APPROXIMATION OF WEAK GEODESICS AND SUBHARMONICITY OF MABUCHI ENERGY

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1. Introduction

In a recent paper [4], R. Berman and B. Berndtsson established the convexity of the Mabuchi energy functional $\mathcal{M}$ along the so-called *weak geodesics*, answering (affirmatively) to a conjecture proposed by the first named author of this article. Given two Kähler metrics in the same cohomology class, it is well-known (cf. [15], [9]) that in general one cannot find a *smooth geodesic* connecting them: this is a major source of difficulties while dealing e.g. with the aforementioned convexity question.

In this article we explore in a systematic way two techniques of approximation of weak geodesics. The first and most natural one is given by the *ε-geodesics*, obtained in [6]. The second one consists in using a fiberwise approximation of weak geodesics via a family of well-chosen Monge-Ampère equations. A corollary of the second technique is an alternative proof of the result in [4]. Roughly speaking, our proof can be seen as a "global version" of the local Bergman kernel arguments, so morally we follow the original ideas of [4]; nevertheless, we feel that our approach might be useful in other contexts. For example, the method we are using here allows us to establish the convexity of $\mathcal{M}$ more *directly* than in the original article, where the first step is to show convexity of $\mathcal{M}$ in the sense of distributions.

We equally infer that the Mabuchi functional is *continuous* up to the boundary when evaluated on a weak geodesic. The proof of this second statement is based on semi-continuity properties of the entropy functional.

Another theorem we will establish here is the almost-convexity of the regularized Mabuchi energy along the ε-geodesics cf. [6]. Actually, our hope is that this latter result could be also used in order to provide a proof of the convexity of $\mathcal{M}$ along weak geodesics. We refer to the comments at the end of this note for further support concerning this belief.

This article is organized as follows. We start by recalling the important result of Xiuxiong Chen in [6] concerning the existence of $\mathcal{C}^{1,1}$ solutions of the MA equation describing the geodesic between two Kähler metrics. After a preliminary discussion about the strategy of the proof, the convexity and the continuity of $\mathcal{M}$ are obtained in section 4 via the approximation procedure mentioned above. Finally, the convexity of $\mathcal{M}$ along ε-geodesics and some other results/expectations are treated in section 5.

2. Geodesics

Let X be a compact Kähler manifold; we denote by $\mathcal{K}$ its Kähler cone. Let $\{\omega\} \in \mathcal{K}$ be a Kähler class of X; the notation above means that the representative ω is non-singular and definite positive. Let $\omega_0, \omega_1 \in \{\omega\}$ be two positive definite representatives of the same cohomology class. A *weak geodesic* between ω_0 and ω_1 is a semi-positive definite (1,1)-current

$$\mathcal{G} := \omega + dd^c\varphi \tag{1}$$

on the product $X \times \Sigma$ of the manifold X with the annulus $\Sigma \subset \mathbb{C}$, such that the following requirements are satisfied.

(a) The function φ is $\mathcal{C}^{1,1}$ on $X \times \Sigma$; in particular, the coefficients of $\mathcal{G}$ are bounded.

(b) We have $\mathcal{G}^{n+1} = 0$.

(c) The current $\mathcal{G}$ is rotationally invariant, and it equals ω_0 and ω_1 on the boundary of Σ, respectively.

The existence of $\mathcal{G}$ with the properties stated above was first established in [6], together with important complements in [5].

3. What is to be proved

Let u be a smooth function on X, such that $\omega_u := \omega + dd^c u$ is a Kähler metric. The *energy* functional is given by the expression

$$\mathcal{E}(u) := \sum_{j=0}^{n} \int_X u\omega_u^{n-j} \wedge \omega^j. \tag{2}$$

Given a (1,1)-form α, one introduces the following version of the energy functional

$$\mathcal{E}^\alpha(u) := \sum_{j=0}^{n-1} \int_X u\omega_u^{n-j-1} \wedge \omega^j \wedge \alpha. \tag{3}$$

For a smooth path $\omega_t := \omega + dd^c u_t$ of Kähler metrics depending on the parameter $t \in \Sigma$, one rapidly computes

$$dd^c\mathcal{E}(t) = \int_X \Omega^{n+1}, \quad dd^c\mathcal{E}^\alpha(t) = \int_X \Omega^n \wedge \alpha, \tag{4}$$

where $\Omega := \omega + dd^c u$ is a (1,1)-form on $X \times \Sigma$, and the integration is understood as the push-forward of an $(n+1, n+1)$ form to Σ (we use the same notation for ω and its inverse image on $X \times \Sigma$). An important observation (cf. [7], [4] and the references therein) is that the equalities

(4) still holds true *in the sense of distributions* if the path (u_t) is only assumed to be continuous.

The *Mabuchi functional* $\mathcal{M}$ along $\mathcal{G}$ is defined as follows

$$\mathcal{M}(t) = \frac{S}{n+1}\mathcal{E}(\varphi_t) - \mathcal{E}^{Ric_\omega}(\varphi_t) + \int_X \log \frac{\mathcal{G}_t^n}{\omega^n}\mathcal{G}_t^n \tag{5}$$

where $\mathcal{G}_t := \omega + dd^c\varphi_t$ is the restriction of $\mathcal{G}$ to the slice $X\times\{t\} \subset X\times\Sigma$, and S is the average of the scalar curvature of (X,ω). Unlike the original definition of $\mathcal{M}$, the expression (5) first introduced in [7], has a meaning even if the regularity of φ is only $\mathcal{C}^{1,1}$. We recall further that the convexity of $\mathcal{M}$ along weak geodesics was conjectured in [7].

Ideally, the convexity of $\mathcal{M}$ would follow provided that one is able to produce the following objects.

Let $(\Theta_\varepsilon)_{\varepsilon>0} \subset \{\omega\}$ be a family of closed positive (1,1) currents on $X\times\Sigma$, such that for each positive ε we have.

(a) The potential ϕ_ε of each Θ_ε is of class $\mathcal{C}^{1,1}$, and it converges to φ locally uniformly on $X\times\Sigma$.

(b) The logarithm of the fiberwise determinant of Θ_ε is of class $\mathcal{C}^1$.

(c) The determinant of $\Theta_\varepsilon|_{X\times\{t\}}$ converges a.e. to $\mathcal{G}^n|_{X\times\{t\}}$.

As explained in [4], it is enough show that we can find $(\Theta_\varepsilon)_{\varepsilon>0}$ as above, such that moreover we have

$$dd^c \log\frac{\Theta_\varepsilon^n}{\omega^n}\wedge\mathcal{G}^n \geq \mathrm{Ricci}_\omega \wedge\mathcal{G}^n \tag{6}$$

where the quantity $\log\frac{\Theta_\varepsilon^n}{\omega^n}$ denotes (slightly abusively) a function on $X\times\Sigma$. Indeed, given the relations (4), the inequality (6) implies the convexity in the sense of distributions of the following functional

$$\mathcal{M}(\varepsilon,t) := \frac{S}{n+1}\mathcal{E}(\varphi_t) - \mathcal{E}^{Ric_\omega}(\varphi_t) + \int_X \log\frac{\Theta_\varepsilon^n}{\omega^n}\mathcal{G}_t^n. \tag{7}$$

By the condition (b) the functional $\mathcal{M}(\varepsilon,t)$ is continuous; therefore its convexity in weak sense implies convexity in usual sense. The condition (c) would imply the same property for $\mathcal{M}$, by letting $\varepsilon\to 0$.

An excellent candidate for the family Θ_ε would be the approximation of $\mathcal{G}$ contained in the proof of X.X. Chen, i.e. the ε-*geodesics*. Indeed, the properties (a) and (b) are direct consequences of [6], and the inequality (6) can be checked to hold true on the set $\Lambda_{A,\varepsilon}$ where Θ_ε is uniformly bounded from below by $\exp(-A)\omega$ via a direct computation (this would be enough to conclude, by letting $A\to\infty$). However, it does not seem to be so easy to establish the crucial property (c) (we refer the the paper [8], section 6, in order to have a glimpse at the difficulties/consequences of such a statement).

In order to overcome this issue, we will consider in the next paragraph a different approximation of $\mathcal{G}$, obtained by solving a family of MA equations.

A version of the convexity of $\mathcal{M}$ along the ε-geodesics will be treated in the last part of our note.

4. Fiber-wise approximation of $\mathcal{G}$

In order to construct the family of currents $(\Theta_\varepsilon)_{\varepsilon>0}$ with the properties stated in the previous section we recall the following result.

Theorem 4.1. ([16]) *Let $p : X \to Y$ be a holomorphic submersion. We consider a semi-positive class $\{\beta\} \in H^{1,1}(X, \mathbb{R})$, such that the adjoint class $c_1(K_{X_y}) + \{\beta\}|_{X_y}$ is Kähler for any $y \in Y$. Then the relative adjoint class*

$$c_1(K_{X/Y}) + \{\beta\}$$

contains a closed positive current Ξ, whose restriction to each fiber X_y is a positive definite form.

We specialize here to the case of the trivial submersion $X \times \Sigma \to \Sigma$, so the relative canonical bundle equals the inverse image of K_X.

As a consequence of the previous result we infer the next statement; we recall that $\mathcal{G}$ denotes the (weak) geodesic between the two metrics ω_0 and ω_1.

Theorem 4.2. *For each $t \in \Sigma$ and for each $0 < \varepsilon \ll 1$, we consider the equation*

$$\big(\Theta_\omega(K_X) + \varepsilon^{-1}\mathcal{G} + dd^c\phi_{t,\varepsilon}\big)^n = \varepsilon^{-n}\exp(\phi_{t,\varepsilon})\omega^n; \tag{8}$$

on $X \times \{t\}$. It has a unique $\mathcal{C}^{1,1}\big(X \times \{t\}\big)$-solution $\phi_{t,\varepsilon}$; the resulting function ϕ_ε on $X \times \Sigma$ is continuous on the interior points of $X \times \Sigma$, and moreover we have

$$\Theta_\omega(K_X) + \varepsilon^{-1}\mathcal{G} + dd^c\phi_\varepsilon \geq 0 \tag{9}$$

on the product manifold $X \times \Sigma$.

Proof. We proceed in a very standard manner, namely by an approximation argument. Let $(\mathcal{G}_\delta)$ be a family of (1,1)-forms on the open set

$$X \times \Sigma' \subset X \times \Sigma$$

obtained by considering the convolution of the potential φ with a convolution kernel K_δ, cf. e.g. [10]. Here Σ' stands for a compact subset of the annulus Σ. The resulting (1,1) forms are approximating our weak geodesic $\mathcal{G}$, as follows.

(i) The forms $\mathcal{G}_\delta$ are non-singular, and we have
$$\mathcal{G}_\delta \geq -C\delta(\omega + \sqrt{-1}dt \wedge d\overline{t})$$
on $X \times \Sigma'$.

(ii) The coefficients of $\mathcal{G}_\delta$ are uniformly bounded, and they converge in L^p norm to the coefficients of $\mathcal{G}$, so that if we write
$$\mathcal{G}_\delta := \omega + dd^c\varphi_\delta$$
then we have
$$|dd^c\varphi_\delta - dd^c\varphi|_{L^p(X\times\{t\})} \to 0$$
as $\delta \to 0$, uniformly with respect to $t \in \Sigma'$ and for any p.

(iii) We have
$$\sup_{t\in\Sigma'} \|\varphi_\delta - \varphi\|_{\mathcal{C}^1(X\times\{t\})} \to 0$$
as $\delta \to 0$.

The properties (ii) and (iii) of the approximation family $(\mathcal{G}_\delta)$ hold true thanks to the explicit construction of $(\mathcal{G}_\delta)$ by using the convolution of the potential of $\mathcal{G}$ with a regularizing kernel ([10]), combined with the fact that the potential of $\mathcal{G}$ is $\mathcal{C}^{1,1}$.

For each $\eta := (\varepsilon, \delta)$ such that ε and δ are positive and small enough we define the semi-positive form
$$\beta_\eta := \frac{1}{\varepsilon}\big(\mathcal{G}_\delta + C\delta(\omega + \sqrt{-1}dt \wedge d\overline{t})\big). \tag{10}$$
The class $c_1(K_X) + \{\beta_\eta\}|_{X\times\{t\}}$ is clearly Kähler; by the classical result of S.-T. Yau we infer that there exists a function $\phi_{t,\eta}$ such that we have
$$\Theta_\omega(K_X) + \beta_\eta|_{X\times\{t\}} + dd^c\phi_{t,\eta} > 0 \tag{11}$$
together with
$$\big(\Theta_\omega(K_X) + \beta_\eta + dd^c\phi_{t,\eta}\big)^n = \varepsilon^{-n}\exp(\phi_{t,\eta})\omega^n \tag{12}$$
on the fiber $X \times \{t\}$.

According to the proof of Theorem 4.1, we infer that
$$\Xi_\eta := \Theta_\omega(K_X) + \beta_\eta + dd^c\phi_\eta > 0, \tag{13}$$
as a smooth positive $(1,1)$ form on $X \times \Sigma'$–actually, this is the only reason why we need to consider the regularization of $\mathcal{G}$ with respect to the parameter "t" as well.

We show next that the family ϕ_η is equicontinuous *for each ε fixed*; prior to this, we introduce the a few notations.

Let $\tau_\eta := \varepsilon\phi_\eta + \varphi_\delta$; it is a smooth function defined on $X \times \Sigma'$. Then for each $t_0, t_1 \in \Sigma'$ we have
$$\big(\Psi_{t_0,\eta} + dd^c(\tau_\eta(t_1) - \tau_\eta(t_0))\big)^n = e^{\frac{1}{\varepsilon}(\tau_\eta(t_1) - \tau_\eta(t_0) - \varphi_\delta(t_1) + \varphi_\delta(t_0))}\Psi^n_{t_0,\eta} \tag{14}$$
where $\Psi_{t,\eta} := \varepsilon\Xi_\eta|_{X\times\{t\}}$.

The maximum principle, combined with the property (iii) above (concerning the uniformity properties of the sequence φ_δ on $X \times \Sigma'$) shows that we have

$$\sup_{x\in X} |\phi_\eta(t_0, x) - \phi_\eta(t_1, x)| \leq C\varepsilon^{-1}|t_0 - t_1| \tag{15}$$

The same kind of arguments (i.e. the maximum principle applied on fibers $X \times \{t\}$) show that in fact we have

$$|\phi_\eta(t_0, x_0) - \phi_\eta(t_1, x_1)| \leq C\varepsilon^{-1}(|t_0 - t_1| + \mathrm{dist}(\mathrm{x}_0, \mathrm{x}_1)) \tag{16}$$

where C is a constant independent of $\eta = (\varepsilon, \delta)$. This proves the claimed equicontinuity.

As a consequence, the limit $\phi_\varepsilon := \lim_{\delta\to 0} \phi_{\varepsilon,\delta}$ is continuous on the interior of $X \times \Sigma$, and we have

$$\Theta_\omega(K_X) + \frac{1}{\varepsilon}\mathcal{G} + dd^c\phi_\varepsilon \geq 0. \tag{17}$$

which finishes the proof of Theorem 4.2, except for the $\mathcal{C}^{1,1}\big(X \times \{t\}\big)$-regularity of the solution $\phi_{t,\varepsilon}$; this will be treated in out next result. □

In our next statement we will use another type of regularization of the geodesic $\mathcal{G}$, borrowed from [3]. We define $\mathcal{G}'_\delta := \omega + dd^c\varphi_\delta$ on $X \times \Sigma$, such that for each $t \in \Sigma$, the function φ_δ is the regularization of $\varphi|_{X\times\{t\}}$ by a global convolution kernel. The properties of the resulting form which will be relevant for us are as follows.

(1) The forms $\mathcal{G}'_\delta$ are non-singular when restricted to each fiber $X \times \{t\}$, and moreover there exists a constant $C > 0$ such that we have
$$|dd^c\varphi_\delta| \leq C$$
for any $\delta > 0$. We equally have
$$\mathcal{G}'_\delta \geq -C\delta(\omega + \sqrt{-1}dt \wedge d\bar{t})$$
on $X \times \Sigma$.

(2) The coefficients of $\mathcal{G}'_\delta$ are uniformly bounded, and they converge in L^p norm to the coefficients of $\mathcal{G}$, so that if we write
$$\mathcal{G}'_\delta := \omega + dd^c\varphi_\delta$$
then we have
$$|dd^c\varphi_\delta - dd^c\varphi|_{L^p(X\times\{t\})} \to 0$$
as $\delta \to 0$, uniformly with respect to $t \in \Sigma$ and for any p.

(3) We have
$$\lim \|\varphi_\delta - \varphi\|_{\mathcal{C}^0(X\times\Sigma)} = 0$$
as $\delta \to 0$.

We consider the Monge-Ampère equation

$$\big(\Theta_\omega(K_X) + \beta'_\eta + dd^c\phi_{t,\eta}\big)^n = \varepsilon^{-n}\exp(\phi_{t,\eta})\omega^n \tag{18}$$

on the fiber $X\times\{t\}$, where

$$\beta_\eta := \frac{1}{\varepsilon}\big(\mathcal{G}'_\delta + C\delta(\omega + \sqrt{-1}dt\wedge d\bar t)\big). \tag{19}$$

The regularity/uniformity properties of the functions $(\phi_{t,\eta})_\eta$ are stated in the following result.

Theorem 4.3. *The following assertions hold true.*

(a) *For each fixed ε, the family ϕ_η obtained by piecing together the fiber-wise solutions $\phi_{t,\eta}$ is equicontinuous.*

(b) *There exists a constant $C>0$, independent of η such that*

$$\sup_X \phi_{\varepsilon,\delta} \le C,\quad -\varepsilon\inf_X \phi_{\varepsilon,\delta} \le C,\quad |\varepsilon dd^c\phi_{\varepsilon,\delta}| \le C.$$

on the fiber $X\times\{t\}$.

(c) *Therefore for each fixed $\varepsilon>0$, we can extract a limit*

$$\lim_{\delta\to 0}\phi_{\varepsilon,\delta} = \phi_\varepsilon,$$

strongly in $\mathcal{C}^0$, where the restriction of ϕ_ε to $X\times\{t\}$ is the unique $\mathcal{C}^{1,1}$ solution of the degenerate Monge-Ampère equation

$$\big(\Theta_\omega(K_X) + \varepsilon^{-1}\mathcal{G} + dd^c\phi_{t,\varepsilon}\big)^n = \varepsilon^{-n}\exp(\phi_{t,\varepsilon})\omega^n \tag{20}$$

on the fiber $X\times\{t\}$.

(d) *The measures*

$$\exp(\phi_{t,\varepsilon})\omega^n$$

are converging to $\mathcal{G}|^n_{X\times\{t\}}$ weakly in L^p for any p, as $\varepsilon\to 0$.

Proof. Except maybe for the point (d), the arguments of the proof rely on basic results in MA theory, so we will be very sketchy.

The point (a) was basically discussed during the proof of Theorem 4.2. In addition, we remark that by th esame procedure we obtain the equicontinuity continuity of (ϕ_η) on $X\times\Sigma$ up to the boundary of $X\times\Sigma$: this is a consequence of the property (2) of $\mathcal{G}'_\delta$.

Concerning the point (c), the upper bound of the potentials is a consequence of the maximum principle since we can rewrite the equation (18) as

$$\big(\varepsilon\Theta_\omega(K_X) + \mathcal{G}_\delta + dd^c(\varepsilon\phi_{t,\eta})\big)^n = \exp(\phi_{t,\eta})\omega^n, \tag{21}$$

then take $\phi_{t,\eta}(p) = \max_X \phi_{t,\eta}$, which implies $dd^c\phi_{t,\eta}(p) \leqslant 0$. But notice the form $\varepsilon\Theta_\omega(K_X) + \mathcal{G}_\delta$ is strictly positive at the point p thanks to the inequality (11), hence

$$\phi_{t,\eta} \leqslant \phi_{t,\eta}(p) \leqslant \log\frac{(\varepsilon\Theta_\omega(K_X)+\mathcal{G}_\delta)^n}{\omega^n}(p) \leqslant C.$$

On the other hand, the lower bound is obtained by considering

$$(\omega + \varepsilon\Theta_\omega(K_X) + dd^c\tau_{t,\eta})^n = e^{\frac{1}{\epsilon}(\tau_{t,\eta}-\varphi_\delta)}\omega^n \tag{22}$$

where $\tau_{t,\eta} = \varepsilon\phi_{t,\eta} + \varphi_\delta$, so by choosing $\tau_{t,\eta}(q) = \min_X \tau_{t,\eta}$, the minimum principle says

$$\varepsilon\phi_{t,\eta} + \varphi_\delta \geq \varepsilon \log \frac{(\omega + \varepsilon\Theta_\omega(K_X))^n}{\omega^n}(q) + \varphi_\delta(q) \geqslant -\varepsilon C + \varphi_\delta(q),$$

hence $\varepsilon\phi_{t,\eta} \geqslant -C$ for some uniform constant C. Now the bound of $dd^c\tau_{t,\eta}$ follows from the usual Laplacian estimate for the Monge-Ampére equations, cf. [5]. For fixed η, let $\omega_\varepsilon = \omega + \varepsilon\Theta_\omega(K_X)$, and notice that it is a smooth non-degenerate approximation of ω, then we can write $\Delta\tau = g_\varepsilon^{j\bar{k}}\tau_{j\bar{k}}$, and set

$$\alpha := \log(n + \Delta\tau) - A\tau,$$

where $A > 0$ is some constant determined later. Consider the point p where the maximum of α is obtained, and take $u = G_\varepsilon + \tau$, where the function G_ε is the local potential of ω_ε in a normal coordinate ball of p, then the standard maximum principle argument implies at the point p

$$0 \geqslant \frac{1}{\Delta u}\Big\{ - B\Delta u \sum_p \frac{1}{u_{p\bar{p}}} + \Delta\phi_{t,\eta} - S\Big\} + A\sum_p \frac{1}{u_{p\bar{p}}} - nA, \tag{23}$$

where $B > 0$ is the lower bound of bisectional curvature, and S is the upper bound of scalar curvature of (X, ω_ε). Now take $A = B$, then we have

$$nB\Delta u + S \geq \frac{1}{\varepsilon}\Delta(\tau - \varphi_\delta),$$

hence

$$n + \Delta\varphi_\delta + S\varepsilon \geq (1 - \varepsilon nB)\Delta u,$$

and finally

$$\Delta u(p) \leq \frac{C}{1 - \varepsilon nB} < 2C$$

for some uniform constant C, when $\varepsilon < \frac{1}{2nB}$. Then suppose there is some uniform constant C' such that $osc(\tau) < C'$, we infer

$$\Delta u \leq 2Ce^{2BC'}$$

The statement (d) is a consequence of the elliptic regularity results, cf. [11]. Moreover, since the solution $\phi_{t,\varepsilon}$ belongs to the space $\mathcal{C}^{1,1}$ we infer that the equality (20) holds almost everywhere on $X \times \{t\}$ (in the sense of L^∞ functions).

In order to establish the point (e), we re-write the equation (20) as follows

$$(\omega + \varepsilon\Theta_\omega(K_X) + dd^c\tau_\varepsilon)^n = e^{\frac{1}{\varepsilon}(\tau_\varepsilon - \varphi)}\omega^n \tag{24}$$

where we recall that $\mathcal{G} = \omega + dd^c\varphi$. In the relation (24), we denote $\tau_\varepsilon := \varepsilon\phi_\varepsilon + \varphi$, so we will be done if we can prove that

$$\varepsilon\phi_\varepsilon \to 0.$$

In any case, thanks to the estimates (c), there exists some function $\rho \in \mathcal{C}^{1,1}$ such that we have

$$\varepsilon\phi_\varepsilon \to \rho \tag{25}$$

strongly in $\mathcal{C}^{1,\alpha}$ for any $\alpha < 1$ as $\varepsilon \to 0$, so that the limit τ extracted from τ_ε verifies the inequality

$$\tau \leq \varphi. \tag{26}$$

as it is clear from the second part of estimate (c).

In particular, the convergence statement in (25) implies that we have

$$(\omega + \varepsilon\Theta_\omega(K_X) + dd^c\tau_\varepsilon)^n \to (\omega + dd^c\tau)^n \tag{27}$$

is weak sense in L^p for any p (this is due to the fact all currents in the concern have uniformly bounded L^∞ coefficients).

Let $\Omega_\delta := \{\tau < \varphi - \delta\}$; it is an open subset of X, and we claim that $\mathcal{G}^n|_{\Omega_\delta} = 0$, for each $\delta > 0$. Indeed, by the comparison principle [14], [13] we have

$$\int_{\Omega_\delta} \mathcal{G}^n \leq \int_{\Omega_\delta} (\omega + dd^c\tau)^n. \tag{28}$$

However, as we can see from equation (24), we have $\displaystyle\int_{\Omega_\delta} (\omega + dd^c\tau)^n = 0$, simply because on the set Ω_δ the inequality $\dfrac{\tau_\varepsilon - \varphi}{\varepsilon} < -\dfrac{\delta}{2\varepsilon}$ as soon as ε is small enough –remark that this is a consequence of the uniform convergence in (25)– so our claim is proved.

But then it follows that $\mathcal{G}^n|_{\overline{\Omega}} = 0$, where $\overline{\Omega}$ is the closure of the open set $\{\tau < \varphi\} \subset X$. Indeed, the set $\overline{\Omega} \setminus \Omega$ has measure zero, and the coefficients of $\mathcal{G}$ are bounded. We infer that we have

$$\mathcal{G}^n = (\omega + dd^c\tau)^n \tag{29}$$

since the complement of $\overline{\Omega}$ is an open set where τ coincides with φ.

We invoke next the uniqueness result in [14] concerning the solutions of MA equations whose right hand side member has a density in L^p, for $p > 1$, so that $\rho = 0$ and the point (e) of our lemma follows. □

Remark 4.4. The fiber-wise convergence (e) was proved in [1] in a slightly different setting; the main argument in that article relies on the variational approach developed in [2]. However, we have chosen to give a direct proof here, for the sake of variation.

4.1. **Convexity.** Let $t_0, t_1, t_2 \in [0,1]$ be three arbitrary points. By the property (e) combined with a result due to Banach-Saks one can find a sequence $\varepsilon_k \to 0$ as $k \to \infty$ such that

$$\frac{1}{k}\sum_{j=1}^{k}\exp(\phi_{t_p,\varepsilon_j}) \to \exp(\varphi|_{X\times\{t_p\}})$$

in $L^1(X\times\{t_p\})$, for each $p = 0,1,2$. The sequence (ε_k) depends on the triple (t_p), but fortunately this does not matter for the rest of the proof.

For each $t \in \Sigma$, let $\omega_{t,k} \in \{\omega\}$ be the Kähler metric such that

$$\omega_{t,k}^n = \frac{1}{k}\sum_{j=1}^{k}\exp(\phi_{t,\varepsilon_j})\omega^n \tag{30}$$

Let $\mathcal{M}_k$ be the Mabuchi functional evaluated on the weak geodesic $\mathcal{G}$, with the entropy term modified by using $\log \omega_{t,k}^n$ instead of $\log \mathcal{G}^n$, i.e.

$$\mathcal{M}_k(t) = \frac{S}{n+1}\mathcal{E}(\varphi_t) - \mathcal{E}^{Ric_\omega}(\varphi_t) + \int_X \log\frac{\omega_{t,k}^n}{\omega^n}\mathcal{G}^n \tag{31}$$

Then we have the following statement.

Lemma 4.5. *For each $k \geq 1$ the functional $\mathcal{M}_k$ is a continuous convex function on* $[0,1]$.

Proof. Our first observation is that the functional $\mathcal{M}_k$ is continuous. Indeed, this is the case given the regularity we have already established in Theoren 4.3 for each ϕ_k, combined with the stability theorem due to S. Kolodziej, cf. [14]; we do not give further details here.

It is therefore enough to show that $\mathcal{M}_k$ is convex in weak sense; this boils down to the inequality

$$dd^c \log\Big(\frac{1}{k}\sum_{j=1}^{k}\exp(\phi_{\varepsilon_j})\Big)\wedge\mathcal{G}^n \geq \mathrm{Ricci}_\omega \wedge\mathcal{G}^n. \tag{32}$$

This is immediately seen to be true, as follows. We have

$$dd^c \log\frac{1}{k}\sum_{j=1}^{k}\exp(\phi_{\varepsilon_j}) \geq \sum_{j=1}^{k}\frac{\exp(\phi_{\varepsilon_j})}{\sum_{i=1}^{k}\exp(\phi_{\varepsilon_i})}dd^c\phi_{\varepsilon_j} \tag{33}$$

by a direct computation, and moreover the point (b) of Theorem 4.3 shows that we have

$$-\mathrm{Ricci}_\omega + \varepsilon_j^{-1}\mathcal{G} + dd^c\phi_{\varepsilon_j} \geq 0. \tag{34}$$

By using this inequality in (33), we obtain

$$dd^c \log\frac{1}{k}\sum_{j=1}^{k}\exp(\phi_{\varepsilon_j})\wedge\mathcal{G}^n \geq \sum_{j=1}^{k}\frac{\exp(\phi_{\varepsilon_j})}{\sum_{i=1}^{k}\exp(\phi_{\varepsilon_i})}\mathrm{Ricci}_\omega \wedge\mathcal{G}^n \tag{35}$$

which proves the lemma. □

Corollary 4.6. *The Mabuchi functional $\mathcal{M}$ is a convex function on the interval* $[0,1]$.

Proof. Let $t_p \in [0,1]$ be three arbitrary points; we can apply the convexity inequality for each $\mathcal{M}_k$ corresponding to the points $(t_p)_{p=0,1,2}$ and we let $k \to \infty$; the convexity of $\mathcal{M}$ follows. $\square$

4.2. **Continuity at the boundary.** Given that $\mathcal{M}$ is a convex function, it is automatically continuous on the open interval $]0,1[$. In this subsection we will show that the continuity property holds up to the boundary.

To this end we recall that the entropy functional H is defined as

$$H(\varphi) = \int_X f_\varphi \log f_\varphi d\mu,$$

where

$$f_\varphi = \frac{\omega_\varphi^n}{\omega^n}$$

is an L^∞ function provided that the potential $\varphi \in \mathcal{C}^{1,1}$, and the probability measure $d\mu$ equals ω^n. In order to study the semi-continuity property of H, we will follow [8],

Lemma 4.7. *Let φ_i, φ and f_i, f be as above, and suppose f, f_i are uniformly bounded non-negative functions, such that $f_i \to f$ weakly in L^1, then*

$$\lim_i \int_X (f_i \log f_i - f \log f) d\mu \geqslant 0$$

Proof. First assume f has positive lower bound, i.e. $f > \delta$ for some small $\delta > 0$, since we can replace f by $f + \delta$, and let δ converges to zero. Now put $F_i(t) = \mathcal{F}(tf_i + (1-t)f) = \mathcal{F}(at+b)$, where $a = f_i - f$ and $b = f$, then

$$F_i'(t) = a(\log u_t + 1),$$

where $u_t = tf_i + (1-t)f$, and

$$F_i''(t) = \frac{a^2}{u_t} \geqslant \frac{a^2}{C},$$

for some constant uniform constant C such that f_i and f are smaller than C. Hence

$$\int_X (f_i \log f_i - f \log f) d\mu = \int_X \left(\int_0^1 \int_0^t F''(s) ds dt + \int_0^1 F'(0) dt \right) d\mu \tag{36}$$
$$\geq \frac{1}{C} \int_X (f_i - f)^2 d\mu + \int_X F_i'(0) d\mu.$$

However,

$$\lim_{t \to 0} \int_X F_i'(t) d\mu = \lim_{t \to 0} \int_X (f_i - f)\{\log(tf_i + (1-t)f) + 1\} d\mu \tag{37}$$

$$= \int_X (f_i - f)(\log f + 1) d\mu,$$

where we have used the fact $f_i \to f$ weakly. Then obviously

$$\lim_i \lim_{t \to 0} \int_X F_i' d\mu = 0,$$

Finally, remember we are dealing with $f + \delta$ instead of f, then

$$\int_X F_i'(0) d\mu = \int_X (f_i - f - \delta) \log(f + \delta) + o(\delta), \tag{38}$$

and the limit will converge to zero when $\delta \to 0$, since

$$\begin{aligned} \lim_i \int_X F'(0) d\mu &= \int_X (f - f - \delta) \log(f + \delta) + o(\delta) \\ &= o(\delta), \end{aligned} \tag{39}$$

which completes the proof. □

We treat next a version of the previous lemma, which will be very useful later on.

Let χ be a continuous function; we define $h_A = \exp(\chi - A)$, and we consider the following truncated version of the entropy functional

$$H_A(\varphi) := \int_X f_i \log \max(f_i, h_A) d\mu.$$

Our claim is as follows.

Lemma 4.8. *The truncated entropy functional has the following semi-continuity type property*

$$\lim_i H_A(\varphi_i) - H_A(\varphi) \geq -\delta(A)$$

for some uniform constant $\delta(A) > 0$ such that $\delta(A) \to 0$ as A tends to ∞. Here the sequence (f_i) is assumed to verify the hypothesis of the preceding lemma.

Proof. First define

$$\tilde{H}_A(\varphi) = \int_X \max(f, h_A) \log \max(f, h_A) d\mu.$$

then set $\Omega_A = \{f < h_A\}$, and $\Omega_{i,A} = \{f_i < h_A\}$

$$\begin{aligned} H_A(\varphi) - \tilde{H}_A(\varphi) &= \int_X \big(f - \max(f, h_A)\big) \log \max(f, h_A) d\mu \\ &= \int_{\Omega_A} (f - h_A) \log h_A d\mu, \end{aligned}$$

then notice that $0 > f - h_A \geq -h_A$ on Ω_A and $\log h_A < 0$ for A large enough,

$$|H_A(\varphi) - \tilde{H}_A(\varphi)| \leq -\int_X (\chi - A)e^{\chi - A} d\mu = C_1(A).$$

Now run the same argument as in lemma (4.7) with f replaced by $\max(f, h_A)$(here we have positive lower bound automatically from this truncation), then obtain

$$H(\varphi_i) - \tilde{H}_A(\varphi) \geq \int_X F_i'(0) d\mu,$$

from equation (36), then by equation (37) it's enough to estimate

$$\int_X \big(f_i - \max(f, h_A)\big) \log\big(\max(f, h_A) + 1\big) d\mu \tag{40}$$

$$= \int_{X - \Omega_A} (f_i - f) \log(f+1) d\mu + \int_{\Omega_A} (f_i - h_A) \log(h_A + 1) d\mu,$$

the first term will converges to zero as $i \to +\infty$ as before, and the second term is bounded from below by

$$\int_{\Omega_A \cap \Omega_{i,A}} (f_i - h_A) \log(h_A + 1) d\mu \geq -\int_{\Omega_A \cap \Omega_{i,A}} h_A \log(1 + h_A) d\mu$$

$$\geq -\int_X h_A \log(1 + h_A) d\mu$$

$$\geq -2 \int_X e^{\chi - A} d\mu = -C_2(A).$$

Finally observe that

$$H_A(\varphi_i) - H(\varphi_i) = \int_X f_i \big(\log \max(f_i, h_A) - \log f_i\big) \geq 0,$$

hence we can decompose

$$H_A(\varphi_i) - H_A(\varphi) = \big(H_A(\varphi_i) - H(\varphi_i)\big) + \big(H(\varphi_i) - \tilde{H}_A(\varphi)\big)$$
$$+ \big(\tilde{H}_A(\varphi) - H_A(\varphi)\big),$$

then the limit of $H_A(\varphi_i) - H_A(\varphi)$ is bounded below by

$$-\delta(A) := -C_1(A) - C_2(A),$$

which converges to zero when A is large. □

The Mabuchi functional can be written as

$$\mathcal{M}(\varphi) = E(\varphi) + H(\varphi) \tag{41}$$

where E is the energy part of the Mabuchi functional, and H is the entropy part. As a consequence of our previous results, we establish here the following statement.

Theorem 4.9. *The Mabuchi functional $\mathcal{M}(\varphi(t))$ is a continuous convex function on* $[0,1]$.

Proof. The proof results immediately from our previous considerations. Indeed, since $\mathcal{M}$ is a convex function, we automatically have

$$\mathcal{M}(0) \geq \limsup_{t\to 0} \mathcal{M}(t) \tag{42}$$

On the other hand, the semi-continuity properties of the entropy functional in Lemma 4.7 show that in fact we have

$$\mathcal{M}(0) \leq \liminf_{t\to 0} \mathcal{M}(t) \tag{43}$$

and the proof of Theorem 4.9 is completed. Indeed, the *energy* part of the Mabuchi functional is continuous, given the regularity of the potential of $\mathcal{G}$. □

5. Almost convexity along ε-geodesics

In [6], the geodesic $\mathcal{G}$ is obtained by the continuity method, and as a by-product of the proof, for each $\varepsilon > 0$ one has a smooth, positive (1,1)-form

$$\omega_\varepsilon := \omega + dd^c\rho_\varepsilon \tag{44}$$

on $X\times\Sigma$ such that the next identity holds

$$\omega_\varepsilon^{n+1} = \varepsilon\sqrt{-1}dt\wedge d\bar{t}\wedge\omega \tag{45}$$

Let $\mathcal{M}_{\varepsilon,A} : \Sigma \to \mathbb{R}$ be the regularization of the Mabuchi functional evaluated on ω_ε. By definition, this equals

$$\mathcal{M}_{\varepsilon,A}(t) = \frac{S}{n+1}\mathcal{E}(\varphi_t) - \mathcal{E}^{Ric_\omega}(\rho_{\varepsilon,t}) + \int_X \max\Big(\log\frac{\omega_\varepsilon^n}{\omega^n}, \log\frac{h_A}{\omega^n}\Big)\omega_\varepsilon^n \tag{46}$$

where h_A is a volume element on X whose associated curvature is greater than $-C\mathcal{G}$ for some positive constant C. Then we can compute the Hessian of $\mathcal{M}_{\varepsilon,A}$ on the approximate geodesic as

$$\begin{aligned}
\int_{X\times\Sigma}\mathcal{M}_{\varepsilon,A}d_td_t^c\tau &= \int_{X\times\Sigma}\tau(\omega_\varepsilon^{n+1} - Ric(\omega)\wedge\omega_\varepsilon^n)\\
&+ \int_{X\times\Sigma}\log\max\big(\frac{\omega_\varepsilon^n}{\omega^n}, \frac{h_A}{\omega^n}\big)\omega_\varepsilon^n\wedge d_td_t^c\tau\\
&= \varepsilon - \int_{X\times\Sigma} Ric(\omega)\wedge\omega_\varepsilon^n + \int_{X\times\Sigma}\tau dd^c\log\max\big(\frac{\omega_\varepsilon^n}{\omega^n}, \frac{h_A}{\omega^n}\big)\omega_\varepsilon^n. \qquad (47)
\end{aligned}$$

where $\tau(t)$ is any test function on Σ.

Now we have the following result.

Theorem 5.1. *For each positive constant A, there is a uniform constant $C_A > 0$ such that the function $\mathcal{M}_{\varepsilon,A} - \varepsilon C_A t(1-t)$ is convex.*

Proof. It would be enough to prove that for any $A \gg 0$ there exists a constant C_A such that we have

$$\int_{X\times\Sigma} \tau dd^c \max\Big(\log\frac{\omega_\varepsilon^n}{\omega_0^n}, \log\frac{h_A}{\omega_0^n}\Big)\wedge\omega_\varepsilon^n \geq \int_{X\times\Sigma}\tau\,\mathrm{Ricci}_{\omega_0}\wedge\omega_\varepsilon^n - \varepsilon C_A\int_\Sigma \tau\sqrt{-1}dt\wedge d\bar{t} \tag{48}$$

for any positive test function τ on Σ. Indeed, once this is done we infer that the second convexity of $\mathcal{M}_\varepsilon$ follows. The inequality (48) is established in the next paragraph by a direct computation.

We recall the following statement, which will be useful during the arguments in the following section.

Lemma 5.2. *Let u and v be two smooth functions on a complex manifold Z, and let ω be a Kähler metric on Z. We assume that v is subharmonic with respect to ω, and that we equally have $\Delta_\omega(u)\geq 0$ on the set $u > v-1$. Then the ω-Laplacian of the function* $\max(u,v)$ *is positive.*

Indeed this follows from the fact that a smooth function is subharmonic if and only if it satisfies the mean value inequality (is this context, the usual Lebesgue measure is replaced with the harmonic measure on balls); we refer to [12] and the references therein for a complete account of these facts. In the next section we will have to deal with functions whose Laplacian is greater than $-C$. Then we have a similar statement, since locally we can construct functions with strictly positive Laplacian (e.g. the potential of the metric ω).

5.1. **The computation.** In order to simplify the notations, we define the function $f_\varepsilon : X\times\Sigma\to\mathbb{R}$ by the equality

$$\frac{\omega_\varepsilon^n}{\omega_0^n}\Big|_{X\times\{t\}} = e^{f_\varepsilon(t,\cdot)}. \tag{49}$$

We introduce the set

$$\Omega_{\varepsilon,A} := \Big\{(z,t)\in X\times\Sigma \text{ such that } f_\varepsilon(t,z) > \log\frac{h_A}{\omega_0^n}(t,z)\Big\} \tag{50}$$

so that we have

$$\max\Big(\log\frac{\omega_\varepsilon^n}{\omega_0^n}, \log\frac{h_A}{\omega_0^n}\Big) = f_\varepsilon(t,z) \tag{51}$$

on $\Omega_{\varepsilon,A}$. This reveals the importance of considering the functional $\mathcal{M}_{\varepsilon,A}$: on the set $\Omega_{\varepsilon,A}$ the distortion function f_ε is bounded from below *by a quantity which is independent of* ε. This simple remark will play a crucial role in the next considerations.

We will proceed next to the evaluation of the integral

$$\int_{\Omega_{\varepsilon,A}} \tau\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)\wedge\omega_\varepsilon^n. \tag{52}$$

Locally near a point $(z,t)\in X\times\Sigma$ we write the metric ω_ε as follows

$$\begin{aligned}\omega_\varepsilon &= \sqrt{-1}g_{t\bar t}dt\wedge d\bar t+\sqrt{-1}g_{t\overline{\alpha}}dt\wedge dz^{\overline{\alpha}}+\sqrt{-1}g_{\alpha\bar t}dz^\alpha\wedge d\bar t\\ &+ \sqrt{-1}g_{\gamma\overline{\alpha}}dz^\gamma\wedge dz^{\overline{\alpha}}\end{aligned} \tag{53}$$

where the coefficients g in the expression above depend on ε as well.

The equation (49) satisfied by ω_ε can be written in local coordinates as

$$\begin{aligned}c(\varphi_\varepsilon) &:= g_{t\bar t}-g^{\gamma\overline{\alpha}}g_{\gamma\bar t}g_{t\overline{\alpha}}\\ &= \varepsilon e^{-f_\varepsilon}.\end{aligned} \tag{54}$$

Given this, we rewrite locally the metric ω_ε as follows

$$\omega_\varepsilon = c(\varphi_\varepsilon)\sqrt{-1}dt\wedge d\bar t+\rho_\varepsilon \tag{55}$$

where ρ_ε has the same expression as ω_ε, except that we replace $g_{t\bar t}$ with $g^{\gamma\overline{\alpha}}g_{\gamma\bar t}g_{t\overline{\alpha}}$. We note that although ρ_ε may not be closed, it is positive definite on each slice $X\times\{t\}$ and it satisfies

$$\rho_\varepsilon^{n+1}=0. \tag{56}$$

We have the equality

$$\omega_\varepsilon^n=\rho_\varepsilon^n+nc(\varphi_\varepsilon)\sqrt{-1}dt\wedge d\bar t\wedge\rho_\varepsilon^{n-1} \tag{57}$$

which is the same as

$$\omega_\varepsilon^n=\rho_\varepsilon^n+nc(\varphi_\varepsilon)\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1} \tag{58}$$

by the definition of the form ρ_ε.

The factor

$$nc(\varphi_\varepsilon)\big(dd^c f_\varepsilon-\mathrm{Ricci}_\omega\big)\wedge\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1} \tag{59}$$

is analyzed as follows.

We observe that we have

$$nc(\varphi_\varepsilon)dd^c f_\varepsilon\wedge\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1}=c(\varphi_\varepsilon)\Delta_{\omega_\varepsilon}(f_\varepsilon)\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^n \tag{60}$$

and by the equality (54) we have

$$nc(\varphi_\varepsilon)dd^c f_\varepsilon\wedge\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1}=\varepsilon\Delta_{\omega_\varepsilon}(f_\varepsilon)\sqrt{-1}dt\wedge d\bar t\wedge\omega_0^n \tag{61}$$

The other term in the equality (59) is bounded in L^1 norm by εC_A, given that on the set $\Omega_{\varepsilon,A}\cap X\times\{t\}$ the eigenvalues of ω_ε are bounded from below (and above) by a constant independent of ε, so that the trace of Ricci_ω with respect to ω_ε is bounded by some constant C_A.

Therefore, we have

$$nc(\varphi_\varepsilon)\frac{\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)\wedge\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1}}{\sqrt{-1}dt\wedge d\bar t\wedge\omega_0^n}\geq \varepsilon\Delta_{\omega_\varepsilon}(f_\varepsilon)-\varepsilon C_A \tag{62}$$

This can be re-written in the following way

$$\begin{aligned} nc(\varphi_\varepsilon)\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)\wedge\sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^{n-1} &\geq \Delta_{\omega_\varepsilon}(f_\varepsilon)\omega_\varepsilon^{n+1}\\ &- C_A\omega_\varepsilon^{n+1}\end{aligned} \tag{63}$$

The evaluation of the main term

$$\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)\wedge\rho_\varepsilon^n \tag{64}$$

goes as follows. Let

$$v := \frac{\partial}{\partial t} - g^{\gamma\overline{\alpha}}g_{t\overline{\alpha}}\frac{\partial}{\partial z^\gamma} \tag{65}$$

be the gradient of the t derivative of φ_ε. Then one can check by a direct computation that the vector field v generates the kernel of ρ_ε, and then we have

$$\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)\wedge\rho_\varepsilon^n = \big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)(v,\overline{v})\sqrt{-1}dt\wedge d\bar t\wedge\rho_\varepsilon^n. \tag{66}$$

Indeed, this is a matter of liner algebra: we have

$$\beta_1\wedge\beta_2^n = \frac{\beta_1(v,\overline{v})}{\lambda(v,\overline{v})}\lambda\wedge\beta_2^n$$

on a vector space of dimension $n+1$, where λ, β_j are (1,1)-forms, such that v is in the kernel of β_2, and such that $\lambda(v,\overline{v})\neq 0$.

A straightforward calculation which we will detail in a moment shows that we have

$$\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)(v,\overline{v})\geq \varepsilon\Delta_{\omega_\varepsilon}\big(e^{-f_\varepsilon}\big) \tag{67}$$

Since we have $\Delta_{\omega_\varepsilon}\big(e^{-f_\varepsilon}\big)\geq -e^{-f_\varepsilon}\Delta_{\omega_\varepsilon}(f_\varepsilon)$, the inequality (67) combined with (63) finishes the proof. Indeed, we first remark that we have

$$\sqrt{-1}dt\wedge d\bar t\wedge\rho_\varepsilon^n = \sqrt{-1}dt\wedge d\bar t\wedge\omega_\varepsilon^n;$$

by (67) we obtain

$$\big(dd^c f_\varepsilon - \mathrm{Ricci}_\omega\big)(v,\overline{v})\geq -\varepsilon\Delta_{\omega_\varepsilon}(f_\varepsilon)\sqrt{-1}dt\wedge d\bar t\wedge\omega_0^n \tag{68}$$

and we observe that the right hand side of (68) is nothing but

$$-\Delta_{\omega_\varepsilon}(f_\varepsilon)\omega_\varepsilon^{n+1}.$$

Thus, we infer the inequality

$$dd^c\max\Big(\log\frac{\omega_\varepsilon^n}{\omega_0^n},\log\frac{h_A}{\omega_0^n}\Big)\wedge\omega_\varepsilon^n\geq -C_A\omega_\varepsilon^{n+1} \tag{69}$$

globally on $X\times\Sigma$.

We prove next the inequality (67); before that, we remark that the following approach is quite standard in the theory of the homogeneous Monge-Ampère equations, cf. [6], [7], [8]... Also, the inequality (67) is very similar to the positivity of the curvature along the leaves of the foliation (which does not exists in our case...), cf. [8].

The next computations are done with respect to a geodesic coordinate system at (X, z); we have

$$\overline{\partial}\log\det(g_{\alpha\overline{\beta}}) = g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\overline{t}}d\overline{t} + g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\overline{\gamma}}dz^{\overline{\gamma}} \tag{70}$$

and thus

$$\begin{aligned}\partial\overline{\partial}\log\det(g_{\alpha\overline{\beta}}) &= \big(g^{\alpha\overline{\beta}}_{,t}g_{\alpha\overline{\beta},\overline{t}} + g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},t\overline{t}}\big)dt\wedge d\overline{t}\\ &+ g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{t}}dz^{\gamma}\wedge d\overline{t} + g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},t\overline{\gamma}}dt\wedge dz^{\overline{\gamma}}\\ &+ g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{\tau}}dz^{\gamma}\wedge dz^{\overline{\tau}}\end{aligned} \tag{71}$$

Since the metric ω_ε is locally given by the Hessian of a function, the following commutation relations

$$g_{\alpha\overline{\beta},t\overline{t}} = g_{t\overline{t},\alpha\overline{\beta}} \tag{72}$$

hold true on X. Given that

$$g^{\alpha\overline{\beta}}_{,t} = -g^{\alpha\overline{\gamma}}g^{\delta\overline{\beta}}g_{\delta\overline{\gamma},t} \tag{73}$$

the equality (71) become

$$\begin{aligned}\partial\overline{\partial}\log\det(g_{\alpha\overline{\beta}}) &= \big(g^{\alpha\overline{\beta}}g_{t\overline{t},\alpha\overline{\beta}} - g^{\alpha\overline{\gamma}}g^{\delta\overline{\beta}}g_{\delta\overline{\gamma},\overline{t}}g_{\alpha\overline{\beta},t}\big)dt\wedge d\overline{t}\\ &+ g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{t}}dz^{\gamma}\wedge d\overline{t} + g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},t\overline{\gamma}}dt\wedge dz^{\overline{\gamma}}\\ &+ g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{\tau}}dz^{\gamma}\wedge dz^{\overline{\tau}}.\end{aligned} \tag{74}$$

We evaluate this in the v-direction, and we get

$$\begin{aligned}\partial\overline{\partial}\log\det(g_{\alpha\overline{\beta}})(v,\overline{v}) &= g^{\alpha\overline{\beta}}g_{t\overline{t},\alpha\overline{\beta}} - g^{\alpha\overline{\gamma}}g^{\delta\overline{\beta}}g_{\gamma\overline{\delta},\overline{t}}g_{\alpha\overline{\beta},t}\\ &- g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{t}}g^{\gamma\overline{\mu}}g_{t\overline{\mu}} - g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},t\overline{\gamma}}g^{\mu\overline{\gamma}}g_{\mu\overline{t}}\\ &+ g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{\tau}}g^{\gamma\overline{\mu}}g^{\rho\overline{\tau}}g_{t\overline{\mu}}g_{\rho\overline{t}}.\end{aligned} \tag{75}$$

The equation satisfied by the metric ω_ε reads as

$$g_{t\overline{t}} - g^{p\overline{q}}g_{p\overline{t}}g_{t\overline{q}} = \varepsilon e^{-f_\varepsilon} \tag{76}$$

so that we have

$$
\begin{aligned}
g^{\alpha\overline{\beta}}g_{t\bar{t},\alpha\overline{\beta}} - \varepsilon\Delta_{\omega_\varepsilon}(e^{-f_\varepsilon}) &= g^{\alpha\overline{\beta}}g^{p\overline{q}}_{,\alpha\overline{\beta}}g_{p\bar{t}}g_{t\overline{q}} \\
&+ g^{\alpha\overline{\beta}}g^{p\overline{q}}g_{p\bar{t},\alpha\overline{\beta}}g_{t\overline{q}} + g^{\alpha\overline{\beta}}g^{p\overline{q}}g_{p\bar{t}}g_{t\overline{q},\alpha\overline{\beta}} \\
&+ g^{\alpha\overline{\beta}}g^{p\overline{q}}g_{p\bar{t},\alpha}g_{t\overline{q},\overline{\beta}} + g^{\alpha\overline{\beta}}g^{p\overline{q}}g_{p\bar{t},\overline{\beta}}g_{t\overline{q},\alpha}
\end{aligned} \tag{77}
$$

By combining (77) with (75) we obtain

$$\partial\overline{\partial}\log\det(g_{\alpha\overline{\beta}})(v,\overline{v}) = |\overline{\partial}v|^2 + \varepsilon\Delta_{\omega_\varepsilon}(e^{-f_\varepsilon}) \tag{78}$$

and the inequality (67) follows. □

5.2. **Further results and comments.** It is very likely that the convexity of $\mathcal{M}$ in the sense of distributions can be derived by the techniques we have developed in the previous section, i.e. using ε–geodesics. One of the motivations to do so is that the resulting proof would be more “self-contained”.

However, we encounter a rather severe difficulty: we ignore whether the fiber-wise sequence of volume elements

$$\omega_\varepsilon^n$$

corresponding to the ε-geodesics is converging almost everywhere to the volume element of the geodesic $\mathcal{G}$.

Nevertheless, we strongly believe that this holds true, based on the following considerations. On the set $\Omega_{\varepsilon,A}$ we have

$$C(A)\omega < \omega_\varepsilon < C\omega \tag{79}$$

where $C(A)$ is a constant depending on A, but uniform with respect to ε, and C is a fixed constant, independent of ε, A. Indeed this is a consequence of the results in [6]. The relation (79) is a uniform *Laplacian estimate* for the metrics $\omega_\varepsilon|_{\Omega_\varepsilon}$. Hence, via Evans-Krilov theory one might hope that it is possible to obtain a higher regularity estimate for the family $(\varphi_\varepsilon|_{\Omega_{\varepsilon,A}})_{\varepsilon>0}$. The problem is that as $\varepsilon \to 0$, the set $\Omega_{\varepsilon,A}$ converges eventually towards a set which is only measurable, and it is a-priori unclear how to implement Evans-Krilov theory in this setting.

However, we show here that the continuity of $\mathcal{M}$ at the endpoints 0 and 1 can be also obtained as a consequence of the results we have established in the previous section (i.e. without knowing a-priori the convexity of $\mathcal{M}$).

Theorem 5.3. *The Mabuchi functional $\mathcal{M}(t)$ is continuous at the boundary points 0 and 1.*

Proof. We identify in what follows τ and its real part $t = Re(\tau)$, since all the functionals involved in the proof only depends on the real part of τ. Also, we will only prove the continuity at 0.

A first observation is that we have

$$\lim_{t\to 0} \mathcal{M}(t) \geq \mathcal{M}(0) \tag{80}$$

thanks to the entropy property recalled in 4.7.

Next, we observe that the lim sup of a sequence of convex functions which are locally bounded above is still convex (unlike subharmonic functions). Hence if we define

$$\limsup_{\varepsilon\to 0} \mathcal{M}_{\varepsilon,A} := \mathcal{M}_A,$$

then $\mathcal{M}_A$ is a convex function on $[0,1]$ by theorem (5.1). And by construction we have $\mathcal{M}_A(0) = \mathcal{M}(0)$ for any value of the regularization parameter A.

Now for every point $\tau \in (0,1)$, we have $\mathcal{M}_A(\tau) \geq \mathcal{M}(\tau) - \delta(A)$, since

$$\limsup_{\varepsilon\to 0} H_A(\varphi_\varepsilon) \geqslant H_A(\varphi) - \delta(A),$$

by lemma (4.8).

We define a new functional

$$\limsup_{A\to +\infty} \mathcal{M}_A := \widetilde{\mathcal{M}},$$

and then $t \to \widetilde{\mathcal{M}}(t)$ is a convex function on $[0,1]$ which still verifies the equality $\widetilde{\mathcal{M}}(0) = \mathcal{M}(0)$.

Then we have $\widetilde{\mathcal{M}}(0) \geq \lim_{t\to 0} \widetilde{\mathcal{M}}(t)$ by convexity, as well as the inequality $\widetilde{\mathcal{M}}(\tau) \geq \mathcal{M}(\tau)$ for each $\tau \in (0,1)$, thanks to the considerations above. We therefore infer that

$$\lim_{t\to 0} \mathcal{M}(t) \leq \mathcal{M}(0) \tag{81}$$

and Theorem 5.3 is proved.

□

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