THE ALGEBRAIC DIMENSION OF COMPACT COMPLEX THREEFOLDS WITH VANISHING SECOND BETTI NUMBER

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1. Introduction

The paper [CDP98] studied compact complex threefolds X such that the second Betti number $b_2(X) = 0$. The main result is based on Lemma 1.5, which is certainly not true in general (but might still hold in the context of the paper). In any case, some of the statements and proofs need to be adapted which is done in this Corrigendum, with special regards to potential complex structures on the 6-sphere.

2. Statement of the results

We prove Theorem 2.1 in [CDP98] in full generality in case X has a meromorphic non-holomorphic map $X \dashrightarrow \mathbb{P}_1$. In the remaining case, X has algebraic dimension 1 and the algebraic reduction $f : X \to C$ is holomorphic. In this case we prove that $c_3(X) \leq 0$; for simplicity, we will assume not only that $b_2(X) = 0$ but slightly stronger that $H^2(X, \mathbb{Z}) = 0$ and moreover that $H^1(X, \mathbb{Z}) = 0$, hence $C \simeq \mathbb{P}_1$. This suffices to treat the main application of complex structures on S^6.

In summary, we shall prove

2.1. Theorem. *Let X be a $3-$dimensional compact complex manifold with $b_2(X) = 0$. Assume that there exists a non-holomorphic meromorphic non-constant map $g : X \dashrightarrow \mathbb{P}_1$. Let B be a holomorphic vector bundle on X. Then*

- *a)* $H^i(X, B \otimes \mathcal{M}) = 0$ *for* $i \geq 0$ *and* $\mathcal{M} \in \mathrm{Pic}^\circ(X)$ *generic.*
- *b)* $\chi(X, B \otimes \mathcal{M}) = 0$ *for all* $\mathcal{M} \in \mathrm{Pic}^\circ(X)$
- *c)* $c_3(X) = 0$*; i.e., either* $b_1(X) = 0$ *and* $b_3(X) = 2$ *or* $b_1(X) = 1$ *and* $b_3(X) = 0$.

Another proof has been given in [LSS18].

Theorem 2.1 takes care of all threefolds X with $1 \leq a(X) \leq 2$ and $b_2(X) = 0$ except of those whose algebraic reduction $f : X \to C$ is holomorphic onto a curve C. In this case the general fiber has Kodaira dimension $\kappa(X_c) \leq 0$.

By topological considerations and surface classification, the general fiber of f is either a torus, a primary Kodaira surface or a surface of type VII; in the latter case it is actually a primary Hopf surface or an Inoue surface, Lemma 4.2 and Lemma 4.3. The case that the general fiber is a Kodaira surface is ruled out in Corollary 5.3. Then we show

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2.2. Theorem. *Let X be a $3-$dimensional compact complex manifold with $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$ and algebraic dimension $a(X) = 1$. Assume that the algebraic reduction $f : X \to C$ is holomorphic. Then*

a) $H^i(X, T_X \otimes \mathcal{M}) = 0$ *for* $i \neq 1$.
b) $\chi(X, T_X \otimes \mathcal{M}) \leq 0$.
c) $c_3(X) \leq 0$.

As a consequence we deduce

2.3. Corollary. *Let X be a 3-dimensional compact complex manifold homeomorphic to S^6. Then $a(X) = 0$.*

Proof. Obviously, $a(X) \neq 3$, otherwise X is Moishezon and therefore $b_2(X) \neq 0$. If $a(X) = 2$, then there exists a meromorphic non-holomorphic map $g : X \to \mathbb{P}_1$. Then we conclude by Theorem 2.1 that $c_3(X) = 0$. By Hopf's theorem, $c_3(X) = \chi_{\mathrm{top}}(S^6) = 2$, a contradiction. If $a(X) = 1$ and the algebraic reduction $g : X \dashrightarrow C$ is not holomorphic, then $C \simeq \mathbb{P}_1$, and we conclude again by Theorem 2.1. If $a(X) = 1$ and the algebraic reduction $g : X \dashrightarrow C$ is holomorphic, then we apply Theorem 2.2 and obtain the same contradiction as before. □

We comment on the strategy to prove Theorem 2.2. The arguments of [CDP98] show the following

2.4. Proposition. *Let X be a 3-dimensional compact complex manifold with $b_2(X) = 0$ and algebraic dimension $a(X) = 1$. Assume that the algebraic reduction $f : X \to C$ is holomorphic. If*

$$R^2 f_*(T_X \otimes \mathcal{L}) = 0 \tag{1}$$

for some (or general) $\mathcal{L} \in \mathrm{Pic}(X)$, then the assertions of Theorem 2.2 hold.

Equation (1) is equivalent to the vanishing

$$H^2(X_c, T_X \otimes \mathcal{L}|X_c) = 0$$

for all complex-analytic fibers X_c (with the natural fiber structure) of f, equivalently,

$$H^0(X_c, \Omega^1_X \otimes \mathcal{L}^*|X_c) = 0.$$

The key is to show that the restriction $\mathcal{L}|X_c$ of some or the general line bundle $\mathcal{L}$ to any fiber is never torsion. Then we compute directly on X_c; here the case when X_c is singular, in particular non-normal, needs special care.

For further informations on the problem of complex structures on S^6, we refer to [Et15] and to volume 57 of the journal Differential Geometry and its Applications.

3. Proof of Theorem 2.1

Instead of simply pointing out the additions in the proof of [CDP98, Theorem 2.1] to be made, we give full details for the benefit of the reader. Notice first that (2.1)(b) follows from (2.1)(a) since $\chi(X, B \otimes \mathcal{M})$ does not depend on $\mathcal{M}$, and (2.1)(c) is a consequence of (2.1)(b) by applying Riemann-Roch to $B = T_X$ and $\mathcal{M} = \mathcal{O}_X$. So it remains to prove (2.1)(a). By Serre duality, (2.1)(a) needs only to be shown for

$i = 0$ and $i = 2$. The case $i = 0$ follows from [CDP98, Cor1.3], since X does carry effective non-zero divisors. Thus it remains to prove

$$H^2(X, B \otimes \mathcal{M}) \tag{2}$$

for generic, equivalently one, line bundle $\mathcal{M}$. Let $g : X \dashrightarrow \mathbb{P}_1$ be a non-constant non-holomorphic meromorphic map and $\sigma : \hat{X} \to X$ be a resolution of indeterminacies of g. Let

$$f : \hat{X} \to C \simeq \mathbb{P}_1$$

be the fiber space given by the Stein factorization of the holomorphic map $\sigma \circ g$. Replacing g by the induced meromorphic map $X \dashrightarrow C$, we may assume from the beginning that g has connected fibers, hence no Stein factorization has to be taken. Note that the canonical map

$$H^2(X, B \otimes \mathcal{M}) \to H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}))$$

is injective (by the Leray spectral sequence). Hence Equation (2) follows from

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) = 0. \tag{3}$$

Fix an ample divisor A on C. Then $f^*(\mathcal{O}_C(A))$ can be written as

$$f^*(\mathcal{O}_C(A)) = \sigma^*(\mathcal{L}) \otimes \mathcal{O}_{\hat{X}}(-E) \tag{4}$$

with a line bundle $\mathcal{L}$ on X and a suitable effective divisor E which is supported on the exceptional set of σ and projects onto C. To verify Equation (3) it suffices to show that

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}) \otimes \mathcal{O}_{\hat{X}}(-tE)) = 0 \tag{5}$$

for some effective divisor E supported on the exceptional locus of σ and some $t \geq 0$. In fact, consider the exact sequence

$$H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M}) \otimes \mathcal{O}_{\hat{X}}(-tE)) \to H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M})) \to H^2(tE, \sigma^*(B \otimes \mathcal{M}))$$

and note that

$$H^2(tE, \sigma^*(B \otimes \mathcal{M})) = 0.$$

This last vanishing is seen as follows: let Z_t be the complex subspace of X defined by the ideal sheaf $\sigma_*(\mathcal{O}_{\hat{X}}(-tE))$. Then by

$$R^q(\sigma_{|tE})_*(\mathcal{O}_{tE}) = 0$$

for $q = 1, 2$ and the Leray spectral sequence,

$$H^2(tE, \sigma^*(B \otimes \mathcal{M})) = H^2(Z_t, B \otimes \mathcal{M}).$$

Now the last group vanishes since $\dim Z_t = 1$.

By replacing $\mathcal{M}$ by $\mathcal{M} \otimes \mathcal{L}^{t+k}$ for some positive integer k and using (4), Equation (5) reads

$$\begin{aligned} H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^{t+k} \otimes \mathcal{O}_{\hat{X}}(-tE))) &= \\ = H^2(\hat{X}, \sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k \otimes f^*(\mathcal{O}_C(tA))). \end{aligned} \tag{6}$$

By the Leray spectral sequence applied to f, Equation (6) comes down to verify

$$R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) = 0 \tag{7}$$

and

$$H^1(C, R^1 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) \otimes \mathcal{O}_C(tA)) = 0 \tag{8}$$

for suitable positive integers k and t and some line bundle $\mathcal{M}$.

To prove (7), let $C^* \subset C$ be the finite set of points $c \in C$ such that some component of the fiber X_c does not meet E. In particular, if X_c is irreducible, then $c \in C \setminus C^*$. Notice also that

$$R^2 f_*(\sigma^*(B \otimes \mathcal{L}^k))|\{c\} \simeq H^2(\hat{X}_c, \sigma^*(B \otimes \mathcal{L}^k)),$$

by the standard base change theorem. Applying Serre duality, we obtain

$$H^2(\hat{X}_c, \sigma^*(B \otimes \mathcal{L}^k)) \simeq H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{L}^{-k}) \otimes \omega_{\hat{X}_c}) \simeq$$
$$\simeq H^0(\hat{X}_c, \sigma^*(B^*) \otimes \mathcal{O}_{\hat{X}_c}(-kE) \otimes \omega_{\hat{X}_c}).$$

We claim first that there is a number k_0 such that for $k \geq k_0$,

$$\text{supp } R^2 f_*(\sigma^*(B \otimes \mathcal{L}^k)) \subset C^*. \tag{9}$$

This is equivalent to saying that

$$H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{M}^*) \otimes \mathcal{O}_{\hat{X}_c}(-kE) \otimes \omega_{\hat{X}_c}) = 0$$

for $c \notin C^*$. Fixing any point $c_0 \in C^*$, this number $k_0 = k_0(c)$ clearly exists; in case X_c is reducible, we apply [CDP98, Prop.1.1]. Hence the support of the direct image sheaf $R^2 f_*(\sigma^*(B \otimes \mathcal{L}^{k_0}))$ is contained in a finite set C_{k_0} in C. Since $\sigma^*(\mathcal{L})|X_c$ is effective, it follows that $C_k \subset C_{k_0}$. Thus, enlarging k_0 if necessary, (9) is verified.

Hence we only need to consider the fibers $\hat{X}_c$ with $c \in C^*$. Let P the set of line bundles $\mathcal{M}$ of the form

$$\mathcal{M} = \mathcal{O}_X(\sum m_i S_i)$$

with m_i positive integers and S_i fiber components of f not meeting E (the S_i considered as surfaces in X).

Since all line bundles $\mathcal{M} \in P$ are trivial on $X \setminus f^{-1}(C^*)$, our previous considerations imply the existence of a number k_0 such that for all $k \geq k_0$ and all $\mathcal{M} \in P$,

$$\text{supp } R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) \subset C^*.$$

We are now going to construct a line bundle $\mathcal{M} \in P$ such that

$$R^2 f_*(\sigma^*(B \otimes \mathcal{M} \otimes \mathcal{L}^k)) = 0$$

for a suitable number k. Fix a point $c \in C^*$. Let $F_0 \subset X_c$ be the sub-divisor of $\hat{X}_c$ consisting of all components meeting E; let further $F_1 \subset \hat{X}_c$ be the sub-divisor consisting of all components which meet F_1 but not E. Continuing in this way we obtain a decomposition

$$\hat{X}_c = \sum_{i=0}^{r} F_r$$

of sub-divisors $F_j \subset X_c$ who pairwise do not have common components and which have the property that all components of F_j meet F_{j-1} but do not meet F_k for $k < j-1$. Now choose $m_r \gg 0$ such that

$$H^0(F_r, \mathcal{O}_{\hat{X}}(-m_r F_{r-1}) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|F_r) = 0$$

This is possible by our construction. Next choose $m_{r-1} \geq m_r$ such that

$$H^0(F_{r-1}, \mathcal{O}_{\hat{X}}(-m_{r-1} F_{r-2}) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|F_{r-1}) = 0.$$

Since $\text{supp}(F_{r-2}) \cap \text{supp}(F_r) = \emptyset$, we obtain

$$H^0(F_{r-1} + F_r, \mathcal{O}_{\hat{X}}(-m_{r-1}(F_{r-2} + F_{r-1}) \otimes \sigma^*(B^*) \otimes \omega_{\hat{X}_c}|F_{r-1}) = 0.$$

Continuing in this way, we obtain a line bundle

$$\mathcal{M}'(c) = \mathcal{O}_X(\sum_{i=2}^{r} m_i F_i)$$

(having in mind that the divisors $F_i, i \geq 1$ do not meet the exceptional locus of σ), such that

$$H^0\big(\sum_{i=2}^{r} F_i, \sigma^*(B^* \otimes \mathcal{M}'(c)) \otimes \omega_{\hat{X}_c} |(\sum_{i=2}^{r} F_i)\big) = 0.$$

Since the component F_0 meets E, it needs a special treatment. We observe that

$$\mathcal{O}_{\hat{X}}(F_0) \simeq \sigma^*\big(\mathcal{O}_X(\sigma(F_0))\big) \otimes \mathcal{O}_{\hat{X}}(-E')$$

with some effective σ-exceptional divisor E'. Hence, choosing $m_1 \gg 0$ and setting

$$\mathcal{M}(c) = \mathcal{M}'(c) \otimes \mathcal{O}_X(m_1 \sigma(F_0)),$$

then

$$H^0\big(\sum_{i=1}^{r} F_i, \sigma^*(B^* \otimes \mathcal{M}(c)^*) \otimes \omega_{\hat{X}_c} |(\sum_{i=1}^{r} F_i)\big) = 0.$$

Finally, enlarging k, we get

$$H^0(\hat{X}_c, \sigma^*(B^* \otimes \mathcal{M}(c)^*) \otimes \mathcal{O}_{\hat{X}}(-kE) \otimes \omega_{\hat{X}_c} |X_c) = 0.$$

Setting

$$\mathcal{M} = \bigotimes_{c \in C^*} \mathcal{M}(c),$$

this settles (7).

As to (8), we fix k as in (7) and then apply Serre's vanishing theorem to the ample divisor A to obtain t.

4. General structure of the fibers

From now on - for the rest of the paper - we let X be a compact complex manifold of dimension 3 with

$$H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0$$

and holomorphic algebraic reduction $f : X \to C$ to the curve $C \simeq \mathbb{P}_1$. We will freely use the theory of compact complex surfaces, in particular of non-Kähler surfaces, and refer to [BHPV04] as general reference.

An application of Riemann-Roch gives

4.1. Lemma. *$\chi(D, \mathcal{O}_D) = 0$ for all effective divisors D on X.*

Proof. By Riemann-Roch,

$$\chi(X, \mathcal{O}_X(-D)) = \chi(X, \mathcal{O}_X),$$

hence

$$\chi(D, \mathcal{O}_D) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-D)) = 0.$$

□

4.2. Lemma. *Let s be the number of singular fibers and r be the numbers of irreducible components of the singular fibers. Then*

$$r = s - 1 + b_1(X_c),$$

where X_c is a smooth fiber. Moreover $H_1(X_c, \mathbb{Z})$ is torsion free for all smooth fibers X_c.

Proof. The first assertion is [CDP98, Lemma 3.2]. For the second, fix a smooth fiber X_c and let $A \subset$ the union of all singular fibers of f and set $X' = X \setminus A$. As seen in the proof of [CDP98, Lemma 3.2],

$$H_1(X', \mathbb{Z}) \simeq H_1(C \setminus f(A), \mathbb{Z}) \oplus H_1(X_c, \mathbb{Z}),$$

hence it suffices to show that $H_1(X', \mathbb{Z})$ is torsion free. To do this, we consider the cohomology sequence for pairs,

$$0 = H^4(X, \mathbb{Z}) \to H^4(A, \mathbb{Z}) \to H^5(X, A, \mathbb{Z}) \to H^5(X, \mathbb{Z}) \to 0.$$

Notice first that $H^4(A, \mathbb{Z})$ is torsion free. Further, $H^5(X, \mathbb{Z})$ is torsion free by the universal coefficent theorem, since $H_4(X, \mathbb{Z})$ is torsion free: by Poincaré duality,

$$H_4(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) = 0.$$

Actually, $H^5(X, \mathbb{Z}) = 0$. Consequently,

$$H^5(X, A, \mathbb{Z}) \simeq H_1(X', \mathbb{Z})$$

is torsion free. □

4.3. Lemma. *Let X_c be a smooth fiber of f. Then X_c is either a primary Hopf surface, an Inoue surface, a torus or a primary Kodaira surface with torsion free first homology group.*

Proof. Note first that K_{X_c} is topologically trivial, since K_X is topologically trivial, due to $b_2(X) = 0$. Then we use the tangent sequence

$$0 \to T_{X_c} \to T_X | X_c \to N_{X_c/X} \simeq \mathcal{O}_{X_c} \to 0$$

and observe that $c_2(X) = 0$, since $b_4(X) = 0$. Thus $c_2(X_c) = 0$. Since the (sufficiently) general fiber of an algebraic reduction has non-positive Kodaira dimension, [Ue75, Thm. 12.1], so does every smooth fiber (see e.g. [BHPV04, VI.8.1]. Then we conclude by surface classification and using the torsion freeness of $H_1(X_c, \mathbb{Z})$, Lemma 4.2.

□

4.4. Corollary. *All fibers of f are irreducible unless the general fiber of f is a torus or a primary Kodaira surface with torsion free first homology.*

We fix some notations for the rest of the paper.

4.5. Notation. Let $S \subset X$ be an irreducible reduced surface. In particular, S is Gorenstein. We denote by ω_S the dualizing sheaf, which is a line bundle. Let

$$\eta : \tilde{S} \to S$$

be the normalization of S; denote by $N \subset S$ the non-normal locus, equipped with the complex structure given by the conductor ideal. Let $\tilde{N} \subset \tilde{S}$ be the complex-analytic preimage. Let

$$\pi : \hat{S} \to \tilde{S}$$

be a minimal desingularization and

$$\sigma : \hat{S} \to S_0$$

be a minimal model. For the class of ω_S we write K_S, analogously for $\omega_{\tilde{S}}$, etc.

4.6. Lemma. *In the notations of (4.5), we have*

$$\omega_{\tilde{S}} \simeq \eta^*(\omega_S) \otimes \mathcal{I}_{\tilde{N}}$$

and

$$\omega_{\hat{S}} \simeq \pi^*\eta^*(\omega_S) \otimes \pi^*(\mathcal{I}_{\tilde{N}}) \otimes \mathcal{O}_{\hat{S}}(E) = \pi^*\eta^*(\omega_S) \otimes \mathcal{O}_{\hat{S}}(-\hat{N}) \otimes \mathcal{O}_{\hat{S}}(-\hat{E})$$

with an effective divisor E supported on the exceptional locus of π and $\hat{N}$ the strict transform of $\tilde{N}$ in $\hat{S}$. Moreover, there are exact sequences

$$0 \to \mathcal{O}_S \to \eta_*(\mathcal{O}_{\tilde{S}}) \to \omega_S^{-1} \otimes \omega_N \to 0$$

and

$$0 \to \mathcal{O}_N \to \eta_*(\mathcal{O}_{\tilde{N}}) \to \omega_S^{-1} \otimes \omega_N \to 0$$

Proof. [Mo82, chap.3, sect.8]. □

As an immediate consequence, we note

4.7. Proposition. *Let S be any irreducible reduced compact Gorenstein surface with $\omega_S \equiv 0$ and $\chi(S, \mathcal{O}_S) = 0$. Then*

a) $\chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) = \chi(N, \omega_S^{-1} \otimes \omega_N) = -\chi(N, \mathcal{O}_N)$;
b) $\chi(\tilde{N}, \mathcal{O}_{\tilde{N}}) = 0$.

Proof. The first equation in (a) follows from Lemma 4.6. As to the second equation in (a), observe by Serre duality

$$\chi(N, \omega_S^{-1} \otimes \omega_N) = -\chi(N, \omega_{S|N}) = -\chi(N, \mathcal{O}_N),$$

since $\omega_S \equiv 0$. For the same reasons

$$\chi(\tilde{N}, \mathcal{O}_{\tilde{N}}) = \chi(N, \mathcal{O}_N) + \chi(N, \omega_S^{-1} \otimes \omega_N) = 0.$$

□

4.8. Proposition. *Let X_c be a smooth fiber of f. Then*

$$H^0(X_c, T_{X|X_c}) \neq 0.$$

Proof. Consider the exact sequence

$$0 \to H^0(X_c, T_{X_c}) \to H^0(X_c, T_{X|X_c}) \xrightarrow{\kappa} H^0(X_c, N_{X_c/X}) \simeq \mathbb{C}.$$

If $H^0(X_c, T_{X_c}) \neq 0$, the assertion is clear. So it remains to treat the case that X_c has no vector fields. By Lemma 4.3 and [In74], X_c is an Inoue surface of type S_M or S_N^-, in which cases $H^1(X_c, T_{X_c}) = 0$. Thus X_c is rigid and κ is surjective, so that $H^0(X_c, T_X|X_c) \neq 0$ also in these cases.

□

4.9. Corollary. *Let $X_c = \lambda S$ be a fiber with S an irreducible singular surface and $\lambda \geq 1$. Then there exists a finite étale cover $S' \to S$ such that $H^0(S', T_{S'}) \neq 0$.*

Proof. We consider the tangent sequence

$$0 \to T_S \to T_{X|S} \xrightarrow{\kappa} N_{S/X}.$$

If $\lambda = 1$, then $N_{S/X} \simeq \mathcal{O}_S$. However, κ is not surjective, since S is singular. Hence

$$H^0(S, T_S) = H^0(S, T_{X|S}).$$

By semicontinuity and Proposition 4.8, $H^0(S, T_X|S) \neq 0$ and we conclude.
If $\lambda \geq 2$, arguing in the same way, we obtain a torsion line bundle $\mathcal{L}$ on X such that

$$H^0(S, T_S \otimes \mathcal{L}_{|S}) \neq 0.$$

Then we pass to a finite étale cover $\tilde{S} \to S$ to trivialize $\mathcal{L}_{|S}$.

□

4.10. Remark. A vector field $v \in H^0(S, T_S)$ induces canonically a vector field $v_0 \in H^0(S_0, T_{S_0})$. We shortly say that v_0 comes from S.

4.11. Proposition. *Let $X_c = \lambda S$ with S singular, irreducible. Then S is non-normal.*

Proof. Suppose S normal and let $\pi : \hat{S} \to S$ be a minimal desingularization and $\sigma : \hat{S} \to S_0$ a minimal model.
(a) Suppose that S has only rational singularities, hence S has only rational double points. Then $K_{\hat{S}} \equiv 0$, in particular $\hat{S}$ is a minimal surface containing (-2)-curves. By surface classification, $\hat{S}$ is either a K3 surface, an Enriques surface, of type VII or non-Kähler of Kodaira dimension $\kappa(\hat{S}) = 1$. The first two cases are impossible since

$$\chi(\hat{S}, \mathcal{O}_{\hat{S}}) = \chi(S, \mathcal{O}_S) = 0.$$

If $\hat{S}$ is of type VII, then it must be a Hopf surface or an Inoue surface, since $K_{\hat{S}} \equiv 0$, but these surfaces do not contain (-2)-curves. If $\kappa(\hat{S}) = 1$, then, since $\hat{S}$ has a vector field, it does not any rational curve, see e.g. [GH90, Satz 1].

(b) Suppose now that S has at least one irrational singularity. Then $\chi(\hat{S}, \mathcal{O}_{\hat{S}}) < \chi(S, \mathcal{O}_S) = 0$, hence

$$h^1(S_0, \mathcal{O}_{S_0}) = h^1(\hat{S}, \mathcal{O}_{\hat{S}}) \geq 2.$$

Suppose first that S_0 is not Kähler. By classification, S_0 has to be a primary Kodaira surface or $\kappa(S_0) = 1$. By Corollary 4.9, $H^0(S_0, T_{S_0}) \neq 0$ (up to finite étale cover). Choose a nonzero vector field v_0 coming from S; then v_0 does not have zeroes by classification, [GH90, Satz 1]; note that in case $\kappa(S_0) = 1$, S_0 is an elliptic bundle over a curve of genus at least 2. Hence we must have $\hat{S} = S_0$. But then $\hat{S}$ does not contain contractible curves, so that S is smooth, a contradiction.
Thus S_0 is Kähler. Since $K_{\hat{S}} = \pi^*(K_S) - E$ with E a non-zero effective divisor, $\kappa(S_0) = -\infty$ and S_0 is a ruled surface over a curve B of genus $g = g(B) \geq 2$. Since S has an irrational singularity, π must contract an irrational curve whose normalization necessarily has genus at least g. Thus

$$h^0(S, R^1\pi_*(\mathcal{O}_{\hat{S}})) \geq g$$

and therefore

$$1 - g = \chi(\hat{S}, \mathcal{O}_{\hat{S}}) \leq \chi(S, \mathcal{O}_S) - g = -g,$$

which is absurd. □

5. Kodaira surfaces and tori

In this section we consider the case that the general fiber of f is a Kodaira surface or a torus. We rule out the case of Kodaira fibers and show in the torus case that for general line bundles $\mathcal{L}$ on X, the restriction $\mathcal{L}_{|X_c}$ to any fiber is never torsion.

5.1. Proposition. *Assume that the general fiber of f is a Kodaira surface or a torus. Then $R^j f_*(\mathcal{O}_X)$ is locally free for all j, in fact, $h^j(X_c, \mathcal{O}_{X_c})$ is independent on $c \in C$.*

Proof. It suffices to show that $h^2(X_c, \mathcal{O}_{X_c})$ is independent of c. Indeed, since $h^0(X_c, \mathcal{O}_{X_c}) = 1$ for all c and since $\chi(X_c, \mathcal{O}_{X_c})$ is constant, $h^1(X_c, \mathcal{O}_{X_c})$ does not depend on c as well, and the assertions follow by Grauert's theorem. By Serre duality,

$$H^2(X_c, \mathcal{O}_{X_c}) = H^0(X_c, \omega_{X_c}) = H^0(X_c, \omega_X|X_c).$$

Setting $\mathcal{L} = f_*(\omega_X)$, a locally free sheaf of rank one, we obtain

$$\omega_X = f^*(\mathcal{L}) \otimes \mathcal{O}_X(\sum_i (m_i - 1)F_i),$$

where F_i are the non-reduced fiber components. In particular, $\omega_{X|X_c} = \mathcal{O}_{X_c}$ for all reduced fibers X_c and therefore

$$h^0(X_c, \omega_{X_c}) = 1$$

for all those c. So let X_c be a non-reduced fiber and set $Y = \mathrm{red}(X_c)$. We consider the complex subspace

$$Z = \sum (m_i - 1)F_i$$

of X_c and have an induced exact sequence

$$0 \to \mathcal{I}_{Z/X_c} \otimes \omega_{X|X_c} \simeq \mathcal{O}_{X_c} \to \omega_{X|X_c} \to \omega_{X|Z} \to 0.$$

Applying f_* and observing that

$$f_*(\mathcal{O}_{X_c}) = f_*(\omega_{X|X_c}) = \mathcal{O}_{\{c\}},$$

shows that the restriction map

$$H^0(X_c, \omega_{X|X_c}) \to H^0(Z, \omega_{X|Z})$$

vanishes. Since

$$\dim H^0(X_c, \omega_{X|X_c}) = \dim H^0(X_c, \mathcal{O}_{X_c}) = 1,$$

we conclude $h^0(X_c, \omega_{X_c}) = 1$.

□

5.2. Corollary. *Let F be a general smooth fiber of f. Then the restriction map*

$$H^1(X, \mathcal{O}_X) \to H^1(F, \mathcal{O}_F)$$

is surjective.

Proof. This is Theorem 3.1 in [CDP98]; the proof works since we now know that $R^1 f_*(\mathcal{O}_X)$ is locally free. □

As a consequence, we obtain

5.3. Corollary.
The general fiber of f cannot be a Kodaira surface.

Proof. This is Proposition 3.6 in [CDP98]. In the proof of Proposition 3.6, Theorem 3.1 is used which is now established by Corollary 5.2. Notice that in Step 2 of the proof of Proposition 3.6 in [CDP98], the local freeness of $R^j f_*(\tilde{\mathcal{L}})$ is used only generically. □

5.4. Remark. The same arguments also rule out Hopf surfaces of algebraic dimension one.

From now - for the remainder of this section - we assume that the general of f is a torus.

5.5. Proposition. $R^1 f_*(\mathcal{O}_X)) \simeq \mathcal{O}_C(b_1) \oplus \mathcal{O}_C(b_2)$ *with* $b_j \geq 0$.

Proof. By Proposition 5.1, the sheaf $R^1 f_*(\mathcal{O}_X)$ is locally free of rank two. Write

$$R^1 f_*(\mathcal{O}_X) = \mathcal{O}_C(b_1) \oplus \mathcal{O}_C(b_2).$$

We observe that $R^1 f_*(\mathcal{O}_X)$ is generically spanned by Corollary 5.2, since

$$H^1(X, \mathcal{O}_X) = H^0(C, R^1 f_*(\mathcal{O}_X)).$$

Hence $b_j \geq 0$.

□

5.6. Proposition. *For general $\mathcal{L} \in \mathrm{Pic}(X)$, the restriction $\mathcal{L}_{|X_c}$ is not torsion for all $c \in C$.*

Proof. By Proposition 5.5, $h^1(X, \mathcal{O}_X) \geq 2$ and the restriction

$$H^1(X, \mathcal{O}_X) \to H^1(X_c, \mathcal{O}_{X_c})$$

is surjective for all c. Consequently, the kernel of the restriction

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X) \to \mathrm{Pic}^\circ(X_c) = H^1(X_c, \mathcal{O}_{X_c})/H^1(X_c, \mathbb{Z})$$

is discrete for all c plus a linear subspace of codimension 2. Since $\dim C = 1$, it follows that for $\mathcal{L} \in \mathrm{Pic}(X)$ general, the restriction $\mathcal{L}_{|X_c}$ is never trivial and thus also not torsion. □

6. Hopf and Inoue surfaces

In this section we assume that the general fiber of f is a Hopf or Inoue surface and show that for general line bundles $\mathcal{L}$ on X, the restriction $\mathcal{L}_{|X_c}$ is never torsion.

6.1. Proposition. *Assume that the general fiber of f is a Hopf or Inoue surface. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be general. Then $\mathcal{L}_{|X_c}$ is not torsion for all $c \in C$, and the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_c)$ is surjective for any smooth fiber X_c.*

Proof. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1$$

and our assumptions give

$$H^1(X, \mathbb{C}^*) = 0.$$

Moreover, $H^2(X, \mathbb{C}^*)$ is torsion. Consider the canonical morphism

$$\lambda : H^0(C, R^1 f_*(\mathbb{C}^*)) \to H^2(C, \mathbb{C}^*) \simeq \mathbb{C}^*.$$

Then by the Leray spectral sequence, λ is injective and the cokernel is torsion. Hence

$$H^0(C, R^1 f_*(\mathbb{C}^*)) \simeq \mathbb{C}^*.$$

Choose

$$1 \neq u \in H^0(C, R^1 f_*(\mathbb{C}^*))$$

non-torsion. This section defines an inclusion

$$\iota : \mathbb{C}^* \to R^1 f_*(\mathbb{C}^*).$$

Let C_0 be the smooth locus of f in C. We claim that

$$R^1 f_*(\mathbb{C}^*)|C_0 = R^1 f_{0*}(\mathbb{C}^*) \simeq \mathbb{C}^*. \tag{10}$$

Suppose first that Claim (10) holds. Then we conclude as follows. Certainly, ι is an isomorphism over C_0. Thus we obtain a sequence

$$0 \to \mathbb{C}^* \to R^1 f_*(\mathbb{C}^*) \to Q \to 0$$

where Q is supported on the finite set $C \backslash C_0$. Since $H^0(C, R^1 f_*(\mathbb{C}^*)) \simeq \mathbb{C}^*$ and since $H^1(C, \mathbb{C}^*) = 0$, it follows $H^0(C, Q) = 0$, hence $Q = 0$. Thus ι is an isomorphism everywhere and consequently u never takes value one, nor does - by our choice of u - any multiple u^m. Hence u defines a line bundle $\mathcal{L}$ such that $\mathcal{L}_{|X_c}$ is not torsion for all $c \in C$.

It remains to prove Claim (10). As before, set $\Delta = C \setminus C_0$, $A = f^{-1}(\Delta)$ and $X_0 = X \setminus A$. Then, as in the proof of Lemma 4.2,

$$H^4(A, \mathbb{C}^*) = H^1(X_0, \mathbb{C}^*) = H^1(C_0, \mathbb{C}^*) \oplus H^0(C_0, R^1 f_{0*}(\mathbb{C}^*)).$$

Since $H^4(A, \mathbb{C}^*) \simeq (\mathbb{C}^*)^s$, it follows

$$H^0(C_0, R^1 f_{0*}(\mathbb{C}^*)) \simeq \mathbb{C}^*.$$

Since $R^1 f_{0*}(\mathbb{C}^*)$ is locally constant of rank one, the claim follows.

□

As a consequence we obtain

6.2. Corollary.

a) $R^1 f_*(\mathcal{O}_X) \simeq \mathcal{O}_C$;
b) $R^2 f_*(\mathcal{O}_X) = 0$;
c) for general $\mathcal{L} \in \mathrm{Pic}(X)$ *and all* $c \in C$, *we have*

$$H^0(X_c, \mathcal{L}_{|X_c}) = H^0(X_c, \mathcal{L}^*_{|X_c}) = 0.$$

Proof. (a) Since $R^1 f_*(\mathcal{O}_X)$ has rank one, we may write

$$R^1 f_*(\mathcal{O}_X) \simeq \mathcal{O}_C(a) \oplus \text{torsion}.$$

By Proposition 6.1, $a = 0$. So it remains to show that $R^1 f_*(\mathcal{O}_X)$ is torsion free. If not, there exists a line bundle $\mathcal{M}$, such that $\mathcal{M}_{|X_{c_0}}$ is not torsion for some c_0 but $\mathcal{M}_{|X_c} \simeq \mathcal{O}_{X_c}$ for $c \neq c_0$. Write $S = \mathrm{red}(X_{c_0})$. Then

$$\mathcal{M} = f^* f_*(\mathcal{M}) \otimes \mathcal{O}_X(D),$$

with an effective divisor D supported on S. Since S is irreducible, $D = mS$ and therefore $\mathcal{O}_X(D)_{|X_c}$ is a torsion line bundle, contradiction.
(b) By (a), $h^1(X, \mathcal{O}_X) = 1$. Since the general fiber of f having negative Kodaira dimension, we have

$$H^3(X, \mathcal{O}_X) = H^0(X, \omega_X) = 0.$$

Thus we conclude from $\chi(X, \mathcal{O}_X) = 0$, that

$$H^2(X, \mathcal{O}_X) = 0.$$

Hence, by the Leray spectral sequence, $R^2 f_*(\mathcal{O}_X)$ must be torsion free, therefore

$$R^2 f_*(\mathcal{O}_X) = 0.$$

(c) As a consequence of (b), $R^2 f_*(\mathcal{L}) = 0$ for general $\mathcal{L}$, hence

$$H^2(X_c, \mathcal{L}_{X_c}) = 0$$

for all c. Thus

$$H^0(X_c, \mathcal{L}_{X_c}) = 0$$

for general $\mathcal{L}$ and all c as well. In summary, we may say that

$$H^0(X_c, \mathcal{L}_{|X_c}) = H^0(X_c, \mathcal{L}^*_{|X_c})$$

for general $\mathcal{L}$ and all c.
Now ω_X^m defines a section $t_m \in H^0(C, R^1 f_*(\mathcal{O}_X^*))$. Notice that for $c \in C_0$, the smooth locus of f, the bundle $\omega^m_{X|X_c} = \omega^m_{|X_c})$ is never trivial and thus t_m does not take value 1 on C_0. Since $R^1 f_*(\mathcal{O}_X^*) \simeq \mathcal{O}_C^*$, the section never takes value 1, hence our claim follows. □

We will further need the following basic statement on Hopf and Inoue surfaces.

6.3. Proposition.
Let S be a primary Hopf surface. Assume that

$$H^0(S, \Omega^1_S \otimes \mathcal{L}) \neq 0$$

for some line bundle $\mathcal{L}$ on S. Then

$$H^0(S, \mathcal{L}) \neq 0.$$

Proof. Choose a vector field v on S and let C be the zero locus of v which is purely one-dimensional. We obtain an exact sequence

$$0 \to \mathcal{O}_S(C) \to T_S \to \mathcal{O}_S(-C) \otimes \omega_S^{-1} \to 0.$$

Dualizing,

$$0 \to \mathcal{O}_S(C) \otimes \omega_S \to \Omega^1_S \otimes \mathcal{O}_S(-C) \to 0.$$

Hence

$$H^0(S, \mathcal{O}_S(C) \otimes \omega_S \otimes \mathcal{L}) \neq 0$$

or

$$H^0(S, \mathcal{O}_S(-C) \otimes \mathcal{L}) \neq 0,$$

In the latter case, the claim is clear. In the first case we observe that

$$H^0(S, \omega_S^{-1} \otimes \mathcal{O}_S(-C)) \neq 0.$$

Indeed, there exists another vector field v', and $v \wedge v'$ is a section of ω_S^{-1} vanishing on C.

□

6.4. Proposition. *Let S be an Inoue surface. Then there is a unique line bundle $\mathcal{L}$ such that*

$$H^0(S, \Omega^1_S \otimes \mathcal{L}) \neq 0.$$

Moreover, one of the following statements holds.

a) either $H^0(S, T_S) \neq 0$ and $\mathcal{L} = \omega_S^{-1}$
b) or $H^0(S, T_S) = 0$ and $\mathcal{L}^{\otimes 2} \simeq \omega_S^{-1}$.

Proof. The existence of $\mathcal{L}$ is classical, [In74].
If S has a non-zero vector field v, necessarily without zeroes, then v induces an exact sequence

$$0 \to \mathcal{O}_S \to T_S \to \omega_S^* \to 0,$$

and the claim is immediate, since S has no curves and since $H^0(S, \Omega^1_S) = 0$. If $H^0(S, T_S) = 0$, consider the exact sequence

$$0 \to \mathcal{L}^* \to \Omega^1_S \to \mathcal{L} \otimes \omega_S \to 0.$$

Since the sequence does not split,

$$H^1(S, \omega_S^{-1} \otimes \mathcal{L}^{\otimes -2}) \neq 0.$$

Hence either $\omega_S^{-1} \simeq \mathcal{O}_{\otimes 2}$ or $\omega_S \simeq \mathcal{L}$, [In74, Lemma 1]. The second case however cannot happen, since

$$H^0(S, \Omega^1_S \otimes \otimes \omega_S \simeq H^2(S, T_S) \neq 0,$$

[In74, Prop.2].

□

7. Proof of Theorem 2.2

As already said in the introduction, it suffices to prove Proposition 2.4. Thus we need to show that

$$H^2(X_c, T_{X|X_c} \otimes \mathcal{L}_{|X_c}) = 0$$

for all $c \in C$. By Serre duality, this comes down to show that

$$H^0(X_c, \Omega^1_{X|X_c} \otimes \mathcal{L}_{|X_c}) = 0$$

for some $\mathcal{L} \in \mathrm{Pic}(X)$ and for all $c \in C$.
We first consider irreducible fibers. Let

$$S = \mathrm{red} X_c.$$

Using the (co-)tangent sheaf sequence

$$0 \to N_S^* \to \Omega^1_{X|S} \to \Omega^1_S \to 0$$

it is immediate that it suffices to show - provided $\mathcal{L}_{|S}$ is not torsion - the following statement

$$H^0(S, \tilde{\Omega}^1_S \otimes \mathcal{L}_{|S}) = 0, \tag{11}$$

where

$$\tilde{\Omega}^1_S = \Omega^1_S / \mathrm{torsion}.$$

We first treat smooth fibers $S = X_c$.

7.1. Proposition. *Equations (11) holds for smooth fibers S (independent on the structure of the general fiber), i.e., for $\mathcal{L} \in \mathrm{Pic}(X)$ general*

$$H^0(S, \Omega^1_S \otimes \mathcal{L}_{|S}) = 0$$

simultaneously for all smooth fibers S.

Proof. (a) First, if S is a torus, then

$$H^0(S, T_S \otimes \mathcal{A}) = H^0(S, \Omega^1_S \otimes \mathcal{A}) = 0$$

for all non-trivial $\mathcal{A}$, hence we may take any $\mathcal{L}$ such that $\mathcal{L}_{|X_c}$ is never trivial, Proposition 5.6.
(b) If S is a Hopf surface, then by Corollary 6.2, $H^0(S, \mathcal{L}_{|S}) = 0$ for general $\mathcal{L}$, hence we conclude by Propositions 6.1 and 6.3.
(c) If S is an Inoue surface with a vector field, then for $\mathcal{L}$ general, also $\mathcal{L}^* \otimes \omega_X$ is general, hence

$$H^0(S, \mathcal{L}^*_{|S} \otimes (\omega_X)_{|S}) = H^0(S, \mathcal{L}^* \otimes \omega_S)) = 0,$$

hence we conclude by Propositions 6.1 and 6.4.
(d) Finally, assume that S is an Inoue surface without vector field. Then we argue as in (c), observing that $(\mathcal{L}^*)^{\otimes 2} \otimes \omega_X$ is general for general $\mathcal{L}$. □

7.2. Remark. Since the conormal bundle of a multiple fiber is torsion, the arguments also apply to fibers $X_c = \lambda S$ with $\lambda \geq 2$ and S smooth.

7.3. Proposition. *Equation (11) holds for singular reduced fibers.*

Proof. Recall the notations 4.5. By Lemma 4.6 and Proposition 4.11, $\kappa(S_0) = -\infty$, the surface S is non-normal and

$$H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^0(\tilde{S}, \omega_{\tilde{S}}) = 0.$$

Arguing by contradiction, there exists a one-dimensional family $\mathcal{L}_t$ of line bundles on X such that

$$H^0(S, \tilde{\Omega}^1_S \otimes \mathcal{L}_{t|S}) \neq 0.$$

Passing to a desingularization and then to a minimal model S_0 as in Notation 4.5, there are numerically trivial line bundles $\mathcal{M}_t$ on S_0 such that

$$\mathcal{M}_t = \sigma_* \pi^* \eta^* (\mathcal{L}_t)^{**}$$

with a one-dimensional family of sections in

$$H^0(S_0, \Omega^1_{S_0} \otimes \mathcal{M}_t).$$

Thus

$$H^0(S_0, \Omega^1_{S_0} \otimes \mathcal{M}_t) \neq 0. \tag{12}$$

Observe that all line bundles $\mathcal{M}_t$ might be trivial.

Step 1. Suppose first that S_0 is Kähler. Then by (12), S_0 must be ruled over a curve of genus at least two.
Claim. $\tilde{S}$ has rational singularities, only.
Proof of the Claim. Assume to the contrary that $\tilde{S}$ has an irrational singularity. We claim that $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$. In fact, π must contract a curve B_0 projecting onto B. Thus $h^1(B_0, \mathcal{O}_{B_0}) \geq g$ and therefore $h^0(\tilde{S}, R^1\pi_*(\mathcal{O}_{\hat{S}})) \geq g$. Since $H^2(\tilde{S}, \omega_{\tilde{S}}) = 0$, the Leray spectral sequence yields $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$ (and $h^0(\tilde{S}, R^1\pi_*(\mathcal{O}_{\hat{S}})) = g$).

Thus all line bundles $\eta^*(\mathcal{L}_t)$ are trivial and we obtain a one-dimensional family $\tilde{\omega}_t$ of holomorphic one-forms on $\tilde{S}$. Moreover there exists a one-dimensional family ω_t of one-forms on B such that

$$\sigma^* p^*(\omega_t) = \pi^*(\tilde{\omega}_t),$$

where $p : S_0 \to B$ is the ruling. Since $p(\sigma(B_0)) = B$, we have $\iota^*_{B_0}\sigma^* p^*(\omega_t) \neq 0$. On the other hand, since π contracts B_0, it follows that $\iota^*_{B_0}(\pi^*(\omega_t)) = 0$, a contradiction. This proves the *Claim* and thus $\tilde{S}$ has rational singularities, only.

In this case the morphism $p_0 : S_0 \to B$ induced a morphism $\tilde{p} : \tilde{S} \to B$. In the language of divisors and using the notations of (4.5) and (4.6) we have

$$-K_{\hat{S}} \equiv \hat{N} + \hat{E},$$

where $\hat{N}$ is the strict transform of $\tilde{N}$ in $\hat{S}$. Set

$$N_0 = \sigma_*(\hat{N}).$$

We are now using the theory of ruled surfaces as in [Ha77, section V.2], taking over also the notations from [Ha77]. In particular we have the invariant e and a section C_0 with minimal self-intersection $C_0^2 = -e$. Moreover,

$$-K_{S_0} \equiv 2C_0 + (e + 2 - 2g)F,$$

where F is a fiber of p_0 and $g = g(B)$ the genus of B. Since $\tilde{S}$ has rational singularities, π cannot contract any curve projecting onto B. Hence we must have

$$N \equiv 2C_0 + aF$$

with $a \leq e + 2 - 2g$. Taking into account the numerical description of irreducible curves in S_0, as given in [Ha77, section V.2], it follows immediately that $e > 0$ and that

$$N = 2C_0 + R$$

with an effective divisor $R \sim aF$ (note that the curve C_0 is the unique contractible curve in S_0). Consequently, $\tilde{N}$ has a unique component, say $\tilde{N}_1$, projecting onto B, and this component has multiplicity two. The map $\tilde{p} : \tilde{S} \to B$ induces a holomorphic map $p : S \to B'$ and a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \eta\ } & S \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\ \tau\ ]{} & B'. \end{array}$$

The general fiber S_b of p is a reduced Gorenstein curve with

$$\omega_{S_b} \equiv 0$$

whose normalization of S_b is a disjoint union of smooth rational curves. Thus, if S_b is irreducible, then S_b is a rational curve with one node or cusp, and if S_b is reducible, it is a cycle of smooth rational curves. In case S_b has a node or is a cycle, the normalization map η is generically $2:1$ along $\tilde{N}_1$. In these cases however, $\tilde{N}_1$ would be reduced, see [KW88], a contradiction. In the remaining case, η has degree one along $\tilde{N}_1$, hence τ has degree one, too. Unless τ is an isomorphism and

$g(B) = 2$, we have $h^1(B', \mathcal{O}_{B'}) \geq 3$, hence $h^1(S, \mathcal{O}_S) \geq 3$. Since $\chi(S, \mathcal{O}_S) = 0$, we conclude

$$h^0(S, \omega_S) = h^2(S, \mathcal{O}_S) \geq 2.$$

Since S is Moishezon and $\omega_S \equiv 0$, this is impossible. Alternatively, apply Proposition 5.5 or Corollary 6.2, respectively.

Hence τ is biholomorphic, i.e., p maps to the smooth curve B of genus two and $h^1(S, \mathcal{O}_S) = h^1(B, \mathcal{O}_B) = 2$. Moreover, $h^0(S, \omega_S) = 1$, and therefore $\omega_S \simeq \mathcal{O}_S$. The map p being flat, $R^1 p_*(\mathcal{O}_S)$ is locally free of rank one, and by relative duality,

$$R^1 p_*(\mathcal{O}_S) \simeq p_*(\omega_{S/B})^* = \omega_B,$$

hence

$$H^0(B, R^1 p_*(\mathcal{O}_S)) \neq 0.$$

But then $h^1(S, \mathcal{O}_S) > h^1(B, \mathcal{O}_B)$, a contradiction. This shows that $g(B) \geq 2$ is impossible and concludes the proof in the Kähler case.

Step 2. We thus are reduced to the case that S_0 is not Kähler.
If S_0 is of type VII, then $H^0(S_0, \Omega^1_{S_0}) = 0$, hence $H^0(S_0, \Omega^1_{S_0} \otimes \mathcal{M}) = 0$ for $\mathcal{M}$ general, contradicting (12).
The same argument applies to a secondary Kodaira surface S_0. If S_0 is a primary Kodaira surface, then the cotangent sequence reads

$$0 \to \mathcal{O}_{S_0} \to \Omega^1_{S_0} \to \mathcal{O}_{S_0} \to 0,$$

which immediately gives a contradiction by tensorizing with $\mathcal{M}_t$.
It remains to exclude the case $\kappa(S_0) = 1$. Since $H^0(S_0, T_{S_0}) \neq 0$, (4.9) and (4.10), the Iitaka fibration $h_0 : S_0 \to B$ is an elliptic bundle over a curve of genus $g(B) \geq 2$, [GH90, Satz 1] and, as already noticed, the induced vector field v_0 has no zeroes. Hence $\tilde{S} = \hat{S} = S_0$. Since $\omega_{\tilde{S}} = \mathcal{I}_{\tilde{N}} \otimes \eta^*(\omega_S)$, we have

$$h^2(S, \mathcal{O}_S) \geq h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}),$$

hence $h^2(S, \mathcal{O}_S) \geq 2$, contradicting Proposition 5.5 or Corollary 6.2, respectively.

□

7.4. Remark. If the fiber $X_c = \lambda S$ with S an irreducible reduced singular surface and $\lambda \geq 2$, we argue in the same way, passing to a finite étale cover.

Finally, we have to treat reducible fibers:

7.5. Proposition. *Equation (11) holds for reducible fibers.*

Proof. Let

$$F = \sum a_i S_i$$

be a reducible fiber. Arguing by contradiction, there is a one-dimensional family $\mathcal{L}_t$ of line bundles on X such that

$$H^0(F, \Omega^1_X | F \otimes (\mathcal{L}_t)_{|F}) \neq 0.$$

Hence there exists a number i_0 such that

$$H^0(S_{i_0}, \tilde{\Omega}^1_X | S_{i_0} \otimes (\mathcal{L}_t)_{|S_{i_0}}) \neq 0$$

for all t, and therefore

$$H^0(S_{i_0}, \tilde{\Omega}^1_{S_{i_0}} \otimes (\mathcal{L}_t)_{|S_{i_0}}) \neq 0$$

Now we argue as in Proposition 7.3 to obtain a contradiction. One might also use the line bundle $\mathcal{O}_X(-kS_{i_1})$ for $k \gg 0$, where the surface S_{i_1} meets in S_{i_0} in a curve. $\square$

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