

On deformation of extremal metrics

Xiuxiong Chen, Mihai Păun and Yu Zeng

June 16, 2015

1 Introduction

In 1950's, E. Calabi (cf. [5], [6]) proposed a program aiming to construct "the best" metrics one could expect to find in a given Kähler class: these objects are currently called *extremal metrics*. To this end he has introduced a functional (the Calabi energy) so that the said metrics are obtained as critical points of it. The Kähler-Einstein metrics (and more generally, the constant scalar curvature Kähler metrics referred as cscK hereafter) are both special cases of extremal metrics.

The main questions concerning the existence and uniqueness of Kähler-Einstein metrics on manifolds whose first Chern class is negative or zero have been clarified in the 80's thanks to the fundamental contributions of Aubin, Calabi and Yau (cf. [1], [4] and [25], respectively). The remaining Fano case has been only recently settled by the crucial work of Chen-Donaldson-Sun (cf. [10], [11], [12]).

After this major achievement, the study of extremal metrics should naturally be *the* dominant subject in the field. However, even the most basic existence questions concerning these metrics seem to be excessively difficult, given that the resulting partial differential equation one has to deal with is of order four. Of course, the equation corresponding to a metric with prescribed Ricci curvature is of order four as well, but one can reduce it easily to a fully non-linear second order equation. This is no longer possible e.g. in the cscK case for general Kähler manifolds.

The continuity method is a very powerful technique in PDE theory. It was successfully used by Aubin and Yau in their respective articles on Kähler-Einstein metrics. In [8], the first named author proposed a continuity path which is very well adapted to the category of extremal metrics (regardless to their Kähler classes). One can see that if all the geometric objects involved belong to a multiple of the canonical class, then the path in [8] is obtained by taking the trace with respect to the solution metric of the continuity path used by Aubin and Yau. In this sense, it represents a natural extension of their techniques. We refer to [8] for the proof of the basic facts about this new approach, including a crucial openness result and a few conjectural pictures.

In the present article we are pursuing this circle of ideas by establishing two deformation results about the cscK and extremal metrics, respectively. Let (M, ω) be a compact complex manifold endowed with a Kähler metric; we denote by $[\omega] \in H^{1,1}(X, \mathbb{R})$ the cohomology class

corresponding to ω . We define the following space of potentials

$$\mathcal{H}^\infty(M) = \left\{ \varphi \in C^\infty(M) : \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \right\}.$$

If $\varphi \in \mathcal{H}^\infty(M)$, then we denote by R_φ the scalar curvature of the corresponding metric ω_φ , and by \underline{R} its average. i.e.

$$\underline{R} := \frac{1}{\text{Vol}(M, \omega)} \int_M R_\varphi \omega_\varphi^n.$$

Our first result states as follows.

Theorem 1.1. *Let (M, ω) be a compact Kähler manifold such that there exists a cscK metric $\omega_{\varphi_0} \in [\omega]$. Then there exist $\epsilon > 0$ and a smooth function $\phi : (1 - \epsilon, 1] \times M \rightarrow \mathbb{R}$ such that $\varphi_t := \phi(t, \cdot) \in \mathcal{H}^\infty(M)$ and such that the corresponding metric verifies the equation*

$$R_{\varphi_t} - \underline{R} - (1 - t)(\text{tr}_{\varphi_t} \omega - n) = 0. \quad (1)$$

Moreover, there exists a holomorphic automorphism f of M such that $\omega_{\varphi_1} = f^* \omega_{\varphi_0}$.

Following the terminology introduced by J. Fine [16] and J. Stoppa cf. [24], a metric verifying the condition (1) is called *twisted constant scalar curvature metric*.

The generalization of this notion in the context of extremal metrics was formulated in [8] as follows. A metric ω_φ is called *twisted extremal Kähler metric* if there exists $t \in (0, 1)$ such that the vector field

$$\nabla_\varphi^{1,0} (R_\varphi - (1 - t) \text{tr}_\varphi \omega)$$

is holomorphic. The result we obtain within this framework states as follows.

Theorem 1.2. *Let M be a compact complex manifold, and let ω be a Kähler metric on X , whose class $[\omega]$ contains an extremal Kähler metric ω_{φ_0} . Then there exists $\epsilon > 0$ together with a smooth function $\phi :]1 - \epsilon, 1] \times M \rightarrow \mathbb{R}$ such that $\varphi_t := \phi(t, \cdot) \in \mathcal{H}^\infty(M)$ and such that the corresponding metric ω_{φ_t} is a twisted extremal metric, i.e.*

$$\nabla_{\varphi_t}^{1,0} (R_{\varphi_t} - (1 - t) \text{tr}_{\varphi_t} \omega)$$

is a holomorphic vector field.

As a direct consequence of Theorem 1.2 we obtain a new proof of the following statement.

Corollary 1.3. *Let (M, ω) be a compact Kähler manifold. Given two extremal metrics $(\omega_j)_{j=1,2} \subset [\omega]$ there exists a holomorphic automorphism f of M such that $f^* \omega_2 = \omega_1$.*

The uniqueness problem above has long history which goes back to E. Calabi. Among the important articles generated by this question since then we refer to [2], [7], [13], [15], [23] as well as the recent paper [3]. To our knowledge, the fact that the original ideas of Bando-Mabuchi could be successfully used in order to establish the uniqueness of cscK/extremal metrics first appears in the paper of Berman-Berndsson cf. [3]; in addition they inject new convexity techniques in the field (our arguments for the corollary above follow a similar approach).

For example, in the cscK case, the Corollary 1.3 can be derived as follows. We consider the one parameter family of twisted K-energy functional¹

$$\frac{dE_t}{ds} = \int_M \frac{d\varphi}{ds} (-t(R_\varphi - \underline{R}) + (1-t)(\text{tr}_\varphi \omega - n)) \omega_\varphi^n, \quad \forall t \in [0, 1]$$

and then we observe that the twisted cscK metrics are precisely the critical points of E_t (up to a change of parameter). Next, we recall that thanks to [7], any two metrics in $[\omega]$ can be connected by a $C^{1,1}$ geodesic; on the other hand, if $t < 1$ then the functional E_t above is *strictly convex* along $C^{1,1}$ geodesics, as a consequence of [3] and [14], together with the strict convexity of the J functional established in [9]. The convexity of the functional E_1 along smooth geodesics is due to T. Mabuchi; the fact that this result still holds in the setting of $C^{1,1}$ geodesics is crucial for the proof. By Theorem 1.1, we can deform the initial metrics ω_1 and ω_2 (modulo the action of a holomorphic automorphism of M) to twisted cscK metrics for which the corresponding parameter t is strictly less than 1, so the corollary follows.

In fact, the results 1.1 and 1.2 above represent the cscK version and the extremal version respectively of the Bando-Mabuchi work [2]. The proof we will present next is based on the *bifurcation technique* developed in their celebrated article [2] concerning the uniqueness up to biholomorphism of Kähler-Einstein metrics (see also Tian-Zhu [26] for an analogous result in the context of Kähler-Ricci solitons). Even if in our proof (and in [3] likewise) one can easily recognize the main steps of the approach by Bando-Mabuchi, the techniques we had to develop/adapt in what follows are much more involved than the ones used in the Kähler-Einstein context. Moreover, the results above are important in their own right, because tightly connected with the program launched by E. Calabi.

Another motivation of the present article arise from the following conjecture, cf. [8].

Conjecture 1.4. *For any $\chi > 0$ and $\chi \in [\omega]$, if $(M, [\omega], J)$ is destabilized by $(M, [\omega], J')$ where the later admits a cscK metric, then for any $s < 1$ but sufficiently close to 1 there exists a twisted cscK metric for the triple structure $([\omega], \chi, s)$.*

This conjecture would give a new criteria for deciding whether a class $(M, [\omega], J)$ is semi-stable or not. If Conjecture 1.4 holds, then given a semi-stable manifold, one would be able to find a continuous family of twisted cscK metrics for any $1 - \varepsilon \leq t < 1$. In principle, this should be sufficient for many geometric applications. Also, we remark that this is perfectly analog to the classical Kähler-Einstein equation: along the continuity path, one can solve the Monge-Ampère equation provided that the condition of K-semistability is satisfied. The conjecture above essentially states that the same phenomenon should occur in the context of the cscK metrics. Even though we cannot solve this conjecture yet, our main results here represent a slightly weaker existence result, which will hopefully lead to the solution of the conjecture itself in a near future.

This article is organized as follows. In the first part we define a map between the Lie group of holomorphic automorphisms of M , and the space of normalized Kähler potentials. In the Kähler-Einstein setting, the image of the differential of this map was computed by Bando-Mabuchi, cf. [2], paragraph §6. Here we obtain an analog result, first for orbits corresponding

¹For $s = \frac{1}{2}$, this is already studied by a number of authors, [24], [20] etc.

to cscK metrics (cf. Proposition 2.1), and then in general, for orbits of extremal metrics in section 4. We remark that the case of extremal metrics is quite delicate, basically because of the fact that the Lichnerowicz operator (denoted by \mathcal{D} here) is not real in general. An important ingredient of the proof is a result due to Calabi, concerning the structure of the algebra of holomorphic vector fields on Kähler manifolds admitting an extremal metric. We equally establish a Leibniz-type identity for the operator \mathcal{D} ; it is an elementary result, modulo the computations in the proof which are really involved. The complete arguments for our main results (i.e. the two theorems stated above) are given in sections three and four, respectively. We are using a version of the implicit function theorem, again modeled after [2], but with many additional difficulties along the way.

2 Preliminaries

Let (M, ω) be a compact Kähler manifold of dimension n ; we denote by $[\omega] \in H^{1,1}(X, \mathbb{R})$ its corresponding cohomology class. We denote by $\text{Aut}_0(M)$ the connected component of the Lie group of holomorphic automorphisms of M containing the identity map, and let $\text{Iso}_0(M, \omega) \subset \text{Aut}_0(M)$ be the group of holomorphic isometries of (M, ω) .

It is well known that the quotient

$$\mathcal{O} = \text{Aut}_0(M) / \text{Iso}_0(M, \omega) \quad (2)$$

is a homogeneous manifold. Let $\Gamma \subset H^0(M, T_M)$ be the vector space of holomorphic vector fields X such that the Lie derivative $\mathcal{L}_X \omega$ vanishes. Then the tangent space of \mathcal{O} is expressed as follows

$$T_{\mathcal{O}} \simeq H^0(M, T_M) / \Gamma. \quad (3)$$

Along the next lines we will construct an embedding of \mathcal{O} into the space of potentials of the Kähler metric ω , and we will identify the image of the corresponding tangent space.

Let $g \in \text{Aut}_0(M)$ be a holomorphic automorphism of M . Then we have $g^* \omega \in [\omega]$, so that there exists a real-valued function $\varphi \in C^\infty(M)$ such that

$$g^* \omega = \omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi. \quad (4)$$

The function φ is unique up to normalization; we introduce the normalized space of Kähler potentials

$$\widetilde{\mathcal{H}} := \{ \varphi \in C^\infty(M) : \omega_\varphi > 0 \text{ and } \int_M \varphi \omega^n = 0 \}. \quad (5)$$

Then we have a well-defined map $\Psi^\omega : \text{Aut}_0(M) \rightarrow \widetilde{\mathcal{H}}$, such that $\Psi^\omega(g) := \varphi$ where the function φ is uniquely defined by (4) together with the normalization in the definition of $\widetilde{\mathcal{H}}$ in (5). Moreover, we have $\Psi^\omega(g) = 0$ for any $g \in \text{Iso}(M, \omega)$, and thus we obtain a map

$$\Psi^\omega : \mathcal{O} \rightarrow \widetilde{\mathcal{H}}. \quad (6)$$

If the metric ω is cscK, then we can describe the image $\Psi^\omega_\star(T_{\mathcal{O},g})$ in a very simple manner, as follows.

Proposition 2.1. *Let (M, ω) be a compact Kähler manifold, such that ω is cscK, and let $g \in \text{Aut}_0(M)$ be an automorphism. Then the image of the tangent space $\Psi_*^\omega(T_{\mathcal{O},g})$ coincides with the space generated by the real-valued functions $f \in \mathcal{C}^\infty(M)$, such that $\nabla_\varphi^{1,0} f$ is holomorphic, where $\varphi := \Psi^\omega(g)$. In addition, the imaginary part of the vector field $\nabla_\varphi^{1,0} f$ is Killing with respect to the metric ω_φ .*

Proof. To start with, let g_t be a smooth path in \mathcal{O} , such that $g_0 = g$ and such that the derivative $\left. \frac{dg_t}{dt} \right|_{t=0}$ identifies with a holomorphic vector field which we denote by X . There exists a smooth family $(\varphi_t) \subset \widetilde{\mathcal{H}}$ with $\varphi_0 = \varphi$ so that we have

$$g_t^* \omega := \omega_{\varphi_t} \quad (7)$$

for any parameter t . Since ω is cscK, so is ω_{φ_0} . By [6], we can decompose the holomorphic vector X as

$$X = X_a + \nabla_{\varphi_0}^{1,0}(f + \sqrt{-1}g), \quad (8)$$

where X_a is the autoparallel component of X , f and g are real-valued functions such that $\nabla_{\varphi_0}^{1,0} f$ and $\nabla_{\varphi_0}^{1,0} g$ are holomorphic (notice that here we are using the cscK condition) We differentiate the relation (7) at $t = 0$ and we obtain

$$\sqrt{-1} \partial \bar{\partial} \dot{\varphi}_0 = \mathcal{L}_{X_{\mathbb{R}}} \omega_{\varphi_0} = \frac{1}{2} (\mathcal{L}_X \omega_{\varphi_0} + \mathcal{L}_{\bar{X}} \omega_{\varphi_0}) = \sqrt{-1} \partial \bar{\partial} f. \quad (9)$$

Thus, we know that $\dot{\varphi}_0 = f - \int f \omega^n$. The fact that the imaginary part of the holomorphic vector field $\nabla_{\varphi_0}^{1,0} \dot{\varphi}_0$ is Killing can be seen as a consequence of the fact that the function $\dot{\varphi}_0$ is real-valued, so we will not detail this point any further. \square

The next paragraph of this section is crucial: among the ω -potentials belonging to the image $\Psi^\omega(\mathcal{O})$, we have to choose one which will enable us later to use the implicit function theorem in the proof of our main results. This is completely analogue to the paragraph §6 in [2].

For any positive (1, 1)-form χ , the J_χ functional introduced in [9] is defined as follows

$$\frac{dJ_\chi}{dt} = \int_M \text{tr}_\varphi \chi \frac{d\varphi}{dt} \frac{\omega_\varphi^n}{n!}, \quad \forall \varphi \in \mathcal{H}^\infty(M).$$

We consider the functional $\iota := J_\omega - nI$, and by a direct computation, we obtain

$$\frac{d}{dt} \iota(\varphi_t) = \int_M (\text{tr}_{\varphi_t} \omega - n) \dot{\varphi}_t \frac{\omega_{\varphi_t}^n}{n!}. \quad (10)$$

as well as

$$\frac{d^2}{dt^2} \iota(\varphi_t) = \int (\ddot{\varphi} - |\nabla \dot{\varphi}|_{\varphi_t}^2) (\text{tr}_{\varphi_t} \omega - n) \omega_{\varphi_t}^n + \int \dot{\varphi}_{,\alpha} \dot{\varphi}_{,\bar{\beta}} \omega_{\bar{\alpha}\beta} \omega_{\varphi_t}^n > 0.$$

so that in particular the functional ι is strictly convex along smooth geodesics.

As a consequence, we infer the following result, corresponding to [2], Lemma 6.2.

Lemma 2.2. *The functional $\iota|_{\Psi^\omega(\mathcal{O})}$ is proper, and the minimum point of this restriction is unique.*

Proof. Let X be a holomorphic vector field, such that $X \in \Psi_\star^\omega(T_{\mathcal{O},g})$; by a result due to T. Mabuchi (cf. [22], page 238), if we define

$$g_t := \exp(tX_{\mathbb{R}}) \quad (11)$$

where $X_{\mathbb{R}}$ is the real part of the vector X , then the map

$$t \rightarrow \Psi^\omega(g_t) \quad (12)$$

is a smooth geodesic. Therefore, our statement is a consequence of the strict convexity properties of the functional ι , combined with Proposition 2.1. \square

We recall a few notations and results taken from [6]. Let f be a smooth function on M ; we define

$$Lf := \bar{\partial}\nabla_\varphi^{1,0}f \quad (13)$$

which written in coordinates gives

$$Lf = \frac{\partial}{\partial z^{\bar{\beta}}} \left(g^{\alpha\bar{\mu}} \frac{\partial f}{\partial z^{\bar{\mu}}} \right) \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}}, \quad (14)$$

and let L^\star be the adjoint operator. Let $\mathcal{D}_\varphi := L^\star L$ be the Lichnerowicz operator. Then \mathcal{D}_φ is a self-adjoint elliptic operator on the space of smooth complex functions of M , which can be written as

$$\mathcal{D}_\varphi f = \Delta_\varphi^2 f + \langle \sqrt{-1}\partial\bar{\partial}f, \text{Ric}_\varphi \rangle_{\omega_\varphi} + \langle \partial R_\varphi, \bar{\partial}f \rangle_{\omega_\varphi} \quad (15)$$

where Ric_φ denotes the Ricci curvature of the metric ω_φ , and R_φ is its trace, namely the scalar curvature. Also, Δ_φ is the Laplace operator corresponding to ω_φ .

Notice that if the metric ω_φ is cscK, the \mathcal{D}_φ is real, self-adjoint operator. And we have the following result, consequence of the general elliptic theory.

Lemma 2.3. *The operator $\mathcal{D}_\varphi : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ has the following properties.*

- (1) *Its kernel coincides with the subspace of functions f such that $\nabla_\varphi^{1,0}f$ is holomorphic.*
- (2) *The image $\mathcal{D}_\varphi(\mathcal{C}^\infty(M)) \subset \mathcal{C}^\infty(M)$ is closed, and we have the orthogonal decomposition*

$$\mathcal{C}^\infty(M) = \mathcal{D}_\varphi(\mathcal{C}^\infty(M)) \oplus \text{Ker}(\mathcal{D}_\varphi). \quad (16)$$

The point (1) is due to the compactness of M ; the fact that the image of \mathcal{D}_φ is closed follows from Sobolev and Gårding results for which we refer to L. Hörmander [17].

For any function $\varphi \in \mathcal{H}^\infty(M)$, we define a bilinear operator $B_\varphi(\cdot, \cdot)$ acting on $u, v \in \mathcal{C}^\infty(M)$ as follows

$$\begin{aligned} B_\varphi(u, v) := & \langle \partial\bar{\partial}v, \partial\bar{\partial}\Delta_\varphi u \rangle_\varphi + \Delta_\varphi \langle \partial\bar{\partial}v, \partial\bar{\partial}u \rangle_\varphi + \langle \partial\bar{\partial}\Delta_\varphi v, \partial\bar{\partial}u \rangle_\varphi \\ & + u_{,\bar{\alpha}p}v_{,\beta\bar{p}}(\text{Ric}_\varphi)_{\alpha\bar{\beta}} + u_{,\bar{p}\beta}v_{,p\bar{\alpha}}(\text{Ric}_\varphi)_{\alpha\bar{\beta}}. \end{aligned}$$

We end this introductory paragraph with the following statement, which will play an important role in our proof. In order to simplify the writing, we will use the following notation

$$\langle \partial v_1, \bar{\partial} v_2 \rangle_\varphi = \sum_{\alpha, \beta} \frac{\partial v_1}{\partial z_\alpha} \frac{\partial v_2}{\partial \bar{z}_\beta} g^{\alpha\bar{\beta}} \quad (17)$$

where v_1, v_2 are smooth functions on M . We have the next technical statement.

Lemma 2.4. *Let $\omega_\varphi \in [\omega]$ be an extremal metric, and let v, ξ be real-valued two smooth functions, such that $\mathcal{D}_\varphi v = \bar{\mathcal{D}}_\varphi v = 0$; then we have the next identity.*

$$\mathcal{D}_\varphi \langle \partial v, \bar{\partial} \xi \rangle_\varphi = \langle \partial v, \bar{\partial} \mathcal{D}_\varphi \xi \rangle_\varphi + B_\varphi(v, \xi). \quad (18)$$

Proof. We check next the validity of (18) by a brute-force computation; we first assume that the scalar curvature of metric ω_φ is constant. Then we have the following long sequence of relations together with some explanations when passing from one line to another.

$$\mathcal{D}_\varphi \langle \partial v, \bar{\partial} \xi \rangle_\varphi = \Delta_\varphi^2 \langle \partial v, \bar{\partial} \xi \rangle_\varphi + \langle \sqrt{-1} \partial \bar{\partial} \langle \partial v, \bar{\partial} \xi \rangle_\varphi, Ric_\varphi \rangle_\varphi \quad (19)$$

$$= (v, \delta \xi, \bar{\delta})_{, \alpha \bar{\alpha} \beta \bar{\beta}} + (v, \delta \xi, \bar{\delta})_{, \beta \bar{\beta} \alpha \bar{\alpha}} (Ric_\varphi)_{\alpha \bar{\beta}} \quad (20)$$

$$= (v, \delta \xi, \bar{\delta} \alpha)_{, \bar{\alpha} \beta \bar{\beta}} + (v, \delta \xi, \bar{\delta} \beta)_{, \bar{\alpha}} (Ric_\varphi)_{\alpha \bar{\beta}} \quad (21)$$

$$= (v, \delta \bar{\alpha} \xi, \bar{\delta} \alpha + v, \delta \xi, \bar{\delta} \alpha \bar{\alpha})_{, \beta \bar{\beta}} + (v, \delta \bar{\alpha} \xi, \bar{\delta} \beta + v, \delta \xi, \bar{\delta} \beta \bar{\alpha}) (Ric_\varphi)_{\alpha \bar{\beta}} \quad (22)$$

$$\text{(since } \xi, \bar{\delta} \alpha \bar{\alpha} = \xi, \alpha \bar{\alpha} \bar{\delta}; \xi, \bar{\delta} \beta \bar{\alpha} = \xi, \beta \bar{\delta} \bar{\alpha} = \xi, \beta \bar{\alpha} \bar{\delta}) \quad (23)$$

$$= (v, \delta \bar{\alpha} \xi, \bar{\delta} \alpha + v, \delta \xi, \alpha \bar{\alpha} \bar{\delta})_{, \beta \bar{\beta}} + v, \delta \bar{\alpha} \xi, \bar{\delta} \beta (Ric_\varphi)_{\alpha \bar{\beta}} + v, \delta \xi, \beta \bar{\alpha} \bar{\delta} (Ric_\varphi)_{\alpha \bar{\beta}} \quad (24)$$

$$= \Delta_\varphi \langle \partial \bar{\partial} v, \partial \bar{\partial} \xi \rangle_\varphi + \langle \partial \bar{\partial} v, \partial \bar{\partial} \Delta_\varphi \xi \rangle_\varphi + v, \delta \xi, \alpha \bar{\alpha} \bar{\delta} \beta \bar{\beta} + v, \delta \bar{\alpha} \xi, \bar{\delta} \beta (Ric_\varphi)_{\alpha \bar{\beta}} \quad (25)$$

$$+ v, \delta (\xi, \beta \bar{\alpha} (Ric_\varphi)_{\alpha \bar{\beta}})_{, \bar{\delta}} - v, \delta \xi, \beta \bar{\alpha} (Ric_\varphi)_{\alpha \bar{\beta}, \bar{\delta}} \quad (26)$$

$$\text{(since } \xi, \alpha \bar{\alpha} \bar{\delta} \beta \bar{\beta} = (\Delta_\varphi \xi)_{, \bar{\delta} \beta \bar{\beta}} = (\Delta_\varphi \xi)_{, \beta \bar{\delta} \bar{\beta}} = (\Delta_\varphi \xi)_{, \beta \bar{\beta} \bar{\delta}} = (\Delta_\varphi^2 \xi)_{, \bar{\delta}}) \quad (27)$$

$$= \Delta_\varphi \langle \partial \bar{\partial} v, \partial \bar{\partial} \xi \rangle_\varphi + \langle \partial \bar{\partial} v, \partial \bar{\partial} \Delta_\varphi \xi \rangle_\varphi + v, \delta (\Delta_\varphi^2 \xi + \xi, \beta \bar{\alpha} (Ric_\varphi)_{\alpha \bar{\beta}})_{, \bar{\delta}} \quad (28)$$

$$+ v, \delta \bar{\alpha} \xi, \bar{\delta} \beta (Ric_\varphi)_{\alpha \bar{\beta}} - v, \delta \xi, \beta \bar{\alpha} (Ric_\varphi)_{\alpha \bar{\beta}, \bar{\delta}} \quad (29)$$

As a consequence, we infer that we have

$$\mathcal{D}_\varphi \langle \partial v, \bar{\partial} \xi \rangle_\varphi - \langle \partial v, \bar{\partial} \mathcal{D}_\varphi \xi \rangle_\varphi \quad (30)$$

$$= \Delta_\varphi \langle \partial \bar{\partial} v, \partial \bar{\partial} \xi \rangle_\varphi + \langle \partial \bar{\partial} v, \partial \bar{\partial} \Delta_\varphi \xi \rangle_\varphi + v, \delta \bar{\alpha} \xi, \bar{\delta} \beta (Ric_\varphi)_{\alpha \bar{\beta}} - v, \delta \xi, \beta \bar{\alpha} (Ric_\varphi)_{\alpha \bar{\beta}, \bar{\delta}} \quad (31)$$

$$= B_\varphi(v, \xi). \quad (32)$$

In order to establish this equality, we have used the identity

$$\begin{aligned} \langle \partial \bar{\partial} \Delta_\varphi v, \partial \bar{\partial} \xi \rangle_\varphi &= v, \alpha \bar{\alpha} \beta \bar{\delta} \xi, \bar{\beta} \bar{\delta} \\ &= (v, \alpha \beta \bar{\alpha} - R_{\beta \bar{\alpha} \alpha \bar{l}} v, l)_{, \bar{\delta} \xi, \bar{\beta} \bar{\delta}} \\ &= -R_{\beta \bar{l} v, l \bar{\delta} \xi, \bar{\beta} \bar{\delta}} - R_{\beta \bar{l}, \bar{\delta} v, l \xi, \bar{\beta} \bar{\delta}}. \end{aligned}$$

This completes the proof of the lemma, in the case of a cscK metric (we remark that we are only using this hypothesis in the expression of the operator \mathcal{D}_φ in the first line of the long string of equalities above).

In the preceding computations, if ω_φ is any Kähler metric (i.e. no curvature assumptions), then the expression of \mathcal{D}_φ has an additional a term, containing the derivative of scalar curvature. In order to complete the proof, we still have to check that we have

$$(R_\varphi)_{,\delta}(\langle \partial v, \bar{\partial} \xi \rangle_\varphi)_{,\bar{\delta}} - \langle \partial v, \bar{\partial}((R_\varphi)_{,\delta} \xi_{,\bar{\delta}}) \rangle_\varphi = 0.$$

Here we will use the curvature assumption, namely that ω_φ is extremal, because then we have

$$(R_\varphi)_{,\delta}(v_{,\alpha} \xi_{,\bar{\alpha}})_{,\bar{\delta}} - v_{,\alpha}((R_\varphi)_{,\delta} \xi_{,\bar{\delta}})_{,\bar{\alpha}} \quad (33)$$

$$= (R_\varphi)_{,\delta} v_{,\alpha} \xi_{,\bar{\alpha}\bar{\delta}} + (R_\varphi)_{,\delta} v_{,\alpha\bar{\delta}} \xi_{,\bar{\alpha}} - v_{,\alpha} (R_\varphi)_{,\delta} \xi_{,\bar{\delta}\bar{\alpha}} - v_{,\alpha} (R_\varphi)_{,\delta\bar{\alpha}} \xi_{,\bar{\delta}} \quad (34)$$

$$= ((R_\varphi)_{,\delta} v_{,\alpha\bar{\delta}} - v_{,\delta} (R_\varphi)_{,\alpha\bar{\delta}}) \xi_{,\bar{\alpha}} \quad (35)$$

$$= ((R_\varphi)_{,\delta} v_{,\bar{\delta}\alpha} - R_{\varphi,\delta\alpha} v_{,\bar{\delta}} - (v_{,\delta} R_{\varphi,\bar{\delta}})_{,\alpha}) \xi_{,\bar{\alpha}} \quad (36)$$

$$= -R_{\varphi,\delta\alpha} v_{,\bar{\delta}} \xi_{,\bar{\alpha}} + (\mathcal{D}_\varphi v - \bar{\mathcal{D}}_\varphi v)_{,\alpha} \xi_{,\bar{\alpha}} = 0 \quad (37)$$

The proof of Lemma 2.4 is therefore finished. \square

3 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1, concerning the deformations of Kähler metrics with constant scalar curvature. This will be achieved by the implicit function theorem; to start with, we define the functional space

$$\mathcal{H}^{4,\alpha}(M) = \{\varphi \in C^{4,\alpha}(M, \mathbb{R}) \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0\}.$$

The continuity path we will use is the same as the one in [8], namely

$$\begin{aligned} \mathcal{F} : \mathcal{H}^{4,\alpha}(M) \times [0, 1] &\longrightarrow C^\alpha(M) \times [0, 1] \\ \mathcal{F}(\varphi, t) &= (R_\varphi - \underline{R} - (1-t)(\text{tr}_\varphi \omega - n), t) \end{aligned}$$

where R_φ is the scalar curvature of ω_φ and

$$\underline{R} = \frac{1}{\text{Vol}(X, \omega)} \int_X R_\varphi \omega_\varphi^n$$

is the average of the scalar curvature (which is easily seen to be a cohomological quantity). The first component of \mathcal{F} will be denoted in what follows by F , i.e.

$$F(\varphi, t) := R_\varphi - \underline{R} - (1-t)(\text{tr}_\varphi \omega - n). \quad (38)$$

In this section, our main result states as follows.

Theorem 3.1. *Let (M, ω) be a compact Kähler manifold, such that the scalar curvature of ω is constant. We denote by $\varphi_1 \in \widetilde{\mathcal{H}}$ the potential for which the restriction $\iota|_{\Psi^\omega(\mathcal{O})}$ is minimal; let $\omega_{\varphi_1} \in [\omega]$ be the corresponding metric. Then there exists $\epsilon > 0$, such that for any $1 - \epsilon < t \leq 1$, there exists $\varphi_t = \varphi(t, \cdot)$ satisfying*

$$F(\varphi_t, t) = 0,$$

and such that $\phi(1, \cdot)$ coincides with the potential φ_1 .

Proof. As we have already mentioned, we intend to use the implicit function theorem, so the first thing to do would be to compute the differential of \mathcal{F} at the point $(\varphi_1, 1)$ for which we have $\mathcal{F}(\varphi_1, 1) = 0$. A standard calculation (which will not be detailed here) shows that we have

$$\begin{aligned} d\mathcal{F}_{(\varphi_1, 1)} : C^{4, \alpha}(M) \times \mathbb{R} &\longrightarrow C^\alpha(M) \times \mathbb{R} \\ (u, s) &\longmapsto (-\mathcal{D}_{\varphi_1} u + s(\text{tr}_{\varphi_1} \omega - n), s), \end{aligned}$$

where \mathcal{D}_{φ_1} is the Lichnerowicz operator with respect to ω_{φ_1} defined in the previous section (we are using here the fact that the scalar curvature of ω_{φ_1} is constant).

Let u_0 be a smooth function such that $\mathcal{D}_{\varphi_1}(u_0) = 0$ (we notice that in our set-up, the kernel of \mathcal{D}_{φ_1} has strictly positive dimension); then we have $d\mathcal{F}_{(\varphi_1, 1)}(u_0, 0) = 0$. Also, we remark that thanks to the minimality property of φ_1 , we have

$$\int u(\text{tr}_{\varphi_1} \omega - n) \omega_{\varphi_1}^n = 0 \tag{39}$$

for any $u \in \text{Ker}(\mathcal{D}_{\varphi_1})$: this is a consequence of Proposition 2.1, combined with the relation (10). In conclusion, $d\mathcal{F}_{(\varphi_1, 1)}$ is neither injective nor surjective.

Let k be a positive integer; we introduce the following notations.

$$\begin{aligned} \mathcal{H}_{\varphi_1} &= \{u \in C^\infty(M) \mid \mathcal{D}_{\varphi_1}(u) = 0, \int u \omega_{\varphi_1}^n = 0\} \\ \mathcal{H}_{\varphi_1, k}^\perp &= \{u \in C^{k, \alpha}(M) \mid \int u \omega_{\varphi_1}^n = 0, \int uv \omega_{\varphi_1}^n = 0, \text{ for all } v \in \mathcal{H}_{\varphi_1}\}. \end{aligned}$$

Thus we have the decomposition $C^{k, \alpha}(M) = \mathbb{R} \oplus \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_1, k}^\perp$. By using these notations the relation (39) becomes

$$\text{tr}_{\varphi_1} \omega - n \in \mathcal{H}_{\varphi_1, 0}^\perp. \tag{40}$$

Consider the following projection map

$$\begin{aligned} \Pi : (\mathbb{R} \oplus \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_1, 4}^\perp) \times [0, 1] &\longrightarrow (\mathbb{R} \oplus \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_1, 0}^\perp) \times [0, 1] \\ (a + u + w, t) &\longmapsto (a + u + \pi_2 \circ F(\varphi_1 + a + u + w, t), t), \end{aligned}$$

where π_2 is the projection from $C^\alpha(M)$ to $\mathcal{H}_{\varphi_1, 0}^\perp$. The derivative of Π at $(0, 1)$ equals

$$\begin{aligned} d\Pi_{(0, 1)} : (\mathbb{R} \oplus \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_1, 4}^\perp) \times \mathbb{R} &\longrightarrow (\mathbb{R} \oplus \mathcal{H}_{\varphi_1} \oplus \mathcal{H}_{\varphi_1, 0}^\perp) \times \mathbb{R} \\ (a + u + w, s) &\longmapsto (a + u - \mathcal{D}_{\varphi_1} w + s(\text{tr}_{\varphi_1} \omega - n), s). \end{aligned}$$

The relation $\text{tr}_{\varphi_1}\omega - n \in \mathcal{H}_{\varphi_1,0}^\perp$ combined with Lemma 2.3 show that $d\Pi|_{(\varphi_1,1)}$ is bijective. By the inverse function theorem, given any $\|u\|_{C^\alpha(M)} < \epsilon$ and $|t-1| < \epsilon$ we obtain $\psi(u, t)$ such that

$$\pi_2 \circ F(\varphi_1 + u + \psi(u, t), t) = 0. \quad (41)$$

The equality (41) shows that we have

$$-\mathcal{D}_{\varphi_1} \frac{\partial \psi}{\partial t} \Big|_{(0,1)} + \text{tr}_{\varphi_1}\omega - n = 0 \quad (42)$$

by differentiating with respect to t . Also, the derivative of (41) with respect to u gives

$$\frac{\partial \psi}{\partial u} \Big|_{(0,1)}(v) = 0, \quad (43)$$

for any $v \in \mathcal{H}_{\varphi_1}$.

We introduce the functional

$$P(u, t) := \pi_1 \circ F(\varphi_1 + u + \psi(u, t), t) \quad (44)$$

where π_1 is the projection onto the factor \mathcal{H}_{φ_1} . In order to finish the proof, it remains to solve the equation

$$P(u_t, t) = 0$$

for each $1 - \epsilon < t \leq 1$. However, we cannot apply the implicit function theorem, because it turns out that $P(u, 1) = 0$ for any $u \in \mathcal{H}_{\varphi_1}$. Indeed, the differential of P with respect to u vanishes at each point $(u, 1)$ (this is a consequence of (43), combined with the fact that $P(0, 1) = 0$).

Then we consider the "first derivative"

$$\tilde{P}(u, t) := \frac{P(u, t)}{t-1} \quad (45)$$

and we observe that $\tilde{P}(u, t)$ can be extended as a continuous function on $\mathcal{H}_{\varphi_1} \times [0, 1]$, because of the equality

$$\tilde{P}(u, 1) = \lim_{t \rightarrow 1^-} \frac{P(u, t)}{t-1} = \frac{\partial P}{\partial t} \Big|_{(u,1)}.$$

Our next observation is that it would be enough to solve the equation $\tilde{P}(u_t, t) = 0$, and so we will compute the partial derivative $\frac{\partial \tilde{P}}{\partial u} \Big|_{(0,1)}$ and we will show that it is invertible. Prior to this, we re-write the expression of \tilde{P} as follows.

$$\begin{aligned} \tilde{P}(u, 1) &= \frac{\partial}{\partial t} P|_{(u,1)} = \pi_1 \left[-\mathcal{D}_{\varphi_1+u+\psi_{u,1}} \frac{\partial \psi}{\partial t} \Big|_{(u,1)} + \text{tr}_{\varphi_1+u+\psi_{u,1}}\omega - n \right] \\ &= \pi_1 \left[-\Delta_{\varphi_1+u+\psi_{u,1}}^2 \frac{\partial \psi}{\partial t} \Big|_{(u,1)} - \left(\frac{\partial \psi}{\partial t} \Big|_{(u,1)} \right)_{,\bar{\alpha}\beta} (\text{Ric}_{\varphi_1+u+\psi_{u,1}})_{\alpha\bar{\beta}} \right. \\ &\quad \left. + \text{tr}_{\varphi_1+u+\psi_{u,1}}\omega - n \right] \end{aligned}$$

We compute

$$\begin{aligned} \frac{\partial}{\partial u} \tilde{P}|_{(0,1)}(v) &= \pi_1 \{ \langle \partial \bar{\partial} v, \partial \bar{\partial} \Delta_{\varphi_1} \xi \rangle_{\varphi_1} + \Delta_{\varphi_1} \langle \partial \bar{\partial} v, \partial \bar{\partial} \xi \rangle_{\varphi_1} + \langle \partial \bar{\partial} \Delta_{\varphi_1} v, \partial \bar{\partial} \xi \rangle_{\varphi_1} + \xi_{,\bar{\alpha}p} v_{,\bar{p}\beta} (Ric_{\varphi_1})_{\alpha\bar{\beta}} \\ &\quad + \xi_{,\bar{p}\beta} v_{,\bar{p}\bar{\alpha}} (Ric_{\varphi_1})_{\alpha\bar{\beta}} - \langle \partial \bar{\partial} v, \chi \rangle_{\varphi_1} - \mathcal{D}_{\varphi_1} \frac{\partial^2 \psi}{\partial u \partial t} |_{(0,1)}(v) \} \\ &= \pi_1 [B_{\varphi_1}(v, \xi) - \langle \partial \bar{\partial} v, \chi \rangle_{\varphi_1}] \end{aligned}$$

where $\xi = \frac{\partial \psi}{\partial t} |_{(0,1)}$ and $B_{\varphi_1}(v, \xi)$ is the operator in Lemma 2.4. The previous string of equalities combined with Lemma 2.4 imply that we have

$$\begin{aligned} \frac{\partial}{\partial u} \tilde{P}|_{(0,1)}(v) &= \pi_1 [\mathcal{D}_{\varphi_1}(\langle \partial v, \bar{\partial} \xi \rangle_{\varphi_1}) - \langle \partial v, \bar{\partial} \mathcal{D}_{\varphi_1} \xi \rangle_{\varphi_1} - \langle \partial \bar{\partial} v, \omega \rangle_{\varphi_1}] \\ &= \pi_1 (-\langle \partial v, \bar{\partial}(\text{tr}_{\varphi_1} \omega - n) \rangle_{\varphi_1} - \langle \partial \bar{\partial} v, \omega \rangle_{\varphi_1}). \end{aligned}$$

Then we see that the scalar product

$$\begin{aligned} \int \frac{\partial \tilde{P}}{\partial u} |_{(0,1)}(v) v \omega_{\varphi}^n &= \int (-\langle \partial v, \bar{\partial}(\text{tr}_{\varphi_1} \omega - n) \rangle_{\varphi_1} v - \langle \partial \bar{\partial} v, \omega \rangle_{\varphi_1} v) \omega_{\varphi_1}^n \\ &= \int v_{,\bar{\alpha}v,\beta} \omega_{\alpha\bar{\beta}} \omega_{\varphi_1}^n \geq 0, \end{aligned}$$

is positive, and it is equal to zero if and only if $v = 0$ in \mathcal{H}_{φ_1} . Therefore, $\frac{\partial \tilde{P}}{\partial u} |_{(0,1)}$ is injective and therefore bijective. The implicit function theorem shows that there exists u_t such that $P(u_t, t) = 0$ for t sufficiently close to 1; when combined with (41), this implies

$$F(\varphi_1 + u_t + \psi(u_t, t), t) = 0$$

which is what we wanted to prove. □

The uniqueness of constant scalar curvature metrics follows almost immediately.

Corollary 3.2. *Suppose there exists two cscK metrics $\omega_{\varphi_1}, \omega_{\varphi_2} \in [\omega]$. Then there exists an element $\sigma \in \text{Aut}_0(M)$ such that $\sigma^* \omega_{\varphi_1} = \omega_{\varphi_2}$.*

Proof. We argue by contradiction: suppose we have two cscK orbits \mathcal{O}_1 and \mathcal{O}_2 such that $\mathcal{O}_1 \neq \mathcal{O}_2$. Then we consider the Kähler potentials φ_1 and φ_2 for which the restriction of ι to \mathcal{O}_1 and \mathcal{O}_2 is reached, respectively,

By Theorem 3.1, we obtain two paths $\varphi_i(t)$ with $\varphi_k(1, \cdot) = \varphi_k$, for $k = 1, 2$; moreover, we obtain

$$R_{\varphi_k(t)} - \underline{R} - (1-t)(\text{tr}_{\varphi_k(t)} \omega - n) = 0. \quad (46)$$

As explained in the introduction, for fixed $t < 1$, the solution of equation (46) is unique. Thus, for any $1 - \epsilon < t < 1$, $\varphi_1(t) = \varphi_2(t)$. In particular $\varphi_1 = \varphi_2$. Therefore, we are done. □

4 Twisted extremal Kähler metrics

We start with a general discussion about the proof of Theorem 1.2 which will follow; hopefully, this will clarify a few facts/choices which will appear shortly.

Let $\omega_{\varphi_1} \in [\omega]$ be an extremal metric. In order to prove Theorem 1.2, our strategy will be to determine the path $\varphi_t := \varphi(t, \cdot)$ by solving the equation

$$\nabla_{\varphi_t}^{1,0}(R_{\varphi_t} - (1-t) \operatorname{tr}_{\varphi_t} \omega) = X_1 \quad (47)$$

where $X_1 := \nabla_{\varphi_1}^{1,0}(R_{\varphi_1})$ is a holomorphic vector field.

We show next that the order of differentiation in the expression (47) can be reduced. Indeed we have

$$i_{X_1} \omega_{\varphi_t} := \sqrt{-1} \partial \bar{\partial} \rho_t(X_1) \quad (48)$$

for a unique function $\rho_t(X_1) : M \rightarrow \mathbb{C}$ normalized such that

$$\int_X \rho_t(X_1) \omega_{\varphi_t}^n = 0. \quad (49)$$

(this can be seen by writing $\omega_{\varphi_t} = \omega_{\varphi_1} + \sqrt{-1} \partial \bar{\partial} \phi_t$). By combining (47) and (48), the equation we have to solve is equivalent to

$$R_{\varphi_t} - \underline{R} - (1-t)(\operatorname{tr}_{\varphi_t} \omega - n) = \rho_t(X_1). \quad (50)$$

The equation (50) above is very similar to the one we had to deal with in the previous section. We could then simply follow the same procedure as in the proof of Theorem 1.1 (i.e. start with an extremal metric whose potential minimizes the functional ι and so on) in order to conclude, even if the presence of the factor $\rho_t(X_1)$ complicates a bit the situation, as we will see next. However in doing so, we would not be able to obtain the uniqueness statement Corollary 1.3, for a simple reason which will become obvious at the end of this section (basically we need the holomorphic gradient of the scalar curvature corresponding to ω_1 and ω_2 to coincide). Also the term $\rho_t(X_1)$ would in general be complex valued and we don't want to choose our image space to be complex valued functions.

Luckily, it is possible to bypass these difficulties by using the following results; the first is due to E. Calabi.

Theorem 4.1. [6] *For any extremal Kähler metric g in a compact complex manifold M , the identity component $\operatorname{Iso}_0(M, g)$ of the group of holomorphic isometries of (M, g) coincides with a maximal compact connected subgroup of $\operatorname{Aut}_0(X)$.*

The following statement is a reformulation of a result due to Futaki-Mabuchi, cf. [18], in which we are using Theorem 4.1.

Theorem 4.2. [18] *Let $g_j \in [\omega]$ be two extremal metrics, such that*

$$\operatorname{Iso}_0(M, g_1) = \operatorname{Iso}(M, g_2).$$

Then we have $\nabla_{g_1}^{1,0}(R_{g_1}) = \nabla_{g_2}^{1,0}(R_{g_2})$.

We assume next that ω is an extremal metric, and we denote by $K := \text{Iso}(M, \omega)$ the corresponding group of holomorphic isometries. The next step would be to consider the minimum ω_{φ_1} of the restriction of the functional ι to the space of potentials $\Psi^\omega(\mathcal{O})$ corresponding to ω ; in doing so, it is possible that the isometry group of ω_{φ_1} is different from K .

In order to prevent this to happen, we will restrict the functional ι to the space of ω -potentials which are K -invariant, defined as follows

$$\begin{aligned}\mathcal{H}_K^\infty(M) &= \{\varphi \in \mathcal{H}^\infty(M) \mid \varphi = \varphi \circ \sigma \text{ for any } \sigma \in K\}; \\ \mathcal{H}_K^{k,\alpha}(M) &= \{\varphi \in \mathcal{H}^{k,\alpha}(M) \mid \varphi = \varphi \circ \sigma \text{ for any } \sigma \in K\};\end{aligned}$$

we equally consider the space

$$C_K^{k,\alpha}(M) = \{u \in C^{k,\alpha}(M) \mid u = u \circ \sigma \text{ for any } \sigma \in K\}.$$

Let \mathcal{O}_K be the quotient N_K/K , where we denote by N_K the normalizer of K in $\text{Aut}_0(M)$, that is to say the group consisting of $g \in \text{Aut}_0(M)$ such that $gKg^{-1} = K$.

We have the following statement, which is the analogue of Proposition 2.1.

Proposition 4.3. *Let (M, ω) be a compact Kähler manifold, such that ω is extremal. Then the image of the tangent space $(\Psi^\omega)_*(T_{\mathcal{O}_K, g})$ coincides with the space generated by the real-valued functions $f \in C^\infty(M)$ which are K -invariant, such that $\nabla_\varphi^{1,0} f$ is holomorphic, where $\varphi := \Psi^\omega(g)$.*

Proof. First, we have to check that the image of $\Psi^\omega|_{\mathcal{O}_K}$ consists of K -invariant potentials. Let $g \in N_K$; we have

$$g^*\omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi. \quad (51)$$

and let $\sigma \in K$. Since g belongs to the normalizer of K , we have

$$\sigma^*g^*\omega = g^*\omega$$

hence by (51) we obtain $\varphi \circ \sigma = \varphi$.

Let g_t be a smooth path in N_K , such that $g_0 = g$ and such that the derivative $\frac{dg_t}{dt}|_{t=0}$ identifies with a holomorphic vector field which we denote by X . There exists a smooth family $(\varphi_t) \subset \widetilde{\mathcal{H}}$ with $\varphi_0 = \varphi$ so that we have

$$g_t^*\omega := \omega_{\varphi_t} \quad (52)$$

for any parameter t . Since ω is extremal and K -invariant, so is ω_{φ_0} . By Calabi's theorem[6], we can decompose a holomorphic (1,0) vector field as follows

$$T_e \text{Aut}_0(M) = a(M) \oplus \nabla_{\varphi_0}^{1,0} E,$$

where $a(M)$ are autoparallel vectors on (M, ω_{φ_0}) and E is the kernel of Lichnerowicz derivative, i.e. $E = \{f \in C^\infty(M, \mathbb{C}) \mid \mathcal{D}_{\varphi_0} f = 0\}$.

We denote by $\bar{\mathcal{D}}_{\varphi_0}$ the conjugate of the operator \mathcal{D}_{φ_0} . Since the metric ω_{φ_0} is extremal, we have

$$[\mathcal{D}_{\varphi_0}, \bar{\mathcal{D}}_{\varphi_0}] = 0$$

i.e. the two operators commute. In particular, we can further decompose the space E according to the eigenspaces of $\bar{\mathcal{D}}_{\varphi_0}|_E$, so that we have

$$T_e \text{Aut}_0(M) = a(M) \oplus \nabla_{\varphi_0}^{1,0} E_0 \oplus \sum_{\lambda > 0} \nabla_{\varphi_0}^{1,0} E_\lambda,$$

where E_λ represents the λ -eigenspace of $\bar{\mathcal{D}}_{\varphi_0}$. Notice here that the eigenvalues above are real and nonnegative.

By the above discussion, we can write

$$X = X_a + \nabla_{\varphi_0}^{1,0}(f_0 + \sum_{\lambda > 0} f_\lambda)$$

where $X_a \in a(M)$ and $f_\lambda \in E_\lambda$ for $\lambda \geq 0$. Notice here the sum is finite since $\text{Aut}_0(M)$ is finite dimensional. Since $g_t \in N_K$, it implies that for any $\sigma \in K$,

$$g_t^* \sigma^* (g_t^{-1})^* \omega_{\varphi_0} = \omega_{\varphi_0}.$$

Differentiate with respect to t ,

$$\begin{aligned} 0 &= \left(\frac{d}{dt} g_t^* \sigma^* (g_t^{-1})^* \omega_{\varphi_0} \right) |_{t=0} = \sigma^* \left(\frac{d}{dt} (g_t^{-1})^* \omega_{\varphi_0} \right) |_{t=0} + \left(\frac{d}{dt} g_t^* \sigma^* \omega_{\varphi_0} \right) |_{t=0} \\ &= \sqrt{-1} \partial \bar{\partial} [(f + \bar{f}) - (f + \bar{f}) \circ \sigma] \end{aligned}$$

where $f = f_0 + \sum_{\lambda > 0} f_\lambda$. Hence we get for any $\sigma \in K$

$$f + \bar{f} - (f + \bar{f}) \circ \sigma = 0. \quad (53)$$

Applying $\bar{\mathcal{D}}_{\varphi_0}$ on both hand sides of (53) k times, we get that

$$\sum_{\lambda > 0} \lambda^k (f_\lambda - f_\lambda \circ \sigma) = 0.$$

Thus we infer that $f_\lambda - f_\lambda \circ \sigma = 0$ for any $\sigma \in K$. Consider

$$\Xi_{\varphi_0} := \text{Im}(\nabla_{\varphi_0}^{1,0} R_{\varphi_0}) = \frac{\sqrt{-1}}{2} [g_{\varphi_0}^{\alpha\bar{\beta}} R_{\varphi_0, \bar{\beta}} \frac{\partial}{\partial z_\alpha} - g_{\varphi_0}^{\alpha\bar{\beta}} R_{\varphi_0, \alpha} \frac{\partial}{\partial \bar{z}_\beta}],$$

and $\exp(t\Xi_{\varphi_0})$ is a one parameter subgroup of K . Since f_λ is K -invariant,

$$0 = \frac{d}{dt} \exp(t\Xi_{\varphi_0})^* f_\lambda = \Xi_{\varphi_0}(f_\lambda) = \frac{\sqrt{-1}}{2} [R_{\varphi_0, \bar{\delta}} f_{\lambda, \delta} - R_{\varphi_0, \delta} f_{\lambda, \bar{\delta}}].$$

Hence

$$\lambda f_\lambda = \bar{\mathcal{D}}_{\varphi_0} f_\lambda = -(\mathcal{D}_{\varphi_0} - \bar{\mathcal{D}}_{\varphi_0}) f_\lambda = R_{\varphi_0, \bar{\delta}} f_{\lambda, \delta} - R_{\varphi_0, \delta} f_{\lambda, \bar{\delta}} = 0.$$

Therefore, $f_\lambda = 0$ for any $\lambda > 0$. Thus,

$$X = X_a + \nabla_{\varphi_0}^{1,0} f_0$$

where $f_0 \in \text{Ker } \mathcal{D}_{\varphi_0} \cap \text{Ker } \bar{\mathcal{D}}_{\varphi_0}$ is a K -invariant complex-valued function. Therefore, $\text{Re}(f_0)$ and $\text{Im}(f_0)$ are both K -invariant and belong to $\text{Ker } \mathcal{D}_{\varphi_0} \cap \text{Ker } \bar{\mathcal{D}}_{\varphi_0}$. By differentiating (52) at $t = 0$, we get that

$$\sqrt{-1} \partial \bar{\partial} \dot{\varphi}_0 = \sqrt{-1} \partial \bar{\partial} \text{Re}(f_0).$$

The rest of the argument follows from Proposition 2.1 and it ends the proof. \square

Precisely as in Lemma 2.2, the restriction $\iota|_{\Psi^\omega(\mathcal{O}_K)}$ is proper. Let $\varphi_1 \in \Psi^\omega(\mathcal{O}_K)$ be the potential for which its minimum is reached; we denote by ω_{φ_1} the resulting (extremal) metric. We note that we have the equality

$$\text{Iso}(M, \omega_{\varphi_1}) = K$$

by Theorem 4.1.

Let $X_1 = \nabla_{\varphi_1}^{1,0} R_{\varphi_1}$ be the holomorphic vector field corresponding to the metric ω_{φ_1} . We define the functional $\mathcal{F}_K : \mathcal{H}_K^{4,\alpha}(M) \times [0, 1] \rightarrow C_K^{0,\alpha}(M)$ by the formula

$$\mathcal{F}_K(\varphi, t) = R_\varphi - \underline{R} - (1-t)(\text{tr}_\varphi \omega - n) - \rho_\varphi(X_1) \quad (54)$$

where we recall that $\rho_\varphi(X_1)$ is uniquely determined by

$$i_{X_1} \omega_\varphi = \sqrt{-1} \bar{\partial} \rho_\varphi(X_1), \quad \int_M \rho_\varphi(X_1) \omega_\varphi^n = 0. \quad (55)$$

Remark. If $\varphi \in \mathcal{H}_K^{4,\alpha}(M)$, then $\rho_\varphi(X_1)$ is real-valued. This is because

$$\sqrt{-1} \bar{\partial} \rho_\varphi(X_1) = i_{X_1} \omega_\varphi = i_{X_1} \omega_{\varphi_1} + i_{X_1} (\sqrt{-1} \bar{\partial} (\varphi - \varphi_1)) \quad (56)$$

$$= \sqrt{-1} \bar{\partial} (R_{\varphi_1} + X_1(\varphi - \varphi_1)). \quad (57)$$

Thus

$$\rho_\varphi(X_1) = R_{\varphi_1} + X_1(\varphi - \varphi_1) - \int_M (R_{\varphi_1} + X_1(\varphi - \varphi_1)) \omega_\varphi^n. \quad (58)$$

And the imaginary part of $\rho_\varphi(X_1)$ is given by

$$\text{Im}(\rho_\varphi(X_1)) = \text{Im}(X_1)(\varphi - \varphi_1) - \int_M \text{Im}(X_1)(\varphi - \varphi_1) \omega_\varphi^n. \quad (59)$$

On the other hand, we know that $\text{Im}(X_1)$ is in the Lie algebra of $\text{Iso}(M, \omega_{\varphi_1}) = K$. Since $(\varphi - \varphi_1)$ is K -invariant, we obtain $\text{Im}(\rho_\varphi(X_1)) = 0$.

By the discussion at the beginning of this section, the following perturbation theorem implies Theorem 1.2.

Theorem 4.4. *Under the notations and conventions above, for any $t \in (0, 1)$ sufficiently close to 1, there exists $\varphi(t, \cdot) = \varphi_t \in \mathcal{H}_K^{4,\alpha}(M)$ such that $\mathcal{F}_K(\varphi_t, t) = 0$ and such that $\varphi(1, \cdot)$ is the potential φ_1 achieving the minimum of ι .*

Proof. The arguments are very similar to the ones used in the proof of Theorem 1.1; for the convenience of the reader, we review here the slight differences. To start with, the expression of the linearization at $(\varphi_1, 1)$ of \mathcal{F}_K has an additional term, which we now compute.

By differentiating the first term of (55), we obtain

$$\bar{\partial} \dot{\rho}_\varphi(X_1) = \bar{\partial} X_1(\dot{\varphi}) \quad (60)$$

so that $\dot{\rho}_\varphi(X_1) - X_1(\dot{\varphi})$ is constant. On the other hand, by differentiating the second term of (55) we infer that we have

$$\int_M \left(\dot{\rho}_\varphi(X_1) + \rho_\varphi(X_1) \Delta_\varphi(\dot{\varphi}) \right) \omega_\varphi^n = 0. \quad (61)$$

Integration by parts together with the relation (55) gives

$$\int_M \dot{\rho}_\varphi(X_1) - X_1(\dot{\varphi}) \omega_\varphi^n = 0 \quad (62)$$

so in conclusion, we have $\dot{\rho}_\varphi(X_1) = X_1(\dot{\varphi})$. Given the definition of X_1 , this is equivalent to

$$\dot{\rho}_\varphi(X_1) = \langle \partial \dot{\varphi}, \bar{\partial} R_\varphi \rangle_{\omega_\varphi}. \quad (63)$$

Then the derivative of \mathcal{F}_K at $(\varphi_1, 1)$ has the following expression

$$\begin{aligned} d\mathcal{F}|_{(\varphi_1, 1)} : C_K^{4, \alpha}(M) \times \mathbb{R} &\longrightarrow C_K^{0, \alpha}(M) \\ (u, s) &\longmapsto -\mathcal{D}_{\varphi_1} u + s(\text{tr}_{\varphi_1} \omega - n). \end{aligned}$$

where –exactly as in the case of cscK metrics– the operator \mathcal{D}_φ is the Lichnerowicz operator. We define the following functional spaces:

$$\begin{aligned} \mathcal{H}_{K, \varphi_1} &:= \{u \in \mathcal{H}_K^\infty(M) \mid \mathcal{D}_{\varphi_1} u = 0, \int u \omega_{\varphi_1}^n = 0\}, \\ \mathcal{H}_{K, \varphi_1, k}^\perp &:= \{u \in C_K^{k, \alpha}(M) \mid \int u v \omega_{\varphi_1}^n = 0 \text{ for any } v \in \mathcal{H}_{K, \varphi_1}, \int u \omega_{\varphi_1}^n = 0\}. \end{aligned}$$

and then we have the following statement.

Lemma 4.5. *We have the orthogonal decomposition*

$$C_K^{k, \alpha}(M) = \mathbb{R} \oplus \mathcal{H}_{K, \varphi_1} \oplus \mathcal{H}_{K, \varphi_1, k}^\perp.$$

Proof. Indeed, this is a consequence of the fact that the operator \mathcal{D}_{φ_1} is K -invariant and self adjoint on $C_K^\infty(M)$. And we have $C_K^{0, \alpha}(M) = \mathcal{D}_{\varphi_1} \left(C_K^{4, \alpha}(M) \right) \oplus \text{Ker}(\mathcal{D}_{\varphi_1})$ which can be derived from the elliptic operators theory. \square

The rest of the proof of Theorem 4.4 is strictly identical to the one presented in the previous section, so we will not discuss it further here. \square

We prove next Corollary 1.3.

Proof. We begin with a few reductions. By Theorem 4.1, combined with the fact that the maximal compact subgroups of $\text{Aut}_0(M)$ are conjugate (by a result of Matsushima), we can assume that we have

$$\text{Iso}(M, \omega_1) = \text{Iso}(M, \omega_2).$$

We can equally assume that φ_j is the minimum point of the functional $\iota|_{\Psi_{\omega_j}}$, for $j = 1, 2$. Then we still have

$$\text{Iso}(M, \omega_{\varphi_1}) = \text{Iso}(M, \omega_{\varphi_2}) \quad (64)$$

and by Theorem 4.2 we have $X_1 = X_2 := X$.

Theorem 4.4 shows that there exists two paths of twisted extremal metrics, $\varphi_{k,t}$ with $\varphi_{k,1} = \varphi_k$ for $k = 1, 2$ satisfying

$$\nabla_{\varphi_{k,t}}^{1,0} (R_{\varphi_{k,t}} - (1-t)\text{tr}_{\varphi_{k,t}}\chi) = X_k.$$

Hence, we get two smooth families ($t \in (1 - \epsilon, 1]$) of solutions to the equation

$$R_{\varphi_t} - \underline{R} - \rho_{\varphi_t}(X) - (1-t)(\text{tr}_{\varphi_t}\omega - n) = 0. \quad (65)$$

We prove next that for fixed $t \in (0, 1)$, the K -invariant smooth solution of (65) is unique. First, we introduce the modified K-energy (c.f. [18]) on \mathcal{H}_K^∞

$$\frac{d\mathcal{E}_K}{dt} = \int_M \left(- (R_\varphi - \underline{R}) + \rho_\varphi(X) \right) \frac{d\varphi}{dt} \omega_\varphi^n.$$

And \mathcal{E}_K is weakly convex along any K -invariant $C^{1,1}$ geodesic segment by [3] and [14]. Moreover, ι is strictly convex along $C^{1,1}$ geodesic segments. Therefore, for $t \in (0, 1)$

$$\mathcal{E}_K + (1-t)\iota \quad (66)$$

is strictly convex along any K -invariant $C^{1,1}$ geodesic segment. Also note that any two K -invariant Kähler potentials can be joined by a K -invariant $C^{1,1}$ geodesic.

By the strict convexity, we can conclude the K -invariant solution of (65) is unique. Hence $\varphi_{1,t} = \varphi_{2,t}$ for $t \in (1 - \epsilon, 1)$. As $t \rightarrow 1$ we get that $\varphi_1 = \varphi_2$, which is a contradiction, and the proof of Corollary 1.3 is finished. \square

References

- [1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*. Bull. Sci. Math. (2) 102 (1978), no. 1, 63–95
- [2] S. Bando and T. Mabuchi. *Uniqueness of Einstein Kähler Metrics Modulo Connected Group Actions*. In Algebraic Geometry, Advanced Studies in Pure Mathematics, 1987.
- [3] R. Berman and B. Berndtsson. *Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics*. <http://arxiv.org/abs/1405.0401>
- [4] E. Calabi. *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*. Michigan Math. J. Volume 5, Issue 2 (1958), 105-126.
- [5] E. Calabi. *Extremal Kähler metrics*. Seminar on Differential Geometry, volume 16 of 102, pages 259-290, Ann. of Math Studies, University Press, 1982.
- [6] E. Calabi. *Extremal Kähler Metrics II*. Differential Geometry and Complex Analysis, pages 96-114, Springer, 1985.
- [7] X. X. Chen. *Space of Kähler metrics*. Journal of Differential Geometry, 56(2):189-234, 2000.

- [8] X. X. Chen. *On the existence of constant scalar curvature Kähler metric: a new perspective*, preprints.
- [9] X. X. Chen. *On the lower bound of the Mabuchi energy and its application*, Int Math Res Notices (2000) Vol. 2000 607-623.
- [10] X. X. Chen, S. Donaldson and S. Sun. *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*. J. Amer. Math. Soc. 28 (2015), pp. 183-197 (I).
- [11] X. X. Chen, S. Donaldson and S. Sun. *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π* . J. Amer. Math. Soc. 28 (2015), pp. 199-234.
- [12] X. X. Chen, S. Donaldson and S. Sun. *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof*. J. Amer. Math. Soc. 28 (2015), pp. 235-278.
- [13] X. X. Chen and G. Tian. *Foliation by holomorphic discs and its application in Kähler geometry, 2003*. Publ. Math. Inst. Hautes tudes Sci. No. 107(2008), 1-107.
- [14] X. X. Chen, L. Long and M. Paun. *Approximation of weak geodesics and subharmonicity of Mabuchi energy*. <http://arxiv.org/abs/1409.7896>.
- [15] S. K. Donaldson. *Scalar curvature and projective embeddings I*. J. Differential Geom. 59 (2001), no. 3, 479-522.
- [16] J. Fine. *Constant scalar curvature Kähler metrics on fibred complex surfaces*. J. Differential Geom. Volume 68, Number 3(2004), 397-432.
- [17] L. Hörmander. *The Analysis of Linear Partial Differential Operators*. Springer-Verlag.
- [18] A. Futaki and T. Mabuchi. *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*. Math. Ann. 301, 199-210 (1995).
- [19] Peter Li. *Lecture notes on geometric analysis*, Lecture Notes Series, vol. 6, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1993. MR1320504 (96m:58269)
- [20] M. Lejmi and G. Székelyhidi. *The J-flow and stability*. <http://arxiv.org/abs/1309.2821v1>.
- [21] C. LeBrun and R. Simanca. *Extremal Kähler Metrics and Complex Deformation Theory*. Geom. Func. Analysis 4 (1994) 298-336.
- [22] T. Mabuchi. *Some Symplectic geometry on compact Kähler metrics*. J. Differential Geom. 21(1985) 73-77.
- [23] T. Mabuchi. *Uniqueness of extremal Kähler metrics for an integral Kähler class*. International Journal of Mathematics Vol. 15, No. 6 (2004) 531-546.

- [24] J. Stoppa. *Twisted constant scalar curvature Kähler metrics and Kähler slope stability*. J. Differential Geom. Volume 83, Number 3(2009), 663-691.
- [25] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*. Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.
- [26] G. Tian and X. H. Zhu. *Uniqueness of Kähler-Ricci solitons*. Acta Math., 184 (2000), 271-305.