_d/d!$, $a_d \in \mathbb{Z}$, $a_d > 0$. We assume that $P(m) \geq 0$ for all $m \geq m_0$. Then*

a) *For all $N \geq 0$, there exists $m \in [m_0, m_0 + Nd]$ such that $P(m) \geq N$.*
b) *For all $k \in \mathbb{N}$, there exists $m \in [m_0, m_0 + kd]$ such that $P(m) \geq a_d k^d / 2^{d-1}$.*
c) *For all $N \geq 2d^2$, there exists $m \in [m_0, m_0 + N]$ such that $P(m) \geq N$.*

Proof. a) Each one of the N equations $P(m) = 0,\ P(m) = 1, \ldots, P(m) = N-1$ has at most d roots, therefore there is necessarily an integer $m \in [m_0, m_0+dN]$ which is not a root of these equations.
b) By virtue of Newton's formula for the iterated differences $\Delta P(m) = P(m+1) - P(m)$, we obtain

$$\Delta^d P(m) = \sum_{1 \leq j \leq d} (-1)^j \binom{d}{j} P(m+d-j) = a_d, \quad \forall m \in \mathbb{Z}.$$

Consequently, if $j \in \{0, 2, 4, \ldots, 2\lfloor\, d/2 \rfloor\} \subset [0, d]$ is the even integer realizing the maximum of $P(m_0 + d - j)$ on this finite set, we obtain

$$2^{d-1} P(m_0 + d - j) = \left(\binom{d}{0} + \binom{d}{2} + \cdots \right) P(m_0 + d - j) \geq a_d,$$

whereby we obtain the existence of an integer $m \in [m_0, m_0 + d]$ with $P(m) \geq a_d/2^{d-1}$. The result is therefore proven for $k = 1$. In the general case, we apply this particular result to the polynomial $Q(m) = P(km - (k-1)m_0)$, for which the leading coefficient is $a_d k^d / d!$

c) If $d = 1$, part a) already gives the result. If $d = 2$, a glance at the parabola shows that

$$\max_{m\in[m_0,m_0+N]} P(m) \geq \begin{cases} a_2N^2/8 & \text{if } N \text{ is even,} \\ a_2(N^2-1)/8 & \text{if } N \text{ is odd;} \end{cases}$$

therefore $\max_{m\in[m_0,m_0+N]} P(m) \geq N$ whenever $N \geq 8$. If $d \geq 3$, we apply b) with k equal to the smallest integer satisfying $k^d/2^{d-1} \geq N$, i.e. $k = \lceil 2(N/2)^{1/d} \rceil$, where $\lceil x \rceil \in \mathbb{Z}$ denotes the greater integer. Then

$$kd \leq (2(N/2)^{1/d} + 1)d \leq N$$

so long as $N \geq 2d^2$, as one sees after a short calculation. □

We now apply the Nadel vanishing theorem in an analogous way to that of Siu [**Siu96**], with some simplifications in the technique and some improvements for the bounds. Their method simultaneously gives a simple proof of a fundamental classical result due to Fujita.

16.4. THEOREM (Fujita). *If L is an ample line bundle on a projective manifold X of dimension n, then $K_X + (n+1)L$ is nef.*

Using the theory of Mori and the *"base point free theorem"* ([**Mor82**], [**Kaw84**]), one can show in fact that $K_X + (n+1)L$ is semi-ample, and that there exists a positive integer m such that $m(K_X + (n+1)L)$ is generated by its sections (see [**Kaw85**] and [**Fuj87**]). The proof is based on the observation that $n+1$ is the maximum length of the extremal rays of smooth projective varieties of dimension n. Their proof of (16.4) is different and was obtained at the same time as the proof of th. (16.5) below.

16.5. THEOREM. *Let L be an ample line bundle and let G be a nef line bundle over a projective manifold X of dimension n. Then the following properties hold.*

a) *$2K_X + mL + G$ simultaneously generates the jets of order $s_1, \ldots, s_p \in \mathbb{N}$ at arbitrary points $x_1, \ldots, x_p \in X$, i.e., there exists a surjective map*

$$H^0(X, \mathcal{O}(2K_X + mL + G)) \twoheadrightarrow \bigoplus_{1\leq j\leq p} \mathcal{O}(2K_X + mL + G) \otimes \mathcal{O}_{X,x_j}/\mathfrak{m}_{X,x_j}^{s_j+1},$$

so long as $m \geq 2 + \sum_{1\leq j\leq p} \binom{3n+2s_j-1}{n}$.
In particular $2K_X + mL + G$ is very ample for $m \geq 2 + \binom{3n+1}{n}$.

b) *$2K_X + (n+1)L + G$ simultaneously generates the jets of order $s_1, \ldots, s_p$ at arbitrary points $x_1, \ldots, x_p \in X$ so long as the intersection numbers $L^d \cdot Y$ of L on all the algebraic subsets Y of X of dimension d are such that*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1\leq j\leq p} \binom{3n+2s_j-1}{n}.$$

PROOF. The proofs of (16.4) and (16.5a, b) are completely parallel, that is why we will present them simultaneously (in the case of (16.4), it is simply agreed that $\{x_1, \ldots, x_p\} = \emptyset$). The idea is to find an integer (or a rational number) m_0 and a singular Hermitian metric h_0 on $K_X + m_0L$ for which the curvature current is strictly positive, $\Theta_{h_0} \geq \epsilon\omega$, such that $V(\mathcal{J}(h_0))$ is of dimension 0 and such that the weight φ_0 of h_0 satisfies $\nu(\varphi_0, x_j) \geq n + s_j$ for all j. Since L and G are nefs, 15.18 c) implies that $(m-m_0)L+G$ has for all $m \geq m_0$ a metric h' for which the curvature $\Theta_{h'}$ has an arbitrarily small negative part, say $\Theta_{h'} \geq -\frac{\epsilon}{2}\omega$. Then $\Theta_{h_0}+\Theta_{h'} \geq \frac{\epsilon}{2}\omega$ is positive definite. An application of Cor. 15.9 to $F = K_X+mL+G = (K_X+m_0L)+((m-m_0)L+G)$ with metric $h_0 \otimes h'$ guarantees the existence of sections of $K_X+F = 2K_X+mL+G$ producing the desired jets for $m \geq m_0$.

Fix an embedding $\Phi_{|\mu L|}: X \to \mathbb{P}^N, \mu \gg 0$, given by the sections $\lambda_0,\ldots,\lambda_N \in H^0(X,\mu L)$, and let h_L be the associated metric on L, with positive definite curvature form $\omega = \Theta(L)$. To obtain the desired metric h_0 on K_X+m_0L, one fixes an integer $a \in \mathbb{N}^*$ and one uses a process of double induction to construct singular metrics $(h_{k,\nu})_{\nu\geq 1}$ on aK_X+b_kL, for a decreasing sequence of positive integers $b_1 \geq b_2 \geq \cdots \geq b_k \geq \cdots$. Such a sequence is necessarily stationary and m_0 will be precisely the stationary limit $m_0 = \lim b_k/a$. The metrics $h_{k,\nu}$ are chosen to be the type that satisfy the following properties:

α) $h_{k,\nu}$ is an “algebraic” metric of the form

$$||\xi||^2_{h_{k,\nu}} = \frac{|\tau_k(\xi)|^2}{(\sum_{1\leq i\leq \nu, 0\leq j\leq N} |\tau_k^{(a+1)\mu}(\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i})|^2)^{1/(a+1)\mu}},$$

defined by the sections $\sigma_i \in H^0\big(X, \mathcal{O}((a+1)K_X+m_iL)\big)$, $m_i < \frac{a+1}{a}b_k$, $1 \leq i \leq \nu$, where $\xi \mapsto \tau_k(\xi)$ is an arbitrary local trivialization of aK_X+b_kL. Observe that $\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i}$ is a section of

$$a\mu((a+1)K_X+m_iL)+((a+1)b_k-am_i)\mu L = (a+1)\mu(aK_X+b_kL).$$

β) $\mathrm{ord}_{x_j}(\sigma_i) \geq (a+1)(n+s_j)$ for all i,j;

γ) $\mathcal{J}(h_{k,\nu+1}) \supset \mathcal{J}(h_{k,\nu})$ and $\mathcal{J}(h_{k,\nu+1}) \neq \mathcal{J}(h_{k,\nu})$ as long as the variety of zeros $V(\mathcal{J}(h_{k,\nu}))$ is positive dimensional.

The weight $\varphi_{k,\nu} = \frac{1}{2(a+1)\mu}\log\sum\left|\tau_k^{(a+1)\mu}(\sigma_i^{a\mu}\cdot\lambda_j^{(a+1)b_k-am_i})\right|^2$ of $h_{k,\nu}$ is plurisubharmonic and the condition $m_i < \frac{a+1}{a}b_k$ implies $(a+1)b_k-am_i \geq 1$, therefore the difference $\varphi_{k,\nu} - \frac{1}{2(a+1)\mu}\log\sum|\tau(\lambda_j)|^2$ is also plurisubharmonic. Consequently $\frac{\mathrm{i}}{2\pi}\Theta_{h_{k,\nu}}(aK_X+b_kL) = \frac{\mathrm{i}}{\pi}d'd''\varphi_{k,\nu} \geq \frac{1}{(a+1)}\omega$. Moreover, condition β) clearly implies that $\nu(\varphi_{k,\nu},x_j) \geq a(n+s_j)$. Finally, condition γ) combined with the strong Noetherian property of coherent sheaves guarantees that the sequence $(h_{k,\nu})_{\nu\geq 1}$ will eventually produce a subscheme $V(\mathcal{J}(h_{k,\nu}))$ of dimension 0. One can check that the sequence $(h_{k,\nu})_{\nu\geq 1}$ terminates at this point, and we set $h_k = h_{k,\nu}$ to be the final metric thus reached, such that $\dim V(\mathcal{J}(h_k)) = 0$.

For $k=1$, it is clear that the desired metrics $(h_{1,\nu})_{\nu\geq 1}$ exist if b_1 is chosen large enough. (For example, such that $(a+1)K_X+(b_1-1)L$ generates the jets of order $(a+1)(n+\max s_j)$ at every point. Then the sections $\sigma_1,\ldots,\sigma_\nu$ can be chosen such that $m_1 = \cdots = m_\nu = b_1-1$.) We assume that the metrics $(h_{k,\nu})_{\nu\geq 1}$ and h_k are already constructed, and proceed with the construction of $(h_{k+1,\nu})_{\nu\geq 1}$. We use again induction on ν, and assume that $h_{k+1,\nu}$ is already constructed and that $\dim V(\mathcal{J}(h_{k+1,\nu})) > 0$. We begin our induction with $\nu = 0$, and let us declare in this case that $\mathcal{J}(h_{k+1,0}) = 0$ (this corresponds to an infinite metric of weight identically equal to $-\infty$). By virtue of the Nadel vanishing theorem applied to $F_m = aK_X+mL = (aK_X+b_kL)+(m-b_k)L$ for the metric $h_k \otimes (h_L)^{\otimes m-b_k}$, we obtain

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{J}(h_k)) = 0 \quad \text{for } q \geq 1, m \geq b_k.$$

Since $V(\mathcal{J}(h_k))$ is of dimension 0, the sheaf $\mathcal{O}_X/\mathcal{J}(h_k)$ is a skyscraper sheaf and the exact sequence $0 \to \mathcal{J}(h_k) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}(h_k) \to 0$ twisted by the invertible sheaf $\mathcal{O}((a+1)K_X + mL)$ shows that

$$H^q(X, \mathcal{O}((a+1)K_X + mL)) = 0 \quad \text{for } q \geq 1,\ m \geq b_k.$$

Analogously, we find

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{J}(h_{k+1,\nu})) = 0 \quad \text{for } q \geq 1,\ m \geq b_{k+1}$$

(it is therefore true for $\nu = 0$, since $\mathcal{J}(h_{k+1,0}) = 0$), and when

$$m \geq \max(b_k, b_{k+1}) = b_k,$$

the exact sequence $0 \to \mathcal{J}(h_{k+1,\nu}) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}(h_{k+1}, \nu) \to 0$ implies

$$H^q(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{J}(h_{k+1,\nu})) = 0 \quad \text{for } q \geq 1,\ m \geq b_k.$$

In particular, since the group H^1 above is zero, any section u' of $(a+1)K_X + mL$ on the sub-scheme $V(\mathcal{J}(h_{k+1,\nu}))$ has an extension u to X. Fix a basis $u'_1, \ldots, u'_N$ of sections of this sheaf on $V(\mathcal{J}(h_{k+1,\nu}))$ and take arbitrary extensions $u_1, \ldots, u_N$ to X. Consider the linear map allotting to each section u on X the collection of jets of order $(a+1)(n+s_j) - 1$ at the points x_j, i.e.

$$u = \sum_{1 \leq j \leq N} a_j u_j \mapsto \bigoplus J_{x_j}^{(a+1)(n+s_j)-1}(u).$$

Since the rank of the bundle of s-jets is $\binom{n+s}{n}$, the target space is of dimension

$$\delta = \sum_{1 \leq j \leq p} \binom{n + (a+1)(n+s_j) - 1}{n}.$$

To obtain a section $\sigma_{\nu+1} = u$ satisfying condition β) and having a non-trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{J}(h_{k+1,\nu}))$, we need at least $N = \delta + 1$ independent sections $u'_1, \ldots, u'_N$. This condition will be realized by applying Lemma (16.3) to the numerical polynomial

$$\begin{aligned} P(m) &= \chi(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{J}(h_{k+1,\nu})) \\ &= h^0(X, \mathcal{O}((a+1)K_X + mL) \otimes \mathcal{O}_X/\mathcal{J}(h_{k+1,\nu})) \geq 0, \quad m \geq b_k. \end{aligned}$$

The polynomial P is of degree $d = \dim V(\mathcal{J}(h_{k+1,\nu})) > 0$. We therefore obtain the existence of an integer $m \in [b_k, b_k + \eta]$ such that $N = P(m) \geq \delta + 1$, for some explicit integer $\eta \in \mathbb{N}$. (For example, $\eta = n(\delta + 1)$ is always appropriate according to (16.3 a), but it will be equally important to use the other possibilities to optimize the choices.) We then find a section $\sigma_{\nu+1} \in H^0(X, (a+1)K_X + mL)$ having a non-trivial restriction $\sigma'_{\nu+1}$ to $V(\mathcal{J}(h_{k+1,\nu}))$, vanishing to order $\geq (a+1)(n+s_j)$ at each point x_j. Now set $m_{\nu+1} = m$, and the condition $m_{\nu+1} < \frac{a+1}{a} b_{k+1}$ is realized if $b_k + \eta < \frac{a+1}{a} b_{k+1}$. This shows that one can choose recursively

$$b_{k+1} = \left\lfloor \frac{a}{a+1}(b_k + \eta) \right\rfloor + 1.$$

By definition, $h_{k+1,\nu} \leq h_{k+1,\nu}$, therefore $\mathcal{J}(h_{k+1,\nu+1}) \supset \mathcal{J}(h_{k+1,\nu})$. It is the case that $\mathcal{J}(h_{k+1,\nu+1}) \neq \mathcal{J}(h_{k+1,\nu})$, because $\mathcal{J}(h_{k+1,\nu+1})$ contains the sheaf of ideals associated to the divisor of zeros of $\sigma_{\nu+1}$, whereas $\sigma_{\nu+1}$ is not identically zero on $V(\mathcal{J}(h_{k+1,\nu}))$. Now, an easy calculation shows that the iterated sequence $b_{k+1} = \lfloor \frac{a}{a+1}(b_k + \eta) \rfloor + 1$ stabilizes to the limit value $b_k = a(\eta + 1) + 1$, for any initial value b_1 greater than this limit. In this way, we obtain a metric h_∞ with positive definite curvature on $aK_X + (a(\eta+1)+1)L$, such that $\dim V(\mathcal{J}(h_\infty)) = 0$ and $\nu(\varphi_\infty, x_j) \geq a(n + s_j)$ at each point x_j.

Proof of (16.4). In this case, the set $\{x_j\}$ is taken to be the empty set, therefore $\delta = 0$. By virtue of (16.3 a), the condition $P(m) \geq 1$ is realized for at least one integer $m \in [b_k, b_k + n]$, therefore one can take $\eta = n$. Since μL is very ample, μL has a metric having an isolated logarithmic pole of Lelong number 1 at each given point (for example, the algebraic metric defined by the sections of μL vanishing at x_0). Therefore

$$F'_a = aK_X + (a(n+1)+1)L + n\mu L$$

has a metric h'_a such that $V(\mathcal{J}(h'_a))$ is of dimension zero and contains $\{x_0\}$. By virtue of Cor. (15.9), we conclude that

$$K_X + F'_a = (a+1)K_X + (a(n+1)+1+n\mu)L$$

is generated by its sections, in particular $K_X + \frac{a(n+1)+1+n\mu}{a+1}L$ is nef. By letting a tend to $+\infty$, we deduce that $K_X + (n+1)L$ is nef. □

Proof of (16.5 a). It suffices here to choose $a = 1$. Then

$$\delta = \sum_{1 \leq j \leq p} \binom{3n + 2s_j - 1}{n}.$$

If $\{x_j\} \neq \emptyset$, one has $\delta + 1 \geq \binom{3n-1}{n} + 1 \geq 2n^2$ for $n \geq 2$. Lemma (16.3 c) shows that $P(m) \geq \delta + 1$ for at least one $m \in [b_k, b_k + \eta]$ with $\eta = \delta + 1$. We begin the induction procedure $k \mapsto k+1$ with $b_1 = \eta + 1 = \delta + 2$, because the only necessary property for the induction step is the vanishing property

$$H^q(X, 2K_X + mL) = 0 \quad \text{for } q \geq 1,\ m \geq b_1,$$

which is realized according to Kodaira's vanishing theorem and the ampleness property of $K_X + b_1 L$. (We use here the result of Fujita (16.4), by observing that $b_1 > n+1$.) The recursive formula $b_{k+1} = \lfloor \frac{1}{2}(b_k+\eta) \rfloor + 1$ then gives $b_k = \eta+1 = \delta+2$ for all k, and (16.5 a) follows. □

Proof of (16.5 b). Completely similar to (16.5 a), except that we choose $\eta = n$, $a = 1$ and $b_k = n + 1$ for all k. By applying Lemma (16.3 b), we have $P(m) \geq a_d k^d / 2^{d-1}$ for at least one integer $m \in [m_0, m_0 + kd]$, where $a_d > 0$ is the leading degree coefficient of P. By virtue of Lemma (16.2), we have $a_d \geq \inf_{\dim Y = d} L^d \cdot Y$. Take $k = \lfloor n/d \rfloor$. The condition $P(m) \geq \delta + 1$ can then be realized for an integer $m \in [m_0, m_0 + kd] \subset [m_0, m_0 + n]$, provided that

$$\inf_{\dim Y = d} L^d \cdot Y \lfloor n/d \rfloor^d / 2^{d-1} > \delta,$$

that which is equivalent to the condition in (16.5 b). □

The big disadvantage of the described technique is that one must necessarily utilize multiples of L to avoid the zeros of the Hilbert polynomial, in particular it is not possible to directly obtain a criterion of large ampleness for $2K_X + L$ in the statement of (16.5 b). Such a criterion can nevertheless be obtained with the aid of the following elementary lemma.

16.6. LEMMA. *Suppose that there exists an integer $\mu \in \mathbb{N}^*$ such that μF simultaneously generates all the jets of order $\mu(n + s_j) + 1$ at every point x_j of a subset $\{x_1, \dots, x_p\} \subset X$. Then $K_X + F$ simultaneously generates all the jets of order s_j at the point x_j.*

PROOF. Choose the algebraic metric on F defined by a basis $\sigma_1, \dots, \sigma_N$ of the space of sections of μF which vanish to order $\mu(n + s_j) + 1$ at each point x_j. Since we are still free to choose the homogenous term of degree $\mu(n + s_j) + 1$ in the Taylor expansion of these sections at the points x_j, we see that $x_1, \dots, x_p$ are isolated zeros of $\cap \sigma_j^{-1}(0)$. If φ is the weight of the metric of F about x_j, we therefore have $\varphi(z) \sim (n + s_j + \frac{1}{\mu}) \log |z - x_j|$ in suitable coordinates. Replace φ in a neighbourhood of x_j by

$$\varphi'(z) = \max\big(\varphi(z), |z|^2 - C + (n + s_j) \log |z - x_j|\big)$$

and we leave φ unchanged everywhere else (this is possible by taking $C > 0$ sufficiently large). Then $\varphi'(z) = |z|^2 - C + (n + s_j) \log |z - x_j|$ in a neighbourhood of x_j, in particular φ' is strictly plurisubharmonic near x_j. In this way, we obtain a metric h' on F with semi-positive curvature everywhere on X, and has positive definite curvature in a neighbourhood of $\{x_1, \dots, x_p\}$. The resulting conclusion then is a direct application of the L^2 estimates (14.2). $\square$

16.7. THEOREM. *Let X be a projective manifold of dimension n and L an ample line bundle on X. Then $2K_X + L$ simultaneously generates the jets of order $s_1, \dots, s_p$ at arbitrary points $x_1, \dots, x_p \in X$ so long as the intersection numbers $L^d \cdot Y$ of L on all the algebraic subsets $Y \subset X$ of dimension d satisfy*

$$L^d \cdot Y > \frac{2^{d-1}}{\lfloor n/d \rfloor^d} \sum_{1 \le j \le p} \binom{(n+1)(4n + 2s_j + 1) - 2}{n}, \quad 1 \le d \le n.$$

PROOF. Lemma (16.6) applied with $F = K_X + L$ and $\mu = n + 1$ shows that the desired property for the jets of $2K_X + L$ occurs if $(n + 1)(K_X + L)$ generates the jets of order $(n + 1)(n + s_j) + 1$ at the points x_j. Lemma (16.6) applied again with $F = pK_X + (n + 1)L$ and $\mu = 1$ shows by descending induction on p that it suffices that F generates all the jets of order $(n+1)(n+s_j)+1+(n+1-p)(n+1)$ at the points x_j. In particular, for $2K_X + (n + 1)L$ it suffices to obtain all the jets of order $(n + 1)(2n + s_j - 1) + 1$. Th. (16.5 b) then gives the desired condition. $\square$

We conclude by mentioning some immediate consequences of th. 16.5, obtained by taking $L = \pm K_X$.

16.8. COROLLARY. *Let X be a projective manifold of general type, with K_X ample and* $\dim X = n$. *Then mK_X is very ample for $m \ge m_0 = \binom{3n+1}{n} + 4$.*

16.9. COROLLARY. *Let X be a Fano variety (that is, a projective manifold such that $-K_X$ is ample), of dimension n. Then $-mK_X$ is very ample for $m \ge m_0 = \binom{3n+1}{n}$.*

17. An effective version of Matsusaka's big theorem

We encounter here the problem of finding an explicit integer m_0 such that mL is very ample for $m \geq m_0$. The existence of such a bound m_0, depending only on the dimension and the coefficients of the Hilbert polynomial of L, was first established by Matsusaka [**Mat72**]. Further Kollár and Matsusaka [**KoM83**] have shown that one could indeed find a bound $m_0 = m_0(n, L^n, K_X \cdot L^{n-1})$ dependent only on $n = \dim X$ and on the first two coefficients. Recently, Siu [**Siu93**] has obtained an effective version of the same result furnishing an explicit "reasonable" bound m_0 (although this bound is unfortunately still far from being optimal). We explain here the method of Siu, starting from some simplifications and improvements suggested in [**Dem96**]. The starting point is the following lemma.

17.1. LEMMA. *Let F and G be nef line bundles on X. If $F^n > nF^{n-1} \cdot G$, then any positive multiple $k(F-G)$ admits a non-trivial section for $k \geq k_0$ sufficiently large.*

PROOF. The lemma can be proven as a special case of the holomorphic Morse inequalities (see [**Dem85**], [**Tra91**], [**Siu93**], [**Ang95**]). We give here a simple proof, following a suggestion of F. Catanese. We can assume that F and G are very ample (if not, it suffices to replace F and G by $F' = pF + A$ and $G' = pG + A$ with A very ample and sufficiently positive to ensure large ampleness of any sum with an nef bundle, then to choose $p > 0$ large enough for which F' and G' satisfy the same numerical hypothesis as F and G). Then $\mathcal{O}(k(F-G)) \simeq \mathcal{O}(kF - G_1 - \cdots - G_k)$ for arbitrary elements $G_1, \ldots, G_k$ of the linear system $|G|$. If we choose such elements G_j in general position, the lemma follows from the Riemann-Roch formula applied to the restriction morphism $H^0(X, \mathcal{O}(kF)) \to \bigoplus H^0(G_j, \mathcal{O}(kF_{\restriction G_j}))$. □

17.2. COROLLARY. *Let F and G be nef line bundles over X. If F is big and if $m > nF^{n-1} \cdot G / F^n$, then $\mathcal{O}(mF - G)$ can be given a (possibly singular) Hermitian metric h, having a positive definite curvature form, i.e. such that $\Theta_h(mF - G) \geq \epsilon\omega$, $\epsilon > 0$, for a Kähler metric ω.*

PROOF. In fact, if A is ample and $\epsilon \in \mathbb{Q}_+$ is small enough, Lemma (17.1) implies that a certain multiple $k(mF - G - \epsilon A)$ admits a section. Let E be the divisor of this section and let $\omega = \Theta(A) \in c_1(A)$ be a Kähler metric representing the curvature form of A. Then $mF - G \equiv \epsilon A + \frac{1}{k}E$ can be given a singular metric h with curvature form $\Theta_h(mF - G) = \epsilon\Theta(A) + \frac{1}{k}[E] \geq \epsilon\omega$. □

We now consider the problem of obtaining a non-trivial section of mL. The idea of [**Siu93**] is to obtain a more general criterion for the ampleness of $mL - B$ when B is nef. In this way, we will be able to subtract from mL any undesired multiple of K_X that would be added to L, by application of the Nadel Vanishing Theorem (for this, we simply replace B, by B plus a multiple of $K_X + (n+1)L$).

17.3. PROPOSITION. *Let L be an ample line bundle on a projective manifold X of dimension n, and let B be an nef line bundle on X. Then $K_X + mL - B$ admits a non-zero section for an integer m satisfying*

$$m \leq n \frac{L^{n-1} \cdot B}{L^n} + n + 1.$$

PROOF. Let m_0 be the smaller integer $> n\frac{L^{n-1}\cdot B}{L^n}$. Then $m_0L - B$ can be given a singular Hermitian metric h with positive definite curvature. By virtue of the Nadel vanishing theorem, we obtain

$$H^q(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{J}(h)) = 0 \quad \text{for } q \geq 1,$$

therefore $P(m) = h^0(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{J}(h))$ is a polynomial for $m \geq m_0$. Since P is a polynomial of degree n which is not identically zero, there exists an integer $m \in [m_0, m_0 + n]$ which is not a root. Therefore there exists a non-trivial section of

$$H^0(X, \mathcal{O}(K_X + mL - B)) \supset H^0(X, \mathcal{O}(K_X + mL - B) \otimes \mathcal{J}(h))$$

for some $m \in [m_0, m_0 + n]$, as stated. □

17.4. COROLLARY. *If L is ample and B is nef, then $mL - B$ has a non-zero section for at least one integer*

$$m \leq n\Big(\frac{L^{n-1}\cdot B + L^{n-1}\cdot K_X}{L^n} + n + 1\Big).$$

PROOF. According to the result of Fujita (16.4), $K_X + (n+1)L$ is nef. We can therefore replace B by $B + K_X + (n+1)L$ in Prop. (17.3). Corollary (17.4) follows. □

17.5. REMARK. We do not know if the bound obtained in the above corollary is optimal, but it is certainly not very far from being it. Indeed, even for $B = 0$, the multiplicative factor n cannot be replaced by a number smaller than $n/2$. To see this, take for example for X a product $C_1 \times \cdots \times C_n$ of curves C_j of large enough genus g_j, and $L = \mathcal{O}(a_1[p_1]) \otimes \cdots \otimes \mathcal{O}(a_n[p_n])$, $B = 0$. Our sufficient condition so that $|mL| \neq \emptyset$ becomes in this case $m \leq \sum(2g_j - 2)/a_j + n(n+1)$, while for a generic choice of p_j the bundle mL admits sections only if $ma_j \geq g_j$ for all j. The inaccuracy of our inequality thus plays more on one multiplicative factor 2 when $a_1 = \cdots = a_n = 1$ and $g_1 \gg g_2 \gg \cdots \gg g_n \to +\infty$. In addition, the additive constant $n+1$ is already the best possible when $B = 0$ and $X = \mathbb{P}^n$. □

Up to this point, the method was not really sensitive to the presence of singularities (Lemma (17.1) is still true in the singular case as is easily seen by passing to a desingularization of X). In the same way, as we observed with remark (15.16), the Nadel vanishing theorem still remains essentially valid. Prop. (17.3) can then be generalized as follows:

17.6. PROPOSITION. *Let L be an ample line bundle on a projective manifold X of dimension n, and let B be an nef line bundle on X. For any (reduced) algebraic subvariety Y of X of dimension p, there exists an integer*

$$m \leq p\frac{L^{p-1}\cdot B\cdot Y}{L^p\cdot Y} + p + 1$$

such that the sheaf $\omega_Y \otimes \mathcal{O}_Y(mL - B)$ has a non-zero section. □

By applying a suitable induction procedure relying on the results above, we can now improve the effective bound obtained by Siu [**Siu93**] for Matsusaka's big theorem. Our statement will depend on the choice of a constant λ_n such that

$m(K_X+(n+2)L)+G$ is very ample for $m \ge \lambda_n$ and all nef line bundles G. Theorem (0.2 c) shows that $\lambda_n \le \binom{3n+1}{n} - 2n$ (a more elaborate argument concerning the recent results of Angehrn-Siu [**AS94**] allows us in fact to see that $\lambda_n \le n^3-n^2-n-1$ for $n \ge 2$). Of course, one expects with this that $\lambda_n = 1$ for all n, if one believes that the conjecture of Fujita is true.

17.7. Effective version of Matsusaka's Big Theorem. *Let L and B be nef line bundles on a projective manifold X of dimension n. Assume that L is ample and let $H = \lambda_n(K_X + (n+2)L)$. Then $mL - B$ is very ample for*

$$m \ge (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1}\cdot(B+H))^{(3^{n-1}+1)/2}(L^{n-1}\cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}}.$$

In particular mL is very ample for

$$m \ge C_n (L^n)^{3^{n-2}} \left(n+2+\frac{L^{n-1}\cdot K_X}{L^n}\right)^{3^{n-2}(n/2+3/4)+1/4}$$

with $C_n = (2n)^{(3^{n-1}-1)/2}(\lambda_n)^{3^{n-2}(n/2+3/4)+1/4}$.

Proof. We utilize Th. (3.1) and Prop. (17.6) to construct by induction a sequence of algebraic subvarieties (not necessarily irreducible) $X = Y_n \supset Y_{n-1} \supset \cdots \supset Y_2 \supset Y_1$ such that $Y_p = \cup_j Y_{p,j}$ is of dimension p, Y_{p-1} being obtained for each $p \ge 2$ as the union of the set of zeros of the sections

$$\sigma_{p,j} \in H^0(Y_{p,j}, \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B))$$

for suitable integers $m_{p,j} \ge 1$. We proceed by induction on the decreasing values of the dimension p, and we seek to obtain with each step an upper bound m_p for the integer $m_{p,j}$.

By virtue of Cor. (17.4), we can find an integer m_n such that $m_nL - B$ admits a non-trivial section σ_n for

$$m_n \le n\frac{L^{n-1}\cdot(B+K_X+(n+1)L)}{L^n} \le n\frac{L^{n-1}\cdot(B+H)}{L^n}.$$

Now suppose that the sections $\sigma_n, \ldots, \sigma_{p+1,j}$ have already been constructed. One then obtains by induction a p-cycle $\tilde Y_p = \sum \mu_{p,j}Y_{p,j}$ defined by $\tilde Y_p$ = sum of the divisors of zeros of the sections $\sigma_{p+1,j}$ on the components $\tilde Y_{p+1,j}$, where the multiplicity $\mu_{p,j}$ of $Y_{p,j} \subset Y_{p+1,k}$ is obtained by multiplying the corresponding multiplicity $\mu_{p+1,k}$ by the order of vanishing of $\sigma_{p+1,k}$ along $Y_{p,j}$. We obtain the equality of cohomology classes

$$\tilde Y_p \equiv \sum (m_{p+1,k}L - B)\cdot(\mu_{p+1,k}Y_{p+1,k}) \le m_{p+1}L\cdot \tilde Y_{p+1}.$$

By induction, we then obtain the numerical inequality

$$\tilde Y_p \le m_{p+1}\cdots m_n L^{n-p}.$$

Now, for each component $Y_{p,j}$, Prop. (17.6) shows that there exists a section of $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ for a certain integer

$$m_{p,j} \le p\frac{L^{p-1}\cdot B\cdot Y_{p,j}}{L^p\cdot Y_{p,j}} + p + 1 \le pm_{p+1}\cdots m_n L^{n-1}\cdot B + p + 1.$$

We have used here the obvious lower bound $L^{p-1} \cdot Y_{p,q} \geq 1$ (this bound is besides undoubtly one of weak points of the method...). The degree $Y_{p,q}$ by comparison to H admits the upper bound

$$\delta_{p,j} := H^p \cdot Y_{p,j} \leq m_{p+1} \cdots m_n H^p \cdot L^{n-p}.$$

The Hovanski-Teissier concavity inequality gives

$$(L^{n-p} \cdot H^p)^{\frac{1}{p}} (L^n)^{1-\frac{1}{p}} \leq L^{n-1} \cdot H$$

([**Hov79**], [**Tei79, 82**], also see [**Dem93**]), which makes it possible to express our bounds in terms of only the intersection numbers L^n and $L^{n-1} \cdot H$. We then obtain

$$\delta_{p,j} \leq m_{p+1} \cdots m_n \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}}.$$

We have need of the following lemma, which will be proven shortly.

17.8. Lemma. *Let H be a very ample line bundle on a projective algebraic manifold X, and let $Y \subset X$ be an irreducible algebraic subvariety of dimension p. If $\delta = H^p \cdot Y$ is the degree of Y with support in H, the sheaf $\mathcal{H}om\big(\omega_Y, \mathcal{O}_Y((\delta - p - 2)H)\big)$ has a non-trivial section.*

According to Lemma (17.8), there exists a non-trivial section of

$$\mathcal{H}om\big(\omega_{Y_{p,j}}, \mathcal{O}_{Y_{p,j}}((\delta_{p,j} - p - 2)H)\big).$$

By combining this section with the section of $\omega_{Y_{p,j}} \otimes \mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ already constructed, we obtain a section of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B + (\delta_{p,j} - p - 2)H)$ on $Y_{p,j}$. We do not want H appearing at this stage, which is why we will replace B by $B + (\delta_{p,q} - p - 2)H$. We obtain then a section $\sigma_{p,j}$ of $\mathcal{O}_{Y_{p,j}}(m_{p,j}L - B)$ for a certain integer $m_{p,j}$ such that

$$\begin{aligned} m_{p,j} &\leq p m_{p+1} \cdots m_n L^{n-1} \cdot (B + (\delta_{p,j} - p - 2)H) + p + 1 \\ &\leq p m_{p+1} \cdots m_n \delta_{p,j} L^{n-1} \cdot (B + H) \\ &\leq p(m_{p+1} \cdots m_n)^2 \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} L^{n-1} \cdot (B + H). \end{aligned}$$

Consequently, by setting $m = nL^{n-1} \cdot (B + H)$, we obtain the descending inductive relation

$$m_p \leq M \frac{(L^{n-1} \cdot H)^p}{(L^n)^{p-1}} (m_{p+1} \cdots m_n)^2 \quad \text{for } 2 \leq p \leq n-1,$$

on the basis of the initial value $m_n \leq M/L^n$. Let $(\overline{m}_p)$ be the sequence of numbers obtained by this inductive formula by replacing the respective inequalities by equalities. Thus we have $m_p \leq \overline{m}_p$ with $\overline{m}_{n-1} = M^3 (L^{n-1} \cdot H)^{n-1}/(L^n)^n$ and

$$\overline{m}_p = \frac{L^n}{L^{n-1} \cdot H} \overline{m}_{p+1}^2 \overline{m}_{p+1}$$

for $2 \leq p \leq n-2$. Then by induction

$$m_p \leq \overline{m}_p = M^{3^{n-p}} \frac{(L^{n-1} \cdot H)^{3^{n-p-1}(n-3/2)+1/2}}{(L^n)^{3^{n-p-1}(n-1/2)+1/2}}.$$

We now show that $m_0L - B$ is nef for

$$m_0 = \max(m_2, m_3, \ldots, m_n, m_2 \cdots m_n L^{n-1} \cdot B).$$

Indeed, let $C \subset X$ be an arbitrary irreducible curve. Alternatively, $C = Y_{1,j}$ for a certain j, or else there exists an integer $p = 2, \ldots, n$ such that C is contained in $Y_p \backslash Y_{p-1}$. If $C \subset Y_{p,j} \backslash Y_{p-1}$, then $\sigma_{p,j}$ is not identically zero on C. Therefore $(m_{p,j}L - B)_{\restriction C}$ is of positive degree or zero and

$$(m_0L - B) \cdot C \geq (m_{p,j}L - B) \cdot C \geq 0.$$

In addition, if $C = Y_{1,j}$, then

$$(m_0L - B) \cdot C \geq m_0 - B \cdot \tilde{Y}_1 \geq m_0 - m_2 \cdots m_n L^{n-1} \cdot B \geq 0.$$

According to the definition of λ_n (and the proof where such a constant exists, cf. (0.2c)), $H + G$ is very ample for any nef line bundle G, in particular $H + m_0L - B$ is very ample. We again replace B by $B + H$. This substitution has the effect of replacing M by the new constant $m = n(L^{n-1} \cdot (B + 2H))$ and m_0 by

$$m_0 = \max(m_n, m_{n-1}, \ldots, m_2, m_2 \cdots m_n L^{n-1} \cdot (B + H)).$$

The latter term being the largest estimation of $\overline{m}_p$ implies

$$\begin{aligned} m_0 &\leq M^{(3^{n-1}-1)/2} \frac{(L^{n-1} \cdot H)^{(3^{n-2}-1)(n-3/2)/2+(n-2)/2} L^{n-1} \cdot (B+H))}{(L^n)^{(3^{n-2}-1)(n-1/2)/2+(n-2)/2+1}} \\ &\leq (2n)^{(3^{n-1}-1)/2} \frac{(L^{n-1} \cdot (B+H))^{(3^{n-1}+1)/2} (L^{n-1} \cdot H)^{3^{n-2}(n/2-3/4)-1/4}}{(L^n)^{3^{n-2}(n/2-1/4)+1/4}} \end{aligned}$$

□

Proof of lemma (17.8). Let $X \subset \mathbb{P}^N$ be the embedding given by H, so that $H = \mathcal{O}_X(1)$. There exists a projective linear map $\mathbb{P}^n \to \mathbb{P}^{p+1}$ for which the restriction $\pi : Y \to \mathbb{P}^{p+1}$ to Y is a finite and birational morphism of Y onto an algebraic hypersurface Y' of degree δ in $\mathbb{P}^{p+1}$. Let $s \in H^0(\mathbb{P}^{p+1}, \mathcal{O}(\delta))$ be the polynomial of degree δ defining Y'. We claim that for any small Stein open subset $W \subset \mathbb{P}^{p+1}$ and any holomorphic p-form u, L^2 on $Y' \cap W$, there exists a holomorphic $(p+1)$-form $\tilde{u}$, L^2 on W, with values in $\mathcal{O}(\delta)$, such that $\tilde{u}_{\restriction Y' \cap W} = u \wedge ds$. In fact, this is precisely the conclusion of the L^2 extension theorem of Ohsawa-Takegoshi [**OT87**], [**Ohs88**] (also see [**Man93**] for a more general version of this result). One can equally invoke standard arguments in local algebra (see Hartshorne [**Har77**], th. III-7.11). Since $K_{\mathbb{P}^{p+1}} = \mathcal{O}(-p-2)$, the form $\tilde{u}$ can be considered as a section of $\mathcal{O}(\delta - p - 2)$ on W, consequently the morphism of sheaves $u \mapsto u \wedge ds$ extends to a global section of $\mathcal{H}om\big(\omega_{Y'}, \mathcal{O}_{Y'}(\delta - p - 2)\big)$. The inverse image of π^* furnishes a section of $\mathcal{H}om\big(\pi^*\omega_{Y'}, \mathcal{O}_Y((\delta - p - 2)H)\big)$. Since π is finite and generically $1:1$, it is easy to see that $\pi^*\omega_{Y'} = \omega_Y$. The lemma follows. □

17.9. Remark. In the case of surfaces ($n = 2$), we can take $\lambda_n = 1$ according to the result of I. Reider [**Rei88**], and the arguments developed above ensure that mL is very ample for

$$m \geq 4 \frac{(L \cdot (K_X + 4L))^2}{L^2}.$$

By working through the proof more carefully, it can be shown that the multiplicative factor 4 can be replaced by 2. In fact, Fernandez del Busto has recently shown that

mL is very ample for

$$m > \frac{1}{2}\left[\frac{(L\cdot(K_X+4L)+1)^2}{L^2}+3\right],$$

and an example of G. Xiao shows that this bound is essentially optimal (see [**FdB94**]).

Matsusaka's big theorem yields a number of other important finiteness results. One of the prototypes of these results is the following statement.

17.10. COROLLARY. *There exists only a finite number of families of deformations of polarized projective manifolds (X, L) of dimension n, where L is an ample line bundle for which the intersection numbers L^n and $K_X \cdot L^{n-1}$ are fixed.*

PROOF. Indeed, since L^n and $K_X \cdot L^{n-1}$ are fixed, there in fact exists a calculable integer m_0 such that $m_0 L$ is very ample. We then obtain an embedding $\Phi = \Phi_{|m_0 L|} : X \to \mathbb{P}^N$ such that $\Phi^*\mathcal{O}(1) = \pm m_0 L$. The image $Y = \Phi(X)$ is of degree

$$\deg(Y) = \int_Y c_1\big(\mathcal{O}(1)\big)^n = \int_X c_1(\pm m_0 L)^n = m_0^n L^n.$$

This implies that Y is a point of one of the components of the Chow scheme of algebraic subvarieties Y of a given dimension and degree in $\mathbb{P}^N$ for which $\mathcal{O}(1)_{\restriction Y}$ is divisible by m_0. More precisely a point of an open set corresponding to a nonsingular subvariety. Since the open set in question is a Zariski open set, it can have only a finite number of irreducible components, whence the corollary. □

We can also show from Matsusaka's Theorem (or even directly from Cor. (16.9)) that there is only a finite number of families of deformations of Fano varieties of a given dimension n. We use for this a fundamental result obtained independently by Kollár-Miyaoka-Mori [**KoMM92**] and Campana [**Cam92**], showing that the discriminant K_X^n is bounded by a constant C_n dependent only on n. The effective bound obtained for very ample line bundles furnishes then (at the expense of some effort!) an effective bound for the number of Fano varieties.

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Frobenius and Hodge Degeneration

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In [D-I], the Hodge degeneration theorem and the Kodaira-Akizuki-Nakano vanishing theorem for smooth projective varieties over a field of characteristic zero are shown by methods of algebraic geometry in characteristic $p > 0$. These present notes will serve as an introduction to the subject, with the intention of keeping the non-specialist in mind (who will be able to also consult the presentation of Oesterlé [**O**]). Thus we will assume known by the reader only some rudiments of the theory of schemes (EGA I 1-4, [**H2**] II 2-3). On the other hand, we require of the reader a certain familiarity with homological algebra. The results of [**D-I**] are expressed simply in the language of derived categories. Although it is possible to avoid there the recourse, see for example [**E-V**], we prefer to place it in its context, which appears more natural. However, to help the beginner, we recall in n°4 the basic definitions and some essential points.

0. Introduction

Let $\mathfrak{X}$ be a complex analytic manifold. By the Poincaré Lemma, the de Rham complex $\Omega^\bullet_{\mathfrak{X}}$ of holomorphic forms on $\mathfrak{X}$ is a resolution of the constant sheaf $\mathbb{C}$. As a result, the augmentation $\mathbb{C} \to \Omega^\bullet_{\mathfrak{X}}$ defines an isomorphism (for all n)

$$H^n(\mathfrak{X}, \mathbb{C}) \xrightarrow{\sim} H^n_{\mathrm{DR}}(\mathfrak{X}) = H^n(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}}), \tag{0.1}$$

where the second term, called the *de Rham cohomology* of $\mathfrak{X}$ (in degree n), is the n-th hypercohomology group of $\mathfrak{X}$ with values in $\Omega^\bullet_{\mathfrak{X}}$. The first spectral sequence of hypercohomology abuts to the de Rham cohomology of $\mathfrak{X}$

$$E_1^{p,q} = H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}}) \Rightarrow H^{p+q}_{\mathrm{DR}}(\mathfrak{X}), \tag{0.2}$$

which is called the *Hodge to de Rham spectral sequence* (or *Hodge-Frölicher*) (cf. [**De**] n°9). Let us assume $\mathfrak{X}$ is compact. Then, by the finiteness theorem of Cartan-Serre, the $H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}})$, and therefore all the terms of the spectral sequence (0.2) are finite dimensional $\mathbb{C}$-vector spaces. If we set

$$b_n = \dim H^n_{\mathrm{DR}}(\mathfrak{X}) = \dim H^n(\mathfrak{X}, \mathbb{C})$$

(n-th Betti number of $\mathfrak{X}$) and

$$h^{p,q} = \dim H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}})$$

(Hodge number), we have

$$b_n \leq \sum_{p+q=n} h^{pq}, \tag{0.3}$$

with equality for all n if and only if (0.2) degenerates at E_1. Suppose in addition that $\mathfrak{X}$ is *Kähler*. Then by Hodge theory, the Hodge spectral sequence of $\mathfrak{X}$ degenerates at E_1 : this is the *Hodge degeneration theorem* ([**De**] 9.9). Denote by

$$0 = F^{n+1} \subset F^n \subset \cdots \subset F^p = F^p H^n_{\mathrm{DR}}(\mathfrak{X}) \subset \cdots \subset F^0 = H^n_{\mathrm{DR}}(\mathfrak{X})$$

the resulting filtration of the Hodge spectral sequence (*Hodge filtration*). By degeneration, one has a canonical isomorphism

$$E_1^{p,q} = H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}}) \simeq E_\infty^{pq} = F^p/F^{p+1}. \tag{0.4}$$

We put

$$H^{p,q} = F^p \cap \overline{F}^q,$$

where the bar denotes complex conjugation on $H^n_{\mathrm{DR}}(\mathfrak{X})$, defined by means of (0.1), and the isomorphism $H^n(\mathfrak{X}, \mathbb{C}) \simeq H^n(\mathfrak{X}, \mathbb{R}) \otimes \mathbb{C}$. It follows that

$$H^{p,q} = \overline{H^{q,p}}.$$

Further, Hodge theory furnishes the following results ([**De**] 9.10):
(a) the composite homomorphism

$$H^{p,q} \hookrightarrow F^p H^{p+q}_{\mathrm{DR}}(\mathfrak{X}) \twoheadrightarrow F^p/F^{p+1}$$

is an isomorphism (i.e. $H^{p,q}$ is a complement of F^{p+1} in F^p); whence, by composing with (0.4), determines an isomorphism

$$H^{p,q} \simeq H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}}); \tag{0.5}$$

(b) one has, for all n,

$$H^n_{\mathrm{DR}}(\mathfrak{X}) = \bigoplus_{p+q=n} H^{p,q}, \tag{0.6}$$

(*Hodge decomposition*). These results apply in particular to the complex analytic manifold $\mathfrak{X}$ associated to a smooth projective scheme X over $\mathbb{C}$. The difference between (a) and (b), which is of a transcendental nature, utilizes complex conjugation in an essential way. The Hodge degeneration can in this case be formulated in a purely algebraic manner. The de Rham complex of $\mathfrak{X}$ is indeed the complex of analytic sheaves associated to the algebraic de Rham complex $\Omega^\bullet_X$ of X over $\mathbb{C}$ (a complex of sheaves in the Zariski topology, for which the components are locally free coherent sheaves). The canonical morphism (of ringed spaces) $\mathfrak{X} \to X$ induces homomorphisms on the Hodge and de Rham cohomologies

$$H^q(X, \Omega^p_X) \to H^q(\mathfrak{X}, \Omega^p_{\mathfrak{X}}), \tag{0.7}$$

$$H^n_{\mathrm{DR}}(X) \to H^n_{\mathrm{DR}}(\mathfrak{X}), \tag{0.8}$$

where $H^n_{\mathrm{DR}}(X) = H^n(X, \Omega^\bullet_X)$. We make use of the Hodge to algebraic de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{DR}}(X), \tag{0.9}$$

and a morphism of (0.9) in (0.2) inducing (0.7) and (0.8) respectively on the initial terms and the abutment. By the comparison theorem of Serre [**GAGA**], (0.7) is an isomorphism, and therefore the same holds for (0.8). Consequently, the degeneration at E_1 of (0.2) is equivalent to that of (0.9). In other words, if one sets

$$h^{p,q}(X) = \dim H^q(X, \Omega^p_X), \quad h^n(X) = \dim H^n_{\mathrm{DR}}(X),$$

the Hodge degeneration theorem for $\mathfrak{X}$ is expressed by the (purely algebraic) relation

$$h^n(X) = \sum_{p+q=n} h^{p,q}(X). \tag{0.10}$$

More generally, if X is a smooth and proper scheme over a field k, one can consider the de Rham complex $\Omega^\bullet_{X/k}$ of X over k, and one still has a Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_{X/k}) \Rightarrow H^{p+q}_{\mathrm{DR}}(X/k) \tag{0.11}$$

(where $H^n_{\mathrm{DR}}(X/k) = H^n(X, \Omega^\bullet_{X/k})$), formed of finite-dimensional k-vector spaces. If k is of characteristic zero, the Hodge degeneration theorem implies the degeneration of (0.11) at E_1 : standard techniques (cf. n°6) indeed make it possible to go back initially to $k = \mathbb{C}$, then with the aid of Chow's Lemma and of the resolution of singularities one reduces the proper case to the projective case ([**D0**]). There are those who have long sought for a purely algebraic proof of the degeneration of (0.11) at E_1 for k of characteristic zero. Faltings [**Fa1**] was the first to give a proof of it independent of Hodge theory[2]. A simplification of crystalline techniques due to Ogus [**Og1**], Fontaine-Messing [**F-M**] and Kato [**Ka1**] led, shortly thereafter, to the elementary proof presented in [**D-I**]. We refer to the introduction of [**D-I**] and to [**O**] for a broad overview. We only indicate that the degeneration of (0.11) (for k of characteristic zero) is proven by reduction to the case where k is of characteristic $p > 0$, where, however, it can happen that the degeneration is automatic! This proof is based however on the help of some additional hypothesis on X (upper bound of the dimension, liftability) which is sufficient for our purposes (see 5.6 for a precise statement). We explain in n°6 the well-known technique which allows us to go from characteristic $p > 0$ to characteristic zero. The degeneration theorem in characteristic $p > 0$ to which we have just alluded follows from a decomposition theorem (5.1), relying on some classical properties of differential calculus in characteristic $p > 0$ (Frobenius endomorphism and Cartier isomorphism), which we recall in n°3, after having summarized, in n°1 and 2, the formalism of differentials and smoothness on schemes. The aforementioned decomposition theorem furnishes at the same time an algebraic proof of the Kodaira-Akizuki-Nakano vanishing theorem for the smooth projective varieties over a field of characteristic zero (6.10 and [**De**] 11.7). The last two sections are of a more technical nature: We outline the evolution of the subject since the publication of [**D-I**], and, in the appendix, we describe some complementary results due to Mehta-Srinivas [**Me-Sr**] and Nakkajima [**Na**].

1. Schemes: differentials, the de Rham complex

We recall here the definition and basic properties of differential calculus over schemes. The reader will find a complete treatment in (EGA IV 16.1-16.6); also see [**B-L-R**] 2.1 and [**H2**] II 8 for an introduction.

1.1. We say that a morphism of schemes $i : T_0 \to T$ is a *thickening of order* 1 (or by abuse, that T is a thickening of order 1 of T_0) if i is a closed immersion defined by an ideal of $\mathcal{O}_T$ of square zero. If T and T_0 are affine, with rings A and A_0, such a morphism corresponds to a surjective homomorphism $A \to A_0$ for which the kernel is an ideal of square zero. The schemes T and T_0 have the same underlying space, and the ideal $\mathfrak{a}$ of i, annihilated by $\mathfrak{a}$, is a quasi-coherent $\mathcal{O}_{T_0}$ ($= \mathcal{O}_T/\mathfrak{a}$)-module.

Let $j : X \to Z$ be an immersion, with ideal I (by definition, j is an isomorphism of X onto a closed subscheme $j(X)$ of a larger open subset U of Z, and I is the quasi-coherent sheaf of ideals of U defining $j(X)$ in U, (EGA I 4.1, 4.2)). Let Z_1 be the subscheme[3] of Z, with the same underlying space as X, defined by the ideal I^2. Then j factors (in a unique way) into

$$X \xrightarrow{j_1} Z_1 \xrightarrow{h_1} Z$$

[2]The purists observe that this proof, which rests on the existence of the Hodge-Tate decomposition for p-adic étale cohomology of a smooth and proper variety over a local field of unequal characteristic, is not entirely "algebraic", in the sense of where it uses the comparison theorem of Artin-Grothendieck between étale cohomology and Betti cohomology for smooth and proper varieties over $\mathbb{C}$.

[3]At the expense of some abuse of notation, we will allow ourselves the flexibility of interchanging "immersion" (resp. "closed immersion") and "subscheme" (resp. "closed subscheme"); that amounts here to neglecting the isomorphism of X onto $j(X)$.

where h_1 is an immersion, and j_1 is a thickening of order 1, with ideal I/I^2; one says that (j_1, h_1), or more simply Z_1, is the *first infinitesimal neighbourhood* of j (or of X in Z). The ideal I/I^2 (which is a quasi-coherent $\mathcal{O}_X$-module) is called the *conormal sheaf* of j (or of X in Z). We denote it by $\mathcal{N}_{X/Z}$.

1.2. Let $f : X \to Y$ be a morphism of schemes, and let $\Delta : X \to Z := X \times_Y X$ be the diagonal morphism. This is an immersion (closed if and only if X is separated over Y) (EGA I 5.3). The conormal sheaf of Δ is called the sheaf of *Kähler* 1-*differentials* of f (or of X over Y) and is denote by $\Omega^1_{X/Y}$; we sometimes write $\Omega^1_{X/A}$ instead of $\Omega^1_{X/Y}$ if Y is affine with ring A. Thus we have a quasi-coherent $\mathcal{O}_X$-module, defined by

$$\Omega^1_{X/Y} = I/I^2, \tag{1.2.1}$$

where I is the ideal of Δ. Let $X \xrightarrow{\Delta_1} Z_1 \to Z$ be the first infinitesimal neighbourhood of Δ. The two projections of $Z = X \times_Y X$ on X induce, by composition with $Z_1 \to Z$, two Y-morphisms $p_1, p_2 : Z_1 \to X$, which retract Δ_1. The sheaf of rings of the scheme Z_1, which has the same underlying space as X, is called *the sheaf of principal parts of order* 1 of X over Y, and is denoted by $\mathcal{P}^1_{X/Y}$. We have, by construction, an exact sequence of abelian sheaves

$$0 \to \Omega^1_{X/Y} \to \mathcal{P}^1_{X/Y} \to \mathcal{O}_X \to 0, \tag{1.2.2}$$

split by each of the ring homomorphisms $j_1, j_2 : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$ induced from p_1, p_2. The difference $j_2 - j_1$ is a homomorphism of abelian sheaves of $\mathcal{O}_X$ in $\Omega^1_{X/Y}$, which is called the *differential*, and which is denoted by

$$d_{X/Y} \quad (\text{or } d) : \quad \mathcal{O}_X \to \Omega^1_{X/Y}. \tag{1.2.3}$$

If M is an $\mathcal{O}_X$-module, a Y-*derivation* of $\mathcal{O}_X$ in M is any homomorphism of sheaves of $f^{-1}(\mathcal{O}_Y)$-modules $D : \mathcal{O}_X \to M$ (where f^{-1} denotes the inverse image functor for abelian sheaves) such that

$$D(ab) = aDb + bDa$$

for all local sections a, b of $\mathcal{O}_X$. We denote by $\mathrm{Der}_Y(\mathcal{O}_X, M)$, the set of Y-derivations of $\mathcal{O}_X$ in M, which is in a natural way an abelian group. The differential $d_{X/Y}$ is a Y-derivation of $\mathcal{O}_X$ in $\Omega^1_{X/Y}$. One shows that it is *universal*, in the sense that for any Y-derivation D of $\mathcal{O}_X$ in an $\mathcal{O}_X$-module M (not necessarily quasi-coherent), there exists a unique homomorphism of $\mathcal{O}_X$-modules $u : \Omega^1_{X/Y} \to M$ such that $u \circ d_{X/Y} = D$, i.e. the homomorphism

$$\mathrm{Hom}(\Omega^1_{X/Y}, M) \to \mathrm{Der}_Y(\mathcal{O}_X, M), \quad u \mapsto u \circ d_{X/Y} \tag{1.2.4}$$

is an isomorphism. The sheaf $\mathcal{H}om(\Omega^1_{X/Y}, \mathcal{O}_X)$ is called the *tangent sheaf* of f (or of X over Y), and is denoted by

$$T_{X/Y} \tag{1.2.5}$$

(or sometimes $\Theta_{X/Y}$). For any open subset U of X, (1.2.4) gives an isomorphism $\Gamma(U, T_{X/Y}) \simeq \mathrm{Der}_Y(\mathcal{O}_U, \mathcal{O}_U)$. Recall that one calls a Y-*point* of X a Y-morphism $T \to X$. By definition, $X \times_Y X$ "parameterizes" the set of pairs of

Y-points of X (i.e. represents the corresponding functor on the category of Y-schemes). The geometric significance of the first infinitesimal neighbourhood Z_1 of the diagonal of X over Y is that *it parameterizes the pairs of Y-points of X neighbouring of order* 1 (i.e. congruent modulo an ideal of square zero): More precisely, if $i : T_0 \to T$ is a thickening of order 1, with ideal $\mathfrak{a}$, where T is a Y-scheme, and if $t_1,\ t_2 : T \to X$ are two Y-points of X which coincide modulo $\mathfrak{a}$ (i.e. such that $t_1 i = t_2 i = t_0 : T_0 \to X$), then there exists a unique Y-morphism $h : T \to Z_1$ such that $p_1 h = t_1$ and $p_2 h = t_2$. Moreover, if $t_1^*,\ t_2^* : \mathcal{O}_X \to t_{0*}\mathcal{O}_T$ [4] are the homomorphisms of sheaves of rings associated to t_1 and t_2, $t_2^* - t_1^*$ is a Y-derivation of X with values in $t_{0*}\mathfrak{a}$, such that

$$(t_2^* - t_1^*)(s) = h^*(ds) \tag{1.2.6}$$

for any local section s of $\mathcal{O}_X$, where $h^* : \Omega^1_{X/Y} \to t_0^*\mathfrak{a}$ is the homomorphism of $\mathcal{O}_X$-modules induced by h (on the corresponding conormal sheaves of X in Z_1 and T_0 in T). If f is a morphism of affine schemes, corresponding to a ring homomorphism $A \to B$, then $Z = \operatorname{Spec} B \otimes_A B$, Δ corresponds to the ring homomorphism sending $b_1 \otimes b_2$ onto $b_1 b_2$, with kernel $J = \Gamma(Z, I)$. We have $\Gamma(X, \mathcal{P}^1_{X/Y}) = (B \otimes_A B)/J^2$, and we set

$$\Gamma(X, \Omega^1_{X/Y}) = \Omega^1_{B/A}. \tag{1.2.7}$$

The B-module $\Omega^1_{B/A} = J/J^2$, for which the associated quasi-coherent sheaf is $\Omega^1_{X/Y}$, is called the *module of Kähler* 1*-differentials* of B over A. The map $d = d_{B/A} = \Gamma(X, d_{X/Y}) : B \to \Omega^1_{B/A}$ is an A-derivation, satisfying a universal property that we leave to the reader to formulate. The homomorphisms $j_1, j_2 : B \to (B \otimes_A B)/J^2$ of 1.1 are given by $j_1 b =$ class of $b \otimes 1$, $j_2 b =$ class of $1 \otimes b$. Since J is generated by $1 \otimes b - b \otimes 1$, $\Omega^1_{B/A}$ is generated, as a B-module, by the image of d. It follows from this that if f is any given morphism of schemes, $\Omega^1_{X/Y}$ is generated, as an $\mathcal{O}_X$-module, by the image of d.

1.3. Any commutative square

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{h} & Y \end{array} \tag{1.3.1}$$

defines in a canonical way, a homomorphism of $\mathcal{O}_{X'}$-modules

$$g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}, \tag{1.3.2}$$

which sends $1 \otimes g^{-1}(d_{X/Y} s)$ onto $d_{X'/Y'}(1 \otimes g^{-1}(s))$. (If E is an $\mathcal{O}_X$-module, by definition $g^*E = \mathcal{O}_{X'} \otimes_{g^{-1}(\mathcal{O}_X)} g^{-1}(E)$.) This is an *isomorphism* if the square (1.3.1) is *cartesian*, i.e. if the morphism $X' \to Y' \times_Y X$ is an isomorphism. Moreover, in this case, the canonical homomorphism

$$f'^*\Omega^1_{Y'/Y} \oplus g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y} \tag{1.3.3}$$

is an isomorphism.

[4] Recall that T and T_0 have the same underlying space.

1.4. Let
$$X \xrightarrow{f} Y \xrightarrow{g} S$$
be morphisms of schemes. Then the canonical sequence of homomorphisms

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{1.4.1}$$

is exact.

1.5. Let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

be a commutative triangle, where i is an immersion, with ideal I. The differential $d_{Z/Y}$ induces a homomorphism $d : \mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y}$, and the sequence

$$\mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0 \tag{1.5.1}$$

is exact.

1.6. Let $X = \mathbb{A}^n_Y = Y[T_1, \ldots, T_n]$ be the affine space of dimension n over Y. The $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is free, with basis dT_i $(1 \le i \le n)$. If Y is affine, with ring A, and if $s \in A[T_1, \ldots, T_n]$, then $ds = \sum(\partial s/\partial T_i)dT_i$, where the $\partial s/\partial T_i$ are the usual partial derivatives.

Properties 1.3 to 1.6, for which the verification is completely standard, are fundamental. It is by virtue of these that we can "calculate" the modules of differentials. For more details, see the indicated references above.

1.7. Let $f : X \to Y$ be a morphism of schemes. For $i \in \mathbb{N}$, we denote by
$$\Omega^i_{X/Y} = \Lambda^i\Omega^1_{X/Y}$$
the i-th exterior product of the $\mathcal{O}_X$-module $\Omega^1_{X/Y}$. (It is agreed that $\Omega^0_{X/Y} = \mathcal{O}_X$.) One shows that there exists a unique family of maps $d : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ satisfying the following conditions:

(a) d is a Y-anti-derivation of the exterior algebra $\bigoplus \Omega^i_{X/Y}$, i.e. d is $f^{-1}(\mathcal{O}_Y)$-linear and $d(ab) = da \wedge b + (-1)^i a \wedge db$ for a homogenous of degree i,

(b) $d^2 = 0$,

(c) $da = d_{X/Y}(a)$ for a of degree zero.

The corresponding complex is called the *de Rham complex* of X over Y and is denoted by

$$\Omega^\bullet_{X/Y} \tag{1.7.1}$$

(or $\Omega^\bullet_{X/A}$ if Y is affine with ring A). It depends functorially on f : A square (1.3.1) gives a homomorphism of complexes (which is also a homomorphism of algebras)

$$\Omega^\bullet_{X/Y} \to g_*\Omega^\bullet_{X'/Y'}. \tag{1.7.2}$$

However, one must be aware that even if for each i, the homomorphism $\Omega^i_{X/Y} \to g_*\Omega^i_{X'/Y'}$ is the adjoint of a homomorphism $g^*\Omega^i_{X/Y} \to \Omega^i_{X'/Y'}$, one cannot in general define a *complex* $g^*\Omega^\bullet_{X/Y}$ for which the differential is a Y'-anti-derivation compatible with that of $\Omega^\bullet_{X'/Y'}$.

2. Smoothness and liftings

There are a number of ways of presenting the theory of smooth morphisms. We follow (or rather, summarize) here the presentation of EGA, where smoothness is defined by the existence of infinitesimal liftings (EGA IV 17). In addition to its elegance, this definition has the advantage of transposing itself to other contexts, for example that of the geometric logarithm (cf. [**I6**]). Other points of view are adopted in (SGA 1 II and III), where the emphasis is placed on the notion of an étale morphism, and [**B-L-R**] 2.2, where this is the jacobian criterion (cf. 2.8), which is taken as the starting point.

2.1. Let $f : X \to Y$ be a morphism of schemes. We say that f is *locally of finite type* (resp. *locally of finite presentation*) if, for any point x of X, there exists an affine open neighbourhood U of x and an affine open neighbourhood V of $y = f(x)$ such that $f(U) \subset V$ and that the homomorphism of rings $A \to B$ associated to $U \to V$ makes B an A-algebra of finite type (i.e. a quotient of an algebra of polynomials $A[t_1, \dots, t_n]$) (resp. of finite presentation (i.e. a quotient of an algebra of polynomials $A[t_1, \dots, t_n]$ by an ideal of finite type)). If Y is locally Noetherian, "locally of finite type" is equivalent to "locally of finite presentation", and if it is, then it follows that X is locally Noetherian.

If $f : X \to Y$ is locally of finite presentation, the $\mathcal{O}_X$-module $\Omega^i_{X/Y}$ is of finite type for all i, therefore coherent if Y is locally Noetherian.

2.2. Let $f : X \to Y$ be a morphism of schemes. We say that f is *smooth* (resp. *net* (or *non-ramified*), resp. *étale*) if f is locally of finite presentation and if the following condition is satisfied:

For any commutative diagram

$$\begin{array}{ccc} & & X \\ & g_0 \nearrow & \downarrow f \\ T_0 & \xrightarrow{i} T & \to Y \end{array} \tag{2.2.1}$$

where i is a thickening of order 1 (1.1), there exists, locally in the Zariski topology on T, a (resp. at most one, resp. a unique) Y-morphism $g : T \to X$ such that $gi = g_0$. It follows immediately from the definition that the composite of two smooth morphisms (resp. net, resp. étale) is smooth (resp. net, resp. étale), and that if $f : X \to Y$ is smooth (resp. net, resp. étale), it is the same with the morphism $f' : X' \to Y'$ induced by a base change $Y' \to Y$. If for $i = 1, 2$, $f_i : X_i \to Y$ is smooth (resp. net, resp. étale), the fiber product $f = f_1 \times_Y f_2 : X_1 \times_Y X_2 \to Y$ is therefore smooth (resp. net, resp. étale). Additionally it is immediate that the projection of the affine line $\mathbb{A}^1_Y = Y[t] \to Y$ is smooth, and it is therefore the same for the projection of the space $\mathbb{A}^n_Y \to Y$.

Remarks 2.3. (a) Because of the uniqueness which allows a gluing together, we can omit in the definition of *étale*, *locally in the Zariski topology*. On the other hand, we cannot do it in the definition of *smooth*. There exist a cohomological obstruction that we will later specify, to the existence of a global extension g of g_0.

(b) If n is an integer ≥ 1, we say that a morphism of schemes $i : T_0 \to T$ is a *thickening of order* n if i is a closed immersion defined by an ideal I such that $I^{n+1} = 0$. If T_m denotes the closed subscheme of T defined by I^{m+1}, i itself factors into a sequence of thickenings of order 1 :

$$T_0 \to T_1 \to \cdots \to T_m \to T_{m+1} \to \cdots \to T_n.$$

In Definition 2.2, we can therefore replace *thickening of order* 1 by *thickening of order* n.

The following proposition summarizes the essential properties of differentials associated to smooth morphisms (resp. net, resp. étale).

PROPOSITION 2.4. (a) *If* $f : X \to Y$ *is smooth (resp. net), the* $\mathcal{O}_X$*-module* $\Omega^1_{X/Y}$ *is locally free of finite type (resp. zero).*

(b) *In the situation of* 1.4, *if* f *is smooth, the sequence* (1.4.1) *extended by a zero to the left*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 \tag{2.4.1}$$

is exact and locally split. In particular, if f *is étale, the canonical homomorphism* $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ *is an isomorphism.*

(c) *In the situation of* 1.5, *if* f *is smooth, the sequence* (1.5.1) *extended by a zero to the left*

$$0 \to \mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0 \tag{2.4.2}$$

is exact and locally split. In particular, if f *is étale, the canonical homomorphism* $\mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y}$ *is an isomorphism.*

2.5. The verification of 2.4 is not difficult (EGA IV 17.2.3), but unfortunately somewhat scattered in (EGA 0_{IV} 20). Here is an outline.

The key ingredient is the following. If $f : X \to Y$ is a morphism of schemes and I a quasi-coherent $\mathcal{O}_X$-module, we call a Y*-extension of* X *by* I, a Y-morphism $i : X \to X'$ which is a thickening of order 1 with ideal I. Two Y-extensions $i_1 : X \to X_1$ and $i_2 : X \to X_2$ of X by I are said to be *equivalent* if there exists a Y-isomorphism g of X_1 onto X_2 such that $gi_1 = i_2$ and that g induces the identity on I. An analogous construction to this is the "Baer sum" for extensions of modules over a ring associated to the set

$$\mathrm{Ext}_Y(X, I)$$

of equivalence classes of Y-extensions of X by I with a structure of an abelian group, with neutral element the trivial extension defined by the algebra of dual numbers $\mathcal{O}_X \oplus I$.

Assertion (c) follows immediately from the definition: The smoothness of f indeed implies that the first infinitesimal neighbourhood i_1 of i retracts locally onto X, and the choice of a retraction r permits the splitting (2.4.2) (by the derivation associated to $\mathrm{Id}_{Z_1} - i_1 \circ r$, cf. (1.2.6)).

Assume f is smooth. If I is a quasi-coherent $\mathcal{O}_X$-module and if $i : X \to Z$ is a Y-extension of X by I, the sequence (2.4.2) is therefore an extension of $\mathcal{O}_X$-modules $e(i)$ of $\Omega^1_{X/Y}$ by I. One can show that $i \mapsto e(i)$ gives an isomorphism

$$\mathrm{Ext}_Y(X, I) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, I) \tag{2.5.1}$$

(cf. **[I1]** I, chap. II, 1.1.9. We define an inverse of (2.5.1) by associating to an extension M of $\Omega^1_{X/Y}$ by I, the Y-extension Z of X defined in the following way: Identify, via j_1, the sheaf of principal parts $\mathcal{P}^1_{X/Y}$ (1.2.2) with the ring of dual numbers $\mathcal{O}_X \oplus \Omega^1_{X/Y}$, and denote by $F = \mathcal{O}_X \oplus M$ the ring of dual numbers over M; the extension M makes F an $f^{-1}(\mathcal{O}_Y)$-extension of $\mathcal{P}^1_{X/Y}$ by I. That is, if $E = F \times_{\mathcal{P}^1_{X/Y}} \mathcal{O}_X$ is the "pull-back" of F by the homomorphism $j_2 = j_1 + d_{X/Y} : \mathcal{O}_X \to \mathcal{P}^1_{X/Y}$, then E is a $f^{-1}(\mathcal{O}_Y)$-extension of $\mathcal{O}_X$ by I, which defines the Y-extension Z). Since f is smooth, any Y-extension of X by I is locally trivial, and therefore by virtue of (2.5.1), it follows from this that the sheaf $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, I)$ (associated to the presheaf $U \mapsto \mathrm{Ext}^1_{\mathcal{O}_U}(\Omega^1_{U/Y}, I_{|U})$) is zero, and therefore also that $\underline{\mathrm{Ext}}^1_{\mathcal{O}_U}(\Omega^1_{U/Y}, J) = 0$ for all open subsets U of X and all quasi-coherent $\mathcal{O}_U$-modules J. Since $\Omega^1_{X/Y}$ is of finite type (2.1), it follows that $\Omega^1_{X/Y}$ is locally free of finite type, which proves the part of (a) relative to the smooth case. (The relative part of the net case is immediate: For any Y-scheme X, if $i : X \to Z$ is the trivial Y-extension of X by a quasi-coherent $\mathcal{O}_X$-module I, the set of Y-retractions of Z on X is identified with $\mathrm{Hom}(\Omega^1_{X/Y}, I)$ by $r \mapsto r - r_0$, where r_0 corresponds to the natural injection of $\mathcal{O}_X$ in $\mathcal{O}_X \oplus I$, cf. (1.2.6).) In particular, it follows from (a) and (2.5.1) that if X is an affine scheme and is smooth over Y, we have $\mathrm{Ext}_Y(X, I) = 0$ for any quasi-coherent $\mathcal{O}_X$-module I. Finally, we arrive at (b), by using, for X, Y, S affine, and any given f, the natural exact sequence (EGA 0_{IV} 20.2.3)

$$\text{(2.5.2)} \qquad 0 \to \mathrm{Der}_Y(\mathcal{O}_X, I) \to \mathrm{Der}_S(\mathcal{O}_X, I) \to \mathrm{Der}_S(\mathcal{O}_Y, f_*I) \to$$
$$\xrightarrow{\partial} \mathrm{Ext}_Y(X, I) \to \mathrm{Ext}_S(X, I) \to \mathrm{Ext}_S(Y, f_*I),$$

where the arrows other than ∂ are the obvious arrows of functoriality, and ∂ associates to an S-derivation $D : \mathcal{O}_Y \to f_*I$ the Y-extension defined by the ring of dual numbers $\mathcal{O}_X \oplus I$ and the homomorphism $a \mapsto f^*a + Da$ of $\mathcal{O}_Y$ in $f_*(\mathcal{O}_X \oplus I)$.

Observe that if $f : X \to Y$ is a morphism locally of finite presentation of affine schemes (i.e. corresponding to a homomorphism of rings $A \to B$ making B an A-algebra of finite presentation), then, for that f is smooth, it is necessary and sufficient that for any quasi-coherent $\mathcal{O}_X$-module I, we have $\mathrm{Ext}_Y(X, I) = 0$ (the sufficiency rises from the definition, and the necessity was already noted above).

Assertions 2.4 (b) and (c) have converses, which furnish a very convenient criteria of smoothness. Their verfication is easy, starting from previous considerations.

PROPOSITION 2.6. (a) *In the situation of* 1.4, *assume gf smooth. If the sequence* (2.4.1) *is exact and locally split, then f is smooth. If the canonical homomorphism $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism, then f is étale.*
(b) *In the situation of* 1.5, *assume g smooth. If the sequence* (2.4.2) *is exact and locally split, then f is smooth. If the canonical homomorphism $\mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y}$ is an isomorphism, then f is étale.*

2.7. Let $f : X \to Y$ be a smooth morphism, assume given x a point of X, and denote by $k(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$. Let $s_1, \dots, s_n$ be sections of $\mathcal{O}_X$ in a neighbourhood of x for which the differentials form a basis of $\Omega^1_{X/Y}$ at x, i.e., chosen such that the images $(ds_i)_x$ of ds_i in $\Omega^1_{X/Y,x}$ form a basis of this module over $\mathcal{O}_{X,x}$, or such that the images $(ds_i)_x$ of ds_i in $\Omega^1_{X/Y} \otimes k(x)$ form a basis of this vector space over $k(x)$. Since $\Omega^1_{X/Y}$ is locally free of finite type, there exists an open neighbourhood U of x such that the s_i are defined over U and that the ds_i form a basis of $\Omega^1_{X/Y|U}$. The s_j then define a Y-morphism of U in the affine space of dimension n over Y:

$$s = (s_1, \ldots, s_n) : U \to \mathbb{A}^n_Y = Y[t_1, \ldots, t_n].$$

According to 1.6 and 2.6 (a), s is *étale*. We say that the s_i form a *local coordinate system* of X on Y over U (or, if U is not specified, at x). A smooth morphism is therefore locally composed of an étale morphism and of the projection of a standard affine space.

2.8. Now assume given the situation of 1.5, by assuming g is smooth, and let x be a point of X. According to 2.4 (c) and 2.6 (b), for that f to be smooth in a neighbourhood of x, it is necessary and sufficient that there exists sections $s_1, \ldots, s_r$ of I in a neighbourhood of x, generating I_x and such that the $(ds_i)(x)$ are linearly independent in $\Omega^1_{Z/Y}(x) = \Omega^1_{Z/Y} \otimes k(x)$ (where $k(x)$ is the residue field of $\mathcal{O}_{Z,x}$, which is also that of $\mathcal{O}_{X,x}$). For this reason, 2.6 (b) is referred to as the *jacobian criterion.*

Suppose f is smooth in a neighbourhood of x (or at x, like one says sometimes), and let $s_1, \ldots, s_r$ be sections of I generating I in a neighbourhood of x. Then, for that the s_i defines a *minimal* system of generators of I_x (i.e. induces a basis of $I \otimes k(x) = I_x/\mathfrak{m}_x I_x$, or still forms a basis of $I/I^2 = \mathcal{N}_{X/Z}$ in a neighbourhood of x), it is necessary and sufficient that the $(ds_i)(x)$ are linearly independent in $\Omega^1_{Z/Y}(x)$[5]. Therefore, wherever this is the case, if we supplement the s_i by sections s_j $(r+1 \leq j \leq r+n)$ of $\mathcal{O}_Z$ in a neighbourhood of x such that the $(ds_i)(x)$ $(1 \leq i \leq r+n)$ form a basis of $\Omega^1_{Z/Y}(x)$, then the s_i $(1 \leq i \leq n)$ define an étale Y-morphism s from an open neighbourhood U of x in Z into the affine space $\mathbb{A}^{n+r}_Y$, such that $U \cap X$ is the inverse image of the linear subspace with equations $t_1 = \cdots = t_r = 0$:

$$\begin{array}{ccc} U \cap X & \to & U \\ \downarrow & & \downarrow f \\ \mathbb{A}^n_Y & \to & \mathbb{A}^{n+r}_Y \end{array}$$

In algebraic geometry, this statement plays the role of the *implicit function theorem.*

2.9. Let k be a field and let $f : X \to Y = \operatorname{Spec} k$ be a morphism. Assuming f smooth, then X is *regular* (i.e. for any point x of X, the local ring $\mathcal{O}_{X,x}$ is regular, i.e. its maximal ideal $\mathfrak{m}_x$ can be generated by a regular sequence of parameters); moreover, if x is a *closed* point, $k(x)$ is a *finite separable* extension of k, and the dimension of $\mathcal{O}_{X,x}$ is equal to the dimension $\dim_x X$ of the irreducible component of X containing x and of the rank of $\Omega^1_{X/Y}$ at x. Conversely, if k is perfect, and if X is regular, then f is smooth.

More generally, we have the following criterion, left as an easy verification from 2.7 and 2.8 :

[5] Or still that the sequence (s_i) is $\mathcal{O}_Z$-regular at x, i.e. that the corresponding Kozul complex is a resolution of $\mathcal{O}_X$ in a neighbourhood of x (cf. (SGA 6 VII 1.4) and (EGA IV 17.12.1)).

PROPOSITION 2.10. *Let $f : X \to Y$ be a morphism locally of finite presentation* (2.1). *The following conditions are equivalent :*
(i) *f is smooth;*
(ii) *f is flat and the geometric fibers of f are regular schemes.*

(We say that f is flat if for any point x of X, $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{Y,f(x)}$. A *geometric fiber* of f is the reduced scheme of a fiber $X_y = X \times_Y \operatorname{Spec} k(y)$ of f at a point y by an extension of scalars to an algebraic closure of $k(y)$.) If $f : X \to Y$ is smooth, and x is a point of X, the integer

$$\dim_x(f) := \dim_{k(x)} \Omega^1_{X/Y} \otimes k(x) = \operatorname{rg}_{\mathcal{O}_{X,x}} \Omega^1_{X/Y,x}$$

is called the *relative dimension* of f at x. By the classical theory of dimension (EGA IV 17.10.2), this is the dimension of the irreducible component of the fiber $X_{f(x)}$ containing x. Since $\Omega^1_{X/Y}$ is locally free of finite type, it is a locally constant function of x. It is zero if and only if f is étale, in other words, f is étale if and only if f is locally of finite presentation, flat and net (it is this criterion which is taken as the definition of an étale in (SGA 1 I)).

If f is smooth and of *pure relative dimension* r, i.e. of constant relative dimension equal to the integer r, then the de Rham complex $\Omega^\bullet_{X/Y}$ (1.7.1) is zero in degree $> r$, and $\Omega^i_{X/Y}$ is locally free of rank $\binom{r}{i}$; in particular, $\Omega^r_{X/Y}$ is an invertible $\mathcal{O}_X$-module.

Smooth morphisms occupy a central place in the theory of infinitesimal deformations. The following two propositions summarize this. They are however of a more technical nature than the preceeding statements, and as they will be useful only in the proof of 5.1, we will advise the reader to refer to it at that time there.

PROPOSITION 2.11. *Assume given a diagram* (2.2.1)*, with f smooth. Let I be the ideal of i.*
(a) *There exists an obstruction*

$$c(g_0) \in \operatorname{Ext}^1(g_0^*\Omega^1_{X/Y}, I)$$

for which the vanishing is necessary and sufficient for the existence of a Y-morphism (global) $g : T \to X$ extending g_0 (i.e. such that $gi = g_0$).
(b) *If $c(g_0) = 0$, the set of extensions g of g_0 is an affine space under* $\operatorname{Hom}(g_0^*\Omega^1_{X/Y}, I)$.

Since $\Omega^1_{X/Y}$ is locally free of finite type, there is a canonical isomorphism

$$\operatorname{Ext}^1(g_0^*\Omega^1_{X/Y}, I) \simeq H^1(T_0, \mathcal{H}om(g_0^*\Omega^1_{X/Y}, I)) \tag{2.11.1}$$

(and $\mathcal{H}om(g_0^*\Omega^1_{X/Y}, I) \simeq g_0^*T_{X/Y} \otimes I$, where $T_{X/Y}$ is the tangent sheaf (1.2.5)). Set $G = \mathcal{H}om(g_0^*\Omega^1_{X/Y}, I)$. According to (1.2.6), if U is an open subscheme of T with corresponding U_0 over T_0, two extensions of $g_{0|U_0}$ to U "differ" by a section of G over U_0 (and being given an extension, one can modify it by "adding" a section of G). Since g_0 locally extends by definition of the smoothness of f, we then conclude that the sheaf P over T_0 associating to U_0 the set of extensions of $g_{0|U_0}$ to U, is a *torsor* under G. Assertions (a) and (b) follow from this: $c(g_0)$ is the class of this torsor. More explicitly, if $(U_i)_{i \in E}$ is an open covering of T and g_i an extension of g_0 over U_i, then, over $U_i \cap U_j$, $g_i - g_j$ is a Y-derivation D_{ij} of $\mathcal{O}_X$ with values in

$g_{0\,*}(I_{|U_i \cap U_j})$, i.e. a homomorphism of $\Omega^1_{X/Y}$ into $g_{0\,*}(I_{|U_i \cap U_j})$, i.e. finally a section of G over $U_i \cap U_j$, and the (g_{ij}) form a cocycle, for which the class is $c(g_0)$.

Note that if T (or what amounts to the same T_0) is affine, then

$$H^1(T_0, \mathcal{H}om(g_0^*\Omega^1_{X/Y}, I)) = 0$$

and consequently g_0 admits a global extension to T.

PROPOSITION 2.12. *Assume given $i : Y_0 \to Y$ a thickening of order 1 with ideal I, and $f_0 : X_0 \to Y_0$ a smooth morphism.*

(a) *There exists an obstruction*

$$\omega(f_0) \in \mathrm{Ext}^2(\Omega^1_{X_0/Y_0}, f_0^*I)$$

for which the vanishing is necessary and sufficient for the existence of a smooth lifting X_0 over Y, i.e. by definition, of a smooth Y-scheme X equipped with a Y_0-isomorphism $Y_0 \times_Y X \simeq X_0$[6].

(b) *If $\omega(f_0) = 0$, the set of isomorphism classes of liftings of X_0 over Y is an affine space under* $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*I)$ *(where by definition, if X_1 and X_2 are liftings of X_0, an isomorphism of X_1 onto X_2 is a Y-isomorphism of X_1 on X_2 inducing the identity on X_0).*

(c) *If X is a lifting of X_0 over Y, the group of automorphisms of X (i.e. Y-automorphisms of X inducing the identity on X_0) is naturally identified with* $\mathrm{Hom}(\Omega^1_{X_0/Y_0}, f_0^*I)$.

Since $\Omega^1_{X_0/Y_0}$ is locally free of finite type, there is, for all $i \in \mathbb{Z}$, a canonical isomorphism

$$\mathrm{Ext}^i(\Omega^1_{X_0/Y_0}, f_0^*I) \simeq H^i(X_0, \mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*I)) \tag{2.12.1}$$

(and $\mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*I) \simeq T_{X_0/Y_0} \otimes f_0^*I$). If X_0 is affine, the second term of (2.12.1) is zero for $i \geq 1$, and consequently there exists a lifting of X_0 over Y, and two such liftings are isomorphic.

2.13. Here is an outline of the proof of 2.12. The data of a lifting X is equivalent to that of a cartesian square

$$\begin{array}{ccc} X_0 & \xrightarrow{j} & X \\ f_0 \downarrow & & \downarrow f \\ Y_0 & \xrightarrow{i} & Y, \end{array}$$

with f smooth. let J be the ideal of thickness j. The flatness of f (2.10) implies that the homomorphism $f_0^*I \to J$ induced from this square is an isomorphism. (It is moreover easy to verify that conversely, if X is a Y-extension of X_0 by J such that the corresponding homomorphism $f_0^*I \to J$ is an isomorphism, then X is automatically a lifting of X_0.) Assertion (c) is therefore a particular case of 2.11 (b). The identification consists of associating with an automorphism u of X the "derivation" $u - \mathrm{Id}_X$. Similarly, if X_1 and X_2 are two liftings of X_0, 2.11 (a) implies that X_1 and X_2 are isomorphisms if X_0 is affine, and that the set of isomorphisms

[6] In this section, when we speak of a *lifting* of a smooth Y_0-scheme, it will be implicit, unless mentioned to the contrary, that we are thinking of it as a *smooth* lifting.

of X_1 over X_2 is then an affine space under $\mathrm{Hom}(\Omega^1_{X_0/Y_0}, f_0^*I)$. Assertions (a) and (b) come about formally. The verification of (b) is analogous to that of 2.11: If X_1 and X_2 are two liftings of X_0, the "difference" of their isomorphism classes is the class of the torsor under $\mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*I)$ of the local isomorphisms of X_1 on X_2. (We also observe that the classes of Y-extensions X_1 and X_2 of X_0 by f_0^*I differ by a unique Y_0-extension of X_0 by f_0^*I, and invoke (2.5.1).) Finally, we indicate the construction of the obstruction $\omega(f_0)$, by assuming for simplicity that X_0 is separated. First of all, by the jacobian criterion (2.8), the existence of a global lifting is assured in the case where X_0 and Y_0 are affine, and f_0 is associated to a homomorphism of rings $A_0 \to B_0$, where B_0 is the quotient of an A_0-algebra of polynomials $A_0[t_1, \dots, t_n]$ by the ideal generated by a sequence of elements $(g_1, \dots, g_r)$ such that the dg_i are linearly independent at every point x of X_0 (to arbitrarily lift the g_i). Since (always according to (2.8)) f_0 is locally of the preceding form, we can choose an open affine covering $\mathcal{U} = ((U_i)_0)_{i\in E}$ of X_0, and for each i, a lifting U_i of $(U_i)_0$ over Y. Since X_0 has been assumed separated, each intersection $(U_{ij})_0 = (U_i)_0 \cap (U_j)_0$ is affine, and consequently, we can choose an isomorphism of liftings u_{ij} of $U_{i|(U_{ij})_0}$ over $U_{j|(U_{ij})_0}$. On a triple intersection $(U_{ijk})_0 = (U_i)_0 \cap (U_j)_0 \cap (U_k)_0$, the automorphism $u_{ijk} = u_{ki}^{-1} u_{jk} u_{ij}$ of $U_{i|(U_{ijk})_0}$ differs from the identity by a section

$$c_{ijk} = u^*_{ijk} - \mathrm{Id}$$

of the sheaf $\mathcal{H}om(\Omega^1_{X_0,Y_0}, f_0^*I)$. One verifies that (c_{ijk}) is a 2-cocycle of $\mathcal{U}$ with values in $\mathrm{Hom}(\Omega^1_{X_0,Y_0}, f_0^*I)$, where the class of this cocycle in

$$H^2(X_0, \mathcal{H}om(\Omega^1_{X_0,Y_0}, f_0^*I))$$

does not depend on the choices, and that it vanishes if and only if on a refinement covering, the u_{ij} can be modified in a way in which they glue on the triple intersection, and also define a global lifting X of X_0. This is the stated obstruction.

Remark 2.14. The theory of *gerbes* [**Gi**] and that of the *cotangent complex* [**I1**], one or the other, allows us to get rid of the separation assumption made above, and especially gives a more conceptual proof of 2.12.

3. Frobenius and Cartier isomorphism

The general references for this section are (SGA 5 XV 1) for the definitions and basic properties of Frobenius morphisms, absolute and relative, and [**K1**] 7 for the Cartier isomorphism (cf. also [**I2**] 0 2 and [**D-I**] 1).

In this section, p denotes a fixed prime number.

3.1. We say that a scheme X is *of characteristic* p if $p\mathcal{O}_X = 0$, i.e. if the morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors (necessary in a unique way) through $\operatorname{Spec} \mathbb{F}_p$. If X is a scheme of characteristic p, we define the *absolute Frobenius morphism* of X (or, simply Frobenius endomorphism, if there is no fear of confusion) to be the endomorphism of X which is the identity over the underlying space of X, and the raising to the p-th power on $\mathcal{O}_X$. We denote it by F_X. If X is affine with ring A, F_X corresponds to the Frobenius endomorphism F_A of $A, a \mapsto a^p$. Let $f : X \to Y$ be a morphism of schemes. Then there is a commutative square

(3.1.1)
$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ f\downarrow & & \downarrow f \\ Y & \xrightarrow{F_Y} & Y. \end{array}$$

Denote by $X^{(p)}$ (or X', if there is no ambiguity) the scheme $(Y, F_Y) \times_Y X$ induced from X by the change of base F_Y. The morphism F_X defines a unique Y-morphism $F = F_{X/Y} : X \to X'$, giving rise to a commutative diagram

(3.1.2)
$$\begin{array}{cccc} X & \xrightarrow{F_X} & X' & \to & X \\ & f\searrow & \downarrow & & \downarrow f \\ & & Y & \xrightarrow{F_Y} & Y, \end{array}$$

where the upper composite is F_X and the square is cartesian. We call F the *relative Frobenius* of X over Y. The morphisms of the upper line induce homeomorphisms on the underlying spaces (F_Y is a "universal homeomorphism", i.e. a homeomorphism and the remainder after any change of base). If Y is affine with ring A, and X is the affine space $\mathbb{A}^n_Y = \operatorname{Spec} B$, where $B = A[t_1, \dots, t_n]$, then $X' = \mathbb{A}^n_Y$[7], and the morphisms $F : X \to X'$ and $X' \to X$ correspond respectively to the homomorphisms $t_i \mapsto t_i^p$ and $at_i \mapsto a^p t_i$ $(a \in A)$.

PROPOSITION 3.2. *Let Y be a scheme of characteristic p, and $f : X \to Y$ a smooth morphism of pure relative dimension n (2.10). Then the relative Frobenius $F : X \to X'$ is a finite and flat morphism, and the $\mathcal{O}_{X'}$-algebra $F_*\mathcal{O}_X$ is locally free of rank p^n. In particular, if f is étale, F is an isomorphism, i.e. the square* (3.1.1) *is cartesian.*

We first treat the case where $n = 0$, which requires some commutative algebra: The point is that F is étale, because according to 2.6 (a), an étale Y-morphism between Y-schemes is automatically étale, and that a morphism which is both étale and radical[8] is an open immersion ((SGA 1 I 5.1) or (EGA IV 17.9.1)). Then the case where X is the affine space $\mathbb{A}^n_Y$ is immediate: The monomials $\prod t_i^{m_i}$, with $0 \le m_i < p - 1$ form a basis of $F_*\mathcal{O}_X$ over $\mathcal{O}_{X'}$. The general case is deduced from 2.7.

REMARKS 3.3. (a) Since, according to 2.10, $\Omega^i_{X/Y}$ is locally free over $\mathcal{O}_X$ of rank $\binom{n}{i}$, it follows from 3.2 that $F_*\Omega^i_{X/Y}$ is locally free over $\mathcal{O}_{X'}$ of rank $p^n\binom{n}{i}$.
(b) The statement of 3.2 relative to $n = 0$ admits a converse: If Y is of characteristic p and if X is a Y-scheme such that the relative Frobenius $F_{X/Y}$ is an isomorphism, then X is étale over Y (SGA 5 XV 1 Prop. 2). When Y is the spectrum of a field, this is "Mac Lane's criteria".

[7] It is not true in general that X and X' are isomorphic as Y-schemes, it is the exceptional case here.

[8] A morphism $g : T \to S$ is said to be radical if g is injective and, for any point t of T, with image in S, the residue field extension $k(s) \to k(t)$ is radical.

3.4. Let Y be a scheme of characteristic p and $f : X \to Y$ a morphism. Set $d = d_{X/Y}$(1.2.3). If s is a local section of $\mathcal{O}_X$, one has $d(s^p) = ps^{p-1}ds = 0$. Since $d(s^p) = F_X^*(ds) = F^*(1 \otimes ds)$, it follows that
(a) the canonical homomorphisms (1.3.2) associated to (F_X, F_Y) and F,

$$F_X^* \Omega^1_{X/Y} \to \Omega^1_{X/Y}, \quad F^* \Omega^1_{X'/Y} \to \Omega^1_{X/Y}$$

are zero;
(b) the differential of the complex $F_* \Omega^\bullet_{X/Y}$ is $\mathcal{O}_{X'}$-linear; in particular, the sheaves of cycles Z^i, with boundaries B^i and the cohomology $\mathcal{H}^i = Z^i/B^i$ of the complex $F_* \Omega^\bullet_{X/Y}$ are $\mathcal{O}_{X'}$-modules, and the exterior product acting on the graded $\mathcal{O}_{X'}$-module $\bigoplus Z^i F_* \Omega^\bullet_{X/Y}$ (resp. $\bigoplus \mathcal{H}^i F_* \Omega^\bullet_{X/Y}$) is a graded anti-commutative algebra.

These facts are at the source of miracles of differential calculus in characteristic p. The principal result is the following theorem, due to Cartier [**C**] :

THEOREM 3.5. *Let Y be a scheme of characteristic p and $f : X \to Y$ a morphism.*
(a) *There exists a unique homomorphism of graded $\mathcal{O}_X$-algebras*

$$\gamma : \bigoplus \Omega^i_{X'/Y} \to \bigoplus \mathcal{H}^i F_* \Omega^\bullet_{X/Y},$$

satisfying the following two conditions :

(i) *for $i = 0$, γ is given by the homomorphism $F^* : \mathcal{O}_{X'} \to F_* \mathcal{O}_X$;*

(ii) *for $i = 1$, γ sends $1 \otimes ds$ to the class of $s^{p-1}ds$ in $\mathcal{H}^1 F_* \Omega^\bullet_{X/Y}$ (where $1 \otimes ds$ denotes the image of the section ds of $\Omega^1_{X/Y}$ in $\Omega^1_{X'/Y}$.*
(b) *If f is smooth, γ is an isomorphism.*

In case (b), γ is called the *Cartier isomorphism*, and is denoted by C^{-1}. Its inverse, or the composite

$$\bigoplus Z^i F_* \Omega^\bullet_{X/Y} \to \bigoplus \Omega^i_{X'/Y}$$

of its inverse with the projection of $\bigoplus Z^i$ onto $\bigoplus \mathcal{H}^i$, where Z^i denotes the sheaf of cycles of $F_* \Omega^\bullet_{X/Y}$ in degree i, is denoted by C. It is this latter homomorphism which was initially defined by Cartier, and which we sometimes call the *Cartier operation*. The adopted presentation in 3.5 is due to Grothendieck (handwritten notes), and detailed in [**K1**] 7.

When Y is a perfect scheme, i.e. such that F_Y is an automorphism, for example if Y is the spectrum of a perfect field, one of the most significant cases for applications is this: If f is smooth, C^{-1} gives by composition with the isomorphism

$$\bigoplus \Omega^i_{X/Y} \to \bigoplus (F_Y)_{X\ *} \Omega^i_{X'/Y}$$

(where $(F_Y)_X ; X' \to X$ is the isomorphism induced from F_Y by change of base) an isomorphism

$$C^{-1}_{\text{abs}} : \bigoplus \Omega^i_{X/Y} \to \bigoplus \mathcal{H}^i F_{X\ *} \Omega^\bullet_{X/Y}$$

that we call the *absolute Cartier isomorphism.*

COROLLARY 3.6. *Let Y be a scheme of characteristic p and $f : X \to Y$ a smooth morphism. Then for any i, the sheaves of $\mathcal{O}_{X'}$-modules*

$$F_* \Omega^i_{X/Y}, \ Z^i F_* \Omega^\bullet_{X/Y}, \ B^i F_* \Omega^\bullet_{X/Y}, \ \mathcal{H}^i F_* \Omega^\bullet_{X/Y}$$

are locally free of finite type (where Z^i resp. B^i denotes the sheaf of cycles resp. boundaries in degree i).

Taking into account 3.3 (a) and the exactness of F_*, it suffices to apply 3.5 (b), while proceeding by *descending* induction on i.

We briefly indicate the proof of 3.5, according to [**K1**] 7. For (a), it amounts to the same, taking into account (1.3.2), to construct the composite of γ with the homomorphism $\bigoplus \Omega^i_{X/Y} \to \bigoplus (F_Y)_X{}_* \Omega^1_{X'/Y}$, i.e. a homomorphism of graded $\mathcal{O}_X$-algebras

$$\gamma_{\text{abs}} : \bigoplus \Omega^i_{X/Y} \to \bigoplus \mathcal{H}^i F_X{}_* \Omega^\bullet_{X/Y}$$

satisfying the analogous conditions to (i) and (ii), i.e. given in degree zero by F_X^*, and in degree 1 sending ds to the class of $s^{p-1}ds$. However the map of $\mathcal{O}_X$ in $\mathcal{H}^1 F_X{}_* \Omega^\bullet_{X/Y}$ sending a local section s of $\mathcal{O}_X$ onto the class of $s^{p-1}ds$ is a Y-derivation (this is a result of the identity $p^{-1}((X+Y)^p - X^p - Y^p) = \sum_{0 \leq i \leq p} p^{-1}\binom{p}{i} X^{p-i}Y^i$ in $\mathbb{Z}[X,Y]$). By (1.2.4), it defines the desired homomorphism $(\gamma_{\text{abs}})^1$. Since the exterior algebra $\bigoplus \Omega^i_{X/Y}$ is strictly anti-commutative ("strictly" means to say that the elements of odd degree are of square zero), it is likewise of its sub-quotient $\bigoplus \mathcal{H}^i F_X{}_* \Omega^\bullet_{X/Y}$, and consequently there exists a unique homomorphism of graded algebras γ_{abs} extending the homomorphisms $(\gamma_{\text{abs}})^0 = F_X^*$ and $(\gamma_{\text{abs}})^1$. For (b), one can assume, according to 2.7, that f factors into

$$X \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{h} Y,$$

where h is the canonical projection and g is *étale*. Given the square (3.1.1) relative to g, being cartesian according to 3.2, it is likewise the same of the analogous square with the relative Frobenius to Y

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ g\downarrow & & \downarrow g' \\ Z & \xrightarrow{F} & Z', \end{array} \tag{3.6.1}$$

where one sets for abbreviation $\mathbb{A}^n_Y = Z$. According to 2.4 (b), the homomorphism $g^*\Omega^i_{Z/Y} \to \Omega^i_{X/Y}$ is an isomorphism. The square (3.6.1) being cartesian and F finite, thus furnishes an isomorphism of complexes of $\mathcal{O}_X$-modules

$$g'^* F_* \Omega^\bullet_{Z/Y} \to F_* \Omega^\bullet_{X/Y}. \tag{3.6.2}$$

Since g' is étale, therefore flat, the homomorphism

$$g'^* \mathcal{H}^i F_* \Omega^\bullet_{Z/Y} \to \mathcal{H}^i F_* \Omega^\bullet_{X/Y} \tag{3.6.3}$$

induced from (3.6.2) is an isomorphism. Since on the other hand $g'^*\Omega^i_{Z'/Y} \to \Omega^i_{X'/Y}$ is an isomorphism (g' being étale), it follows (by functoriality of γ) that it suffices to prove (b) for Z. By analogous arguments (extension of scalars and Künneth) one can easy reduce to $Y = \operatorname{Spec} \mathbb{F}_p$ and $n = 1$, i.e. $Z = \operatorname{Spec} \mathbb{F}_p[t]$. Then $Z' = Z$, the monomials $1, t, \ldots, t^{p-1}$ form a basis of $F_*\mathcal{O}_Z$ over $\mathcal{O}_Z$, and since the differential $d : F_*\mathcal{O}_Z \to F_*\Omega^1_Z = (F_*\mathcal{O}_Z)dt$ sends t^i onto $it^{i-1}dt$, one concludes that $\mathcal{H}^0 F_* \Omega^\bullet_{Z/\mathbb{F}_p}$ (resp. $\mathcal{H}^1 F_* \Omega^\bullet_{Z/\mathbb{F}_p}$) is free over $\mathcal{O}_Z$ with basis 1 (resp. $t^{p-1}dt$), and therefore that γ is an isomorphism.

3.7. There is a close link between Cartier isomorphism and Frobenius lifting. This was known by Cartier, and it serves as motivation for its construction. The decomposition and degeneration theorems of [**D-I**] originates from this, see n°5. It consists of the following.

Let $i : T_0 \to T$ be a thickening of order 1 and $g_0 : S_0 \to T_0$ a flat morphism. By *lifting* to a T_0-scheme S_0 over T one extends a *flat* T-scheme over S equipped with a T_0-isomorphism $T_0 \times_T S \simeq S_0$, i.e. a cartesian square

$$\begin{array}{ccc} S_0 & \xrightarrow{j} & S \\ g_0 \downarrow & & \downarrow g \\ T_0 & \xrightarrow{i} & T \end{array}$$

with g flat. If I (resp. J) is the ideal of thickening i (resp. j), the flatness of g implies that the canonical homomorphism $g_0^* I \to J$ is an isomorphism (cf. 2.13).

Take for i the thickening $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathbb{Z}/p^2\mathbb{Z}$, of the ideal generated by p. Let Y_0 be a scheme of characteristic p, and let Y be a lifting of Y_0 over $\mathbb{Z}/p^2\mathbb{Z}$. The ideal of Y_0 in Y is therefore $p\mathcal{O}_Y$, and the flatness of Y over $\mathbb{Z}/p^2\mathbb{Z}$ implies that multiplication by p induces an isomorphism

$$\mathbf{p} : \mathcal{O}_{Y_0} \xrightarrow{\sim} p\mathcal{O}_Y. \tag{3.7.1}$$

Now let $f_0 : X_0 \to Y_0$ be a *smooth* morphism of $\mathbb{F}_p$-schemes. Denote by

$$F_0 : X_0 \to X_0'$$

the Frobenius of X_0 relative to Y_0. Assume given a (smooth) lifting X (resp. X') of X_0 (resp. X_0') over Y and a Y-morphism $F : X \to X'$ lifting F_0, i.e. such that the square

$$\begin{array}{ccc} X_0 & \to & X \\ F_0 \downarrow & & \downarrow F \\ X_0' & \to & X' \end{array}$$

commutes. (We have seen that there exists obstructions to the existence of X, X', and F, cf. 2.11 and 2.12, and that these objects, whenever they exist, are not unique. We will return to this later.)

PROPOSITION 3.8. *Let $f_0 : X_0 \to Y_0$ and $F : X \to X'$ be given as in* 3.7. *Then:*

(a) *multiplication by p induces an isomorphism*

$$\mathbf{p} : \Omega^1_{X_0/Y_0} \xrightarrow{\sim} p\Omega^1_{X/Y}.$$

(b) *the image of the canonical homomorphism*

$$F^* : \Omega^1_{X'/Y} \to F_*\Omega^1_{X/Y}$$

is contained in $pF_*\Omega^1_{X/Y}$.

(c) *Denote by*

$$\varphi_F : \Omega^1_{X_0^p/Y_0} \to F_{0*}\Omega^1_{X_0/Y_0}$$

the homomorphism "induced from F^* by division by p", i.e. the unique homomorphism rendering the square commutative

$$\begin{array}{ccc} \Omega^1_{X'/Y} & \xrightarrow{F^*} & pF_{0\,*}\Omega^1_{X/Y} \\ \downarrow & & \uparrow \mathbf{p} \\ \Omega^1_{X'_0/Y_0} & \to & F_{0\,*}\Omega^1_{X_0/Y_0}\ . \end{array}$$

Then the image of φ_F is contained in the kernel of the differential of the de Rham complex, i.e. in the sheaf of cycles $Z^1F_{0\,*}\Omega^\bullet_{X_0/Y_0}$, and the composite of φ_F with the projection on $\mathcal{H}^1F_{0\,*}\Omega^\bullet_{X_0/Y_0}$ is the Cartier isomorphism C^{-1} in degree 1 *(cf. 3.5).*

Assertion (a) is trivial, (b) follows from 3.4 (a), and (c) is immediate from (a), (b) and the characterization of the Cartier isomorphism. Indeed, if a is a local section of $\mathcal{O}_X$, with reduction a_0 module p, and a' lifts in $\mathcal{O}_{X'}$ the image a'_0 of a_0 in $\mathcal{O}_{X'_0}$, we have

$$F^*a' = a^p + pb$$

for a local section b of $\mathcal{O}_X$. (This is because the reduction modulo p of F^*a' is $F_0^*a'_0 = a_0^p$.) Consequently

$$F^*(d') = pa^{p-1}da + pdb,$$

hence

$$\varphi_F(da'_0) = a_0^{p-1}da_0 + db_0, \tag{3.8.1}$$

for which (c) follows at once.

3.9. Suppose, as is in practice, and one of the more important cases, that Y_0 is the spectrum of a perfect field k of characteristic p. Then the spectrum Y of the ring $W_2(k)$ of Witt vectors of length 2 over k lifts Y_0 over $\mathbb{Z}/p^2\mathbb{Z}$ (besides, due to an isomorphism, it is the unique (flat) lifting of Y_0). Recall that $W_2(k)$ is the set of pairs (a_1, a_2) of elements of k, equipped with addition and multiplication given by

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, S_2(a, b)), \\ (a_1, a_2)(b_1, b_2) &= (a_1b_1, P_2(a, b)), \end{aligned}$$

where

$$\begin{aligned} S_2(a, b) &= a_2 + b_2 + p^{-1}(a_1^{p-1} + b_1^{p-1} - (a_1 + b_1)^p), \\ P_2(a, b) &= b_1^pa_2 + b_2a_1^p. \end{aligned}$$

The homomorphism $W_2(k) \to k$ is given by $(a_1, a_2) \mapsto a_1$. If $k = \mathbb{F}_p$, then $W_2(k) \simeq \mathbb{Z}/p^2\mathbb{Z}$, the isomorphism being given by $(a_1, a_2) \mapsto \tau(a_1) + p\tau(a_2)$, where τ denotes the multiplicative section of $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{F}_p$. (For an overall discussion of the theory of Witt vectors, see [**S**] II 6, [**D-G**] V.)

In this case, if X_0 is a smooth Y_0-scheme (i.e. a smooth k-scheme), and since the absolute Frobenius of Y_0 is an automorphism, lifting X_0 over $Y = \operatorname{Spec} W_2(k)$ is equivalent to lifting X'_0, and according to 2.12, the obstruction to the existence of such a lifting is found in $\mathrm{Ext}^2(\Omega^1_{X_0}, \mathcal{O}_{X_0}) \simeq H^2(X_0, T_{X_0})$ [9]. If this obstruction is zero, one can choose a lifting X' of X'_0 and a lifting X of X_0, and then the obstruction to a lifting $F : X \to X'$ of the relative Frobenius F_0 is found in $\mathrm{Ext}^1(F_0^*\Omega^1_{X'_0}, \mathcal{O}_{X_0}) \simeq \mathrm{Ext}^1(\Omega^1_{X'_0}, F_{0\,*}\mathcal{O}_{X_0})$ (2.11)[10]. In every case, these two obstructions are locally zero, and even as soon as X_0 is *affine*. The choice of a lifting F furnishes then, according to 3.8, a relatively explicit description of the Cartier isomorphism in degree 1 (and therefore in every degree, by multiplicativity).

4. Derived categories and spectral sequences

There are many reference sources on this subject at various levels. The reader with pressing obligations can consult [**I3**], which can be used as an introduction and contains a broad bibliography. We will limit ourselves here by recalling some fundamental points which we will use in the following section.

4.1. Let A be an abelian category (in practice, A will be the category of $\mathcal{O}_X$-modules of a scheme X). We denote by $C(A)$ the category of A-complexes, with differential of degree 1, and further denote by $L^\bullet$ (or L) for such a complex

$$\cdots \to L^i \to L^{i+1} \to \cdots .$$

We say that L is *with lower bounded degree* (resp. *upper*, resp. with *bounded degree*) if $L^i = 0$ for i sufficiently small (resp. sufficiently large, resp. outside of a bounded interval of $\mathbb{Z}$). We denote by $Z^iL = \mathrm{Ker}\, d : L^i \to L^{i+1}$, $B^iL = \mathrm{Im}\ d : L^{i-1} \to L^i$, $H^iL = Z^iL/B^iL$, respectively the objects of cycles, boundaries and cohomology in degree i. If A is the category of $\mathcal{O}_X$-modules, we write $C(X)$ in place of $C(A)$, and often $\mathcal{H}^iL$ instead of H^iL for an object of $C(X)$ (in order to indicate that it acts on the cohomology *sheaf* in degree i, and not on the global cohomology group $H^i(X, L)$).

For $n \in \mathbb{Z}$, the *naive truncation* $L^{\leq n}$ (resp. $L^{\geq n}$) of a complex L is the quotient (resp. the subcomplex) of L which coincides with L in degree $\leq n$ (resp. $\geq n$) and has zero components elsewhere. The *canonical truncation* $\tau_{\leq n}L$ (resp. $\tau_{\geq n}L$) is the subcomplex (resp. quotient) of L with components L^i for $i < n$, Z^iL for $i = n$ and 0 for $i > n$ (resp: L^i for $i > n$, L^i/B^iL for $i = n$ and 0 for $i < n$). One sets $\tau_{<n}L = \tau_{\leq n-1}L$. The inclusion $\tau_{\leq n}L \hookrightarrow L$ induces an isomorphism on H^i for $i \leq n$. The projection $L \twoheadrightarrow \tau_{\geq n}L$ induces an isomorphism on H^i for $i \geq n$. For $n \in \mathbb{Z}$, the *translate* $L[n]$ of a complex L is the complex with components $L[n]^i = L^{n+i}$ and with differential $d_{L[n]} = (-1)^n d_L$. A complex L is said to be *concentrated in degree* r (resp. *in the interval* $[a, b]$) if $L^i = 0$ for $i \neq r$ (resp. $i \notin [a, b]$). An object E of A is often considered as a complex concentrated in degree zero. The complex $E[-n]$ is then concentrated in degree n, with component E in this degree.

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[9] We omit here, for abbreviation, $/Y_0$ in the notation of differentials.

[10] One can show ([**Me-Sr**] Appendix) that the obstruction to a choice of (X, X', F) such that X' is the inverse image of X by the Frobenius automorphism of $W_2(k)$ is found in $\mathrm{Ext}^1(\Omega_{X'_0}, B^1F_*\Omega^\bullet_{X_0})$; more precisely, such a triplet (X, X', F) exists if and only if the extension class

$$0 \to B^1F_*\Omega^\bullet_{X_0} \to Z^1F_*\Omega^\bullet_{X_0} \xrightarrow{C} \Omega^1_{X'_0} \to 0$$

(particular case $i = 1$ of the Cartier isomorphism 3.5) is zero. See [**Sr**] for an application to another proof of the principal theorem of [**D-I**].

A homomorphism of complexes $u : L \to M$ is called a *quasi-isomorphism* if $H^i u$ is an isomorphism for all i. We say that a complex K is *acyclic* if $H^i K = 0$ for all i.

If $u : L \to M$ is a homomorphism of complexes, the *cone* $N = C(u)$ of u is the complex defined by $N^i = L^{i+1} \oplus M^i$, with differential $d(x, y) = (-d_L x, ux + d_M y)$. For that u to be a quasi-isomorphism, it is necessary and sufficient that $C(u)$ is acyclic.

4.2. Denote by $K(A)$ the category of complexes of A up to homotopy, i.e. the category having the same objects as $C(A)$ but for which the set of arrows of L in M is the set of homotopy classes of morphisms of L into M. The *derived category* of A, denoted by $D(A)$, is the category induced from $K(A)$ by formally reversing the (homotopy classes of) quasi-isomorphisms: The quasi-isomorphisms of $K(A)$ become isomorphisms in $D(A)$ and $D(A)$ is universal for this property. When A is the category of O_X-modules over a ringed space X, we write $D(X)$ instead of $D(A)$. The categories $K(A)$ and $D(A)$ are additive categories, and one has canonical additive functors

$$C(A) \to K(A) \to D(A).$$

The category $D(A)$ *has the same objects* as $C(A)$. Its arrows are calculated "by fractions" from those of $K(A)$: An arrow $u : L \to M$ of $D(A)$ is defined by a couple of arrows of $C(A)$ of the type

$$L \xleftarrow{s} L' \xrightarrow{f} M \quad \text{or} \quad L \xrightarrow{g} M' \xleftarrow{t} M,$$

where s and t are quasi-isomorphisms. More precisely, one shows that the homotopy classes of quasi-isomorphisms with source M (resp. target L) form a filtered category[11] (resp. the opposite of a filtered category) and that one has

$$\mathrm{Hom}_{D(A)}(L, M) = \varinjlim_{t: M \to M'} \mathrm{Hom}_{K(A)}(L, M') = \varinjlim_{s: L' \to L} \mathrm{Hom}_{K(A)}(L', M)$$

as t (resp. s) runs over the preceeding category (resp. its opposite). If L, M are complexes, we set, for $i \in \mathbb{Z}$,

$$\mathrm{Ext}^i(L, M) = \mathrm{Hom}_{D(A)}(L, M[i]) = \mathrm{Hom}_{D(A)}(L[-i], M).$$

The functors H^i and the *canonical truncation* functors $\tau_{\leq i}$, $\tau_{\geq i}$ on $C(A)$ naturally extend to $D(A)$. On the other hand, it is not the same as the naive truncation functors.

4.3. We denote by $D^+(A)$ (resp. $D^-(A)$, resp. $D^b(A)$) the full subcategory of $D(A)$ formed from complexes L *cohomologically bounded below* (resp. *above*, resp. *bounded*), i.e. such that $H^i L = 0$ for i small enough (resp. large enough, resp. outside a bounded interval). If A contains sufficiently many injectives (i.e. if any object of A embeds in an injective), for example if A is the category of $\mathcal{O}_X$-modules over a scheme X, then any object of $D^+(A)$ is isomorphic to a complex, with bounded below degree, formed from injectives, and the category $D^+(A)$ is equivalent to the full subcategory of $K(A)$ formed from such complexes.

[11] A category I is said to be *filtered* if it satisfies the following conditions (a) and (b):
(a) For any two arrows $f, g : i \to j$, there exists an arrow $h : j \to k$ such that $hf = hg$.
(b) Assume given any objects i and j, there exists an object k and arrows $f : i \to k$, $g : j \to k$.

4.4. The categories $K(A)$ and $D(A)$ are not in general abelian, but possess a *triangle category* structure, in the sense of Verdier [**V**]. This structure is defined by the family of *distinguished triangles*. A *triangle* is a sequence of arrows $T = (L \to M \to N \to L[1])$ of $K(A)$ (resp. $D(A)$). A *morphism* of T in $T' = (L' \to M' \to N' \to L'[1])$ is a triplet $(u : L \to L',\ v : M \to M',\ w : N \to N')$ such that the three squares formed with $u,\ v,\ w,\ u[1]$ commute. A triangle is said to be *distinguished* if it is isomorphic to a triangle of the form

$$L \xrightarrow{u} M \xrightarrow{i} C(u) \xrightarrow{p} L[1],$$

where u is the cone of a morphism of complexes u, and i (resp. p) denotes the obvious inclusion (resp. the *opposite* of the projection). Any short exact sequence of complexes $0 \to E \xrightarrow{u} F \to G \to 0$ defines a distinguished triangle $D(A)$, by means of the natural quasi-isomorphism $C(u) \to G$, and any distinguished triangle of $D(A)$ is isomorphic to a triangle of this type.

Any distinguished triangle $T = (L \to M \to N \to L[1])$ of $D(A)$ gives rise to a long exact sequence

$$\begin{aligned}
&\cdots \to H^iL \to H^iM \to H^iN \xrightarrow{d} H^{i+1}L \to \cdots\ ,\\
&\cdots \to \mathrm{Ext}^i(E, L) \to \mathrm{Ext}^i(E, M) \to \mathrm{Ext}^i(E, N) \to \mathrm{Ext}^{i+1}(E, L) \to \cdots\ ,\\
&\cdots \to \mathrm{Ext}^i(N, E) \to \mathrm{Ext}^i(M, E) \to \mathrm{Ext}^i(L, E) \to \mathrm{Ext}^{i+1}(N, E) \to \cdots\ ,
\end{aligned}$$

for $E \in \mathrm{ob}\ D(A)$. If the triangle T is associated to a short exact sequence given explicitly above, the operator d of the first of these sequences is the usual boundary operator (this is the reason for the convention of sign in the definition of p).

4.5. Let L be a complex of A and $i \in \mathbb{Z}$. The quotient $\tau_{\leq i}L/\tau_{\leq i-1}L$ is mapped quasi-isomorphically onto $H^iL[-i]$. Therefore there is a canonical distinguished triangle from $D(A)$

$$\tau_{\leq i-1}L \to \tau_{\leq i}L \to H^iL[-i] \to \tau_{\leq i-1}L[1].$$

We similarly define a canonical distinguished triangle

$$H^{i-1}L[-i+1] \to \tau_{\geq i-1}L \to \tau_{\geq i}L \to H^{i-1}L[-i+2].$$

Finally,

$$\tau_{[i-1,i]}L := \tau_{\geq i-1}\tau_{\leq i}L = \tau_{\leq i}\tau_{\geq i-1}L = (0 \to L^{i-1}/B^iL \to Z^iL \to 0)$$

defines a distinguished triangle

$$H^{i-1}L[-i+1] \to \tau_{[i-1,i]}L \to H^iL[-i] \to,$$

which furnishes a canonical element

$$c_i \in \mathrm{Ext}^2(H^iL, H^{i-1}L). \tag{4.5.1}$$

The triplet $(H^{i-1}L,\ H^iL,\ c_i)$ is an invariant of L in $D(A)$. It permits its reconstruction up to an isomorphism if L is cohomologically concentrated in degree $i-1$ and i. One can show that the c_i universally realizes the differential d_2 of the spectral sequences of derived functors applied to L (cf. Verdier's theorem[12], or [**D3**]).

[12] Which should be appearing soon in Astérique.

4.6. Let L be an object of $D^b(A)$. We say that L is *decomposable* if L is isomorphic, in $D(A)$, to a complex with zero differential. If L is decomposable, and if $u : L' \to L$ is an isomorphism of $D(A)$, with L' having zero differential, then u induces isomorphisms $L'^i \to H^i L$. In particular L' has bounded degree and $L' = \bigoplus L'^i[-i]$ (in $C(A)$) (4.1), therefore

$$L \simeq \bigoplus H^i L[-i] \tag{4.6.1}$$

(in $D(A)$). Conversely, if L satisfies (4.6.1), L is trivially decomposable. If L is decomposable, one calls a *decomposition* of L the choice of an isomorphism (4.6.1) *inducing the identity* on H^i for all i. There exists a finite sequence of obstructions to the decomposability of L: The first are the classes c_i (4.5.1); if the c_i are zero, there are secondary obstructions in $\operatorname{Ext}^3(H^i L, H^{i-2} L)$, etc. In addition, if L is decomposable, L admits in general many decompositions.

In the following section, we are especially interested in the case when L is concentrated in degree 0 and 1 : $L = (L^0 \to L^1)$. In this case:

a) the class $c_1 \in \operatorname{Ext}^2(H^1 L, H^0 L)$ is the obstruction to the decomposability of L;
b) the giving of a decomposition of L is equivalent to that of a morphism $H^1 L[-1] \to L$ inducing the identity on H^1;
c) The set of decompositions of L is an affine space under $\operatorname{Ext}^1(H^1 L, H^0 L)$ ([**D-I**] 3.1).

4.7. We now return for example to [**H1**], II for the definition of the derived functors $\overset{L}{\otimes}$, $R\mathcal{H}om$[13], $R\operatorname{Hom}$, Rf_*, Lf^*, $R\Gamma$ in the derived category $D(X)$, where X is a variable scheme, and the description of certain remarkable relations between these functors. We need only recall that these functors are, compared to each argument, *exact* functors, i.e. transform distinguished triangles to distinguished triangles, and are "calculated" in the following way:

(a) For $E \in \operatorname{ob} D(X)$, $F \in \operatorname{ob} D^-(X)$, $E \overset{L}{\otimes} F \simeq E \otimes F'$ if $F \simeq F'$ in $D(X)$, with F' having upper bounded degree (4.1) and with flat components. For given F, there exists a quasi-isomorphism $F' \to F$ with F' of the preceding type; moreover the homotopy classes of such quasi-isomorphisms form a coinitial system (in the category of classes of quasi-isomorphisms with target F, cf. 4.2).
(b) For $E \in \operatorname{ob} D(X)$, $F \in \operatorname{ob} D^+(X)$, if $F \simeq F'$, with F' having lower bounded degree and with injective components, then $R\mathcal{H}om(E,F) \simeq \mathcal{H}om^\bullet(E,F')$ and $R\operatorname{Hom}(E,F) \simeq \operatorname{Hom}^\bullet(E,F')$. For given F, there exists a quasi-isomorphism $F \to F'$ with F' of the preceding type (and the homotopy classes of such quasi-isomorphisms form a cofinal system).
(c) For $f : X \to Y$ and $E \in \operatorname{ob} D^+(X)$, if $E \simeq E'$, with E' having lower bounded degree and with flasque components (for example, injective), then $Rf_* E \simeq f_* E'$ and $R\Gamma(X,E) \simeq \Gamma(X,E')$. One simply writes $H^i(X,E)$ instead of $H^i R\Gamma(X,E)$; and more generally, one defines in the same way, $Rf_* : D^+(X, f^{-1}(\mathcal{O}_Y)) \to D^+(Y)$, where $D(X, f^{-1}(\mathcal{O}_Y))$ denotes the derived category of the category of complexes of $f^{-1}(\mathcal{O}_Y)$-modules (the de Rham complex $\Omega^\bullet_{X/Y}$ is such a complex).
(d) For $f : X \to Y$ and $F \in \operatorname{ob} D^-(Y)$, $Lf^* F \simeq f^* F'$ if $F \simeq F'$, with F' having upper bounded degrees and with flat components.

[13] An error of sign slipped into the definition of the complex $\operatorname{Hom}^\bullet(L,M)$ in [**H1**] p. 64: For $u \in \operatorname{Hom}(L^i, M^{i+n})$, it necessarily reads $du = d \circ u + (-1)^{n+1} u \circ d$.

4.8. It can be said that spectral sequences are perhaps one of the most avoided objects in mathematics, and yet at the same time, are one of the most useful algebraic tools for cohomology. This is particularly true of derived categories, which sometimes contributes to this, but they remain essential. There are many references, the oldest ([**C-E**], XV) being one of the best. In these notes, we will be especially interested in the *spectral sequence* called *the Hodge to de Rham*, for which we will recall the definition.

Let $T : A \to B$ be an additive functor between abelian categories. Assume that A has sufficiently many injectives. Then T admits a right derived functor

$$RT : D^+(A) \to D^+(B),$$

which is calculated by $RT(K) \simeq T(K')$ if $K \to K'$ is a quasi-isomorphism with K' with bounded below degree and with injective components. The objects of cohomology $H^i \circ RT : D^+(A) \to B$ are denoted by R^iT. For $K \in \mathrm{ob}\ D(A)$, with bounded below degree, there is a spectral sequence

$$E_1^{ij} = R^jT(K^i) \Rightarrow R^*T(K), \tag{4.8.1}$$

called *the first spectral sequence of hypercohomology* of T. It is obtained in the following way: Chooses a resolution $K \to L$ of K by a bicomplex L, such that each column $L^{i\bullet}$ is an injective resolution of K^i. If sL denotes the associated simple complex, the resulting homomorphism of complexes $K \to sL$ is a quasi-isomorphism, therefore $RT(K) \simeq T(sL) = sT(L)$, $RT(K^i) \simeq T(L^{i\bullet})$, and the filtration of $sT(L)$ by the *first* degree of L given rise to (4.8.1).

Let K be a field and X a k-scheme. The group (cf. (1.7.1) and 4.7 (c))

$$H^i_{\mathrm{DR}}(X/k) = H^i(X, \Omega^\bullet_{X/k}) = \Gamma(\operatorname{Spec} k, R^i f_*(\Omega^\bullet_{X/k})) \tag{4.8.2}$$

(where $f : X \to \operatorname{Spec} k$ is the structure morphism) is called i-th *de Rham cohomology group* of X/k. This is a k-vector space. The spectral sequence (4.8.1) relative to the functor $\Gamma(X, \bullet)$ and the complex $\Omega^\bullet_{X/k}$ is called *the Hodge to de Rham spectral sequence* of X/k :

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^*_{\mathrm{DR}}(X/k). \tag{4.8.3}$$

This is a spectral sequence of k-vector spaces. The groups $H^j(X, \Omega^i_{X/k})$ are called *the Hodge cohomology groups* of X over k. If X is *proper* over k ([**H2**] II 4) (for example, projective over k, i.e. a closed subscheme of a projective space $\mathbb{P}^n_k$), and since the $\Omega^i_{X/k}$ are coherent sheaves (2.1), the finiteness theorem of Serre-Grothendieck ([**H2**] III 5.2 in the projective case, (EGA III 3) in the general case) implies that the Hodge cohomology groups of X over k are *finite dimensional* k-vector spaces. By the spectral sequence (4.8.3), it follows from this that the de Rham cohomology groups $H^n_{\mathrm{DR}}(X/k)$ are also finite dimensional over k. Moreover, for each n, one has

$$\sum_{i+j=n} \dim_k H^j(X, \Omega^i_{X/k}) \geq \dim_k H^n_{\mathrm{DR}}(X/k), \tag{4.8.4}$$

with equality for all n if and only if the Hodge to de Rham spectral sequence of X over k degenerates at E_1, i.e. the differential d_r is zero for all $r \geq 1$.

5. Decomposition, degeneration and vanishing theorems in characteristic $p > 0$

In this section, as in n°3, p will denote a fixed prime number.
The main result is the following theorem ([**D-I**] 2.1, 3.7):

THEOREM 5.1. *Let S be a scheme of characteristic p. Assume given a (flat) lifting T of S over $\mathbb{Z}/p^2\mathbb{Z}$* (3.7). *Let X be a smooth S-scheme, and let us denote as in* 3.1, *$F : X \to X'$ the relative Frobenius of X/S. Then if X' admits a (smooth) lifting over T, the complex of $\mathcal{O}_{X'}$-modules $\tau_{<p}F_*\Omega^\bullet_{X/S}$* (4.1) *is decomposable in the derived category $D(X')$ of $\mathcal{O}_{X'}$-modules* (4.6).

5.2. Before beginning the proof, note that a decomposition of $\tau_{<p}F_*\Omega^\bullet_{X/S}$ is equivalent to giving an arrow of $D(X')$

$$\bigoplus_{i<p} \mathcal{H}^i F_*\Omega^\bullet_{X/S}[-i] \to F_*\Omega^\bullet_{X/S}$$

inducing the identity on $\mathcal{H}^i$ for all $i < p$. According to Cartier's theorem (3.5), this data is still equivalent to that of an arrow of $D(X')$

$$\varphi : \bigoplus_{i<p} \Omega^i_{X'/S}[-i] \to F_*\Omega^\bullet_{X/S} \tag{5.2.1}$$

inducing C^{-1} on $\mathcal{H}^i$ for all $i < p$. The proof in fact consists of associating canonically such an arrow φ to each lifting of X' over T. It includes three steps.

Step A. We start by treating the case where F admits a global lifting.

PROPOSITION 5.3. *Under the hypothesis of* 5.1, *assume that $F : X \to X'$ admits a global lifting $G : Z \to Z'$, where Z (resp. Z') lifts X (resp. X') over T. Let*

$$\varphi_G : \bigoplus \Omega^i_{X'/S}[-i] \to F_*\Omega^\bullet_{X/S} \tag{5.3.1}$$

be the homomorphism of complexes, with i-th component φ^i_G, defined in the following way:

$$\varphi^0_G = F^* : \mathcal{O}_X \to F_*\mathcal{O}_X; \quad \varphi^1_G : \Omega^1_{X'/S} \to F_*\Omega^1_{X/S}$$

is the homomorphism "G^*/p" defined in 3.8 (c). *For $i \geq 1$, φ^i_G is composed with $\Lambda^i\varphi^1_G$ and of the product $\Lambda^i F_*\Omega^1_{X/S} \to F_*\Omega^i_{X/S}$. Then φ_G is a quasi-isomorphism, inducing the Cartier isomorphism C^{-1} on $\mathcal{H}^i$ for all i.*

This is immediate.

Step B. This is the principal step. We show that the giving of a lifting Z' of X' over T allows us to define a decomposition of $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}$, i.e. a homomorphism

$$\varphi^1_{Z'} : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$$

of $D(X')$ (and not $C(X')$) inducing C^{-1} over $\mathcal{H}^1$. With this intention, we need to compare the homomorphisms φ^1_G of (5.3.1) associated to any other lifting of F with target Z'.

LEMMA 5.4. *To any pair* $(G_1 : Z_1 \to Z', G_2 : Z_2 \to Z')$ *of liftings of* F *is associated canonically a homomorphism*

$$h(G_1, G_2) : \Omega^1_{X'/S} \to F_*\mathcal{O}_X \tag{5.4.1}$$

such that $\varphi^1_{G_2} - \varphi^1_{G_1} = dh(G_1, G_2)$. *If* $G_3 : Z_3 \to Z'$ *is a third lifting of* F, *one has*

$$h(G_1, G_2) + h(G_2, G_3) = h(G_1, G_3). \tag{5.4.2}$$

Let us suppose initially that Z_1 and Z_2 are isomorphic (in the sense of 2.12 (b)). Choose an isomorphism $u : Z_1 \xrightarrow{\sim} Z_2$. Then G_2u and G_1 lift F, i.e. extend to Z_1 the composite $X \xrightarrow{F} Z \hookrightarrow Z'$. Therefore according to 2.11 (b), they differ by a homomorphism h_u of $F^*\Omega^1_{X'/S}$ in $\mathcal{O}_X$, or what amounts to the same, of $\Omega^1_{X'/S}$ in $F_*\mathcal{O}_X$. If v is a second isomorphism of Z_1 onto Z_2, then taking into account 3.4 (a), it follows from 2.11 (b) that u and v differ by a homomorphism "$u - v$" : $\Omega^1_{X/S} \to \mathcal{O}_X$, therefore G_2u and G_2v differ by the composite of "$u - v$" and the homomorphism $F^*\Omega^1_{X'/S} \to \Omega^1_{X/S}$, which is zero, *a fortiori* $G_2u = G_2v$. Therefore h_u does not depend on the choice of u. Since Z_1 and Z_2 are locally isomorphic according to 2.11 (a), we deduce from this a homomorphism (5.4.1) characterized by the property that if u is an isomorphism of Z_1 onto Z_2 over an open subset U of X (recall still that Z_1, Z_2 and X have the same underlying space), the restriction of $h(G_1, G_2)$ to U is the homomorphism h_u, the "difference" between G_1 and G_2u. The formula $\varphi^1_{G_2} - \varphi^1_{G_1} = dh(G_1, G_2)$ follows from the explicit description of φ^1_G given in (3.8.1), and formula (5.4.2) is immediate.

Now fix the lifting Z' of X' over T. According to 2.11 (a) and 2.12 (a), we can choose an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X in such a way that we have for each i, a lifting Z_i of U_i over T and a lifting $G_i : Z_i \to Z'$ of $F_{|U_i}$. We then arrange for each i, a homomorphism of complexes

$$f_i = \varphi^1_{G_i} : \Omega^1_{X'/S|U_i}[-1] \to F_*\Omega^\bullet_{X/S|U_i}$$ [14]

of (5.3.1), and for each pair (i, j), a homomorphism

$$h_{ij} = h(G_{i|U_{ij}}, G_{j|U_{ij}}) : \Omega^1_{X'/S|U_{ij}} \to F_*\Omega^\bullet_{X/S|U_i}$$

of (5.4.1), where $U_{ij} = U_i \cap U_j$. These datum are connected by

$$\begin{aligned} f_j - f_i &= dh_{ij} \quad (\text{on } U_{ij}), \\ h_{ij} + h_{jk} &= h_{ik} \quad (\text{on } U_{ijk} = U_i \cap U_j \cap U_k). \end{aligned}$$

They make it possible to define a homomorphism of complexes of $\mathcal{O}_{X'}$-modules

$$\varphi^1_{Z',(\mathcal{U},(G_i))} : \Omega^1_{X'/S}[-1] \to \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S}),$$

where $\check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S})$ is the simple complex associated to the Čech bicomplex of the covering $\mathcal{U}$ with values in $F_*\Omega^\bullet_{X/S}$. The components of this complex are given by

$$\check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S})^n = \bigoplus_{a+b=n} \check{\mathcal{C}}^b(\mathcal{U}, F_*\Omega^a_{X/S})$$

[14] We identify the underlying spaces of X and X' by means of F (3.1).

with differential $d = d_1 + d_2$, where d_1 is induced by the differential of the de Rham complex and d_2 is, in bidegree (a, b), equal to $(-1)^a(\sum(-1)^i\partial^i)$ (see [**G**] 5.2 or [**H2**] III 4.2). In particular,

$$\check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S})^1 = \check{\mathcal{C}}^1(\mathcal{U}, F_*\mathcal{O}_X) \oplus \check{\mathcal{C}}^0(\mathcal{U}, F_*\Omega^1_{X/S}).$$

The morphism $\varphi^1_{Z',(\mathcal{U},(G_i))}$ is defined as having for components, (φ_1, φ_2) in degree 1, with

$$(\varphi_1\omega)(i,j) = h_{ij}(\omega)_{|U_{ij}}, \quad (\varphi_2\omega)(i) = f_i(\omega)_{|U_i}.$$

Using the fact that the f_i are morphisms of complexes, together with the above formulas connecting the f_i and the h_{ij}, it follows that $\varphi^1_{Z',(\mathcal{U},(G_i))}$ is thus a well-defined morphism of complexes. We also has at our disposal the natural augmentation

$$\epsilon : F_*\Omega^\bullet_{X/S} \to \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S}),$$

which is a quasi-isomorphism, because for any a, the complex $\mathcal{C}(\mathcal{U}, F_*\Omega^a_{X/S})$ is a resolution of $F_*\Omega^a_{X/S}$ (cf. [**Go**] or [**H2**] loc. cit.). We then define

$$\varphi^1_{Z'} : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$$

to be the arrow of $D(X')$ composed with $\varphi^1_{Z',(\mathcal{U},(G_i))}$ and with the inverse of ϵ (4.2). If $(\mathcal{U} = (U_i)_{i\in I},\ (G_i)_{i\in I})$ and $(\mathcal{V} = (V_j)_{j\in J},\ (G_j)_{j\in J})$ are two choices of systems of Frobenius liftings, then by considering the covering $\mathcal{U}\coprod\mathcal{V}$, indexed by $I\coprod J$, formed from the U_i and from V_j, it follows that $\varphi^1_{Z'}$ does not depend on choices (cf. [**D-I**] p. 253). Moreover $\varphi^1_{Z'}$ induces C^{-1} on $\mathcal{H}^1$: The question is indeed local, therefore we can arrange for a global lifting of F, and apply 5.3. This completes step B.

Step C. We again fix a lifting Z' of X', and show how to extend the decomposition of $\tau_{\leq 1}F_*\Omega^\bullet_{X/S}$ defined by $\varphi^i_{Z'}$ $(i = 0, 1)$ to a decomposition of $\tau_{<p}F_*\Omega^\bullet_{X/S}$. We use for this the multiplicative structure of the de Rham complex. From $\varphi^1_{Z'}$ we deduce, for all $i \geq 1$, an arrow of $D(X')$

$$(\varphi^1_{Z'})^{\overset{L}{\otimes} i} = \varphi^1_{Z'}\overset{L}{\otimes}\cdots\overset{L}{\otimes}\varphi^1_{Z'} : (\Omega^1_{X'/S}[-1])^{\overset{L}{\otimes} i} \to (F_*\Omega^\bullet_{X/S})^{\overset{L}{\otimes} i}.$$

Since $\Omega^1_{X'/S}$ is locally free of finite type, we have (4.7 (a))

$$(*) \qquad (\Omega^1_{X'/S}[-1])^{\overset{L}{\otimes} i} \simeq (\Omega^1_{X'/S})^{\otimes i}[-i],$$

and similarly, since the $F_*\Omega^a_{X/S}$ are locally free of finite type (3.3 (a)),

$$(**) \qquad (F_*\Omega^\bullet_{X/S})^{\overset{L}{\otimes} i} \simeq (F_*\Omega^\bullet_{X/S})^{\otimes i}.$$

We then define for $i < p$,

$$\varphi^i_{Z'} : \Omega^i_{X'/S}[-i] \to F_*\Omega^\bullet_{X/S}$$

as the composite (via $(*)$ and $(**)$) of the standard antisymmetrization arrow

$$\Omega^i_{X'/S}[i] \to (\Omega^1_{X'/S})^{\otimes i}[-i], \quad \omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{i!}\sum_{\sigma\in\mathfrak{S}_i}\mathrm{sgn}(\sigma)\omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(i)}$$

(well defined because of the assumption $i < p$), of the arrow $(\varphi^1_{Z'})^{\overset{L}{\otimes} i}$, and of the product arrow $(F_*\Omega^\bullet_{X/S})^{\otimes i} \to F_*\Omega^\bullet_{X/S}$. Since the antisymmetrication arrow is a section of the projection of $(\Omega^1_{X'/S})^{\otimes i}$ onto $\Omega^i_{X'/S}$, the multiplicative property of the Cartier isomorphism results in $\varphi^i_{Z'}$ inducing C^{-1} over $\mathcal{H}^i$, and this completes the proof of the theorem.

Taking into account 3.9, we then deduce:

COROLLARY 5.5. *Let k be a perfect field of characteristic p, and let X be a smooth scheme over $S = \operatorname{Spec} k$. If X is lifted over $T = \operatorname{Spec} W_2(k)$, then $\tau_{<p} F_*\Omega^\bullet_{X/S}$ is decomposable in $D(X')$. Moreover, if X is of dimension $< p$, then $F_*\Omega^\bullet_{X/S}$ is decomposable.*

REMARK 5.5.1. According to 5.3, if X is smooth over $\operatorname{Spec} k$ and if X and F are lifted over $W_2(k)$, then $F_*\Omega^\bullet_{X/S}$ is decomposable (and this is without the assumption of dimension on X). This is the case for example if X is *affine.* On the other hand, if X is proper, it is rare that X admits a lifting over $W_2(k)$ where F is lifted. One can show that if X and F are lifted, then X is *ordinary,* i.e. satisfies $H^j(X, B^i\Omega^\bullet_{X/S}) = 0$ for all (i,j) (cf. 8.6). The notion of an ordinary variety, which makes sense only in non-zero characteristic, was initially introduced for curves and abelian varieties. It intervenes in rather many questions in algebraic geometry. See [**I4**] for an introduction and the references cited there.

COROLLARY 5.6. *Let k be a perfect field of characteristic p, and let X be a smooth and proper k-scheme, of dimension $< p$. If X is lifted over $W_2(k)$, the spectral sequence of Hodge to de Rham* (4.8.3) *of X over k*

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^*_{\mathrm{DR}}(X/k)$$

degenerates at E_1.

By virtue of the compatibility of Ω^i by a change of base (1.3.2), the absolute Frobenius isomorphism $F_S : S \to S$ (where $S = \operatorname{Spec} k$) induces, for all (i,j), an isomorphism $F_S^* H^j(X, \Omega^i_{X/k}) \xrightarrow{\sim} H^j(X', \Omega^i_{X'/k})$, and in particular, we have

$$\dim_k H^j(X, \Omega^i_{X/K}) = \dim_k H^j(X', \Omega^i_{X'/k}).$$

In addition, since $F : X \to X'$ is a homeomorphism, one has canonically, for all n,

$$H^n(X', F_*\Omega^\bullet_{X/k}) \xrightarrow{\sim} H^n(X, \Omega^\bullet_{X/k}) = H^n_{\mathrm{DR}}(X/k).$$

Finally, if X is lifted over $W_2(k)$, a decomposition $\varphi : \bigoplus \Omega^i_{X'/S}[-i] \xrightarrow{\sim} F_*\Omega^\bullet_{X/S}$ of $F_*\Omega^\bullet_{X/S}$ in $D(X')$ induces, for all n, an isomorphism

$$\bigoplus_{i+j=n} H^j(X', \Omega^i_{X'/k}) \xrightarrow{\sim} H^n(X', F_*\Omega^\bullet_{X/k}).$$

It follows from this that one has, for all n,

$$\sum_{i+j=n} \dim_k H^j(X, \Omega^i_{X/k}) = \dim_k H^n_{\mathrm{DR}}(X/k),$$

and according to 4.8, this results in the degeneration at E_1 of the Hodge to de Rham spectral sequence.

5.7. For the remaining part to follow, the reader can consult [**H2**] II, III. Let k be a ring and X a *projective* k-scheme, i.e. admits a closed k-immersion i in a standard projective space $P = \mathbb{P}^r_k = \mathrm{Proj}_k[t_0, \dots, t_n]$. Let L be an invertible sheaf over X. Recall that:
(i) L is *very ample* if one has $L \simeq i^*\mathcal{O}_P(1)$ for such a closed immersion i, which means that there exists global sections $s_j \in \Gamma(X, L)$ $(0 \leq j \leq r)$ defining a closed immersion $x \mapsto (s_0(x), \dots, s_r(x))$ of X in P;
(ii) L is *ample*, if there exists $n > 0$ such that $L^{\otimes n}$ is very ample.
Assume L ample. Then, according to *Serre's theorem* ([**H2**] II 5.17, III 5.2):
(a) For any coherent sheaf E on X, there exists an integer n_0 such that for any $n \geq n_0$, $E \otimes L^{\otimes n}$ is generated by a finite number of its global sections, i.e. a quotient of $\mathcal{O}_X^N$ for suitable N.
(b) For any coherent sheaf E on X, there exists an integer n_0 such that for any $n \geq n_0$ and all $i \geq 1$, one has

$$H^i(X, E \otimes L^{\otimes n}) = 0.$$

The theorem which follows is an analog in characteristic p, of the Kodaira-Akizuki-Nakano vanishing theorem [**KAN**], [**AkN**]:

Theorem 5.8. *Let k be a field of characteristic p, and let X be a smooth projective k-scheme. Let L be an ample invertible sheaf on X. Then if X is of pure dimension $d < p$ (cf.* 2.10*) and is lifted over $W_2(k)$, we have*

$$H^j(X, L \otimes \Omega^i_{X/k}) = 0 \quad \text{for } i + j > d, \tag{5.8.1}$$

$$H^j(X, L^{\otimes -1} \otimes \Omega^i_{X/k}) = 0 \quad \text{for } i + j < d. \tag{5.8.2}$$

This is a corollary of 5.5, due to Raynaud. The proof is analogous to that of 5.6, starting from 5.5. First of all, by the Serre duality theorem ([**H2**] III 7.7, 7.12), if M is an invertible sheaf on X, and if $i + i' = d = j + j'$, then the finite dimensional k-vector spaces $H^j(X, M \otimes \Omega^i_{X/k})$ and $H^{j'}(X, M^{\otimes -1} \otimes \Omega^{i'}_{X/k})$ are canonically dual. Formulas (5.8.1) and (5.8.2) are therefore equivalent. It will be more convenient to prove (5.8.2). By Serre's vanishing theorem (5.7 (b)), there exists $n \geq 0$ such that $H^j(X, L^{\otimes p^n} \otimes \Omega^i_{X/k}) = 0$ for all $j > 0$ and all i. By Serre duality, it follows that $H^j(X, L^{\otimes -p^n} \otimes \Omega^i_{X/k}) = 0$ for all $j < d$ and all i, and *in particular for all (i, j)* such that $i + j < d$. Proceeding by descending induction on n, it therefore suffices to prove the following assertion:
$(*)$ *if M is an invertible sheaf over X satisfying $H^j(X, M^{\otimes p} \otimes \Omega^i_{X/k}) = 0$ for all (i, j) such that $i + j < d$, then $H^j(X, M \otimes \Omega^i_{X/k}) = 0$ for all (i, j) such that $i + j < d$.*
Note as in 5.1, X' is the scheme induced from X by the change of base by the absolute Frobenius of $S = \operatorname{Spec} k$. If F_X denotes the absolute Frobenius of X, we have a canonical isomorphism $F_X^* M \simeq M^{\otimes p}$, induced by the map $m \mapsto m^{\otimes p}$, and therefore an isomorphism $F'^{\,*}M' \simeq M^{\otimes p}$, where $F : X \to X'$ is the relative Frobenius and M' is the inverse image of M over X'. We deduce, for all i, the following isomorphisms of $\mathcal{O}_{X'}$-modules

$$M' \otimes F_*\Omega^i_{X/k} \simeq F_*(F^*M' \otimes \Omega^1_{X/k}) \simeq F_*(M^{\otimes p} \otimes \Omega^i_{X/k}). \tag{$**$}$$

Let us consider the spectral sequence (4.8.1) relative to the functor $T = \Gamma(X', \bullet)$ and on the complex $K = M' \otimes F_*\Omega^\bullet_{X/k}$:

$$E_1^{ij} = H^j(X', M' \otimes F_*\Omega^i_{X/k}) \Rightarrow H^*(X', M' \otimes F_*\Omega^\bullet_{X/k}).$$

The hypothesis and $(**)$ imply that $E_1^{ij} = 0$ for $i + j < d$. Therefore

$$H^n(X', M' \otimes F_*\Omega^\bullet_{X/k}) = 0 \quad \text{for } n < d.$$

But like, according to 5.5, $F_*\Omega^\bullet_{X/k}$ is decomposable, we have (in $D(X')$)

$$F_*\Omega^\bullet_{X/k} \simeq \bigoplus \Omega^i_{X'/k}[-i],$$

therefore

$$H^n(X', M' \otimes F_*\Omega^\bullet_{X/k}) \simeq \bigoplus_{i+j=n} H^j(X', M' \otimes \Omega^i_{X'/k}),$$

and therefore

$$H^j(X', M' \otimes \Omega^i_{X'/k}) = 0 \quad \text{for } i + j < d.$$

The conclusion $(*)$ follows from this, since we have

$$F_S^* H^j(X, M \otimes \Omega^i_{X/k}) \simeq H^j(X', M' \otimes \Omega^i_{X'/k})$$

(cf. the end of the proof of 5.6).

REMARKS 5.9. The reader will find in [**D-I**] many complements of the aforementioned results. Here are some.

1. Let us assume given the hypothesis of 5.1. Then:
   (a) X' is lifted over T if and only if $\tau_{\leq 1} F_*\Omega^\bullet_{X/S}$ is decomposable in $D(X')$ (or, what amounts to the same, $\tau_{<p} F_*\Omega^\bullet_{X/S}$ is). Recall that there exists an obstruction $\omega \in \mathrm{Ext}^2(\Omega^1_{X'/S}, \mathcal{O}_{X'})$ to the lifting of X' (2.12 (a) and 3.7.1)), and that taking into account the Cartier isomorphism, this is in the same group that is found the obstruction c_1 to the decomposability of $\tau_{\leq 1} F_*\Omega^\bullet_{X/S}$ (4.6(a)): One can show with some convenient conventions of signs, that $\omega = c_1$.
   (b) If X' is lifted over T, the set of isomorphism classes of liftings of X' is an affine space under $\mathrm{Ext}^1(\Omega^1_{X'/S}, \mathcal{O}_X)$ (2.12 (b) and (3.7.1)), and (always taking into account the Cartier isomorphism) the set of decompositions of $\tau_{\leq 1} F_*\Omega^\bullet_{X/S}$ is an affine space under the same group (4.6(c)): One can show that the map $Z' \mapsto \varphi_{Z'}$ constructed in the proof of 5.1 is an affine bijection between these two spaces.
   (c) In fact, there is in [**D-I**] 3.5 a statement covering (a) and (b), by appealing to the theory *gerbes* of Giraud [**Gi**].
2. The degeneration theorem 5.6 has a relative variant. We work under the assumption of 5.1, and denote by $f : X \to S$ the structure morphism. Consider then the spectral sequence (4.8.1) relative to the functor f_* and the complex $\Omega^\bullet_{X/S}$,

$$E_1^{ij} = R^j f_* \Omega^i_{X/S} \Rightarrow R^* f_*(\Omega^\bullet_{X/S}),$$

   which is called *the relative Hodge to de Rham spectral sequence* (of X over S). Then if X is smooth and proper of relative dimension $< p$, and if X'

is lifted over T, this spectral sequence degenerates at E_1 and the sheaves $R^j f_* \Omega^i_{X/S}$ are locally free of finite type. ([**D-I**] 4.1.5).

3. The latter assertion of 5.5 and the conclusions of 5.6 and 5.8 still remain true if one only assumes X of dimension $\leq p$ ([**D-1**] 2.3). This is a consequence of Grothendieck duality for the morphism F.
4. There exists many examples of *smooth and proper surfaces* X over an algebraically closed field k of characteristic p for which the Hodge to de Rham spectral sequence does not degenerate at E_1 and which does not satisfy the vanishing property of Kodaira-Akizuki-Nakano type of 5.8. (Taking into account (3) if $p = 2$, or 5.6 and 5.8 if $p > 2$, these surfaces are therefore not lifted over $W_2(k)$.) See ([**D-I**] 2.6 and 2.10) for a bibliography on this subject.
5. Formulas (5.8.1) and (5.8.2) are still useful if $d = 2 \leq p$, X is liftable over $W_2(k)$ and L is only assumed *numerically positive*, i.e. satisfies $L \cdot L > 0$ and $L \cdot \mathcal{O}(D) \geq 0$ for any effective divisor D, see [**D-I**] 2.

6. From characteristic $p > 0$ to characteristic zero

6.0. There exists a standard technique in algebraic geometry, which allows one to prove certain statements of geometric nature[15], formulas over a base field of characteristic zero, from analogous statements over a field of characteristic $p > 0$, even a finite field. Roughly speaking, it consists of a given base field K, which is in characteristic zero, as an inductive limit of its $\mathbb{Z}$-sub-algebras of finite type A_i: Data on K, provided that they satisfy certain finiteness conditions, arise by extension of scalars from similar data on one of the A_i, say $A_{i_0} = B$. It is then enough to solve the similar problem on $T = \operatorname{Spec} B$, that which is seemingly more difficult. The advantage however, is that the closed points of T are then the spectrum of a finite field, and that in a sense which one can specify, there are many such points, so that it is enough to check the statement posed on T after sufficient specialization to these points. There is the business dealing with a problem of characteristic $p > 0$, where one has the range of corresponding methods (Frobenius, Cartier isomorphism, etc.); moreover one can exploit the fact of being able to choose the characteristic large enough.

The two ingredients of the method are: (a) results of passing to the limit, presented in great generality in (EGA IV 8), allowing the "spreading out" of certain data and properties on K, to similar data and properties on B; (b) density properties of closed points on schemes such that the schemes are of finite type over a field or over $\mathbb{Z}$ (EGA IV 10).

6.1. Let $((A_i)_{i \in I},\ u_{ij} : A_i \to A_j\ (i \leq j))$ be a filtered inductive system of rings, with inductive limit A, and denote by $u_i : A_i \to A$ the canonical homomorphism. The two very important examples are: (i) a ring A written as an inductive limit of its sub-$\mathbb{Z}$-algebras of finite type; (ii) the localization $A_{\mathfrak{p}}$ of a ring A at a prime ideal $\mathfrak{p}$ written as an inductive limit of localizations A_f $(= A[1/f])$ for $f \notin \mathfrak{p}$.

The prototype of problems and results of type (a) above is the following. Let $(E_i) = ((E_i)_{i \in I},\ v_{ij} : E_i \to E_j)$ be an inductive system of A_i-modules, having for inductive limit the A-module E. Let us agree to say that (E_i) is *cartesian* if,

[15] I.e. stable by base extension, as opposed to statements of *arithmetic nature,* where the base plays an essential role.

for any $i \leq j$, v_{ij} (which is an A_i-linear homomorphism of E_i into E_j considered as an A_i-module via u_{ij}) induces, by adjunction, an isomorphism (A_j-linear) of $u_{ij}^* E_i = A_j \otimes_{A_i} E_i$ in E_j. In this case, the canonical homomorphism $v_i : E_i \to E$ induces for all i, an isomorphism $u_i^* E_i$ $(= A \otimes_{A_i} E_i) \xrightarrow{\sim} E$. Let $((F_i)_i \in I, w_{ij})$ be a second inductive system of A_i-modules. If (E_i) is cartesian, the $\mathrm{Hom}_{A_i}(E_i, F_i)$ form an inductive system of A_i-modules: The transition map for $i \leq j$ associated to $f_i : E_i \to F_i$ is the homomorphism $E_j \to F_j$ composed with the inverse of the isomorphism of $A_j \otimes E_i$ in E_j defined by v_{ij}, from $A_j \otimes f_i : A_j \otimes E_i \to A_j \otimes F_i$, and from the map of $A_j \otimes F_i$ in F_j defined by w_{ij}. If F denotes the inductive limit of the F_i, one has analogous maps of $\mathrm{Hom}_{A_i}(E_i, F_i)$ into $\mathrm{Hom}_A(E, F)$, which defines a homomorphism

$$\operatorname{ind\,lim} \mathrm{Hom}_{A_i}(E_i, F_i) \to \mathrm{Hom}_A(E, F). \tag{6.1.1}$$

We can then pose the following two questions :

(1) Being given an A-module E, does there exist $i_0 \in I$ and an A_{i_0}-module E_{i_0} such that E results from E_{i_0} by an extension of scalars of A_{i_0} to A (or, that which amounts to the same, does there exist a cartesian inductive system (E_i), indexed by $\{i \in I | i \geq i_0\}$, for which the limit is E) ?

(2) If there exists i_0 such that (E_i) and (F_i) are cartesian for $i \geq i_0$, is the map (6.1.1) (where the inductive limit is reached for $i \geq i_0$) an isomorphism ?

There is a positive answer to the two questions with the help of hypothesis of finite presentation. (Recall that a module is said to be *finitely presented* if it is the cokernel of a homomorphism between free modules of the finite type.) More precisely, there is the following statement, which can be verified immediately:

LEMMA 6.1.2. *With the preceding notation:*

(a) *If E is a finitely presented A-module, there exists $i_0 \in I$ and an A_{i_0}-module of finite presentation E_{i_0} such that $u_{i_0}^* E_{i_0} \simeq E$.*

(b) *Let (E_i), (F_i) be two inductive systems, cartesian for $i \geq i_0$, with respective inductive limits E and F. Then if E_{i_0} is finitely presented, the map* (6.1.1) *is an isomorphism.*

It follows from this that if E is finitely presented, the E_{i_0} which arises by extension of scalars is essentially unique, in this sense that if E_{i_1} is another choice (E_{i_0} and E_{i_1} both being two finite presentations), there exists i_2 with $i_2 \geq i_1$ and $i_2 \geq i_0$ such that E_{i_0} and E_{i_1} become isomorphisms by extensions of scalars to A_{i_2}.

The $S_i = \operatorname{Spec} A_i$ form a projective system of schemes for which $S = \operatorname{Spec} A$ is the projective limit. If $(X_i, v_{ij} : X_j \to X_i)$ is a projective system of S_i-schemes, we say that this system is *cartesian* for $i \geq i_0$ if, for $i_0 \leq i \leq j$, the transition arrow v_{ij} gives a cartesian square

$$\begin{array}{ccc} X_j & \to & X_i \\ \downarrow & & \downarrow \\ S_j & \to & S_i. \end{array}$$

In this case, the S-scheme induced from X_{i_0} by extension of scalars to S is the projective limit of X_i. If (Y_i) is a second projective system of S_i-schemes, cartesian for $i \geq i_0$, the projective limit Y $(= S \times_{S_{i_0}} Y_{i_0})$ of the $\mathrm{Hom}_{S_i}(X_i, Y_i)$ form a projective system, and one has an analogous map to (6.1.1):

$$\text{(6.1.3)} \qquad \operatorname{proj\,lim} \operatorname{Hom}_{S_i}(X_i, Y_i) \to \operatorname{Hom}_S(X, Y).$$

We can then formulate similar questions to (1) and (2) above. They have similar answers, with the condition of replacing the hypothesis of finite presentation for modules by the hypothesis of finite presentation for schemes (a morphism of schemes $X \to Y$ is said to be a *finite presentation* if it is locally of finite presentation (2.1) and *"quasi-compact and quasi-separated"*, which means that X is a finite union of open affine subsets U_α over an open affine subset V_α of Y and that the intersections $U_\alpha \cap U_\beta$ have the same property; if Y is Noetherian, X is finitely presented over Y if and only if X is of finite type over Y, i.e. locally of finite type over Y (2.1) and Noetherian):

PROPOSITION 6.2. (a) *If X is an S-scheme of finite presentation, there exists $i_0 \in I$ and an S_{i_0}-scheme X_{i_0} of finite presentation for which X is induced by a base change.*
(b) *If (X_i), (Y_i) are two projective systems of S_i-schemes, cartesian for $i \geq i_0$, and if X_{i_0} and Y_{i_0} are finitely presented over S_{i_0}, then the map* (6.1.3) *is bijective.*

As in the preceding, it follows from this that X_{i_0} of 6.2 (a) is essentially unique (two such schemes become S_i-isomorphic for i large enough). Moreover, the usual properties of an S-scheme of finite presentation (or of a morphism between such) are already determined to some extent, over S_i for i large enough. Here are some, which are useful statements in themselves (the reader will find a long list in (EGA IV 8, 11.2, 17.7)):

PROPOSITION 6.3. *Let X be an S-scheme of finite presentation. We assume that X has one of the following properties $\mathcal{P}$: projective, proper, smooth. Then there exists $i_0 \in I$ and an S_{i_0}-scheme X_{i_0} of finite presentation, having the same property $\mathcal{P}$, for which X is induced by base change.*

The case where $\mathcal{P}$ is "projective" is easy: X is the closed subscheme of a standard projective space $P = \mathbb{P}^r_S$ defined by an ideal locally of finite type. It suffices to lift P, and then the closed immersion (i.e. the corresponding quotient of $\mathcal{O}_P$, cf. 6.11). The "proper" case is less immediate, but roughly, it goes back to a classical result, namely *Chow's Lemma* (cf. EGA IV 8.10.5). The "smooth" case is a little more difficult (which uses criterion 2.10), see (EGA IV 11.2.6 and 17.7.8). With regard to the properties of type (b) evoked in 6.0, we will only have need of the following result:

PROPOSITION 6.4. *Let S be a scheme of finite type over $\mathbb{Z}$. Then:*
(a) *If x is a closed point of S, the residue field $k(x)$ is a finite field,*

(b) *All locally closed nonempty components Z of S contain a closed point of S.*

For the proof, we refer to (EGA IV 10.4.6, 10.4.7), or in the case where S is affine, this goes back to (Bourbaki, Alg. Com. V, by 3, n° 4) (this is a consequence of Hilbert's theorem of zeros).

We will need to apply 6.4 (b) to the case where Z is the smooth part of S, S being assumed integral[16] :

[16] A scheme is said to be *integral* if it is reduced and irreducible.

PROPOSITION 6.5. *Let S be an integral scheme of finite type over $\mathbb{Z}$. The set of points x of S for which S is smooth over* $\operatorname{Spec}\mathbb{Z}$ *is a nonempty open set of S. In particular, if A is a $\mathbb{Z}$-algebra of finite type, and integral, there exists $s \in A,\ s \neq 0$, such that* $\operatorname{Spec} A_s$ *is smooth over $\mathbb{Z}$.*

The openness of the set of smooth points of a morphism locally of finite presentation is a general fact, which is a consequence for example of the jacobi criterion 2.6 (a), cf. (EGA IV 12.1.6.). That in the present case this open set is nonempty follows from a local variant of 2.10 and from the fact that the generic fiber of S is smooth over $\mathbb{Q}$ at its generic point, $\mathbb{Q}$ being perfect.

We will finally have to use some standard results of compatability of direct images by a base change (or, as one says sometimes, of *cohomological cleanliness*). Not wanting to weigh down our exposition, we will state them only in the case where it will be useful for us to have, for the Hodge cohomology and the de Rham cohomology.

PROPOSITION 6.6. *Let S be an affine scheme*[17]*, Noetherian, integral, and $f : X \to S$ a smooth and proper morphism.*

(a) *The sheaves $R^j f_* \Omega^i_{X/S}$ and $R^n f_* \Omega^\bullet_{X/S}$ are coherent. There exists a nonempty open set U of S such that, for any (i, j) and any n, the restrictions to U of these sheaves are locally free of finite type.*

(b) *For any $i \in \mathbb{Z}$ and for any morphism $g : S' \to S$, if $f' : X' \to S'$ denotes the induced scheme of X by base change via g, the canonical arrows of $D(S')$ (according to base change)*

$$Lg^* Rf_* \Omega^i_{X/S} \to Rf'_* \Omega^i_{X'/S'} \tag{6.6.1}$$

$$Lg^* Rf_* \Omega^\bullet_{X/S} \to Rf'_* \Omega^\bullet_{X'/S'} \tag{6.6.2}$$

are isomorphisms.

(c) *Fix $i \in \mathbb{Z}$ and assume that for any j, the sheaf $R^j f_* \Omega^i_{X/S}$ is locally free over S, of constant rank h^{ij}. Then for any j, the base change arrow (induced from* (6.6.1)*)*

$$g^* R^j f_* \Omega^i_{X/S} \to R^j f'_* \Omega^i_{X'/S'} \tag{6.6.3}$$

is an isomorphism. In particular, $R^j f'_* \Omega^i_{X'/S'}$ is locally free of rank h^{ij}.

(d) Suppose that for all n, $R^n f_* \Omega^\bullet_{X/S}$ is locally free of constant rank h_n. Then for all n, the change of base arrow (induced from (6.62)*)*

$$g^* R^n f_* \Omega^\bullet_{X/S} \to R^n f'_* \Omega^\bullet_{X'/S'} \tag{6.6.4}$$

is an isomorphism. In particular, $R^n f'_* \Omega^\bullet_{X'/S'}$ is locally free of rank h^n.

Let us briefly indicate the proof. The fact that the $R^j f_* \Omega^i_{X/S}$ are coherent is a particular case of the finiteness theorem of Grothendieck (EGA III 3) (or [**H2**] III 8.8 in the projective case). The coherence of $R^n f_* \Omega^\bullet_{X/S}$ follows from this by the relative Hodge to de Rham spectral sequence (5.9(2)). For the second

[17]The hypothesis "affine" is unnecessary; we use it only to facilitate the proof of (b).

assertion of (a), denote by A the (integral) ring of S, K its field of fractions, which is therefore the local ring of S at its generic point η. We set for abbreviation $R^j f_* \Omega^i_{X/S} = \mathcal{H}^{ij}$, $R^n f_* \Omega^\bullet_{X/S} = \mathcal{H}^n$. The fiber of $\mathcal{H}^{ij}$ (resp. $\mathcal{H}^n$) at η is free of finite type (a K-vector space of finite dimension), and is the inductive limit of $\mathcal{H}^{ij}_{|D(s)}$ (resp. $\mathcal{H}^n_{|D(s)}$), for s transversing A, $D(s)$ denoting "the open complement" of s, i.e. $\operatorname{Spec} A_s = X - V(s)$. By 6.1.2 it follows from this that there exists s such that $\mathcal{H}^{ij}_{|D(s)}$ (resp. $\mathcal{H}^n_{|D(s)}$) are free of finite type. For (b), we choose a finite covering $\mathcal{U}$ of X by open affine sets, denote by $\mathcal{U}'$ the open covering of X' induced from $\mathcal{U}$ by base change. Since S is affine and that X is proper, therefore separated over S, the finite intersections of open sets in $\mathcal{U}$ are affine and similarly the finite intersections of open sets in $\mathcal{U}'$ are (relatively) affine[18] over S'. Consequently (cf. **[H2]** III 8.7), $Rf_* \Omega^i_{X/S}$ (resp. $Rf'_* \Omega^i_{X'/S'}$) is represented by $f_* \check{\mathcal{C}}(\mathcal{U}, \Omega^i_{X/S})$ (resp. $f'_* \check{\mathcal{C}}(\mathcal{U}', \Omega^i_{X'/S'})$), where we denote here by $\check{\mathcal{C}}(\mathcal{U}, \bullet)$ the *alternating* complex of cochains. By the compatability of Ω^i by a base change, there is a canonical isomorphism of complexes

$$g^* f_* \check{\mathcal{C}}(\mathcal{U}, \Omega^i_{X/S}) \xrightarrow{\sim} f'_* \check{\mathcal{C}}(\mathcal{U}', \Omega^i_{X'/S'}).$$

Since the complex $f_* \check{\mathcal{C}}(\mathcal{U}, \Omega^i_{X/S})$ is bounded and with flat components, this isomorphism realizes the isomorphism (6.6.1). Similarly, $Rf_* \Omega^\bullet_{X/S}$ (resp. $Rf'_* \Omega^\bullet_{X'/S'})$) is represented by $f_* \check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S})$ (resp. $f'_* \check{\mathcal{C}}(\mathcal{U}', \Omega^\bullet_{X'/S'})$) (where $\check{\mathcal{C}}$ denotes this time the associated simple complex of the Čech bicomplex), and one has a canonical isomorphism of complexes

$$g^* f_* \check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S}) \xrightarrow{\sim} f'_* \check{\mathcal{C}}(\mathcal{U}', \Omega^\bullet_{X'/S'}),$$

which realizes the isomorphism (6.6.2). Assertions (c) and (d) follow from (b) and from the following lemma, for which we leave the verification to the reader:

LEMMA 6.7. *Let A be a Noetherian ring and E a complex of A-modules such that $H^i(E)$ are projective of finite type for any i and zero for almost all i. Then:*
(a) *E is isomorphic, in $D(A)$, to a bounded complex with projective components of finite type.*
(b) *If E is bounded and with projective components of finite type, for any A-algebra B, and for all i, the canonical homomorphism*

$$B \otimes_A H^i(E) \to H^i(B \otimes_A E)$$

is an isomorphism.

REMARKS 6.8. (a) A complex of A-modules, isomorphic in $D(A)$, to a bounded complex with projective components of finite type is said to be *perfect.* One must be aware that if E is perfect, it is not true in general, that the $H^i(E)$ are projective of finite type. One can show that under the hypothesis of 6.6, the complexes $Rf_* \Omega^i_{X/S}$ and $Rf_* \Omega^\bullet_{X/S}$ are perfect over S (and not only over U). The notion of a perfect complex plays an important role in numerous questions in algebraic geometry.
(b) In the statements of 6.6 concerning $\Omega^i_{X/S}$, one can replace $\Omega^i_{X/S}$ by any locally free $\mathcal{O}_X$-module F of finite type (even coherent and relatively flat over S): The

[18] A morphism of schemes is said to be *affine* if the inverse image of any affine open set is affine.

conclusions of (a), (b) and (c) are still valid on the condition of replacing $\Omega^i_{X'/S'}$ by the inverse image sheaf F' of F over X'. Similarly, the complex Rf_*F is perfect over S.

We are now able to state and prove the promised application of 5.6:

THEOREM 6.9 (Hodge Degeneration Theorem). *Let K be a field of characteristic zero, and X a smooth and proper K-scheme. Then the Hodge spectral sequence of X over K (4.8.3)*

$$E_1^{ij} = H^j(X, \Omega^i_{X/K}) \Rightarrow H^*_{\mathrm{DR}}(X/K)$$

degenerates at E_1.

Set $\dim_K H^j(X, \Omega^i_{X/K}) = h^{ij}$, $\dim H^n_{\mathrm{DR}}(X/K) = h^n$. It suffices to prove that for all n, $h^n = \sum_{i+j=n} h^{ij}$ (cf. (4.8.3)). Write K as an inductive limit of the family $(A_\lambda)_{\lambda \in L}$ of its sub-$\mathbb{Z}$-algebras of finite type. According to 6.3, there exists $\alpha \in L$ and a smooth and proper S_α-scheme X_α (where $S_\alpha = \operatorname{Spec} A_\alpha$) for which X is induced by base change $\operatorname{Spec} K \to S_\alpha$. Even if it means to replace A_α by $A_\alpha[t^{-1}]$ for a suitable nonzero $t \in A_\alpha$, we can assume, according to 6.5, that S_α is smooth over $\operatorname{Spec} \mathbb{Z}$. Abbreviate A_α by A, S_α by S, X_α by $\mathfrak{X}$, and denote by $f : \mathfrak{X} \to S$ the structure morphism. Again by replacing A by $A[t^{-1}]$, we can according to 6.6 (a), assume that the sheaves $R^j f_* \Omega^i_{\mathfrak{X}/S}$ (resp. $R^n f_* \Omega^\bullet_{\mathfrak{X}/S}$) are free of constant rank, necessarily equal then to h^{ij} (resp. h^n) according to 6.6 (c) and (d). Since the relative dimension of $\mathfrak{X}$ over S is a locally constant function and that X is quasi-compact, one can in addition choose an integer d which bounds this dimension at any point of X and therefore the dimension of the fibers of X over S at any point of S. Applying 6.4 (b) to $Z = \operatorname{Spec} A[1/N]$ for suitable N (say, the product of prime numbers $\leq d$), one can choose a closed point s of S, for which the residue field $k = k(s)$ (a finite field) is of characteristic $p > d$. Since S is smooth over $\operatorname{Spec} \mathbb{Z}$, the canonical morphism $\operatorname{Spec} k \to S$ (a closed immersion) is extended (by definition of smoothness (2.2)) to a morphism $g : \operatorname{Spec} W_2(k) \to S$, where $W_2(k)$ is the ring of Witt vectors of length 2 over k (3.9). Denote by $Y = \mathfrak{X}_s$ the fiber of $\mathfrak{X}$ over $s = \operatorname{Spec} k$ and Y_1 the scheme over $\operatorname{Spec} W_2(k)$ induced from $\mathfrak{X}$ by the base change g. We therefore have cartesian squares:

$$\begin{array}{ccccccc}
Y & \to & Y_1 & \to & \mathfrak{X} & \leftarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s & \to & \operatorname{Spec} W_2(k) & \xrightarrow{g} & S & \leftarrow & \operatorname{Spec} K.
\end{array}$$

By construction, Y is a smooth and proper k-scheme of dimension $< p$, lifted over $W_2(k)$. Therefore according to 5.6, the Hodge to de Rham spectral sequence of Y over k degenerates at E_1. We therefore have for all n,

$$\sum_{i+j=n} \dim_k H^j(Y, \Omega^i_{Y/k}) = \dim_k H^n_{\mathrm{DR}}(Y/k).$$

But according to 6.6 (c) and (d), we have for all (i,j) and for all n,

$$\dim_k H^j(Y, \Omega^i_{Y/k}) = h^{ij}, \quad \dim_k H^n_{\mathrm{DR}}(Y/k) = h^n.$$

Therefore $\sum_{i+j=n} h^{ij} = h^n$ for all n, for which the proof follows.

THEOREM 6.10 (Kodaira-Akizuki-Nakano Vanishing Theorem [**KAN**], [**AkN**]). *Let K be a field of characteristic zero, X a smooth projective K-scheme of pure dimension d, and L an ample invertible sheaf on X. Then we have:*

$$H^j(X, L \otimes \Omega^i_{X/K}) = 0 \quad \text{for } i + j > d, \tag{6.10.1}$$

$$H^j(X, L^{\otimes -1} \otimes \Omega^i_{X/K}) = 0 \quad \text{for } i + j < d. \tag{6.10.2}$$

We deduce 6.10 from 5.8 just like 6.9 from 5.6. We need for this a result of passing to the limit for modules, generalizing and clarifying 6.1.2 (cf. (EGA IV 8.5, 8.10.5.2)):

PROPOSITION 6.11. *Assume given, as in 6.1, a filtered projective system of affine schemes $(S_i)_{i \in I}$, with limit S. Let $i_0 \in I$, X_{i_0} be a S_{i_0}-scheme of finite presentation and consider the induced projective cartesian system (X_i) for $i \geq i_0$, with limit $X = S \times_{S_{i_0}} X_{i_0}$.*

(a) *If E is a finitely presented $\mathcal{O}_X$-module, there exists $i \geq i_0$ and a $\mathcal{O}_{X_i}$-module E_i of finite presentation for which E is induced by extension of scalars. If E is locally free (resp. locally free of rank r), there exists $j \geq i$ such that $E_j = \mathcal{O}_{X_j} \otimes_{\mathcal{O}_{X_i}} E_i$ is locally free (resp. locally free of rank r). If X is projective over S and E is an ample invertible $\mathcal{O}_X$-module (resp. very ample)* (5.7)*, there exists $j \geq i$ such that X_j is projective over S_j and E_j is ample invertible (resp. very ample).*

(b) *Let E_{i_0}, F_{i_0} be finitely presented $\mathcal{O}_X$-modules, and consider the systems (E_i), (F_i) which are induced by extension of scalars over the X_i for $i \geq i_0$, as well as the modules E and F which are induced by extension of scalars over X. Then there is a natural map*

$$\operatorname{ind}\lim_{i \geq i_0} \operatorname{Hom}_{\mathcal{O}_{X_i}}(E_i, F_i) \to \operatorname{Hom}_{\mathcal{O}_X}(E, F),$$

which is bijective.

The proof of (b), then of the first two assertions of (a), brings us back to 6.1.2. For the latter part of (a), it suffices treat the case where E is very ample, i.e. corresponds to a closed immersion $h : X \to P = \mathbb{P}^r_S$ such that $h^*\mathcal{O}_P(1) \simeq E$. For i sufficiently large, one lifts h by an S_i-morphism $h_i : X_i \to P_i = \mathbb{P}^r_{S_i}$ and E by invertible E_i over X_i. Even if it means to increase i, h_i is a closed immersion and the isomorphism $h^*\mathcal{O}_P(1) \simeq E$ comes from an isomorphism $h_i^*\mathcal{O}_{P_i}(1) \simeq E_i$; E_i is then very ample.

Proving 6.10. Proceeding as in the proof of 6.9, and moreover applying 6.11, one can find a subring A of K of finite type and smooth over $\mathbb{Z}$, a smooth projective morphism $f : \mathfrak{X} \to S = \operatorname{Spec} A$ of pure relative dimension d, for which $X \to \operatorname{Spec} K$ is induced by base change, and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ for which L is induced by extension of scalars. By virtue of 6.6 and 6.8 (b), one can assume, even if it means to replace A by $A[t^{-1}]$, that the sheaves $R^j f_*(\mathcal{M} \otimes \Omega^i_{\mathfrak{X}}/S)$, where $\mathcal{M} = \mathcal{L}$ (resp. $\mathcal{L}^{\otimes -1}$), are free of finite type, of constant rank, necessarily equal, according to 6.8 (b), to $h^{ij}(L) = \dim_K H^j(X, L \otimes \Omega^i_{X/K})$ (resp. $h^{ij}(L^{\otimes -1}) = H^j(X, L^{\otimes -1} \otimes \Omega^i_{X/K})$). Let us choose then $g : \operatorname{Spec} W_2(k) \to S$ as in the proof of 6.9. The inverse image sheaf $\mathcal{L}_s$ of $\mathcal{L}$ over $Y = X_s$ is ample. According to 6.6 and 6.8 (b), one has $\dim_k H^j(Y, \mathcal{L}_s \otimes \Omega^i_{Y/k}) = h^{ij}(L)$, and $\dim_k H^j(Y, \mathcal{L}_s^{\otimes -1} \otimes \Omega^i_{Y/k}) = h^{ij}(L^{\otimes -1})$. The conclusion then follows from 5.8.

REMARK 6.12. In a similar manner, the Ramanujam vanishing theorem on surfaces [**Ram**] follows from the variant of 5.8 relative to the numerically positive sheaves (cf. 5.9 (5)).

7. Recent developments and open problems

A. Divisors with normal crossings, semi-stable reduction, and logarithmic structures.

7.1. Let S be a scheme, X a smooth S-scheme, and D a closed subscheme of X. We say that D is a *divisor with normal crossings relative to S* (or simply, *relative*) if, "locally for the étale topology on X", the couple (X, D) is "isomorphic" to the couple formed from the standard affine space $\mathbb{A}^n_S = S[t_1, \ldots, t_n]$ and from the divisor $V(t_1 \cdots t_r)$ of the equation $t_1 \cdots t_r = 0$, for $0 \leq r \leq n$ (the case $r = 0$ corresponds to $t_1 \cdots t_r = 1$ and $V(t_1 \cdots t_r) = \emptyset$). This means that there exists an *étale* covering $(X_i)_{i \in I}$ of X (i.e. a family of étale morphisms $X_i \to X$ for which the union of the images is X) such that, if $D_i = X_i \times_X D$ is the closed subscheme induced by D on X_i, there exists an étale morphism $X_i \to \mathbb{A}^n_S$ for which there is a cartesian square

$$\begin{array}{ccc} D_i & \to & X_i \\ \downarrow & & \downarrow \\ V(t_1 \cdots t_r) & \to & \mathbb{A}^n_S \end{array}$$

(n and r dependant on i). In other words, that there exists a coordinate system $(x_1, \ldots, x_n)$ on X_i in the sense of 2.7 (defining the étale morphism $X_i \to \mathbb{A}^n_S$) such that D_i is the closed subscheme of the equation $x_1 \cdots x_r = 0$. This definition is modeled after the analogous definition in complex analytic geometry (cf. [**D1**]), where "locally for the étale topology" is replaced by "locally for the classical topology", and "étale morphism" by "local isomorphism". A standard example of a divisor with normal crossings relative to $S = \operatorname{Spec} k$, k a field of characteristic different from 2, is the cubic with double point $D = \operatorname{Spec} k[x, y]/(y^2 - x^2(x - 1))$ in the affine plane $X = \operatorname{Spec} k[x, y]$. (Observe in this example that there does not exist a system of coordinates (x_j) as above on a *Zariski* open covering of X, an étale extension (extraction of a square root of $x - 1$) being necessary for to make possible such a system in a neighbourhood of the origin.)

The notion of a divisor with the normal crossings $D \hookrightarrow X$ relative to S is stable by étale localization over X and by base change $S' \to S$.

If $D \hookrightarrow X$ is a relative divisor with normal crossings, and if $j : U = X \backslash D \hookrightarrow X$ is the inclusion of the open complement, we define a subcomplex

$$\Omega^{\bullet}_{X/S}(\log D) \tag{7.1.1}$$

of $j_* \Omega^{\bullet}_{U/S}$, called the *de Rham complex of X/S with logarithmic poles along D*, by the condition that a local section ω of $j_* \Omega^i_{U/S}$ belong to $\Omega^i_{X/S}(\log D)$ if and only if ω and $d\omega$ have at most a simple pole along D (i.e. are such that if f is a local equation of D, $f\omega$ (resp. $f\, d\omega$) is a section of $\Omega^i_{X/S}$ (resp. $\Omega^{i+1}_{X/S}$) (NB. f is necessarily a nonzero divisor in $\mathcal{O}_X$)). One easily sees that the $\mathcal{O}_X$-modules $\Omega^i_{X/S}(\log D)$ are locally free of finite type, that $\Omega^i_{X/S}(\log D) = \Lambda^i \Omega^1_{X/S}(\log D)$, and that if as above, $(x_1, \ldots, x_n)$ are coordinates on an X' étale neighbourhood over X where D has for equation $x_1 \cdots x_r = 0$, $\Omega^1_{X/S}(\log D)$ is free with basis $(dx_1/x_1, \ldots, dx_r/x_r, dx_{r+1}, \ldots, dx_n)$.

There is a natural variant in complex analytic geometry of the construction (7.1.1) (cf. [**D1**]). If $S = \operatorname{Spec} \mathbb{C}$ and $D \subset X$ is a(n algebraic) divisor with normal crossings, the complex of analytic sheaves associated to (7.1.1) on the analytic space X^{an} associated to X,

$$\Omega^\bullet_{X/\mathbb{C}}(\log D)^{\mathrm{an}} = \Omega^\bullet_{X^{\mathrm{an}}/\mathbb{C}}(\log D^{\mathrm{an}}),$$

calculates the transcendental cohomology of U with values in $\mathbb{C}$: There is a canonical isomorphism (in the derived category $D(X^{\mathrm{an}}, \mathbb{C})$)

$$Rj_*\mathbb{C} \simeq \Omega^\bullet_{X/S}(\log D)^{\mathrm{an}}, \tag{7.1.2}$$

and consequently an isomorphism

$$H^i(U^{\mathrm{an}}, \mathbb{C}) \simeq H^i(X^{\mathrm{an}}, \Omega^\bullet_{X/S}(\log D)^{\mathrm{an}}) \tag{7.1.3}$$

(loc. cit.). Moreover, if X is proper over $\mathbb{C}$, the comparison theorem of Serre [**GAGA**] allows us to deduce from (7.1.3) the isomorphism

$$H^i(U^{\mathrm{an}}, \mathbb{C}) \simeq H^i(X, \Omega^\bullet_{X/S}(\log D)). \tag{7.1.4}$$

Moreover the filtration F of $H^*(X, \Omega^\bullet_{X/S}(\log D))$, being the outcome of the first spectral sequence of hypercohomology of X with values in $\Omega^\bullet_{X/S}(\log D)$,

$$E_1^{pq} = H^q(X, \Omega^p_{X/S}(\log D) \Rightarrow H^{p+q}(X, \Omega^\bullet_{X/S}(\log D)) \tag{7.1.5}$$

is the *Hodge filtration* of the natural *mixed Hodge structure* of $H^*(U^{\mathrm{an}}, \mathbb{Z})$ defined by Deligne, and the spectral sequence (7.1.5) degenerates at E_1 ([**D2**]).

Just as in the case where $D = \emptyset$ (6.9), this degeneration can be shown by reduction to characteristic $p > 0$. Indeed, we have the following result which generalizes 5.1 and for which the proof is analogous ([**D-I**] 4.2.3):

THEOREM 7.2. *Let S be a scheme of characteristic $p > 0$, $S^\sim$ a flat lifting of S over $\mathbb{Z}/p^2\mathbb{Z}$, X a smooth S-scheme and $D \subset X$ a relative divisor with normal crossings. Denote by $F : X \to X'$ the relative Frobenius of X/S. If the couple (X', D') admits a lifting $(X'^\sim, D'^\sim)$ over $S^\sim$, where $X'^\sim$ is smooth and $D'^\sim \subset X'^\sim$ is a relative divisor with normal crossings, the complex of $\mathcal{O}_X$-modules $\tau_{<p}F_*\Omega^\bullet_{X/S}(\log D)$ is decomposable in the derived category $D(X')$.*

The reader will find in [**D-I**] various complements to 7.2 and in [**E-V**] another presentation of the same results, and some applications of the theorems pertaining to ampleness and vanishing results.

7.3. The preceeding theory extends without much change to a class of morphisms which are no longer smooth, but not far from this, namely the morphisms that are said to be "of semi-stable reduction". Let T be a scheme. The prototype of such morphisms is the morphism

$$s : \mathbb{A}^n_T = T[x_1, \ldots, x_n] \to \mathbb{A}^1_T = T[t], \quad t \mapsto x_1 \cdots x_n \quad (n \geq 1).$$

In other words, if $S = \mathbb{A}^1_T$, the scheme $\mathbb{A}^n_T$, considered as S-scheme by s, is the sub-S-scheme of $\mathbb{A}^n_S = S[x_1, \ldots, x_n] = T[x_1, \ldots, x_n, t]$ with equation $x_1 \cdots x_n = t$.

The morphism s is smooth outside 0 and its fiber at 0 is the divisor D with equation $(x_1 \cdots x_n = 0)$, a divisor with normal crossings relative to T, but not with S (a "vertical" divisor). More generally, if S is a smooth T-scheme of relative dimension 1 and $E \subset S$ a relative divisor with normal crossings (if T is the spectrum of an algebraically closed field, E is therefore simply a finite set of rational points of S), we say that the S-scheme X has *semi-stable reduction along* E if, locally for the étale topology (over X and over S) the morphism $X \to S$ is of the form $s \circ g$, with g smooth, s being the morphism considered above. The divisor $D = X \times_S E \subset X$ is then a divisor with normal crossings relative to T (but not to S)[19]. An elementary example is furnished by the "Legendre family" $X = \operatorname{Spec} k[x,y,t]/(y^2 - x(x-1)(x-t))$ over $S = \operatorname{Spec} k[t]$, ($k$ a field of characteristic $\neq 2$), which has semi-stable reduction on $\{0\} \cup \{1\}$, the fiber at each of these points being isomorphic to the cubic with double point considered above. The interest in the notion of semi-stable reduction comes from the *semi-stable reduction conjecture*, which roughly asserts that locally, after suitable ramification of the base, a smooth morphism can be extended to a morphism with semi-stable reduction. This conjecture was established by Grothendieck-Deligne-Mumford and Artin-Winters ([**G**], [**A-W**], [**D-M**]) in any characteristic but relative dimension 1, and Mumford ([**M**]) in characteristic zero and arbitrary relative dimension.

If $f : X \to S$ has semi-stable reduction along E, we define the de Rham complex with relative logarithmic poles

$$\omega^\bullet_{X/S} = \Omega^\bullet_{X/S}(\log D/E), \tag{7.3.1}$$

with components $\omega^i_{X/S} = \Lambda^i \omega^1_{X/S}$, where $\omega^1_{X/S}$ is the quotient of $\Omega^1_{X/T}(\log D)$ by the image of $f^*\Omega^1_{S/T}(\log E)$ and the differential is induced from that of $\Omega^\bullet_{X/T}(\log D)$ by passing to the quotient. This complex has locally free components of finite type (in the case of the morphism s above, $\omega^1_{X/S}$ is isomorphic to $(\bigoplus \mathcal{O}_X dx_i/x_i)/\mathcal{O}_X(\sum dx_i/x_i)$ (therefore free with basis $dx_i/x_i,\ i \geq 2$)). It induces on the smooth open part U of X over S the usual de Rham complex $\Omega^\bullet_{U/S}$, and one can show that this is the unique extension over X of this complex which has locally free components of finite type. Moreover, if one sets for abbreviation, $\omega^\bullet_{X/T} = \Omega^\bullet_{X/T}(\log D)$, $\omega^\bullet_{S/T} = \Omega^\bullet_{S/T}(\log E)$, there is an exact sequence

$$0 \to \omega^1_{S/T} \otimes \omega^\bullet_{X/S}[-1] \to \omega^\bullet_{X/T} \to \omega^\bullet_{X/S} \to 0, \tag{7.3.2}$$

where the arrow to the left is given by $a \otimes b \mapsto f^*a \wedge b$. This exact sequence plays an important role in *the regularity theorem of the Gauss-Manin connection* (cf. [**K2**] and the article of Bertin-Peters in this volume). There also exists a variant of these constructions in complex analytic geometry. Assume that $T = \operatorname{Spec} \mathbb{C}$, that S is a smooth curve over $\mathbb{C}$, $E \subset S$ the divisor reduced to a point 0, and that $f : X \to S$ is a morphism with semi-stable reduction at $\{0\}$, with fiber Y at 0. (Y is therefore a divisor with normal crossings in X relative to $\mathbb{C}$.) We consider the complex

$$\omega^\bullet_Y = \mathbb{C}_{\{0\}} \otimes_{\mathcal{O}_S} \omega^\bullet_{X/S}, \tag{7.3.3}$$

with components the locally free sheaves of finite type $\omega^i_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \omega^i_{X/S}$. Steenbrink [**St**] has shown that the complex analogue $\omega^\bullet_{Y^{\mathrm{an}}}$ over Y^{an} (which is also the

[19] One can similarly define a notion of semi-stable reduction along E without the hypothesis on the relative dimension of S over T, cf. [**I5**].

complex of sheaves associated to $\omega_Y^\bullet$ over Y^{an}) embodies the *complex of neighbouring cycles* $R\Psi(\mathbb{C})$ of f at 0, so that if moreover f is proper, $H^*(Y,\omega_Y^\bullet)$ "calculates" $H^*(X_t^{\mathrm{an}},\mathbb{C})$ for t "close enough" to 0. Steenbrink also shows (under this extra hypothesis) that the spectral sequences

$$E_1^{pq} = R^q f_* \omega^p_{X/S} \Rightarrow R^{p+q} f_* \omega^\bullet_{X/S} \tag{7.3.4}$$

and

$$E_1^{pq} = H^q(Y,\omega_Y^p) \Rightarrow H^{p+q}(Y,\omega_Y^\bullet) \tag{7.3.5}$$

degenerate at E_1 and that the sheaves $R^q f_*\omega^p_{X/S}$ are locally free of finite type and of formation compatible with any base change. These results form part of the construction of a *limiting mixed Hodge structure* on $H^*(X_t^{\mathrm{an}},\mathbb{Z})$ for t tending to 0 (loc. cit.). They can by themselves, be proven by reduction to characteristic $p>0$ ([**I5**]). For T of characteristic $p>0$, and $f : X \to S$ with semi-stable reduction along $E \subset S$, the complexes $\omega^\bullet_{X/S}$ and

$$\omega_D^\bullet = \mathcal{O}_D \otimes_{\mathcal{O}_S} \omega^\bullet_{X/S} \tag{7.3.6}$$

(where $D = E \times_S X$) indeed give rise to Cartier morphisms (of the type of 3.5), and under the hypothesis of a suitable lifting modulo p^2, $\tau_{<p}F_*\omega^\bullet_{X/S}$ and $\tau_{<p}F_*\omega^\bullet_D$ decompose (in $D(X')$). (See [**I5**] 2.2 for a precise statement, which generalizes 7.2 and other corollaries (degeneration and vanishing statements).)

7.4. The complex $\omega_D^\bullet$ above does not depend only on D, but on X/S. It does depend on it however locally (in a neighbourhood of D). While seeking to elucidate the additional structure on D necessary for the definition, J.-M. Fontaine and the author were led to introduce the notion of *logarithmic structure*. This paved the way to a theory, *logarithmic geometry*, as a natural extension of the theory of schemes. Widely developed by K. Kato and his school, it makes possible to unify the various constructions of complexes with logarithmic poles considered above and to consider the toric varieties of Mumford et al. and the morphisms with semi-stable reduction as particular cases of a novel notion of smoothness. See [**I6**] for an introduction. The preceding decomposition, degeneration and vanishing results admit generalizations in this program, see [**Ka2**] and [**Og2**].

B. Degeneration mod p^n and crystals.

7.5. The decomposition Theorem 5.1 was originally obtained as a by-product of the work of Ogus [**Og1**], Fontaine-Messing [**F-M**] and Kato [**Ka1**] by crystalline cohomology (see [**I4**] for a panorama of this theory). The link (a small technique) between 5.1 and the point of view of crystalline is explicit in [**D-I**] 2.2 (iv). We limit ourselves to a statement of a degeneration result mod p^n ([**F-M**], [**Ka1**]) analogous to 5.6:

THEOREM 7.6. *Let k be a perfect field of characteristic $p>0$, $W=W(k)$ the ring of Witt vectors over k, X a smooth and proper W-scheme of relative dimension $<p$. Then for any integer $n \geq 1$, the Hodge to de Rham spectral sequence*

$$E_1^{ij} = H^j(X_n, \Omega^i_{X_n/W_n}) \Rightarrow H^{i+j}_{\mathrm{DR}}(X_n/W_n) \tag{7.6.1}$$

degenerates at E_1*, where* $W_n = W_n(k) = W/p^nW$ *denotes the ring of Witt vectors of length* n *over* k *and* X_n *the scheme over* W_n *induced from* X *by reduction modulo* p^n *(i.e. by extension of scalars of* W *to* W_n*).*

7.7. For $n = 1$, we have $W_n = W_n(k) = W/p^nW$ and we recover statement 5.6, apart from which in 7.6, we assume given a lifting of X over W (rather than over W_2)[20]. Under the hypothesis of 7.6, it is not true in general, for $n \geq 2$, that the de Rham complex $\Omega^\bullet_{X_n/W_n}$ (which is, a priori, only a complex of sheaves of W_n-modules over X_n (or X_1, X_n and X_1 having the same underlying space)) is decomposable in the corresponding derived category $D(X_1, W_n)$. However, the results of Ogus ([**Og1**] 8.20) imply that if σ denotes the Frobenius automorphism of W_n, $\sigma^*\Omega^\bullet_{X_n/W_n}$ is isomorphic in the derived category $D(X_1, W_n)$ of sheaves of W_n-modules over X_1, to the complex $\Omega^\bullet_{X_n,W_n}(p)$ induced from $\Omega^\bullet_{X_n,W_n}$ by multiplying the differential by p. (NB. For $n = 1$, we have $\Omega^\bullet_{X_n/W_n}(p) = \bigoplus \Omega^i_{X_1/k}[-i]$.) The conclusion of 7.6 comes about easily, like various additional properties of $H^*_{\mathrm{DR}}(X_n/W_n)$ (structure called "of Fontaine-Laffaille" - including in particular the fact that the Hodge filtration is formed from direct factors), see [**F-M**] and [**Ka1**].

7.8. The degeneration and decomposition results for which we discussed until now carry over to de Rham complexes of schemes, possibly with logarithmic poles. More generally, we can consider the de Rham complexes with coefficients in modules with integrable connections. Many generalizations of this type have been obtained: For Gauss-Manin coefficients [**I5**], of sheaves of Fontaine-Laffaille [**Fa2**], of T-crystals [**Og2**] (besides these last objects providing a common generalization of the previous two).

C. Open problems.

7.9. Let k be a perfect field of characteristic $p > 0$, X a smooth k-scheme of dimension d, X' the scheme induced from X by base change by the Frobenius automorphism of k, $F : X \to X'$ the relative Frobenius (3.1). We have seen in 5.9 (1) (a) (with $S = \operatorname{Spec} k$, $T = \operatorname{Spec} W_2(k)$) that the following conditions are equivalent :

(i) X' - or, that which amounts to the same here, X - is lifted (by a smooth and proper scheme) over $W_2(k)$;

(ii) $\tau_{\leq 1} F_*\Omega^\bullet_{X/k}$ is decomposable in $D(X')$ (4.6);

(iii) $\tau_{<p} F_*\Omega^\bullet_{X/k}$ is decomposable in $D(X')$.

We say that X is DR-*decomposable* if $F_*\Omega^\bullet_{X/k}$ is decomposable (in $D(X')$). As we have observed in 5.2, this condition is equivalent, taking into account the Cartier isomorphism (3.5), to the existence of an isomorphism

$$\bigoplus \Omega^i_{X'/k}[-i] \xrightarrow{\sim} F_*\Omega^\bullet_{X/k}$$

of $D(X')$ inducing C^{-1} on $\mathcal{H}^i$. The arguments of 5.6 and 5.8 show that:

[20] In fact, Ogus has shown – albeit more difficult – that being given $n \geq 1$ and Z smooth and proper over W_n of dimension $< p$, then if Z admits a lifting (smooth and proper) over W_{n+1}, the Hodge to de Rham spectral sequence of Z/W_n degenerates at E_1 ([**Og2**] 8.2.6). This result truly generalizes 5.6.

(a) If X is proper over k and DR-decomposable, the Hodge to de Rham spectral sequence of X/k degenerates at E_1.
(b) If X is projective over k, of pure dimension d, and DR-decomposable, and if L is an invertible ample sheaf over X, one has the vanishing results of Kodaira-Akizuki-Nakano (5.8.1) and (5.8.2).

By virtue of the equivalence between conditions (i) and (ii) above, a necessary condition for that X is DR-decomposable is that X is lifted over $W_2(k)$. According to [**D-I**], it is sufficient if $d \leq p$ (5.5 and 5.9 (3)). We are unaware if it is always true in general:

Problem 7.10. Let X be a smooth k-scheme of dimension $d > p$, liftable over $W_2(k)$. Is it the case that X is DR-decomposable ?

7.11. Recall (5.5.1) that if X and F lift over $W_2(k)$, X is DR-decomposable; this is the case if X is affine, or is a projective space over k. As indicated in [**D-I**] 2.6 (iv), if X is liftable over $W_2(k)$ and if, for any integer $n \geq 1$, the product morphism $(\Omega^1_{X/k})^{\otimes n} \to \Omega^n_{X/k}$ admits a section, then X is DR-decomposable (see 8.1 for a proof). This second condition is checked in particular if X is *parallelizable*, i.e. if $\Omega^1_{X/k}$ is a free $\mathcal{O}_X$-module (or, that which amounts to the same, the tangent bundle $T_{X/k}$, dual of $\Omega^1_{X/k}$, is trivial), therefore for example if X is an *abelian variety*. By a theorem of Grothendieck (cf. [**Oo**] and [**I7**] Appendix 1), any abelian variety over k is lifted over $W_2(k)$ (and similarly over $W(k)$). Therefore any abelian variety over k is DR-decomposable. Another interesting class of liftable k-schemes (over $W(k)$) is formed from *complete intersections* in $\mathbb{P}^r_k$ (see the expose of Deligne (SGA 7 XI) for the definitions and basic properties of these objects). But we do not know if those are DR-decomposable. The first unknown case is that of a (smooth) quadric of dimension 3 in characteristic 2. We also don't know if the *Grassmannians*, and more generally, *flag varieties*, which are, albeit liftable over $W(k)$, are DR-decomposable (the only known example is projective space!).

Problem 7.10, with "liftable over $W_2(k)$" replaced by "liftable over $W(k)$", is also an open problem. On the other hand, we can replace "liftable over $W(k)$" by "liftable over A", where A is a *totally ramified* extension of $W(k)$ (= ring of complete discrete valuations, finite and flat over $W(k)$, with residue field k, and of degree > 1 over $W(k)$): Lang [**L**] has indeed constructed in any characteristic $p > 0$, a smooth projective k-surface X liftable over such a ring A of degree 2 over $W(k)$ such that the Hodge to de Rham spectral sequence of X/k does not degenerate at E_1.

7.12. The decomposition statements to which we referred to at the end of 7.3 apply in particular to a *smooth curve* S over $T = \operatorname{Spec} k$ and with a scheme X over S having semi-stable reduction along a divisor with normal crossings $E \subset S$ (therefore étale over k), for which certain hypothesis of liftability modulo p^2 are satisfied. More precisely, if we assume that:

(i) There exists a lifting $(E^\sim \subset S^\sim)$ of $(E \subset S)$ over $W_2 = W_2(k)$ (with $S^\sim$ smooth and $E^\sim$ a relative divisor with normal crossings, i.e. étale over W_2), admitting a lifting $F^\sim : S^\sim \to S^\sim$ of the Frobenius (absolute) of S such that $(F^\sim)^{-1}(E^\sim) = pE^\sim$ [21],

[21] This notation denotes the divisor induced from $E^\sim$ by the raising to the p-th power of its local equations.

(ii) f is lifted by $f^{\sim} : X^{\sim} \to S^{\sim}$ having semi-stable reduction along $E^{\sim}$, then $\tau_{<p}F_*\omega^\bullet_{X/S}$ and $\tau_{<p}F_*\omega^\bullet_D$ (where $D = E \times_S X$) are decomposable, (and therefore $F_*\omega^\bullet_{X/S}$ and $F_*\omega^\bullet_D$ are also if X is of relative dimension $< p$ over S).

7.13. The relative result of $\omega^\bullet_D$ suggests the following problem of a different characteristic. Now denote by S the spectrum of $W = W(k)$, and $E = \operatorname{Spec} k$ the closed point of S. Let X be an S-scheme. By analogy with the definition given in 7.3, we say that X has *semi-stable reduction* if, locally in the étale topology (on X and on S), X is smooth over the subscheme of $\mathbb{A}^n_S = S[x_1, \ldots, x_n]$ with equation $x_1 \cdots x_n = p$. Assume that X has semi-stable reduction. Then X is a regular scheme, its fiber X_K at the generic point $\operatorname{Spec} K$ of S ($K =$ the field of fractions of W) is smooth, and its fiber $D = E \times_S X$ at the closed point is a "divisor with normal crossings" in X. In this situation, we define a complex $\omega^\bullet_{X/S}$ analogous to (7.3.1), which is the unique extension, with locally free components of a finite type of the de Rham complex of U over S, where U is the smooth open part of X over S. If $X = S[x_1, \ldots, x_n]/(x_1 \cdots x_n - p)$, the $\mathcal{O}_X$-module $\omega^1_{X/S}$, considered as a subsheaf of $\Omega^1_{X_K/K}$, is identified with $(\bigoplus \mathcal{O}_X dx_i/x_i)/\mathcal{O}_X(\sum dx_i/x_i)$. The complex $\omega^\bullet_D$, defined by the formula (7.3.6), has locally free components of finite type over D, and coincides with the de Rham complex $\Omega^\bullet_{D/k}$ over the smooth open part of D. One has $\omega^i_{X/S} = \Lambda^i\omega^1_{X/S}$ and $\omega^i_D = \Lambda^i\omega^1_D$. According to a result of Hyodo [**Hy**] (generalized by Kato in [**Ka2**]), the complex $\omega^\bullet_D$ gives rise to a Cartier isomorphism $C^{-1} : \omega^i_{D'} \simeq \mathcal{H}^i F_*\omega^\bullet_D$. The complexes $\omega^\bullet_{X/S}$ and $\omega^\bullet_D$ play an important role in the recent developments of the theory of p-adic periods (cf. [**I4**] for a general view).

Problem 7.14. With the notation of 7.13, let X be a semi-stable S-scheme with fiber D at the closed point of S. Is it the case that $\tau_{<p}F_*\omega^\bullet_D$ is decomposable (in $D(D')$) ?

Note that the decomposability of $\tau_{<p}F_*\omega^\bullet_D$ is equivalent to that of $\tau_{\leq 1}F_*\omega^\bullet_D$ (same argument as in step C of the proof of 5.1). The answer is yes according to 5.5 if X is *smooth* over S. (NB. What was called X (resp. S) in 5.5 is here D (resp. E).) The answer is still yes (for trivial cohomological reasons (cf. 4.6 (a))) if X is affine, or if X is of relative dimension ≤ 1. But the general case is unknown.

7.15. Finally, with regard to the *vanishing theorem*, we cannot prove by the methods of characteristic $p > 0$, the classical results of Grauert-Riemenschneider or Kawamata-Viehmeg. Neither can we generalize 5.9 (5) in dimension > 2. See [**E-V**] for a discussion of these questions.

8. Appendix: parallelizability and ordinary

In this section, k denotes a perfect field of characteristic $p > 0$. We denote by $W_n = W_n(k)$ the ring of Witt vectors of length n over k. We begin by giving a proof of the result mentioned in 7.11:

PROPOSITION 8.1. *Let X be a smooth k-scheme. Assume that X lifts over W_2 and that for any $n \geq 1$, the product morphism $(\Omega^1_{X/k})^{\otimes n} \to \Omega^n_{X/k}$ admits a section. Then X is DR-decomposable* (7.9).

We will have need of the following lemma:

LEMMA 8.2. *Let S and T be as in* 5.1, *X a smooth scheme over S, Z' a (smooth) lifting of X' over T. Let*

$$\varphi^1 : \Omega^1_{X'/S}[-1] \to F_*\Omega^\bullet_{X/S}$$

be the homomorphism $\varphi^1_{Z'}$ of $D(X')$ defined in step B of the proof of 5.1, *and for $n \geq 1$,*

$$\psi^n : (\Omega^1_{X'/S})^{\otimes n}[-n] \to F_*\Omega^\bullet_{X/S}$$

the composite homomorphism $\pi \circ (\varphi^1)^{\overset{L}{\otimes} n}$, where $\pi : (F_*\Omega^\bullet_{X/S})^{\overset{L}{\otimes} n} \to F_*\Omega^\bullet_{X/S}$ is the product homomorphism. Likewise, we denote by $\pi : (\Omega^1_{X'/S})^{\otimes n} \to \Omega^n_{X'/S}$ the product homomorphism. Then for any local section ω of $(\Omega^1_{X'/S})^{\otimes n}$, one has

$$\mathcal{H}^n\psi^n(\omega) = C^{-1} \circ \pi(\omega),$$

where $C^{-1} : \Omega^n_{X'/S} \to \mathcal{H}^n F_*\Omega^\bullet_{X/S}$ is the Cartier isomorphism.

PROOF. It suffices to show this for ω of the form $\omega_1 \otimes \cdots \otimes \omega_n$, where ω_i is a local section of $\Omega^1_{X'/S}$. By functoriality in the $E_i \in D(X')$, of the product

$$\mathcal{H}^1 E_1 \otimes \cdots \otimes \mathcal{H}^1 E_n \to \mathcal{H}^n(E_1 \overset{L}{\otimes} \cdots \overset{L}{\otimes} E_n), \quad a_1 \otimes \cdots \otimes a_n \mapsto a_1 \cdots a_n,$$

one has

$$\mathcal{H}^n((\varphi^1)^{\overset{L}{\otimes} n})(\omega_1 \otimes \cdots \otimes \omega_n) = (\mathcal{H}^1\varphi^1)(\omega_1) \cdots (\mathcal{H}^1\varphi^1)(\omega_n)$$

in $\mathcal{H}^n((F_*\Omega^\bullet_{X/S})^{\overset{L}{\otimes} n})$. Since $\mathcal{H}^1\varphi^1 = C^{-1}$, it follows that

$$\mathcal{H}^n\psi^n(\omega_1 \otimes \cdots \otimes \omega_n) = C^{-1}(\omega_1) \wedge \cdots \wedge C^{-1}(\omega_n)$$

in $\mathcal{H}^n F_*\Omega^\bullet_{X/S}$, and therefore that

$$\mathcal{H}^n\psi^n(\omega_1 \otimes \cdots \otimes \omega_n) = C^{-1}(\omega_1 \wedge \cdots \wedge \omega_n) = C^{-1} \circ \pi(\omega_1 \otimes \cdots \otimes \omega_n)$$

by the multiplicative property of the Cartier morphism.

PROOF OF 8.1. Let us choose for each n, a section s of the product $\pi : (\Omega^1_{X/k})^{\otimes n} \to \Omega^n_{X/k}$. Still let π and s be the morphisms relative to X' which result from this. Let us choose (cf. 3.9) a lifting Z' of X' over W_2, and define ψ^n as in 8.2, with $\varphi^1 = \varphi^1_{Z'}$. Let

$$\varphi^n : \Omega^n_{X'/k}[-n] \to F_*\Omega^\bullet_{X/S}$$

be the composite morphism $\psi^n \circ s$ (where s still denotes, by abuse of notation, the corresponding section of $(\Omega^1_{X/k}[-1])^{\otimes n} \to \Omega^n_{X/k}[-n]$). It is a question of checking that $\mathcal{H}^n\varphi^n = C^{-1}$. However, according to 8.2, if a is a local section of $\Omega^n_{X'/k}$, then

$$\mathcal{H}^n\varphi^n(a) = (\mathcal{H}^n\psi^n)(sa) = C^{-1}(\pi sa) = C^{-1}(a),$$

that which completes the proof.

COROLLARY 8.3. *Let X be a smooth parallelizable k-scheme, i.e. such that the $\mathcal{O}_X$-module $\Omega^1_{X/k}$ is free (of finite type). Then for X to be DR-decomposable, it is necessary and sufficient that X lifts over W_2.*

PROOF. It suffices to prove the sufficiency. One can assume that k is algebraically closed. Let us choose a rational point x of X over k and an *isomorphism* $\alpha : \Omega^1_{X/k} \simeq f^*E$, where $f : X \to \operatorname{Spec} k$ is the projection and E is the k-vector space $x^*(\Omega^1_{X/k})$. Via α, a section of the surjective homomorphism $E^{\otimes n} \to \Lambda^n E$ extends to a section of $(\Omega^1_{X/k})^{\otimes n} \to \Omega^n_{X/k}$. Now apply 8.1.

Recall (5.5.1) the following definition:

DEFINITION 8.4. *Let X be a smooth and proper k-scheme. We say that X is ordinary if for any (i,j), one has $H^j(X, B\Omega^i_{X/k}) = 0$, where $B\Omega^i_{X/k} = d\Omega^{i-1}_{X/k}$ is the sheaf of boundaries in degree i, of the de Rham complex.*

This condition is equivalent to $H^j(X', F_*B\Omega^i_{X/k}) = 0$ for any (i,j). Recall (3.6) that $F_*B\Omega^i_{X/k} = B^iF_*\Omega^\bullet_{X/k}$ and $F_*Z\Omega^i_{X/k} = Z^iF_*\Omega^\bullet_{X/k}$ are locally free $\mathcal{O}_X$-modules of finite type.

8.5. For a given smooth and proper k-scheme X, there exists a link, highlighted by Mehta and Srinivas [**Me-Sr**], between the properties to be DR-decomposable, parallelizable, and ordinary. This link expresses itself by the means of a concept close to that of DR-decomposability, introduced earlier by Mehta and Ramanathan [**Me-Ra**], which is the following. We say that a smooth k-scheme X is *Frobenius-decomposable* ("Frobenius-split") if the canonical homomorphism $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ admits a retraction, i.e. the exact sequence of $\mathcal{O}_{X'}$-modules (cf. 3.5)

$$0 \to \mathcal{O}_{X'} \to F_*\mathcal{O}_X \xrightarrow{d} F_*B\Omega^1_{X/k} \to 0 \tag{8.5.1}$$

is split. We first observe that *if X is Frobenius-decomposable*, X is liftable over W_2 (or, that which amounts to the same (5.9 (1) (a)), $\tau_{<p}F_*\Omega^\bullet_{X/k}$ is decomposable): The obstruction to the lifting, which is the class of the extension

$$0 \to \mathcal{O}_{X'} \to F_*\mathcal{O}_X \xrightarrow{d} F_*Z\Omega^1_{X/k} \xrightarrow{C} \Omega^1_{X'/k} \to 0,$$

composed with (8.5.1) and the extension

$$0 \to F_*B\Omega^1_{X/k} \to F_*Z\Omega^1_{X/k} \xrightarrow{C} \Omega^1_{X'/k} \to 0, \tag{8.5.2}$$

is zero. In general, we are unaware if "Frobenius-decomposable" implies "DR-decomposable". This is the case according to 8.3, if X is parallelizable. But the converse is false. Indeed one has the following result ([**Me-Sr**] 1.1): If X is a smooth and proper k-scheme, parallelizable, then the following conditions are equivalent:
(a) X is Frobenius decomposable;
(b) the extension (8.5.2) is split;
(c) X is ordinary;
(d) (for X of pure dimension d) the homomorphism $F^* : H^d(X', \mathcal{O}_{X'}) \to H^d(X, \mathcal{O}_X)$ induced by the Frobenius is an isomorphism.

In particular, if X is ordinary and parallelizable, X lifts over W_2 (Nori-Srinivas ([**Me-Sr**] Appendix) show in fact that for X projective, X lifts to a smooth projective scheme over W). Moreover – this is the principal result of [**Me-Sr**] – if k is algebraically closed and X connected, there exists a Galois étale lifting $Y \to X$ of order of a power of p such that Y is an abelian variety.

If X is projective and smooth over k, ordinary and parallelizable, Nori-Srinivas (loc. cit.) show more precisely that there exists a unique couple (Z, F_Z), where Z

is a lifting (projective and smooth) of X over W_2 (resp. W_n ($n \geq 2$ given), resp. W) and $F_Z : Z \to Z'$ a lifting of $F : X \to X'$, where Z' is the inverse image of Z by the Frobenius automorphism of W_2 (resp. W_n, resp. W). The existence and uniqueness of this lifting, said *canonical*, was first established by Serre-Tate [**Se-Ta**] in the case of abelian varieties. As indicated in 5.5.1, this result admits a converse, without the assumption of parallelizability.

PROPOSITION 8.6. *Let X be a smooth and proper k-scheme. Assume that there exists schemes Z and Z' lifting respectively X and X' over W_2 and a W_2-morphism $G : Z \to Z'$ lifting $F : X \to X'$* [22]. *Then X is ordinary.*

This result was obtained independently by Nakkajima [**Na**].

8.7. PROOF OF 8.6. Let $G : Z \to Z'$ be a lifting of F and $\varphi = \varphi_G : \bigoplus \Omega^i_{X'}[-i] \to F_*\Omega^\bullet_X$ the associated homomorphism of complexes, defined in (5.3.1) (one omits $/k$ from the notation of differentials). This homomorphism sends $\Omega^i_{X'}$ into $F_*Z\Omega^i_X$ (notation of 8.4) and splits the exact sequence (cf. 3.5)

$$0 \to F_*B\Omega^i_X \to F_*Z\Omega^i_X \xrightarrow{C} \Omega^i_{X'} \to 0. \tag{8.7.1}$$

We prove, by descending induction on i, that $H^*(X, B\Omega^i_X) = 0$ (i.e. that $H^n(X, B\Omega^i_X) = 0$ for all n). For $i > \dim X$, $B\Omega^i_X = 0$. Fix i and assume that we proved $H^*(X, B\Omega^j_X) = 0$ for $j \geq i$. Then we show that $H^*(X, B\Omega^{i-1}_X) = 0$. By the exact cohomology sequence associated to the exact sequence

$$0 \to F_*Z\Omega^{i-1}_X \to F_*\Omega^{i-1}_X \xrightarrow{d} F_*B\Omega^i_X \to 0, \tag{8.7.2}$$

the induction hypothesis implies that for any n, one has

$$H^n(X', F_*Z\Omega^{i-1}_X) \xrightarrow{\sim} H^n(X, \Omega^{i-1}_X),$$

and therefore

$$\dim H^n(X', F_*Z\Omega^{i-1}_X) = \dim H^n(X, \Omega^{i-1}_X) = \dim H^n(X', \Omega^{i-1}_{X'}). \tag{8.7.3}$$

The sequence (8.7.1) (relative to $i-1$) being split, implies that the exact sequence of cohomology gives the short exact sequence

$$0 \to H^n(X', F_*B\Omega^{i-1}_X) \to H^n(X', F_*Z\Omega^{i-1}_X) \xrightarrow{C} H^n(X', \Omega^{i-1}_{X'}) \to 0.$$

The equality (8.7.3) implies that in this situation C is an isomorphism, and therefore that $H^n(X', F_*B\Omega^{i-1}_X) = 0$, which concludes the proof.

REMARK 8.8. The reader familiar with logarithmic structures will have observed that 8.6 and its proof extends to the case where k is replaced by a logarithmic point $\underline{k} = (k, M)$ with underlying point k and X by a log scheme $\underline{X} = (X, L)$ log smooth and of Cartier type over $\underline{k}$ ([**Ka2**]), proper over k. If one assumes that $(\underline{X}, \underline{F})$ is lifted over $W_2(\underline{k})$ (cf. [**Hy-Ka**] 3.1), then $\underline{X}$ is ordinary, i.e. $H^j(X, B\omega^i_{\underline{X}}) = 0$ for all i and all j.

[22] We do not assume that Z' is the inverse image of Z by the Frobenius of W_2.

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