

POSITIVITY OF TWISTED RELATIVE PLURICANONICAL BUNDLES AND THEIR DIRECT IMAGES

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Abstract. Our main goal in this article is to establish a quantitative version of the positivity properties of twisted relative pluricanonical bundles and their direct images. Some of the important technical points of our proof are an $L^{2/m}$ -extension theorem of Ohsawa-Takegoshi type which is derived from the original result by a simple fixed point method, and the notion of “singular Hermitian metric” on vector bundles, together with an appropriate definition of positivity of the associated curvature. Part of this article is based on the joint work of the first named author with Bo Berndtsson [4, Part A], and it can be seen as an expanded and updated version of it.

1. INTRODUCTION

We continue to investigate here the (quantitative version of the) positivity properties of twisted relative pluricanonical bundles and their direct images. A particular case of the main result we establish in this paper is as follows.

Theorem 1.1. *Let $f : X \rightarrow Y$ be a projective surjective morphism of complex manifolds with connected fibers, and let $Y_0 \subset Y$ be the maximum Zariski open subset where f is smooth. Let L be a holomorphic line bundle on X with a continuous Hermitian metric h with semi-positive curvature current. Suppose that the direct image sheaf $f_*(mK_{X/Y} + L)$ is non-zero for a positive integer m . Then*

(1) *the line bundle $mK_{X/Y} + L$ admits a singular Hermitian metric $h^{(m)} = B_m^{-1}$ with semi-positive curvature, and on every fiber X_y over $y \in Y_0$, the restricted dual $B_m|_{X_y}$ is the twisted m -th Bergman kernel metric $B_{m,y}$ of $-(mK_{X_y} + L|_{X_y})$. In particular, if X is compact, $mK_{X/Y} + L$ is pseudo-effective.*

(2) *The torsion free sheaf $f_*(mK_{X/Y} + L)$ admits a singular Hermitian metric g_m with Griffiths semi-positive curvature, and g_m is continuous on Y_0 and $0 < \det g_m < \infty$ on Y_0 , moreover g_m on each fiber $f_*(mK_{X/Y} + L)_y$, which is in fact $H^0(X_y, mK_{X_y} + L|_{X_y})$, with $y \in Y_0$ is the so-called m -th Narashimhan-Simha Hermitian form. In particular, if Y is projective, $f_*(mK_{X/Y} + L)$ is weakly positive at every $y \in Y_0$ in the sense of Nakayama [35, V.3.20].*

We also deal with the case when the metric h is singular. In such a case, the corresponding statements become more complicated naturally (see 5.1.2 for example). Let us add some explanations on the statement being $L = \mathcal{O}_X$ for simplicity. The weak positivity $f_*(mK_{X/Y})$ at every point $y \in Y_m$ for some Zariski open $Y_m \subset Y$ is known classically if Y is projective, by the research of Kawamata [23], Viehweg [46] concerning the Iitaka conjecture, building on earlier works by Griffiths [20] and Fujita [19]. As a corollary of it, the pseudo-effectivity of $mK_{X/Y}$ follows. Actually this field has generated many important works, including the twisted case by a semi-ample line bundle L ; we will only mention here a few of them [1], [7], [18], [22], [24], [25], [26], [28], [29], [32], [33], [34], [47] (and we apologize to the authors we omit ...).

Our main contributions in the statement above are to provide a quantitative counterpart of (1) and (2) above, and to describe the open subset $Y_m \subset Y$ where these properties behave nicely. By this we mean that the semi-positivity property of $mK_{X/Y}$, as well as the (pointwise) weak positivity property of $f_*(mK_{X/Y})$ can be obtained as a consequence of the existence of the natural Bergman and Narashimhan-Simha type metrics we construct on these sheaves. We would also like to stress that the Griffiths semi-positivity of vector bundles or more generally torsion free sheaves is in practice stronger than the weak positivity in algebraic geometry (recall for example, a Griffiths positive vector bundle over a projective manifold is ample, but the converse is not known; or a nef line bundle is not necessarily semi-positive), and that Viehweg's weak positivity does not imply the curvature semi-positivity of the metric $h^{(m)}$ of $mK_{X/Y}$ nor g_m on $f_*(mK_{X/Y})$. The explicit fiberwise description of B_m is crucial in the process of construction of the metric g_m on $f_*(mK_{X/Y})$.

The metrics $h^{(m)}$ and g_m appeared in Theorem 1.1 are both canonically defined as we shall explain next (see §3.1 in general, also [36], [23], [43]). These metrics are defined in a way similar to the Bergman kernel metric, but using L^p -spaces instead of L^2 -spaces, with $p = 2/m$.

To explain this, we let here X be a compact complex manifold with a holomorphic Hermitian line bundle (L, h) . Then the m -th Bergman kernel metric B_m on the dual $(mK_X + L)^*$ is given, at every point $x \in X$ and for every vector $\xi_x \in (mK_X + L)_x^*$, by

$$|\xi_x| = \sup | \xi_x \cdot u(x) |,$$

where the supremum being taken over all $u \in H^0(X, mK_X + L)$ with

$$\|u\|_m := \left(\int_X (cu \wedge \bar{u}h)^{1/m} \right)^{m/2} \leq 1$$

with a constant $c \in \mathbb{C}$ which makes the pairing $cu \wedge \bar{u}h$ positive definite. Then the singular Hermitian metric $h^{(m)}$ on $mK_X + L$ is defined by the dual metric of the m -th Bergman kernel metric, i.e., $h^{(m)} = B_m^{-1}$. Using B_m , we also define a Hermitian form g_m on $H^0(X, mK_X + L)$ given by

$$g_m(u, v) = \int_X (-1)^{n^2/2} u \wedge \bar{v} (B_m^{-1})^{(m-1)/m} h^{1/m},$$

where $n = \dim X$. When $m = 1$, g_1 is nothing but the canonical L^2 -pairing $\int_X (-1)^{n^2/2} u \wedge \bar{v}h$ on $H^0(X, K_X + L)$. Also in general $m \geq 1$, letting $h_{m-1} = (B_m^{-1})^{(m-1)/m} h^{1/m}$ be a singular Hermitian metric on $L_{m-1} = (m-1)K_X + L$, g_m can be seen as the canonical L^2 -pairing on $H^0(X, K_X + L_{m-1})$. Theorem 1.1 asserts that the fiberwise metrics $h_y^{(m)}$, the pairings $g_{m,y}$ respectively, glues together nicely over Y_0 and has curvature positivity, and then extends over Y across the singularities of f keeping the curvature positivity.

The main differences between Theorem 1.1 in its general form and the corresponding previous result [3, 4.2] are as follows: (i) the morphism f can be singular, (ii) the direct image sheaf can merely be torsion free, and (iii) the subset of Y where the metrics $h^{(m)}$ and g_m can be described explicitly is given in a very accurate manner. To obtain these properties, there are some new ingredients which are introduced in this paper. One is the $L^{2/m}$ -version of Ohsawa-Takegoshi's extension theorem for Theorem 1.1(1), another is a notion of singular Hermitian metrics on vector bundles and torsion free sheaves and its curvature positivity/negativity for Theorem 1.1(2).

We recall Ohsawa-Takegoshi's L^2 -extension theorem [37] (cf. [13, 12.9]) in a very basic set-up. Let $\Omega \subset \mathbb{C}^n$ be a ball of radius r and let $\sigma : \Omega \rightarrow \mathbb{C}$ be a holomorphic function on Ω ; $\sigma \in \mathcal{O}(\Omega)$, such that $\sup_{\Omega} |\sigma| \leq 1$, and that the gradient $\partial\sigma$ of σ is nowhere zero on the set $V := \{\sigma = 0\}$. In particular V is a smooth complex hypersurface in Ω . We denote by φ a plurisubharmonic function on Ω such that its restriction to V is well-defined (i.e., $\varphi|_V \not\equiv -\infty$). Then Ohsawa-Takegoshi's theorem states that for any $f \in \mathcal{O}(V)$, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_V = f$ and

$$\int_{\Omega} |F|^2 \exp(-\varphi) d\lambda \leq C_0 \int_V |f|^2 \exp(-\varphi) \frac{d\lambda_V}{|\partial\sigma|^2},$$

where C_0 is an absolute constant, and where $d\lambda$ and $d\lambda_V$ are the volume form induced from the Euclidean metric on \mathbb{C}^n . (The right hand side can formally be $+\infty$.) Recently, the optimal bound of C_0 is obtained by Błocki [6] and Guan-Zhou [21]. Then our generalization in the set-up above is the following (see 4.2.5).

Proposition 1.2. *Let $m \geq 1$ be a real number. Then for any $f \in \mathcal{O}(V)$, there exists $F \in \mathcal{O}(\Omega)$ such that $F|_V = f$ and satisfying the $L^{2/m}$ -bound*

$$\int_{\Omega} |F|^{2/m} \exp(-\varphi) d\lambda \leq C_0 \int_V |f|^{2/m} \exp(-\varphi) \frac{d\lambda_V}{|\partial\sigma|^2},$$

where C_0 is the same constant as in the Ohsawa-Takegoshi theorem.

Once this result is established, the proof of Theorem 1.1(1) runs as follows. For any $y \in Y_0$, all the elements in $H^0(X_y, mK_{X_y} + L|_{X_y})$ extend locally near y , thanks to Siu's invariance plurigenera [41] (see also [38]). Thus over Y_0 (i.e., on $f^{-1}(Y_0) \subset X$), we can apply [3, 4.2] and therefore the logarithm of the m -th Bergman kernel metric has a psh variation. We then use the $L^{2/m}$ -extension result to estimate our metric from above by a uniform constant. Standard results of pluri-potential theory then gives that the metric extends, across the singular fibers of f , to a semi-positively curved metric on all of X .

Aiming to understand the metric properties of direct image sheaves $f_*(mK_{X/Y} + L)$ in Theorem 1.1(2), we explore a few aspects of the notion of *singular Hermitian metric* on vector bundles and more generally on torsion free sheaves, in §2. This notion is well understood and extremely useful in rank 1, i.e. for line bundles. Motivated by the study of direct image sheaves, one would like to develop a similar theory in the context of vector bundles E of rank $(E) \geq 2$. At first glance it may look surprising that there are not so many works dedicated to this subject –with the notable exceptions of [9], [3, §3], [39], [40]. The observation which is central in the present article is that “in practice” what we actually need is to define a notion of negativity or positivity of a (smooth) Hermitian vector bundle (E, h_E) in order to derive some consequences of this property. The positivity in the sense of Griffiths has a very important property: the bundle (E, h_E) is Griffiths positive if and only if the dual Hermitian bundle (E^*, h_{E^*}) is Griffiths negative (unfortunately, the Nakano positivity fails to satisfy this property). It is then easy to see that the Griffiths semi-positivity of (E, h_E) can be characterized as follows. Let $u \in H^0(\Omega, E^*)$ be any non-zero section, where $\Omega \subset X$ is a coordinate open subset. The function $\log |u|_{h_{E^*}}$ is psh on Ω if and only if (E, h_E) is Griffiths semi-positive. Thus, we are able to formulate this notion without an explicit mention of the curvature tensor associated to (E, h_E) .

The above property of (smooth) Hermitian vector bundles is converted into the definition of the Griffiths semi-positivity in the context of singular Hermitian vector bundles, see [3, §3]. By definition, a singular Hermitian metric h_E on the trivial vector bundle E of rank r defined on an open set $\Omega \subset \mathbb{C}^n$ is simply a measurable map defined on Ω with values in the space of positive semi-definite $r \times r$ -Hermitian matrices, such that $0 < \det(h_E) < \infty$ almost everywhere. This later condition is imposed so as to avoid the metric h_E to be everywhere degenerate e.g. at each point of Ω along some directions in E . If E is non-trivial, a singular Hermitian metric h_E on E corresponds to a collection of singular Hermitian metrics on each local trivialization of E , verifying the usual glueing condition. By the discussion above, we can define a notion of Griffiths semi-positivity for (E, h_E) . The corresponding Nakano semi-positivity of a singular Hermitian metric does not seem to be well-understood yet; nevertheless we refer to [39], [40] for the state of arts in this direction.

After treating the basic properties of singular Hermitian metrics (on torsion free sheaves, for example), we obtain the following general result 2.5.2 which is connected with algebraic geometry. In fact we establish a more general Finsler metric version 2.5.3, which gives criteria for the pointwise bigness and the pointwise weak positivity of a torsion free sheaf.

Theorem 1.3. *Let Y be a smooth projective variety, and let \mathcal{E} be a torsion free sheaf on Y . Suppose that \mathcal{E} admits a singular Hermitian metric h with Griffiths semi-positive curvature. Then \mathcal{E} is weakly positive at y in the sense of Nakayama, for every $y \in Y$ where \mathcal{E} is locally free around y and $\det h(y) < +\infty$.*

Going back to the situation in Theorem 1.1, the basic result is due to Berndtsson [1, 1.2]: it states that the Hermitian metric on $f_*(K_{X/Y} + L)|_{Y_0}$ obtained as a family of fiberwise canonical L^2 -metrics has Nakano semi-positive curvature provided the metric h is smooth semi-positive. We then extend this result (but weakening to the semi-positivity in the sense of Griffiths) to $f_*(K_{X/Y} + L)$ with possibly singular Hermitian line bundle (L, h) with mild singular h , for example with trivial multiplier ideal $\mathcal{I}(h) = \mathcal{O}_X$ (see 3.3.5). Finally, regarding $mK_{X/Y} + L = K_{X/Y} + L_{m-1}$ as an adjoint type bundle for $L_{m-1} = (m-1)K_{X/Y} + L$ with a singular Hermitian metric h_{m-1} as we explained above, we deduce Theorem 1.1(2) from all these arguments mentioned above (see 5.2.2).

The organization of this paper is as follows. We will start with §2 a general discussion on singular Hermitian metrics on vector bundles, its curvature properties and its geometric consequences. The positivity of relative adjoint type bundles $K_{X/Y} + L$ has been studied in [3]. We then proceed to the positivity of $f_*(K_{X/Y} + L)$ in §3, the positivity of $mK_{X/Y} + L$ in §4, and finally the positivity of $f_*(mK_{X/Y} + L)$ in §5. Some variants and refinements of Theorem 1.1 (1) and (2) will be given in §4.3 and §5.2 respectively.

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2. SINGULAR HERMITIAN METRICS ON VECTOR BUNDLES

In this first part of our article we introduce a notion of *singular Hermitian metric* on a vector bundle (and more generally, on a torsion free sheaf) and discuss the corresponding notion of positivity/negativity of the associated curvature. There are earlier attempts to do so, cf. [9], [3], [39], [40] to quote only a few.

It turns out that as soon as the rank of vector bundle is greater than 1, we have at our disposal two notions of positivity: in the sense of Griffiths and in the sense of Nakano, respectively. The former corresponds to the type of hypothesis one has in algebraic geometry and has nicer functorial properties, whereas the later is important when one wants to establish vanishing results for higher cohomology. The observation which is central in the present article is that “in practice” what we actually need is to define a notion of negativity (or positivity) of a Hermitian vector bundle (E, h_E) in order to derive some consequences of this property.

Our definition here is tailored in order to be compatible with the notion of *weak positivity* in algebraic geometry; more precisely, one of the results we want to hold true is that a torsion free sheaf which can be endowed with a singular Hermitian metric with semi-positive curvature is automatically weakly positive.

2.1. Prelude. To start with, let X be a complex manifold, and let $E \rightarrow X$ be a vector bundle of rank $r \geq 1$, endowed with a (smooth) Hermitian metric h . We denote by

$$\Theta_h(E) \in \mathcal{C}_{1,1}^\infty(X, \text{End}(E))$$

the curvature form of (E, h) . We refer to [13, Ch. 3, Ch. 10] for basics.

We recall next two notions of positivity, playing a important role in analytic and algebraic geometry respectively. The former is the notion of positivity in the sense of Griffiths, and the later is its (conjectural) counterpart in algebraic geometry.

Definition 2.1.1. Let $x_0 \in X$ be a point, and let (z_1, \dots, z_n) be a system of local coordinates on X centered at x_0 . We consider a local holomorphic frame e_1, \dots, e_r of E near x_0 , such that it is orthonormal at x_0 . The Chern curvature tensor can be locally expressed as follows

$$\Theta_h(E) = \sqrt{-1} \sum c_{j\bar{k}\lambda\bar{\mu}} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

where $j, k = 1, \dots, n$ and $\lambda, \mu = 1, \dots, r$. We say that (E, h) is semi-positive in the sense of Griffiths at x_0 if

$$\sum c_{j\bar{k}\lambda\bar{\mu}} \xi_j \bar{\xi}_k v_\lambda \bar{v}_\mu \geq 0$$

for every $\xi \in T_{X, x_0}$ and $v \in E_{x_0}$. The Hermitian bundle (E, h) is semi-positive in the sense of Griffiths if it satisfies the property above at each point of X . If there exists a Hermitian form ω on X such that

$$\Theta_h(E) \geq \omega \otimes \text{Id}_{\text{End}(E)},$$

then we say that (E, h) is positive (in the sense of Griffiths). The (semi-)negativity is defined in a similar manner.

It is well-known (cf. [13, 10.1]) that (E, h) is semi-positive if and only if its dual (E^*, h^*) is semi-negative in the sense of Griffiths (we recall that the notion of Nakano positivity fails to satisfy this important property, although we do not recall the definition).

Remark 2.1.2. We denote by $\mathbf{P}(E^*)$ the projectivization of the dual of E (we do not follow here the Grothendieck's convention here), and let $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ be the corresponding tautological line bundle. The metric h induces a metric on $\mathcal{O}_{\mathbf{P}(E^*)}(1)$ and (almost by definition) we see that if the bundle (E, h) is semi-negative, then

$$\log |v|_h^2$$

is plurisubharmonic (psh, for short), for any local holomorphic section v of the bundle $\mathcal{O}_{\mathbf{P}(E^*)}(1)$. This remark is crucial from our point of view, since it gives the possibility of *defining* the notion of Griffiths positivity for a bundle (E, h) without referring to the curvature tensor!

On the algebraic geometry side, we recall that a vector bundle E on a projective manifold is said to be *ample* if the associated bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle on $\mathbb{P}(E)$. Then it is well-known that if (E, h) is positive, then E is ample (see [31, 6.1.25]); the converse, known as Griffiths conjecture, is a long standing and deemed difficult problem.

According to [15, 6.1], a vector bundle E on a projective manifold X is *pseudo-effective* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a pseudo-effective line bundle, and if the union of all curves $C \subset \mathbb{P}(E)$ with $\mathcal{O}_{\mathbb{P}(E)}(1) \cdot C < 0$ is contained in a union of subvarieties which does not project onto X . It is shown that in [15, 6.3] that this property is equivalent to the fact that E is *weakly positive*: there exist positive integers k_0 and $m_1(k)$, together with an ample line bundle A such that the map

$$H^0(X, S^m(S^k E \otimes A)) \rightarrow S^m(S^k E \otimes A)$$

is generically surjective, for any $k \geq k_0$ and $m \geq m_1(k)$. This notion can be formulated in the context of torsion free sheaves, and it is relevant in the following context.

Theorem 2.1.3. [46] *Let $f : X \rightarrow Y$ be a proper surjective morphism of projective manifolds with connected fibers. If m is a positive integer, then the direct image sheaf $f_*(mK_{X/Y})$ is weakly positive. Here we denote by $mK_{X/Y}$ the m^{th} tensor power of the relative canonical bundle.*

Our goal in what follows is twofold. We define a metric property of a Hermitian vector bundle E which –in the light of Griffiths conjecture– should correspond to the notion of the weak positivity (we will at least show that it implies the weak positivity). Also, our metric property should be verified by the direct images $f_*(mK_{X/Y})$. The candidate we have for this is presented in the next subsections; we start with its local version.

2.2. Singular metrics on trivial bundles. Here we suppose that $E = X \times \mathbb{C}^r$ is the trivial vector bundle of rank r on a complex manifold X . We denote by $H_r := \{A = (a_{i\bar{j}})\}$ the set of $r \times r$, semi-positive definite Hermitian matrixes. We endow our manifold X with the Lebesgue measure.

We recall the following notion.

Definition 2.2.1. A *singular Hermitian metric* h on E is a measurable map from X to H_r satisfying $0 < \det h < +\infty$ almost everywhere. \square

By ‘‘convention’’, a matrix valued function $h = (h_{i\bar{j}})$ is measurable (resp. smooth, continuous, ...), if all entries $h_{i\bar{j}}$ are measurable (resp. smooth, continuous, ...). We will identify two functions if they coincide almost everywhere (however, there is a risk of confusion that a discontinuous function can be continuous under this convention, for example). We note that in the paper [3, p. 357], one does not require the condition $0 < \det h < +\infty$ almost everywhere in the definition of a *singular Hermitian metric* on a vector bundle. However, as it was highlighted in [39], [40], this additional condition concerning the determinant is important, as it implies that one can define the notion of *curvature tensor* associated to a singular Hermitian metric. A standard notion of singular Hermitian metrics of line bundles is $h = e^{-\varphi}$ with $\varphi \in L_{loc}^1(X, \mathbb{R})$. Our definition in 2.2.1 requires less in general.

A local section v of E means an element $v \in H^0(U, E)$ on some open subset $U \subset X$ (we always assume v to be holomorphic). If h is a singular Hermitian metric and $v \in H^0(U, E)$, then $|v|_h : U \rightarrow \mathbb{R}_{\geq 0}$ is a measurable function given by $|v|_h^2 = {}^t v h \bar{v} = \sum h_{i\bar{j}} v^i \bar{v}^j$ ($v = {}^t(v^1, \dots, v^r)$ is a column vector). Following [3, p. 357], we introduce the curvature positivity/negativity as follows.

Definition 2.2.2. Let h be a singular Hermitian metric on E .

- (1) h is *negatively curved* (or h has Griffiths semi-negative curvature), if for any open subset $U \subset X$ and any $v \in H^0(U, E)$, $\log |v|_h^2$ is psh on U .
- (2) h is *a.e. negatively curved*, if for any open subset $U \subset X$ and any $v \in H^0(U, E)$, there exists a psh function ψ on U such that $\psi = \log |v|_h^2$ almost everywhere on U (we may say $\log |v|_h^2$ is a.e. psh on U).
- (3) h is *positively curved* (or h has Griffiths semi-positive curvature), if the dual singular Hermitian metric $h^* := {}^t h^{-1}$ on the dual vector bundle E^* is negatively curved. \square

When h is smooth and $0 < \det h < +\infty$ everywhere (i.e. if h is a Hermitian metric in the usual sense), the requirement 2.2.2(1) (resp. (3)) is nothing but the classical Griffiths semi-negativity (resp. Griffiths semi-positivity) of h (see, [39, §2]).

We have the following remark, which (hopefully) clarifies some aspects of the notion above.

Remark 2.2.3. (1) If h is negatively curved, then $(h_{i\bar{j}}(x)) \in H_r$ for any $x \in X$ as we now see. For a constant vector $v_1 = {}^t(1, 0, \dots, 0)$, $|v_1|_h^2 = h_{1\bar{1}}$ is semi-positive and psh. Hence it is everywhere defined $h_{1\bar{1}} \in [0, +\infty)$ and locally bounded. For $v = {}^t(1, 1, 0, \dots, 0)$, we have $|v|_h^2 = h_{1\bar{1}} + h_{2\bar{2}} + 2\operatorname{Re} h_{1\bar{2}}$. For $v' = {}^t(1, \sqrt{-1}, 0, \dots, 0)$, we have $|v'|_h^2 = h_{1\bar{1}} + h_{2\bar{2}} + 2\operatorname{Im} h_{1\bar{2}}$. Thus by the same token, $h_{1\bar{2}}$ is everywhere defined, and $|h_{1\bar{2}}|^2 \leq h_{1\bar{1}} h_{2\bar{2}} \leq \frac{1}{2}(h_{1\bar{1}}^2 + h_{2\bar{2}}^2)$.

(2) The argument in (1) also shows that if h is a.e. negatively curved, there exists a unique negatively curved singular Hermitian metric h' such that $h = h'$ almost everywhere. With a suitable convention, we may need not to distinguish negatively curved and a.e. negatively curved metrics. In the case of line bundles, we do not need to. It is only a difference $\varphi \in L_{loc}^1$ with $\sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ in the sense of distribution and φ psh. We could also define a.e. positively curved singular Hermitian metrics. However in the line bundle case, we suppose every singular Hermitian metric h , whose curvature is bounded from below by a continuous form, has an upper-semi-continuous local weight, i.e., $h = e^{-\varphi}$ with quasi-psh φ . \square

Remark 2.2.4. (1) Following [39, Theorem 1.3], we will present next here an example of negatively curved metric on the trivial rank 2 bundle $E = \mathbb{C} \times \mathbb{C}^2$ on \mathbb{C} , for which the curvature tensor cannot be defined in a reasonable way around the origin. Indeed, the coefficients of connection matrix corresponding to the next metric

$$h = \begin{pmatrix} 1 + |z|^2 & z \\ \bar{z} & |z|^2 \end{pmatrix}$$

are not in $L^1(\mathbb{C}, 0)$.

(2) One could define a singular Hermitian metric h on E by requiring that the map $h : X \rightarrow H_r$ is measurable, such that $\log \det h \in L^1_{loc}(X, \mathbb{R})$. The line bundle case, this is the definition. However, as the previous example (h with $\det h = |z|^4$) shows it, the curvature tensor corresponding to such a metric is not well-defined. \square

We collect some basic local properties. The first and most basic one is the following approximation result.

Lemma 2.2.5. [3, 3.1] *Suppose X is a polydisc in \mathbb{C}^n , and suppose h is a singular Hermitian metric on E with negative (resp. positive) curvature. Then, on any smaller polydisc there exists a sequence of smooth Hermitian metrics $\{h_\nu\}_\nu$ decreasing (resp. increasing) pointwise to h whose corresponding curvature tensor is Griffiths negative (resp. positive).*

In the statement above the sequence $\{h_\nu\}_\nu$ is called *decreasing* in the sense that the sequence of functions $\{|s|_{h_\nu}^2\}_\nu$ is decreasing for any constant section s , or equivalently, the matrix corresponding to each difference $h_\nu - h_{\nu+1}$ is semi-positive definite. We note that $\{h_\nu\}_\nu$ is obtained simply by convolution (which is possible due to the hypothesis concerning X).

This result is relevant for example in the following context.

Lemma 2.2.6. [39, Proposition 1.1(ii)] *Suppose a singular Hermitian metric h is negatively curved. Then $\log \det h \in L^1_{loc}(X, \mathbb{R})$ and psh.*

Proof. Let $\{h_\nu\}_\nu$ be the sequence of metrics approximating h , with the properties stated in 2.2.5. We define $\varphi_\nu := \log \det h_\nu$, the weight of the induced metric on the determinant bundle; it is psh (and smooth), and φ_ν is decreasing to $\varphi_h := \log \det h$. Thus φ_h is psh and in particular $\varphi_h \in L^1_{loc}(X, \mathbb{R})$. \square

As we have already mentioned in 2.2.3, a negatively curved singular Hermitian metric is well-defined in each point of X , thus one can define the set of its eigenvalues (with respect to any smooth positive definite Hermitian metric). We present in our next statement a few basic properties of these functions.

Lemma 2.2.7. *Let $h = (h_{i\bar{j}})$ be a singular Hermitian metric on E . Let $U \subset X$ be a relatively compact open subset.*

(1) *Suppose h is negatively curved. Then there exists a constant $C > 0$ such that $|h_{i\bar{j}}| \leq C$ on U for any pair of indices i, j . Let $0 \leq \lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_r(x) < +\infty$ be the eigenvalues of $h(x)$ at each point $x \in X$. Then we have $\lambda_r(x) \leq rC$ on U , $\lambda_1(x) \geq \frac{1}{(rC)^{r-1}} \det h(x)$ on U , and $h - \frac{1}{(rC)^{r-1}} (\det h) I_r$ is a measurable map from U to H_r , where I_r is the $r \times r$ identity matrix and H_r is the set of semi-positive Hermitian forms.*

(2) Suppose h is positively curved. Then there exists a constant $C' > 0$ (independent of $x \in U \setminus \{\det h^* = 0\}$) such that $\lambda_1(x) \geq \frac{1}{rC'}$ and $\lambda_r(x) \leq (rC')^{r-1} \det h(x)$ on $U \setminus \{\det h^* = 0\}$, where $0 \leq \lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_r(x) < +\infty$ are eigenvalues of $h(x)$ at $x \in X \setminus \{\det h^* = 0\}$.

Proof. The first assertion of (1) is a direct consequence of our remark 2.2.3. Given a matrix $A = (a_{ij}) \in M(r, \mathbb{C})$, we denote by $C := \max\{|a_{ij}|; 1 \leq i, j \leq r\}$, the maximum of the absolute value of its entries. Then we have $|\lambda| \leq rC$ for any eigenvalue of A . Note $\det h = \lambda_1 \lambda_2 \dots \lambda_r \leq \lambda_1 (rC)^{r-1}$ at every $x \in U$.

(2) Take dual and apply (1) as long as $\det h^* \neq 0$. Even if $\det h^*(x) = 0$, we can say $\lambda_1(x) \geq \frac{1}{rC'}$ and $\lambda_r(x) = (rC')^{r-1} \det h(x) = +\infty$ under some convention as in the case of line bundles. It can be $\lambda_1(x) = +\infty$ in general for $x \in \{\det h^* = 0\}$. \square

2.3. Global properties. We consider in this subsection a holomorphic vector bundle E of rank r on a complex manifold X .

Definition 2.3.1. (1) A *singular Hermitian metric* h on E is a collection of singular Hermitian metrics on local trivializations $E|_{U_\lambda} \cong U_\lambda \times \mathbb{C}^r$ satisfying the gluing condition as the one in the usual case.

(2) A singular Hermitian metric h on E is *negatively curved* (or h has Griffiths semi-negative curvature), if $\log |v|_h^2$ is psh for any local section v of E ; h is *positively curved* (or h has Griffiths semi-positive curvature), if $h^* = {}^t h^{-1}$ defines a singular Hermitian metric on E^* with negative curvature.

(3) A vector bundle E is negatively (resp. positively) curved, if it admits a negatively (resp. positively) curved singular Hermitian metric h . Unless explicitly stated otherwise, the metric h is allowed to be singular in the sense of (1) of the present definition. \square

We suppose as an example that X is projective, and $E = A \oplus B$, where A is a positive line bundle and B is a negative line bundle. As a singular Hermitian metric of E in the weaker sense [3, p. 357], we can take a direct sum of Hermitian metric $h = h_A \oplus h_B$ with $h_A \equiv 0$ and h_B smooth and negative curvature. Then h is “negatively curved”. This is a good indication that indeed one should add the condition $\det h > 0$ a.e. in the definition of singular Hermitian metric.

We will start next a systematic study of the properties of negatively curved vector bundles as above. Our first statement here concerns the behavior with respect to inverse images.

Lemma 2.3.2. *Let $f : Y \rightarrow X$ be a holomorphic surjective map between two complex manifolds, and let E be a vector bundle on X .*

(1) *We assume that E admits a singular Hermitian metric h with negative (resp. positive) curvature. Then f^*h is a singular Hermitian metric on $f^*(E)$, and it is negatively (resp. positive) curved.*

(2) *Suppose moreover that f is proper. Let X_1 be a non-empty Zariski open subset, and let $Y_1 = f^{-1}(X_1)$ be its inverse image. We consider h_1 a singular Hermitian metric on $E_1 = E|_{X_1}$. We assume that the singular Hermitian metric f^*h_1 on f^*E_1 extends as a singular Hermitian metric h_Y on the inverse image bundle f^*E with negative (resp. positive) curvature. Then*

h_1 extends as a singular Hermitian metric on E with negative (resp. positive) curvature. In particular, if we have $X_1 = X$ then the metric h_1 is negative (resp. positive) curved if and only if f^*h_1 is.

We remark that the statement above holds even if we do not assume that the map f is surjective, provided that the pull-back f^*h is well-defined.

Proof. We will discuss the negatively curved case.

(1) Let U be a coordinate open subset in X such that $E|_U \cong U \times \mathbb{C}^r$, and $U' \subset U$ a smaller open subset so that an approximation result 2.2.5 holds for h . Let $\{h_\nu\}$ be a decreasing sequence towards h as in 2.2.5. We consider an open subset $V \subset f^{-1}(U')$ and a non-zero section $v \in H^0(V, f^*E)$. Then the sequence $\{\log |v|_{f^*h_\nu}^2\}_\nu$ is decreasing to $\log |v|_{f^*h}^2 \not\equiv -\infty$. Since f^*h_ν is smooth and negatively curved, $\log |v|_{f^*h_\nu}^2$ is psh. Thus so is its limit $\log |v|_{f^*h}^2$.

(2) Let $u \in H^0(U, E)$ be any local section. Then $f^*u \in H^0(f^{-1}(U), f^*E)$. Then by assumption, $\log |f^*u|_{h_Y}^2$ is (a.e.) psh on $f^{-1}(U)$. In particular $\log |f^*u|_{h_Y}^2$ is bounded from above on any relatively compact subset of $f^{-1}(U)$. For any $x \in U \cap X_1$ and a point $y \in f^{-1}(x)$, we have $|f^*u|_{h_Y}^2(y) = |u|_{h_1}^2(x)$. Thus we see that the psh function $\log |u|_{h_1}^2$ a-priori defined only on $U \cap X_1$ is bounded from above on $U' \cap X_1$, where $U' \subset U$ is any relatively compact subset. Hence, it can be extended as a psh function on U , so the metric h_1 indeed admits an extension h on X in such a way that the Hermitian bundle (E, h) is negatively curved. \square

The next statements show that the tensor bundles associated to E and the quotient bundles of E inherits its positivity properties.

Lemma 2.3.3. *Let (E, h_E) is positively curved vector bundle. Then the symmetric power $S^m E$ and the wedge product $\Lambda^q E$ are positively curved.*

Proof. The mechanism of the argument is the same as in 2.2.6 (the proof of [39, Proposition 1.1(ii)]), which shows $(\det E, \det h)$ is positive. Basically the statement is well-known for the smooth case, and we are using 2.2.5 in order to obtain the general case. We will not provide further details here. \square

Lemma 2.3.4. *Let h be a singular Hermitian metric on E .*

- (1) *Let $S \subset E$ be a subbundle. Then the restriction $h_S := h|_S$ defines a singular Hermitian metric on S , and h_S is negatively curved if h is.*
- (2) *Let $E \rightarrow Q$ be a quotient vector bundle. Suppose that h is positively curved. Then Q has a naturally induced singular Hermitian metric h_Q with positive curvature.*

We will not present here the arguments for the 2.3.4, since we will deal with this kind of statements in a more general context in 2.4.3. We rather discuss next the connection between 2.3.1 and the notion of positivity in algebraic geometry.

Proposition 2.3.5. *Let $\pi : \mathbb{P}(E) \rightarrow X$ be the projective space bundle, and let $\pi^*E \rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ be the universal quotient line bundle. Suppose E admits a positively curved singular Hermitian metric h . Then*

- (1) *The bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ has a (naturally induced) singular Hermitian metric g whose curvature current is semi-positive.*

(2) Let $\det h$ be the induced singular Hermitian metric of $\det E$ with semi-positive curvature. Let $U \subset \mathbb{P}(E)$ and $W \subset X$ be coordinate neighborhoods respectively such that $U \subset \pi^{-1}(W)$, and regard $g|_U$ and $\det h|_W$ as a function on U and W respectively. Then

$$g \leq C \pi^*(\det h|_W)$$

holds on U for a constant $C = C_{U,W} > 0$.

(2') In particular, if $\det h(x) < +\infty$ at a point $x \in X$, then the restriction $g|_{\mathbb{P}(E_x)}$ is well-defined and the multiplier ideal sheaf satisfies $\mathcal{I}(g^k|_{\mathbb{P}(E_x)}) = \mathcal{O}_{\mathbb{P}(E_x)}$ for any $k > 0$, where $\mathbb{P}(E_x) = \pi^{-1}(x)$. We can express this equality in global terms, namely $\pi(V\mathcal{I}(g^k)) \subset V\mathcal{I}((\det h)^k)$ holds for any $k > 0$, where $V\mathcal{I}(g^k) \subset \mathbb{P}(E)$ is the complex subspace defined by the multiplier ideal sheaf $\mathcal{I}(g^k)$.

(3) If X is projective, then the inclusion

$$\mathbf{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1)) \subset \cup_{k>0} V\mathcal{I}(g^k)$$

holds, where $\mathbf{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))$ is the restricted base locus of $\mathcal{O}_{\mathbb{P}(E)}(1)$ introduced in [16, §1]. In particular, E is pseudo-effective in the sense of [15, 6.1], and therefore E is weakly positive in the sense of Viehweg.

Proof. (1) We will use the notation $\mathcal{O}(1)$ instead of $\mathcal{O}_{\mathbb{P}(E)}(1)$. Our first remark is that π^*h defines a singular Hermitian metric on π^*E with positive curvature (cf. 2.3.2). By the property 2.3.4, the metric g^* on $\mathcal{O}(-1)$ induced by the bundle injection $\mathcal{O}(-1) \rightarrow \pi^*E^*$ has negative curvature, hence $(\mathcal{O}(1), g)$ is positively curved in the sense of currents, where g is the dual of the metric g^* .

(2) We may suppose that X is a coordinate open set, and that $\det E = X \times \mathbb{C}$ is trivial. Then we write $\det h = e^{-\varphi_h}$ where φ_h is a psh function. In this case we also have $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$ where $r = \text{rank } E$. We consider the set $U := X \times U_0$, where $U_0 \subset \mathbb{P}^{r-1}$ is a coordinate open set. Since h^* and g^* are negatively curved, they are bounded in the sense of 2.2.7(1). Let $x \in X$ be an arbitrary point. The metric g^* is defined as the restriction of π^*h^* to a 1-dimensional linear subspace, therefore we have $\lambda_1(g^*, p) \geq \lambda_1(\pi^*h^*, p)$ for any $p \in \mathbb{P}(E_x) \cap U$, where $\lambda_1(g^*, p)$ stands for the smallest eigenvalue of g^* at p (and $\lambda_1(\pi^*h^*, p)$ for π^*h^*). We have $\lambda_1(g^*, p) = e^{\psi(p)}$, and $\lambda_1(\pi^*h^*, p) = \lambda_1(h^*, x) \geq \frac{1}{(rC)^{r-1}} \det h^*(x)$ by 2.2.7 (1), where $C > 0$ is a constant independent of $x \in X$. Thus we have $e^{\psi(p)} \geq \frac{1}{(rC)^{r-1}} e^{\varphi_h(x)}$ for any $p \in \mathbb{P}(E_x) \cap U$.

(2') The first assertion is clear from (2). For the second, what we proved in (2) above is that $g = e^{-\psi} g_1 \leq (rC)^{r-1} e^{-\pi^*\varphi_h}$ on $\mathbb{P}(E) = X \times \mathbb{P}^{r-1}$ (up to -irrelevant- choices of local trivializations). Since $\pi : \mathbb{P}(E) \rightarrow X$ is a product, our claim is a consequence of Fubini theorem.

(3) The pseudo-effectivity of E follows from the properties (1)–(2'), combined with the usual techniques in algebraic/analytic geometry. Then the weak positivity follows from [15, 6.3] as a consequence. \square

Remark 2.3.6. It does not seem to be known whether the weak positivity of E implies the Griffiths semi-positivity in the sense of 2.3.1 (this would be a singular version of Griffiths conjecture). \square

2.4. Metrics on torsion free sheaves. It turns out that very interesting objects, such as direct images of adjoint bundles $\mathcal{E}_m := f_*(mK_{X/Y} + L)$ do not always have a vector bundle structure; nevertheless, we would like to have a notion of “singular Hermitian metric” on \mathcal{E}_m . More generally, we introduce in this subsection a notion of (metric) positivity for torsion free sheaves. It turns out that the theory is essentially the same as the vector bundle case.

Let \mathcal{E} be a torsion free (always coherent) sheaf on a complex manifold X . We will denote by $W_{\mathcal{E}} \subset X$ the maximum Zariski open subset of X such that the restriction of \mathcal{E} to $W_{\mathcal{E}}$ is locally free. Then we have $\text{codim}(X \setminus W_{\mathcal{E}}) \geq 2$, thanks to which the following definition is meaningful.

Definition 2.4.1. (1) A *singular Hermitian metric* h on \mathcal{E} is a singular Hermitian metric on the vector bundle $\mathcal{E}|_{W_{\mathcal{E}}}$.

(2) A singular Hermitian metric h on \mathcal{E} is *negatively curved* (resp. *a.e. negatively curved*), if $h|_{W_{\mathcal{E}}}$ is negatively curved (resp. a.e. negatively curved) in the sense of 2.2.2.

(3) A singular Hermitian metric h on \mathcal{E} is *positively curved*, if there exists a negatively curved singular Hermitian metric g on $\mathcal{E}^*|_{W_{\mathcal{E}}}$ such that $h|_{W_{\mathcal{E}}} = (g|_{W_{\mathcal{E}}})^*$.

(4) \mathcal{E} is *negatively* (resp. *positively*) *curved*, if it admits a negatively (resp. positively) curved singular Hermitian metric. \square

In what follows, we may equally say that h has *Griffiths semi-positive/negative curvature* meaning that the sheaf in question has the property (2) or (3) in the definition above.

Remark 2.4.2. (1) If $W' \subset W_{\mathcal{E}}$ is a non-empty Zariski open and h' is a singular Hermitian metric on $\mathcal{E}|_{W'}$, then h' can be seen as a singular Hermitian metric on $\mathcal{E}|_{W_{\mathcal{E}}}$ by assigning the (arbitrary) value $h' \equiv 0$ on $W_{\mathcal{E}} \setminus W'$. In particular, there is a natural correspondence between the set of singular Hermitian metric on \mathcal{E} and the set of singular Hermitian metric on \mathcal{E}^{**} , via $\mathcal{E}|_{W_{\mathcal{E}}} \cong \mathcal{E}^{**}|_{W_{\mathcal{E}}}$.

(2) If h is a negatively curved singular Hermitian metric on \mathcal{E} , then for any open $U \subset X$ and any $v \in H^0(U, \mathcal{E})$, $\log |v|_h^2$ is psh on $U \cap W_{\mathcal{E}}$. Because of $\text{codim}(X \setminus W_{\mathcal{E}}) \geq 2$, $\log |v|_h^2$ extends as a psh function on U (compare with 2.2.2). Thus in the case under discussion, we have *unique privileged* extension of the metric h , once it is defined on the complement of a set of codimension at least two.

(3) In order to obtain a negatively curved singular Hermitian metric on \mathcal{E} , it is enough to find an a.e. negatively curved singular Hermitian metric as in 2.2.3(2). In conclusion, we see that \mathcal{E} is negatively (resp. positively) curved if and only if its associated double dual \mathcal{E}^{**} is negatively (resp. positively) curved. \square

Lemma 2.4.3. *Let h be a singular Hermitian metric on \mathcal{E} .*

(1) *Let $\mathcal{S} \subset \mathcal{E}$ be a subsheaf. Then the restriction $h_{\mathcal{S}} := h|_{\mathcal{S}}$ defines a singular Hermitian metric on \mathcal{S} . If h is a.e. negatively curved, then so is $h_{\mathcal{S}}$. In particular \mathcal{S} is negatively curved if \mathcal{E} is, by 2.4.2 (3).*

(2) *Let $\mathcal{E} \rightarrow \mathcal{Q}$ be a quotient torsion free sheaf. Suppose that h is positively curved. Then \mathcal{Q} has a naturally induced singular Hermitian metric $h_{\mathcal{Q}}$ with positive curvature.*

(3) Let \mathcal{F} be a torsion free sheaf, and suppose there exists a sheaf homomorphism $a : \mathcal{E} \rightarrow \mathcal{F}$ which is generically surjective. Suppose that h is positively curved. Then \mathcal{F} has a naturally induced singular Hermitian metric $h_{\mathcal{F}}$ with positive curvature.

Proof. (1) By restricting everything on the maximum Zariski open subset where \mathcal{S} is locally free, we may assume \mathcal{S} is locally free. Let W be a Zariski open subset such that \mathcal{E} is locally free and \mathcal{S} is a subbundle of \mathcal{E} . Then $h_{\mathcal{S}} = h|_{\mathcal{S}}$ defines a singular Hermitian metric on \mathcal{S} over W , and hence induces a singular Hermitian metric on \mathcal{S} over X by 2.4.2. Let U be an open subset and $v \in H^0(U, \mathcal{S}) \subset H^0(U, \mathcal{E})$. Then on $U \cap W$, $|v|_{h_{\mathcal{S}}}^2 = |v|_h^2$ (in the right hand side, $|v|_h^2$ is measured as local section of the sheaf \mathcal{E} , i.e. $v \in H^0(U, \mathcal{E})$). If h is a.e. negatively curved, then $\log |v|_h^2$ is a.e. psh on U , and hence $h_{\mathcal{S}}$ is a.e. negatively curved.

(3) By dualizing the map a , we obtain a sheaf injection $\mathcal{F}^* \rightarrow \mathcal{E}^*$. Thanks to the point (1), we obtain a singular Hermitian metric $h^*|_{\mathcal{F}^*}$ on \mathcal{F}^* with a.e. negative curvature. The procedure described in 2.4.2 allows us to modify $h^*|_{\mathcal{F}^*}$ on a Zariski closed set, such that the result is a singular Hermitian metric with negative curvature on \mathcal{F}^* . By taking the dual, we have a singular Hermitian metric $(h^*|_{\mathcal{F}^*})^*$ (not exactly same, but the discrepancies eventually occur on a measure zero set only) on \mathcal{F}^{**} with positive curvature. We then use 2.4.2 again to obtain a singular Hermitian metric on \mathcal{F} . \square

2.5. Griffiths semi-positivity and weak positivity. We show here that a torsion free sheaf endowed with a singular Finsler metric with positive curvature with a condition on the singular locus is *weakly positive in the sense of Nakayama*. To this end we will first recall this notion due to Nakayama which is a refinement of the corresponding definition due to Viehweg.

For a torsion free sheaf \mathcal{F} on a variety Y , we denote by $S^m(\mathcal{F})$ the m -th symmetric tensor product of \mathcal{F} with the convention that $S^0(\mathcal{F}) = \mathcal{O}_Y$, and let $\widehat{S}^m(\mathcal{F})$ be the double dual of the sheaf $S^m(\mathcal{F})$.

Definition 2.5.1. [35, V.3.20] Let Y be a smooth projective variety, and let \mathcal{F} be a torsion free coherent sheaf on Y . Let A be an auxiliary ample divisor on Y . Then

- (1) \mathcal{F} is *weakly positive at a point* $y \in Y$, if for any integer $a > 0$, there exists an integer $b > 0$ such that $\widehat{S}^{ab}(\mathcal{F}) \otimes \mathcal{O}_Y(bA)$ is generated by global sections at y .
- (2) \mathcal{F} is *dd-ample at a point* $y \in Y$ (“ample modulo double-duals”), if $\widehat{S}^m(\mathcal{F}) \otimes \mathcal{O}_Y(-A)$ is generated by global sections at y for an integer $m > 0$.
- (3) \mathcal{F} is *weakly positive* (resp. *big*), if it is weakly positive (resp. dd-ample) at some point. \square

We note that \mathcal{F} is weakly positive over a Zariski open subset $U \subset Y$ in the sense of Viehweg [47, 2.13] implies that \mathcal{F} is weakly positive at every point $y \in U$ in the sense of Nakayama ([35, p. 200, Remark (4)]). Thus the weak positivity in the sense of Viehweg requires global generation property (1) to hold on a Zariski open subset, whereas in the version due to Nakayama the property (1) above could be verified on countable intersection of Zariski open sets only.

If \mathcal{F} is a line bundle, we set $\text{Amp}(\mathcal{F}) = \{y \in Y; \mathcal{F} \text{ is dd-ample at } y\}$. The complement $\text{NAmp}(\mathcal{F}) = Y \setminus \text{Amp}(\mathcal{F})$ is commonly denoted by $\mathbf{B}_+(\mathcal{F})$ and called the *augmented base locus* of \mathcal{F} ([17, §1]).

The following is a generalization of 2.3.5, and its proof goes along the same lines. So we leave it to the interested reader (see 2.5.3 for a more general form of this result).

Theorem 2.5.2. *Let Y be a smooth projective variety, and let \mathcal{E} be a torsion free sheaf on Y . Suppose that \mathcal{E} admits a singular Hermitian metric h with positive curvature. Then \mathcal{E} is weakly positive at y (in the sense of Nakayama 2.5.1), for every $y \in Y$ where \mathcal{E} is locally free around y and $\det h(y) < +\infty$.*

The rest of this subsection is devoted to the proof the following result.

Theorem 2.5.3. *Let Y be a smooth projective variety, and let \mathcal{F} be a torsion free coherent sheaf on Y . Let $\mathbb{P}(\mathcal{F}) = \text{Proj}(\bigoplus_{m \geq 0} S^m(\mathcal{F}))$ be a scheme over Y associated to \mathcal{F} , say $\pi : \mathbb{P}(\mathcal{F}) \rightarrow Y$, and let $\mathcal{O}_{\mathcal{F}}(1)$ be the tautological line bundle on $\mathbb{P}(\mathcal{F})$. Let $Y_1 \subset Y$ be a Zariski open subset on which \mathcal{F} is locally free (in particular $\mathbb{P}(\mathcal{F})$ is smooth over Y_1) and $\text{codim}_Y(Y \setminus Y_1) \geq 2$. Suppose that $\mathcal{O}_{\mathcal{F}}(1)|_{\pi^{-1}(Y_1)}$ admits a singular Hermitian metric g with semi-positive curvature, and that there exists a point $y \in Y_1$ such that $\mathcal{I}(g^k|_{\mathbb{P}(\mathcal{F}_y)}) = \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}$ for any $k > 0$, where $\mathbb{P}(\mathcal{F}_y) = \pi^{-1}(y)$. Then*

- (1) \mathcal{F} is weakly positive at y .
- (2) Assume moreover that there exists an open neighborhood W of y and a Kähler form η on W such that $\sqrt{-1}\Theta_g - \pi^*\eta \geq 0$ on $\pi^{-1}(W)$, then \mathcal{F} is dd -ample at y . In particular \mathcal{F} is big.

As a preliminary discussion, we recall some facts due to [35, V.3.23]. Let $\pi : \mathbb{P}(\mathcal{F}) \rightarrow Y$ be the scheme associated to \mathcal{F} , together with its tautological line bundle $\mathcal{O}_{\mathcal{F}}(1)$. Let $Y_2 \subset Y$ be the maximum Zariski open subset of Y such that the restriction $\mathcal{F}|_{Y_2}$ is locally free; we have $W \subset Y_1 \subset Y_2$. Let $\mathbb{P}'(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{F})$ be the normalization of the component of $\mathbb{P}(\mathcal{F})$ containing $\pi^{-1}(Y_2)$, and let $X \rightarrow \mathbb{P}'(\mathcal{F})$ be a birational morphism from a smooth projective variety that is an isomorphism over $\pi^{-1}(Y_2)$. Let $\mu : X \rightarrow \mathbb{P}'(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{F})$ be the composition, and let $f = \pi \circ \mu : X \rightarrow Y$. We take a morphism $X \rightarrow \mathbb{P}'(\mathcal{F})$ so that $X \setminus f^{-1}(Y_2)$ is a divisor. We set

$$Z = \pi^{-1}(Y_1), \quad X_1 = \mu^{-1}(Z) = f^{-1}(Y_1)$$

and may regard $Z = X_1 \subset X$ by μ .

$$\begin{array}{ccccc} X & \xrightarrow{\mu} & \mathbb{P}(\mathcal{F}) & \supset & Z \\ f \downarrow & & \downarrow \pi & & \downarrow \\ Y & \xlongequal{\quad} & Y & \supset & Y_1 \end{array}$$

Denote by $L_0 = \mu^*\mathcal{O}_{\mathcal{F}}(1)$ a line bundle (or a divisor) on X . There exists a natural inclusion $\mathcal{F} \rightarrow f_*L_0$, which is isomorphic over Y_2 . By [35, III.5.10(3)], there exists an f -exceptional effective divisor E such that $f_*\mathcal{O}_X(m(L_0 + E)) \cong \widehat{S}^m(\mathcal{F})$ for any integer $m > 0$. We fix such a divisor E and let

$$L = L_0 + E.$$

Then 2.5.3 is essentially reduced to the following

Theorem 2.5.4. *In the situation above and under the assumptions in 2.5.3 (2), the line bundle L is big and $f^{-1}(y) \subset \text{Amp}(L)$.*

Proof. The rough outline of our argument goes as follows. By the main result of [17], it is enough to prove that we have

$$\liminf_{\varepsilon \rightarrow 0} \text{vol}_{X|V}(L + \varepsilon A) > 0$$

for any positive-dimensional analytic set V containing a point $x \in f^{-1}(y)$. If the metric g would be known to be globally defined and semi-positively curved on the whole space X , this can be shown to hold true by a simple application of the holomorphic Morse inequalities [13, §8]. However, in our current situation g is only defined on X_1 , so the existence of sections allowing us to show the inequality above will be shown in a direct manner, by the L^2 theory. We start by a few general reduction steps.

(1) Let us take a point $x \in X$. Assume that there exist an integer $\ell_x > 0$ and a (not necessarily ample) line bundle A on X such that the evaluation map to the jet space at x ;

$$H^0(X, \mathcal{O}_X(k\ell_x L + A)) \longrightarrow \mathcal{O}_X(k\ell_x L + A) \otimes \mathcal{O}_X/\mathfrak{m}_{X,x}^{k+1}$$

is surjective for any integer $k > 0$. Here in general $\mathfrak{m}_{X,x}$ stands for the maximal ideal of a point x on a variety X . We then can show that $x \in \text{Amp}(L)$ as follows.

Due to the jet separation property, for any integer $k > 0$ and any positive dimensional subvariety $V \subset X$ containing x , we have $\text{vol}_{X|V}(k\ell_x L + A) \geq k^d$, where $\text{vol}_{X|V}(k\ell_x L + A)$ is the restricted volume and $d = \dim V > 0$ (by [17, 4.1] for example). Thus

$$\text{vol}_{X|V}(L + \frac{1}{k\ell_x} A) \geq 1/\ell_x^d$$

holds (we remark here that ℓ_x is independent of V).

Assume on the contrary that $x \in \text{NAmp}(L)$. Since $\text{NAmp}(L)$ has no isolated points ([17, 1.1]), there exists a subvariety $V \subset X$ of $d := \dim V > 0$, which contains x and is an irreducible component of $\text{NAmp}(L)$. By the preceding argument, we infer that we have $\liminf_{k \rightarrow \infty} \text{vol}_{X|V}(L + \frac{1}{k\ell_x} A) \geq 1/\ell_x^d > 0$. This contradicts to [17, 5.7].

(2) We denote by $\mathbb{P}(\mathcal{F}_y) = \pi^{-1}(y) \subset Z$, $X_y = f^{-1}(y) \subset X$, and the ideal sheaves $\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}$ and \mathcal{I}_{X_y} respectively. We will show in (3) below that there exist an integer $\ell_y > 0$ and a line bundle A on Y such that the restriction map

$$(*) \quad J_y^k : H^0(Z, \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A) \longrightarrow H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A \otimes \mathcal{O}_Z/\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1})$$

is surjective for any integer $k > 0$. Note that the fibration $\pi : Z \rightarrow Y_1$ is geometrically simple enough so that we have an isomorphism

$$H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A \otimes \mathcal{O}_Z/\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1}) \cong H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}(k\ell_y)) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_Y/\mathfrak{m}_{Y,y}^{k+1},$$

and a similar isomorphism for $f : X_1 \rightarrow Y_1$. Taking the surjection $(*)$ for granted, we show that $f^{-1}(y) = X_y \subset \text{Amp}(L)$ as follows.

We first note that every section of $\mathcal{O}_X(k\ell_y L) \otimes f^* A$ over $X_1 (= Z)$ extends to a section of $\mathcal{O}_X(k\ell_y L) \otimes f^* A$ over X , because $H^0(Z, \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A) \cong H^0(Y_1, S^{k\ell_y}(\mathcal{F}) \otimes A) \hookrightarrow H^0(Y, \widehat{S}^{k\ell_y}(\mathcal{F}) \otimes A) \cong H^0(Y, f_*(\mathcal{O}_X(k\ell_y L)) \otimes A) \cong H^0(X, \mathcal{O}_X(k\ell_y L) \otimes f^* A)$. Here we need to explain the inclusion and the next isomorphism. Since $\widehat{S}^{k\ell_y}(\mathcal{F})$ is reflexive on Y and locally free on Y_1 of $\text{codim}_Y(Y \setminus Y_1) \geq 2$, we have a natural inclusion $H^0(Y_1, S^{k\ell_y}(\mathcal{F}) \otimes A) \hookrightarrow H^0(Y, \widehat{S}^{k\ell_y}(\mathcal{F}) \otimes A)$. The isomorphisms $f_*(\mathcal{O}_X(mL)) \cong \widehat{S}^m(\mathcal{F})$ for any $m > 0$ ([35, III.5.10(3)] mentioned above) induce $H^0(Y, \widehat{S}^{k\ell_y}(\mathcal{F}) \otimes A) \cong H^0(Y, f_*(\mathcal{O}_X(k\ell_y L)) \otimes A)$.

Thus, from (*), the restriction map

$$H^0(X, \mathcal{O}_X(k\ell_y L) \otimes f^* A) \longrightarrow H^0(X_y, \mathcal{O}_X(k\ell_y L) \otimes f^* A \otimes \mathcal{O}_X/\mathcal{I}_{X_y}^{k+1})$$

is surjective for any $k > 0$. We take an arbitrary point $x \in X_y$ and show that $x \in \text{Amp}(L)$. Noting $L|_{X_y} \cong \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}(1)$, the restriction map $H^0(X_y, \mathcal{O}_X(k\ell_y L)) \rightarrow \mathcal{O}_{X_y}/\mathfrak{m}_{X_y, x}^{k+1}$ is surjective for any $k > 0$. Thus, as a combination of the above two types of surjections, the restriction map

$$H^0(X, \mathcal{O}_X(k\ell_y L) \otimes f^* A) \longrightarrow \mathcal{O}_X(k\ell_y L) \otimes f^* A \otimes \mathcal{O}_X/\mathfrak{m}_{X, x}^{k+1}$$

is surjective for any $k > 0$. Then by (1), we have $x \in \text{Amp}(L)$.

(3) Let us prove the surjectivity of (*) in (2). Denote by $m = \dim Y$. We take a local coordinate $t = (t_1, \dots, t_m)$ centered at y , and let $|t| = (\sum_{i=1}^m |t_i|^2)^{1/2}$. We take a C^∞ cut-off function $\rho : Y \rightarrow \mathbb{R}$ such that $\rho \equiv 1$ on a neighborhood of y and $\text{supp } \rho \subset W$. We let a function $\phi := (m+1)\rho \log |t|^2 : Y \rightarrow [-\infty, \infty)$. Comparing with the Kähler form η , we can find an integer $\ell_y > 0$ such that $\ell_y \eta + \sqrt{-1} \partial \bar{\partial} \phi > 0$ on W . We then compare with the curvature current $\sqrt{-1} \Theta_g$ of $\mathcal{O}_{\mathcal{F}}(1)|_Z$. Since $\sqrt{-1} \Theta_g \geq \pi^* \eta$ on $\pi^{-1}(W)$, we can also see that $\ell_y \sqrt{-1} \Theta_g + \sqrt{-1} \partial \bar{\partial} \pi^* \phi$ is a semi-positive current on Z . We take an ample line bundle A on Y such that $A \otimes K_Y^{-1} \otimes (\widehat{\det} \mathcal{F})^{-1}$ is ample, where $\widehat{\det} \mathcal{F}$ is the double dual of $\bigwedge^r \mathcal{F}$ and r is the rank of \mathcal{F} . Let h_{K_Y} (resp. $h_{\widehat{\det} \mathcal{F}}$) be a smooth Hermitian metric on K_Y (resp. $\widehat{\det} \mathcal{F}$), and let h_A be a smooth Hermitian metric on A such that $h_A h_{K_Y}^{-1} h_{\widehat{\det} \mathcal{F}}^{-1}$ has positive curvature.

We now take any integer $k > 0$. We consider the following line bundle on Z :

$$M(= M_k) := \mathcal{O}_{\mathcal{F}}(k\ell_y + r)|_Z \otimes \pi^*(A \otimes K_Y^{-1} \otimes (\widehat{\det} \mathcal{F})^{-1})|_Z.$$

We note $\mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A = K_Z \otimes M$ on Z . The line bundle M has a singular Hermitian metric

$$g_M := (g^{\ell_y} \pi^* e^{-\phi})^k \cdot g^r \cdot \pi^*(h_A h_{K_Y}^{-1} h_{\widehat{\det} \mathcal{F}}^{-1})$$

of semi-positive curvature. We note that

$$\sqrt{-1} \Theta_{g_M} \geq r \sqrt{-1} \Theta_g \geq \pi^* \eta \text{ on } \pi^{-1}(W),$$

which is a strict positivity for the normal direction of $\mathbb{P}(\mathcal{F}_y)$ in Z . We also note that we can take an open ball neighborhood $U \subset W$ of y such that $\mathcal{I}(g^{\ell_y k+r}) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}$ on $\pi^{-1}(U)$, in particular

$$\mathcal{I}(g_M) = \mathcal{O}_Z \text{ on } \pi^{-1}(U \setminus \{y\})$$

(as ℓ_y and r are given, U depends only on k). This follows from our assumption $\mathcal{I}(g^{\ell_y k+r}|_{\mathbb{P}(\mathcal{F}_y)}) = \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}$ and the restriction theorem of multiplier ideals, namely Ohsawa-Takegoshi's extension. We also have

$$\mathcal{I}(g_M) \subset \mathcal{I}(e^{-k(m+1)\pi^* \log |t|^2}) = \mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k(m+1)-m+1} \subset \mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1} \text{ on } \pi^{-1}(U).$$

We take a section $\sigma \in H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A \otimes \mathcal{O}_Z/\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1})$. Noting

$$H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A \otimes \mathcal{O}_Z/\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1}) \cong H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}(k\ell_y)) \otimes_{\mathcal{O}_{Y, y}} \mathcal{O}_Y/\mathfrak{m}_{Y, y}^{k+1},$$

we have a section $\sigma' \in H^0(\mathbb{P}(\mathcal{F}|_U), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A)$, which is mapped to σ by the restriction map

$$J_y^k : H^0(\mathbb{P}(\mathcal{F}|_U), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A) \longrightarrow H^0(\mathbb{P}(\mathcal{F}_y), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^* A \otimes \mathcal{O}_Z/\mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1}).$$

The map J_y^k is defined for sections over $\mathbb{P}(\mathcal{F}|_{Y'})$ for any open $Y' \subset Y$ containing y . We use the same notation J_y^k for any open $Y' \subset Y$. We take a C^∞ cut-off function λ such that $\lambda \equiv 1$ around y , say on $U_1 \subset U$, and $\text{supp } \lambda \subset U$. We consider

$$u := \bar{\partial}((\pi^*\lambda)\sigma') = \bar{\partial}(\pi^*\lambda) \wedge \sigma',$$

which can be seen as an M -valued $(n, 1)$ -form on Z via $(\mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^*A)|_Z = K_Z \otimes M$, where $n := \dim Z = m + r - 1$. Thanks to the observation on the multiplier ideal $\mathcal{I}(g_M)$ on $\pi^{-1}(U)$, we see that u is square integrable with respect to g_M and any given complete Kähler metric on Z . (Recall that $Z \cong X_1$ is a Zariski open subset of a smooth projective variety X , in particular Z admits a complete Kähler metric, refer [10, 0.2] if one needs. The quasi-projectivity of Z will be more important for a regularization process.) Then by [10, 5.1], we can solve a $\bar{\partial}$ -equation $\bar{\partial}v = u$ for some $v \in L_{2,loc}^{n,0}(Z, M)$ (an M -valued $(n, 0)$ -form on Z with locally square integrable functions coefficient) and $c_n v \wedge \bar{v} g_M \in L_{loc}^1(Z, \mathbb{R})$ with $c_n = (-1)^{n^2/2}$. Since $u \equiv 0$ on $\pi^{-1}(U_1)$, v is holomorphic on $\pi^{-1}(U_1)$. The integrability $c_n v \wedge \bar{v} g_M \in L_{loc}^1(Z, \mathbb{R})$, in particular $c_n v \wedge \bar{v} e^{-k\pi^*\phi} \in L_{loc}^1(Z, \mathbb{R})$, implies $v|_{\pi^{-1}(U_1)} \in H^0(\mathbb{P}(\mathcal{F}|_{U_1}), \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^*A \otimes \mathcal{I}_{\mathbb{P}(\mathcal{F}_y)}^{k+1})$. Then $\tilde{\sigma} := (\pi^*\lambda)\sigma' - v \in H^0(Z, K_Z \otimes M) = H^0(Z, \mathcal{O}_{\mathcal{F}}(k\ell_y) \otimes \pi^*A)$ and $J_y^k(\tilde{\sigma}) = J_y^k(\sigma') = \sigma$. This proves the surjectivity of $(*)$ in (2). \square

Proof of 2.5.3. We first prove (2). By 2.5.4 and [35, V.3.23], our \mathcal{F} in (2) is dd-ample at y . Then by definition, \mathcal{F} is big.

(1) Let g be a singular Hermitian metric on $\mathcal{O}_{\mathcal{F}}(1)|_{\pi^{-1}(Y_1)}$ in the statement. Let H be an ample line bundle on Y with a Hermitian metric h_H with positive curvature. Let $a > 0$ be an arbitrary integer. By applying the preceding argument in the proof of 2.5.4 for a line bundle $\mathcal{O}_{\mathcal{F}}(a) \otimes \pi^*H$ over $\pi^{-1}(Y_1)$ with a metric $g^a \cdot \pi^*h_H$ instead of $\mathcal{O}_{\mathcal{F}}(1)|_{\pi^{-1}(Y_1)}$ with g , we see that $\widehat{S}^a(\mathcal{F}) \otimes H$ is dd-ample at y . Note that we could take a Kähler form $\eta = \sqrt{-1}\Theta_{h_H}$ on Y . Since $a > 0$ is arbitrary, \mathcal{F} is weakly positive at y ([35, p. 200, Remark (3)]). \square

3. POSITIVITY OF DIRECT IMAGE SHEAVES OF ADJOINT TYPE

3.1. The $L^{2/m}$ metric, absolute version. We recall in this subsection the construction of canonical metrics on the bundle $mK_X + L$ and the vector space $H^0(X, mK_X + L)$, due to Narashimhan-Simha's ([36, §2]). We will deal with the relative version of this metric in the next subsection.

3.1.1. Let X be a compact complex manifold of $\dim X = n$, and L be a holomorphic line bundle on X with a singular Hermitian metric h with semi-positive curvature.

(1) For every $u \in H^0(X, mK_X + L)$, $(cu \wedge \bar{u}h)^{1/m}$ is a real semi-positive $2n$ -form on X with measurable function coefficients, where $c = c_{n,m} = ((-1)^{n^2/2}(-1)^n)^{-m}$. It is often written as $|u|^{2/m}h^{1/m} = (cu \wedge \bar{u}h)^{1/m}$. We set

$$\|u\|_m = \left(\int_X (cu \wedge \bar{u}h)^{1/m} \right)^{m/2}.$$

This is called an $L^{2/m}$ -pseudo-norm on $H^0(X, mK_X + L)$ (note that $\|u\|_p$ is not an L^p -norm). It is a norm (and in fact a metric) only when $m = 1$. In this case we will refer to $\|\cdot\|_1$ as the *canonical L^2 -metric*, and set $g_1(u, v) = \int_X c_n u \wedge \bar{v}h$ the corresponding pairing, where $c_n = c_{n,1} = (-1)^{n^2/2}$. To be more precise, this is defined on a subspace

$$V_m = \{u \in H^0(X, mK_X + L); \|u\|_m < \infty\}.$$

By definition of $\|u\|_m$, if $h^{1/m} \in L_{loc}^1$, then $V_m = H^0(X, mK_X + L)$. We will remark in §4.2 that there exists a coherent ideal sheaf $\mathcal{J}_m(h^{1/m})$, called an $L^{2/m}$ -multiplier ideal sheaf, such that $V_m = H^0(X, (mK_X + L) \otimes \mathcal{J}_m(h^{1/m}))$. Even if our notation does not reflect it, this space depends on h .

(2) We consider a C^∞ volume form dV on X (i.e. a C^∞ Hermitian metric on $-K_X$) and a C^∞ Hermitian metric h_0 on L as reference metrics. Then the quantity $cu \wedge \bar{u}h_0 / (dV)^m$ represents a C^∞ semi-positive function on X . For every $x \in X$, we set

$$B'_m(x) = \sup \left\{ \frac{cu \wedge \bar{u}h_0}{(dV)^m}(x); u \in H^0(X, mK_X + L), \|u\|_m \leq 1 \right\}.$$

We obtain a function B'_m on X , and set $B_m = B'_m(dV)^m h_0^{-1}$. Here B'_m depends on dV and h_0 , but B_m does not. Because of this reason, we often denote by

$$B_m(x) = \sup \{cu \wedge \bar{u}(x); u \in H^0(X, mK_X + L), \|u\|_m \leq 1\}.$$

This B_m defines a singular Hermitian metric on the dual $(mK_X + L)^*$, and is called the (*twisted*) *canonical $L^{2/m}$ metric, or m -th Bergman kernel metric*. For every $x \in X$ and a vector $\xi_x \in (mK_X + L)_x^*$, the length with respect to the metric B_m is given by

$$|\xi_x|_{B_m} := \sup \{|\xi_x \cdot u(x)|; u \in H^0(X, mK_X + L), \|u\|_m \leq 1\},$$

where $\xi_x \cdot u(x)$ is the duality pairing between $(mK_X + L)_x^*$ and $(mK_X + L)_x$. The dual metric

$$h^{(m)} := B_m^{-1}$$

is a singular Hermitian metric on $mK_X + L$, and it is clear that the curvature current of $h^{(m)}$ is semi-positive.

(3) We define the line bundle

$$L_{m-1} = (m-1)K_X + L = \frac{m-1}{m}(mK_X + L) + \frac{1}{m}L$$

and a singular Hermitian metric

$$h_{m-1} = (B_m^{-1})^{(m-1)/m} h^{1/m}.$$

The construction above induces a Hermitian form g_{mNS} (g_m for short slightly) on the vector space $H^0(X, mK_X + L)$ as follows

$$g_{mNS}(u, v) = \int_X c_n u \wedge \bar{v} h_{m-1},$$

which is nothing but the canonical L^2 -metric on $H^0(X, K_X + L_{m-1})$ with respect to h_{m-1} as in (1). This Hermitian form is defined on $H^0(X, (K_X + L_{m-1}) \otimes \mathcal{I}(h_{m-1}))$. In the case $m = 1$, g_{1NS} is nothing but the canonical L^2 -metric g_1 with respect to $h_0 = h$, i.e. $\|\cdot\|_1$ in (1). We may call the Hermitian form g_{mNS} on $H^0(X, mK_X + L)$ as the *(twisted) m -th Narashimhan-Simha Hermitian form*. In a relative situation $f : X \rightarrow Y$, a family of fiberwise $h^{(m)}$ (resp. of g_{mNS}) will define a metric on $mK_{X/Y} + L$ (resp. on $f_*(mK_{X/Y} + L)$).

(4) We note that, although h_{m-1} is “more singular” than $h^{1/m}$, the inclusion

$$H^0(X, (mK_X + L) \otimes \mathcal{I}(h)) \subset V_m \subset H^0(X, (K_X + L_{m-1}) \otimes \mathcal{I}(h_{m-1}))$$

holds true for any integer $m > 0$. For the left inclusion, $|u|^2 h \in L_{loc}^1$ for $u \in H^0(X, mK_X + L)$ implies $|u|^{2/m} h^{1/m} \in L_{loc}^1$ by Hölder’s inequality. For the right inclusion, it is enough to see that $c_n u \wedge \bar{v} h_{m-1} \in L_{loc}^1$ after a local trivialization, where $u \in V_m \subset H^0(X, mK_X + L)$. By definition of B_m^{-1} ,

$$c_n u \wedge \bar{v} h_{m-1} = c_n u \wedge \bar{v} (B_m^{-1})^{(m-1)/m} h^{1/m} \leq C_u (c_{n,m} u \wedge \bar{v} h)^{1/m},$$

where C_u is a positive constant depends only on u . The last term is in L_{loc}^1 for $u \in V_m$.

In particular, if $\mathcal{I}(h^{1/m}) = \mathcal{O}_X$, then $V_m = H^0(X, mK_X + L)$ and hence $V_m = H^0(X, (K_X + L_{m-1}) \otimes \mathcal{I}(h_{m-1})) = H^0(X, mK_X + L)$.

(5) In the case $V_m = 0$ at the beginning, we formally agree that $B_m \equiv 0, h^{(m)} \equiv +\infty, h_{m-1} \equiv +\infty$ (except for $h_{1NS} = h$). \square

3.2. The $L^{2/m}$ metric, relative version.

Set up 3.2.1. Let X and Y be complex manifolds of $\dim X - \dim Y = n$, and let $f : X \rightarrow Y$ be a projective surjective morphism with connected fibers. Let L be a holomorphic line bundle on X with a singular Hermitian metric h with semi-positive curvature. We let X_y the scheme theoretic fiber of $y \in Y$, $L_y := L|_{X_y}, h_y := h|_{X_y}$ the restrictions (it can be $h_y \equiv +\infty$),

$$Y_0 \subset Y$$

the maximum Zariski open subset where f is smooth, $\Delta := Y \setminus Y_0$, and let

$$Y_h = \{y \in Y_0; h_y \not\equiv +\infty\}.$$

For every integer $m \geq 1$ and $y \in Y_h$, we can discuss the Narashimhan-Simha construction on the fiber X_y with (L_y, h_y) as in 3.1.1, and have fiberwise objects $V_{m,y}, B_{m,y}, h_y^{(m)}, h_{m-1,y}, g_{m,y} = g_{mNS,y}$ accordingly (it can be $V_{m,y} = 0$). \square

We remark that the set Y_h is Zariski dense in Y , however may not be Zariski open. The complement $Y \setminus Y_h$ is a pluripolar set and hence can be quite different from algebraic/analytic objects. For example, a pluripolar set may not be closed in Hausdorff topology, and can be Zariski dense. In any case we adopt the conventions that if $h_y \equiv +\infty$ for a point $y \in Y_0$ i.e., $y \in Y_0 \setminus Y_h$, then $V_{m,y} = 0, B_{m,y} \equiv 0, h_{m-1,y} \equiv +\infty$ and so on.

Remark 3.2.2. It is very important to understand the local expression of the metrics we consider above, as they will be needed in order to deduce their extension properties.

We give here an explicit formulation of the (*relative*) *canonical L^2 -metric* $g_{1NS,X/Y}$ ($g_{1,X/Y}$ for short slightly) on $f_*(K_{X/Y} + L)|_{Y_0}$ with respect to h in 3.2.1. For this purpose, we may suppose that Y itself is a coordinate neighborhood.

(1) Let $\eta \in H^0(Y, K_Y)$ be a nowhere vanishing section, trivializing the canonical bundle of Y ; in particular we have $K_Y = \mathcal{O}_Y \eta$. We recall that we have

$$H^0(Y, f_*(K_{X/Y} + L)) = H^0(Y, \mathcal{H}om(K_Y, f_*(K_X + L))),$$

and therefore every section $u \in H^0(Y, f_*(K_{X/Y} + L))$ corresponds to an \mathcal{O}_Y -homomorphism $u : K_Y \rightarrow f_*(K_X + L)$. We still use the same symbol u for the induced homomorphism $u : H^0(Y, K_Y) \rightarrow H^0(Y, f_*(K_X + L)) = H^0(X, K_X + L)$. Thus we can write $u(\eta) \in H^0(X, K_X + L)$. Let $\{U_\lambda\}_\lambda$ be a local coordinate system of X . Regarding $u(\eta)$ as an L -valued top-degree holomorphic form on each U_λ , there exists $\sigma_{u\lambda} \in H^0(U_\lambda \setminus f^{-1}(\Delta), \Omega_X^n \otimes L)$ such that we have

$$u(\eta) = \sigma_{u\lambda} \wedge f^* \eta$$

on $U_\lambda \setminus f^{-1}(\Delta)$ i.e. we can “divide” $u(\eta)$ by $f^* \eta$ where $f^* \eta$ has no zeros. The choice of $\sigma_{u\lambda}$ is not unique (the ambiguity lies in the image of $\Omega_X^{n-1} \otimes f^* \Omega_Y^1$), however the restriction $\sigma_{u\lambda}|_{X_y} \in H^0(U_\lambda \cap X_y, K_{X_y} + L_y)$ on each smooth fiber X_y ($y \in Y_0$) is unique and independent of the local frame η . Finally the collection $\{\sigma_{u\lambda}\}_\lambda$, resp. $\{\sigma_{u\lambda}|_{X_y}\}_\lambda$, glue together as a global section

$$\sigma_u \in H^0(X \setminus f^{-1}(\Delta), K_{X/Y} + L), \quad \sigma_{u_y} \in H^0(X_y, K_{X_y} + L_y)$$

respectively; moreover, the later is the restriction of the former $\sigma_{u_y} = \sigma_u|_{X_y}$ for $y \in Y_0$. This construction independent of the reference metrics.

(2) Then the canonical L^2 -metric $g_{1NS,X/Y}$ for $u, v \in H^0(Y, f_*(K_{X/Y} + L))$ at $y \in Y_0 = Y \setminus \Delta$ is given by the expression

$$g_{1NS,X/Y}(u, v)(y) = \int_{X_y} c_n \sigma_u|_{X_y} \wedge \bar{\sigma}_v|_{X_y} h_y.$$

We may write the integrand as $(c_n \sigma_u \wedge \bar{\sigma}_v h)|_{X_y}$ by an abuse of notations, where $c_n = (-1)^{n^2/2}$. The convergence of the last integral depends on u and v . There may be other ways to express $g_{1NS,X/Y}$, for example $g_{1NS,X/Y} = \{g_{1NS,X/Y,y}\}_{y \in Y_0}$ as a family of fiberwise Hermitian forms, where $g_{1NS,X/Y,y}$ is the restriction of $g_{1NS,y}$ on the image of $f_*(K_{X/Y} + L)_y$ in $H^0(X_y, K_{X_y} + L_y)$ (the latter space can be larger in general), and where $g_{1NS,y}$ is the canonical L^2 -metric on $H^0(X_y, K_{X_y} + L_y)$ with respect to h_y (it can be $g_{1NS,y}(u, u) = +\infty$ for some $u \in H^0(X_y, K_{X_y} + L_y)$). \square

Remark 3.2.3. As a consequence of Ohsawa-Takegoshi's extension theorem, we have the inclusion $\mathcal{I}(h_y) \subset \mathcal{I}(h) \cdot \mathcal{O}_{X_y}$ for any $y \in Y_0$. The next set we will be interested in would be $y \in Y_0$ such that

$$\mathcal{I}(h_y) = \mathcal{I}(h) \cdot \mathcal{O}_{X_y}.$$

In the algebraic case, this holds for any y in a Zariski open subset ([31, 9.5.35]). We show here that this equality holds on a set of full measure on Y (which in general is not Zariski open). To this end, it is enough to show that $\mathcal{I}(h) \cdot \mathcal{O}_{X_y} \subset \mathcal{I}(h_y)$ for a.a. $y \in Y$. We may suppose that Y is a small coordinate neighborhood. Let $U \subset X$ be a local coordinate set, which is isomorphic to a polydisk, such that $f|_U$ is (conjugate to) the projection to a sub-polydisk. We may assume that $\mathcal{I}(h)|_U$ is generated by a finite number of holomorphic functions $s_1, \dots, s_k \in H^0(U, \mathcal{O}_X)$, in particular $|s_i|^2 h \in L_{loc}^1(U)$. As these $s_i|_{X_y}$ generate $\mathcal{I}(h) \cdot \mathcal{O}_{X_y \cap U}$, it is enough to show that each $s_i|_{X_y} \in \mathcal{I}(h_y)|_{X_y \cap U}$ for a.a. $y \in f(U)$. By Fubini's theorem, $|s_i|_{X_y}|^2 h_y \in L_{loc}^1(X_y \cap U)$ for a.a. $y \in f(U)$, i.e., $s_i|_{X_y} \in \mathcal{I}(h_y)|_{X_y \cap U}$ for a.a. $y \in f(U)$.

Remark 3.2.4. We note that $f_*(K_{X/Y} + L)$ on Y and the canonical L^2 -metric $g_1 = g_{1NS, X/Y}$ on Y_0 are invariant under bimeromorphic transform of X . Let $\mu : X' \rightarrow X$ be a proper bimeromorphic morphism from a complex manifold X' , $f' = f \circ \mu : X' \rightarrow Y$, and let $L' = \mu^*L$, $h' = \mu^*h$. Then it is easy to see $f'_*(K_{X'/Y} + L') = f_*(K_{X/Y} + L)$. Suppose that $f' : X' \rightarrow Y$ is smooth over $Y'_0 \subset Y$, and let $g'_1 = g_{1NS, X'/Y}$ be the canonical L^2 -metric with respect to h' on $f'_*(K_{X'/Y} + L')|_{Y'_0}$. Then $g_1 = g'_1$ on $Y_0 \cap Y'_0$ (one should observe this in the case Y is a point, and then the relative case). As a result g_1 and g'_1 define a singular Hermitian metric of $f_*(K_{X/Y} + L)|_{Y_0 \cup Y'_0}$. (We do not know if $Y'_0 \subset Y_0$ holds.)

3.3. Direct image sheaves of adjoint type and their metric properties. In this subsection we show that the direct image sheaves $f_*(K_{X/Y} + L)$ can be endowed with a singular Hermitian metric with semi-positive curvature in the sense we have defined in 2.3.1. We remark that in general, the singularities of the metric we construct cannot be avoided. The main result we obtain here can be seen as a more general and precise form of [3, 3.5], and it implies various notions of weak positivity in algebraic geometry. The set-up and notations in what follows are the same as in 3.2.1.

3.3.1. To start with, we define the following sets, which play an important role for the understanding of the metrics we will define next. We recall that $Y_0 \subset Y$ is the set of regular values of f , and that $Y_h = \{y \in Y_0; h_y = h|_{X_y} \not\equiv +\infty\}$.

$$Y_{1,h,\text{ext}} := \{y \in Y_h; H^0(X_y, (K_{X_y} + L_y) \otimes \mathcal{I}(h_y)) = H^0(X_y, K_{X_y} + L_y)\},$$

$$Y_{1,\text{ext}} := \{y \in Y_0; h^0(X_y, K_{X_y} + L_y) \text{ equals to the rank of } f_*(K_{X/Y} + L)\}.$$

$$Y_{1,\text{free}} := \text{The largest Zariski open subset such that } f_*(K_{X/Y} + L) \text{ is locally free.}$$

We remark that the set $Y_{1,\text{ext}}$ and $Y_{1,\text{free}}$ are independent of the metric h and $Y_{1,\text{ext}} \subset Y_{1,\text{free}}$. We have then the following observation.

Lemma 3.3.2. (1) *We have the inclusion $Y_{1,h,\text{ext}} \subset Y_{1,\text{ext}} \cap Y_h$.*

(2) *Let $y \in Y_{1,h,\text{ext}}$. Then the equalities*

$$f_*(K_{X/Y} + L)_y = H^0(X_y, K_{X_y} + L_y) = H^0(X_y, (K_{X_y} + L_y) \otimes \mathcal{I}(h_y))$$

hold. As a consequence of 3.2.2(2), the restriction of the direct image $f_*(K_{X/Y} + L)$ to the set $Y_{1,h,\text{ext}}$ can be endowed with the canonical L^2 -metric $g_{1NS,y}$ with respect to h_y . Moreover $f_*(K_{X/Y} + L)$ is locally free at y , and the natural inclusion $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is isomorphic at y . Moreover, the inclusion $Y_{1,h,\text{ext}} \subset Y_{1,\text{ext}}$ holds.

(3) If the natural inclusion $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is generically isomorphic, then $Y_{1,h,\text{ext}}$ is not empty and $Y \setminus Y_{1,h,\text{ext}}$ has measure zero.

(4) In conclusion, the sheaf $f_*(K_{X/Y} + L)|_{Y_{1,\text{ext}}}$ can be endowed with the canonical L^2 -metric $\{g_{1NS,y}\}_{y \in Y_{1,\text{ext}}}$; however, we note that we have $\det g_{1NS,y} = +\infty$ if $y \in Y_{1,\text{ext}} \setminus Y_{1,h,\text{ext}}$.

Proof. The first point (1) will be established at the end of the argument for we give next for (2).

(2) By Ohsawa-Takegoshi, every section $u \in H^0(X_y, K_{X_y} + L_y)$ admits an extension

$$\tilde{u} \in H^0(X_W, (K_{X/Y} + L) \otimes \mathcal{I}(h))$$

defined on some neighborhood X_W of X_y . In particular the natural induced map

$$f_*((K_{X/Y} + L) \otimes \mathcal{I}(h))_y \rightarrow H^0(X_y, K_{X_y} + L_y)$$

is surjective (we note that the left-hand side direct image above is contained in $f_*(K_{X/Y} + L)_y$). The cohomology base change theorem implies that $f_*(K_{X/Y} + L)$ is locally free at y , and as a consequence the natural inclusion $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is isomorphic at y . This argument also shows (1).

(3) There exists a non-empty Zariski open subset $W \subset Y$ such that the inclusion map

$$f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$$

is isomorphic on W , and such that

$f_*((K_{X/Y} + L) \otimes \mathcal{I}(h))_y = H^0(X_y, (K_{X_y} + L_y) \otimes \mathcal{I}(h) \cdot \mathcal{O}_{X_y})$, $f_*(K_{X/Y} + L)_y = H^0(X_y, K_{X_y} + L_y)$ for any $y \in W$. Since $\mathcal{I}(h) \cdot \mathcal{O}_{X_y} = \mathcal{I}(h_y)$ for a.a. $y \in Y_0$ in general by 3.2.3, so our assertion follows.

(4) This point follows directly from the preceding ones. \square

We recall next the following result.

Theorem 3.3.3. [3, 3.5] *Let $f : X \rightarrow Y$ and (L, h) be as in 3.2.1. Suppose that f is smooth and $f_*(K_{X/Y} + L)$ is locally free. Then the fiberwise canonical L^2 -metrics*

$$g_{1NS,y}(u, u) := \int_{X_y} c_n u \wedge \bar{u} h_y \leq +\infty$$

for $u \in H^0(X_y, K_{X_y} + L_y)$ induces a singular Hermitian metric $g_{1NS,X/Y}$ on $f_*(K_{X/Y} + L)$ with positive curvature.

The notion of *singular Hermitian metric* h on a vector bundle E adopted in [3, p. 357] is more general than 2.3.1: one only assume that h corresponds locally to a measurable map from the base manifold to the space of semi-positive definite Hermitian matrices (hence, no assumption concerning the determinant metric). The negativity of its curvature is defined by the plurisubharmonicity of $\log |v|_h^2$ for any local section $v \in H^0(U, E)$. We note that stated

in this way, the notion of singular Hermitian metric makes not much sense, since any vector bundle admits $h \equiv 0$ as a singular Hermitian metric with negative curvature.

In our set-up in this paper, the previous result has the following consequence.

Theorem 3.3.4. [3, 3.5] *Let $f : X \rightarrow Y$ and (L, h) be as in 3.2.1. Suppose that f is smooth, $f_*(K_{X/Y} + L)$ is locally free, and that the natural inclusion*

$$f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$$

is generically isomorphic. Then $f_(K_{X/Y} + L)$ admits a singular Hermitian metric $g_{1NS, X/Y} = \{g_{1NS, X/Y, y}\}_{y \in Y}$ with positive curvature and satisfying a base change property on $Y_{1, \text{ext}}$, which means, $g_{1NS, X/Y, y} = g_{1NS, y}$ holds on every fiber $f_*(K_{X/Y} + L)_y$ at $y \in Y_{1, \text{ext}}$, where $g_{1NS, y}$ is the canonical L^2 -metric on $f_*(K_{X/Y} + L)_y = H^0(X_y, K_{X_y} + L_y)$ with respect to h_y in 3.3.2(4).*

The statement concerning the equality $g_{1NS, X/Y, y} = g_{1NS, y}$ on $f_*(K_{X/Y} + L)_y$ for $y \in Y_{1, \text{ext}}$ is slightly more informative than the original one in [3, 3.5]. We stress on the fact that the statement above is implicit in [3], it is not mentioned explicitly.

Our version of the previous result reads as follows.

Theorem 3.3.5. *Let $f : X \rightarrow Y$ and (L, h) be as in 3.2.1. We suppose that the natural inclusion $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is generically isomorphic. Then the canonical L^2 -metric $g_{1NS, X/Y}$ on $f_*(K_{X/Y} + L)|_{Y_0 \cap Y_{1, \text{free}}}$ with respect to h has positive curvature by 3.3.4, and it extends as a singular Hermitian metric $\tilde{g}_{1NS, X/Y}$ on the torsion free sheaf $f_*(K_{X/Y} + L)$ with positive curvature. In particular $\tilde{g}_{1NS, X/Y}$ has a base change property on $Y_{1, \text{ext}}$.*

The rest of this subsection is devoted to the proof of 3.3.5. We start by fixing some notations, and making some standard reductions. Let

$$E := f_*(K_{X/Y} + L), \quad \text{resp. } g := g_{1NS, X/Y},$$

be the direct image of the bundle $K_{X/Y} + L$, resp. the metric on $E|_{Y_0 \cap Y_{1, \text{free}}}$. Noting that we have $\text{codim}(Y \setminus Y_{1, \text{free}}) \geq 2$, it is enough to show that g extends to a singular Hermitian metric on $E|_{Y_{1, \text{free}}}$ with positive curvature by definition of curvature positivity of singular Hermitian metric of torsion free sheaves (see 2.4.2). Thus we may assume from the start that E is locally free, i.e., $Y_{1, \text{free}} = Y$. We also note that the extension of g is a local question on Y . Moreover, we can freely restrict ourselves on a Zariski open subset $Y'' \subset Y$ with $\text{codim}(Y \setminus Y'') \geq 2$, since the extended metric is unique.

In conclusion, it is enough to consider the following local setting, near the general point of a codimension 1 subset $\Delta \subset Y$ (possibly after taking a modification of X along $f^{-1}(\Delta)$; such operations do not affect E and the metric g , see 3.2.4).

Set up 3.3.6. Let $f : X \rightarrow Y$ and (L, h) be as in 3.2.1. In addition to the assumption in 3.3.5, let us assume further the following

(1) The base Y is a unit polydisk in \mathbb{C}^m with coordinates $t = (t_1, \dots, t_m)$. Let $dt = dt_1 \wedge \dots \wedge dt_m \in H^0(Y, K_Y)$ be a global frame of K_Y .

(1.i) f is flat, and its discriminant locus $\Delta \subset Y$ is given by the equation $\Delta = \{t_m = 0\}$ (or $\Delta = \emptyset$), $Y_0 = Y \setminus \Delta$,

(1.ii) the effective divisor $f^*\Delta$ has a normal crossing support, and

- (1.iii) the morphism $\text{Supp } f^*\Delta \rightarrow \Delta$ is relative normal crossing (see 3.3.7).
- (2) (2.i) $E = f_*(K_{X/Y} + L) \cong \mathcal{O}_Y^{\oplus r}$, i.e., globally free and trivialized of rank $r > 0$.
- (2.ii) For every point $y \in Y$, the naturally induced homomorphism $E/(\mathfrak{m}_{Y,y}E) \rightarrow H^0(X_y, (K_{X/Y} + L) \otimes \mathcal{O}_{X_y})$ is injective, where $\mathfrak{m}_{Y,y} \subset \mathcal{O}_Y$ is the maximal ideal and X_y is the scheme theoretic fiber of f ; X_y is defined by the ideal $\mathfrak{m}_{Y,y}\mathcal{O}_X$ (see 3.3.8).
- (3) Let g be the canonical L^2 -metric on E over Y_0 , which is positively curved by 3.3.4.

We may replace Y by slightly smaller polydisks, or may assume everything is defined over a slightly larger polydisk. \square

Definition 3.3.7. (1) In the set-up 3.3.6 above, by *Supp $f^*\Delta \rightarrow \Delta$ is relative normal crossing* we mean that locally near every point $x \in X$, there exists a local coordinate $(U; z = (z_1, \dots, z_{n+m}))$ such that $f|_U$ is given by $t_1 = z_{n+1}, \dots, t_{m-1} = z_{n+m-1}, t_m = z_{n+m}^{b_{n+m}} \prod_{j=1}^n z_j^{b_j}$ with non-negative integers b_j and b_{n+m} .

(2) The map f is said to be *semi-stable*, if moreover $f^*\Delta$ is reduced (in the notations above, this simply means that $b_j \leq 1$). \square

We explain 3.3.6(2.ii) in a slightly general context.

Lemma 3.3.8. *Let $f : X \rightarrow Y$ be a proper surjective morphism of complex manifolds, F a torsion free sheaf on X which is flat over Y , and let $E = f_*F$ be its direct image, which is merely torsion free. Then there exists a Zariski closed subset $Z \subset Y$ with $\text{codim}_Y Z \geq 2$ such that $E|_{Y \setminus Z}$ is locally free, and such that for every point $y \in Y \setminus Z$, the natural homomorphism $E/(\mathfrak{m}_{Y,y}E) \rightarrow H^0(X_y, F_y)$ is injective, where $F|_{X_y} = F \otimes \mathcal{O}_{X_y}$.*

Proof. Our assertion is local on Y around complement of some codimension 2 Zariski closed subsets of Y . Let $m = \dim Y \geq 1$. As E is torsion free, we may suppose E is locally free on Y by removing codimension 2 subsets in Y . By general results (such as the upper-semicontinuity theorem, and the cohomology and base change theorem), there exist Zariski closed subsets $Z \subset Y$ with $\text{codim}_Y Z \geq 1$ such that for every point $y \in Y \setminus Z$, the natural map $E/(\mathfrak{m}_{Y,y}E) \rightarrow H^0(X_y, F_y)$ is isomorphic. By removing codimension 2 subsets in Y and replacing Y , we may suppose that Z is smooth, irreducible and of $\dim Z = m - 1$. In fact, we may suppose $Z = \{s_1 = 0\}$ in a coordinate of $Y \subset (\mathbb{C}^m, (s_1, \dots, s_m))$. Denote by $f_1 : X_1 \rightarrow Z$ be the induced morphism, where X_1 is the scheme theoretic inverse image of Z , namely in this case X_1 is a divisor in X defined by $\{f^*s_1 = 0\}$. In general $F \otimes \mathcal{O}_{X_1}$ is flat over Z , and $E_1 := (f_1)_*(F \otimes \mathcal{O}_{X_1})$ is a coherent sheaf on Z . If Y is a curve, then Z is a point $y \in Y$ and $E_1 = H^0(X_y, F_y)$.

We have an exact sequence $0 \rightarrow F \otimes \mathcal{O}_X(-X_1) \rightarrow F \rightarrow F \otimes \mathcal{O}_{X_1} \rightarrow 0$ on X , and an exact sequence $0 \rightarrow f_*(F \otimes \mathcal{O}_X(-X_1)) \rightarrow f_*F \rightarrow f_*(F \otimes \mathcal{O}_{X_1})$ on Y . Noting that $\mathcal{O}_X(-X_1) = f^*\mathcal{O}_Y(-Z)$ is an invertible sheaf, the last exact sequence turns to be $0 \rightarrow \mathcal{O}_Y(-Z) \otimes E \rightarrow E \rightarrow E_1$. Thus the induced homomorphism $E/(s_1E) \rightarrow E_1$ is injective. If Y is a curve, we are done at this point. We suppose $m \geq 2$.

By removing codimension 1 subsets in Z and replacing Y , we may suppose that E_1 is locally free on Z , more strongly for every point $y \in Z$, the natural map $E_1/(\mathfrak{m}_{Z,y}E_1) \rightarrow H^0(X_y, F_y)$ is isomorphic, and that the map $E' := E/(s_1E) \rightarrow E_1$ is injective as a vector bundle

homomorphism (not only as a sheaf homomorphism). In particular E' can be seen as a direct summand of E_1 (as we are in local situation). In general the rank of E' would be strictly smaller than that of E_1 . For every point $y \in Z$, we thus have an injection $E' / (\mathfrak{m}_{Z,y} E') \rightarrow H^0(X_y, F_y)$ as a composition. As $E' / (\mathfrak{m}_{Z,y} E') = E / (\mathfrak{m}_{Y,y} E)$ (this can be checked by taking a local coordinate), our claim is proved. \square

Lemma 3.3.9. *In order to prove Theorem 3.3.5, we can further suppose that f is semi-stable in addition to 3.3.6.*

Proof. We use the weakly semi-stable reduction (or the semi-stable reduction in codimension 1) ([27, Ch. II] [30, §7.2] [47, §6.4]). Let

$$f^* \Delta = \sum_j b_j B_j$$

be the prime decomposition. Let Y' be another copy of a unit polydisk in \mathbb{C}^m with coordinates $t' = (t'_1, \dots, t'_{m-1}, t'_m)$. Let ℓ be the least common multiple of all b_j . Let $\tau : Y' \rightarrow Y$ be a ramified covering given by $(t'_1, \dots, t'_{m-1}, t'_m) \mapsto (t'_1, \dots, t'_{m-1}, (t'_m)^\ell)$, and $X^\circ = X \times_Y Y'$ be the fiber product. Let $\nu : X' \rightarrow X^\circ$ be the normalization, and $\mu : X'' \rightarrow X'$ be a resolution of singularities, which is biholomorphic on the smooth locus of X' . We encode next the induced morphisms in the following diagram.

$$\begin{array}{ccccccc} X'' & \xrightarrow{\mu} & X' & \xrightarrow{\nu} & X^\circ = X \times_Y Y' & \xrightarrow{\tau^\circ} & X \\ f'' \downarrow & & f' \downarrow & & f^\circ \downarrow & & \downarrow f \\ Y' & \xlongequal{\quad} & Y' & \xlongequal{\quad} & Y' & \xrightarrow{\quad \tau} & Y \end{array}$$

We let also $\tau'' : X'' \rightarrow X$ be the composition, $L'' = \tau''^* L$, and $h'' = \tau''^* h$ the induced singular Hermitian metric on L'' . We set $\Delta' = \text{Supp } \tau^* \Delta = \{t'_m = 0\} \subset Y'$. The fact that we can take a resolution $\mu : X'' \rightarrow X'$ so that $f'' : X'' \rightarrow Y'$ is semi-stable in codimension 1 is by no means “obvious”, but it is established in the references quoted at the beginning of the proof.

By [46, Lemma 3.2] [35, V.3.30] (we note that adding $L'' = \tau''^* L$ is harmless), we then have a natural inclusion homomorphism

$$\alpha : f''_* (K_{X''/Y'} + L'') \longrightarrow \tau^* f_* (K_{X/Y} + L),$$

which is isomorphic over $Y' \setminus \Delta'$. In particular $f''_* (K_{X''/Y'} + L'')$ is locally free over $Y' \setminus \Delta'$. The natural inclusion $f''_* ((K_{X''/Y'} + L'') \otimes \mathcal{I}(h'')) \subset f''_* (K_{X''/Y'} + L'')$ is also generically isomorphic, and $E'' := f''_* (K_{X''/Y'} + L'')$ admits also the canonical L^2 -metric $g' = g_{1NS, X''/Y'}$ over $Y' \setminus \Delta'$ with respect to h'' (by 3.3.4). Moreover the inclusion α induces an isometry over $Y' \setminus \Delta'$, i.e., $g' = \alpha^* \tau^* g$ as a singular Hermitian metric on $E''|_{Y' \setminus \Delta'}$. In fact, since $\tau : Y' \setminus \Delta' \rightarrow Y \setminus \Delta$ is locally biholomorphic, the coordinate free description of the L^2 -metric in 3.2.2 explain the isometry.

By 2.3.2, it is enough to prove that the singular Hermitian metric $\tau^* g$ over $Y' \setminus \Delta'$ can be extend as a singular Hermitian metric on $\tau^* E = \tau^* f_* (K_{X/Y} + L)$ over Y' with positive curvature. To this end, we use 2.4.3(3), which shows that it is enough to prove that the singular Hermitian metric g' over $Y' \setminus \Delta'$ can be extend as a singular Hermitian metric on $E'' = f''_* (K_{X''/Y'} + L'')$ over Y' with positive curvature. Once this is established, we apply 2.4.3(3), which shows that the extension of g' with positive curvature induces a singular Hermitian metric g^+ on

$\tau^*E = \tau^*f_*(K_{X/Y} + L)$ with positive curvature. The metric g^+ extends the pull-back τ^*g , because of $g' = \alpha^*\tau^*g$ on $Y' \setminus \Delta'$ (and because of the proof or construction of 2.4.3(3)). And finally, in order to extend g' over Y' (by removing codimension two subvarieties of Y' and localizing again), we may suppose that f'' is semi-stable and E'' satisfies 3.3.6(2). \square

Proposition 3.3.10. *Theorem 3.3.5 holds true when f is semi-stable (hence in general, by the result above).*

Proof. In what follows, we will assume that the relations (1)–(3) in 3.3.6 are verified, and that the map f is semi-stable. This will allow us to perform a few explicit computations in order to understand the behavior of the metric near Δ .

By 3.3.4, the metric g^* is negatively curved on Y_0 . Our goal is to extend it on Y with negative curvature. We consider any open set $U \subset Y$ and a non-zero $\xi \in H^0(U, E^*)$. We may assume that $U = Y$; then by hypothesis $\log |\xi|_{g^*}^2$ is psh on Y_0 . The conclusion of 3.3.5 is that the function $\log |\xi|_{g^*}^2$ extends as a psh function on Y . As it is well known (by Hartogs theorem) the extension property holds provided that we can show that there exists a constant $c_\xi > 0$ such that $|\xi|_{g^*}^2 < c_\xi$ on Y_0 (possibly after shrinking Y).

We consider a global frame $e_1, \dots, e_r \in H^0(Y, E)$ and we trivialize $E = Y \times \mathbb{C}^r$ via this frame. With respect to this trivialization, the metric g can be written as a matrix valued function $g = (g_{i\bar{j}})$ on Y_0 . For every $t \in Y_0$, we denote by $0 < \lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_r(t) \leq +\infty$ the eigenvalues of $g(t) = (g_{i\bar{j}}(t))$, see 2.2.7. Then we have $\lambda_r(t) < +\infty$ for every $t \in Y_{1,h,\text{ext}}$ by definition 3.3.1 of $h_{1,h,\text{ext}}$. We shall show in 3.3.11 that there exists a constant $c_0 > 0$ independent of $t \in Y_0$ such that $\lambda_1(t) > c_0$ for all $t \in Y_0$ (including the case $\lambda_1(t) = +\infty$). This implies the existence of a constant $c_\xi > 0$ such that $|\xi|_{g^*}^2 < c_\xi$ on Y_0 , and our claim will be proved.

For a constant vector $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, we let $u_s = \sum_{i=1}^r s_i e_i \in H^0(Y, E) = H^0(X, K_{X/Y} + L)$. We denote by $S^{2r-1} = \{s \in \mathbb{C}^r; |s| = (\sum |s_i|^2)^{1/2} = 1\}$ the unit sphere. Since e_1, \dots, e_r generate E over Y , u_s is nowhere vanishing on Y as soon as $s \neq 0$, namely u_s is non-zero in $E/(\mathfrak{m}_{Y,y}E)$ at any $y \in Y$. The map $\mathbb{C}^r \rightarrow H^0(X, K_X + L)$ given by

$$s \mapsto u_s \mapsto u_s(dt) = \sum_{i=1}^r s_i e_i(dt), \quad \mathbb{C}^r \rightarrow H^0(X, K_{X/Y} + L) \cong H^0(X, K_X + L)$$

is continuous, with respect to the standard topology of \mathbb{C}^r and the topology of $H^0(X, K_X + L)$ of uniform convergence on compact sets (recall 3.2.2 for the meaning of $u_s(dt)$). Then the following 3.3.11 shows our claim. \square

Lemma 3.3.11. (cf. [19, 1.12].) *Let $y \in \Delta$. Then there exist a neighborhood W_y of y in Y and a constant $c_0 > 0$, such that $g(u_s, u_s)(t) \geq c_0$ for any $s \in S^{2r-1}$ and any $t \in W_y \setminus \Delta$.*

Since S^{2r-1} is compact, 3.3.11 is a consequence of the next statement.

Lemma 3.3.12. (cf. [19, 1.11].) *In notations above, let $y \in \Delta$ and let $s_0 \in S^{2r-1}$. Then there exist a neighborhood $S(s_0)$ of s_0 in S^{2r-1} , a neighborhood W_y of y in Y and a constant $c_0 > 0$ such that $g(u_s, u_s)(t) \geq c_0$ for any $s \in S(s_0)$ and any $t \in W_y \setminus \Delta$.*

Proof. (1) Let $u \in H^0(Y, E)$, and assume u does not vanish at y . Then we claim that there exists a component B_j in $f^*\Delta = \sum B_j$ such that $u(dt) \in H^0(X, K_X + L)$ does not vanish identically along $B_j \cap f^{-1}(y)$.

The isomorphism $H^0(X, K_{X/Y} + L) \cong H^0(X, K_X + L)$ is as $H^0(Y, \mathcal{O}_Y)$ -modules. In particular, this will show that $u(dt) \in H^0(X, K_X + L)$ does not vanish at $y \in Y$ as an element of an $H^0(Y, \mathcal{O}_Y)$ -module. We have to show that if $u(dt)$ vanishes identically along $f^{-1}(y)$, then u vanishes at y , i.e., $u \in \mathfrak{m}_{Y,y}E$. If $m = 1$, we can check it by a direct computation as in the next paragraph. If $m > 1$, we cut Y and X by $\{t_1 = 0\}$ (note $X \cap \{t_1 = 0\}$ is still smooth) and use the induction hypothesis. More precisely, we let $Y_1 = \{t_1 = 0\} \subset Y$, $X_1 = f^{-1}(Y_1)$, and let $f_1 : X_1 \rightarrow Y_1$ be the induced morphism. We have a restriction map $r : H^0(X, K_X + L) \rightarrow H^0(X_1, (K_X + L)|_{X_1}) \cong H^0(X_1, K_{X_1} + L|_{X_1})$ by the adjunction $K_X|_{X_1} = K_{X_1}$. The kernel is generated by t_1 . If $u(dt)$ vanishes identically along $f^{-1}(y)$, then by induction hypothesis, the image of $u(dt)$ in $H^0(X_y, (K_{X/Y} + L)|_{X_y})$ is zero. Thus by the property 3.3.6(2), we have $u(dt) \in \mathfrak{m}_{Y,y}E$.

Let us suppose $m = 1$, and let $t = t_1$ be the coordinate function on Y . Suppose that $u(dt)$ vanishes identically along $f^{-1}(y)$, which is now a reduced divisor. We shall show that $u \in H^0(X, K_{X/Y} + L)$ is divided by $f^{-1}t$ as a holomorphic section, and obtain a contradiction. To see u is divided by $f^{-1}t$, it is enough to show that u vanishes identically along $f^{-1}(y)$ as a bundle valued continuous function, and it is enough to show that it vanishes identically along the smooth part of $f^{-1}(y)$ by the continuity. So we take a smooth point x of $f^{-1}(y)$ and a local coordinate $(U; z)$ around x such that the map f is given by

$$(z_1, \dots, z_{n+1}) \mapsto t = z_{n+1},$$

and $y = 0$ in this coordinates t . With respect to the coordinates $(U; z)$, (not all, but) a representative for the element u can be written as a $(n, 0)$ -form on U as follows

$$u = \tau dz_1 \wedge \dots \wedge dz_n,$$

where $\tau \in H^0(U, \mathcal{O}_X)$. By this we mean that we have the equality

$$u(dt) = \tau dz_1 \wedge \dots \wedge dz_n \wedge f^*(dt) = \tau dz_1 \wedge \dots \wedge dz_n \wedge dz_{n+1}.$$

as top degree forms defined on U . The vanishing of $u(dt)$ along $f^{-1}(y) \cap U$ is equivalent to the vanishing of τ , as well as of u , along $f^{-1}(y) \cap U$.

(2) For our nowhere vanishing u_{s_0} , we take a component

$$B = B_j$$

in $f^*\Delta$ such that $u_{s_0}(dt)$ does not vanish identically along $B \cap f^{-1}(y)$. We take a general point $x_0 \in B \cap f^{-1}(y)$, and a local coordinate $(V; z = (z_1, \dots, z_{n+m}))$ centered at $x_0 \in X$ such that f is given by $t = (t_1, \dots, t_m) = f(z) = (z_{n+1}, \dots, z_{n+m})$ on V . In particular $(f^*\Delta)|_V = B|_V = \{z_{n+m} = 0\}$. Over V , we may assume that L and h are also trivialized, i.e., $L|_V \cong V \times \mathbb{C}$ and $h = e^{-\varphi}$ with a (quasi-)psh function $\varphi \in L^1_{loc}(V, \mathbb{R})$. In particular $e^{-\varphi} > a$ on V for a constant $a > 0$ (possibly after shrinking V).

(3) For every $s \in S^{2r-1}$, we have $u_s(dt) \in H^0(X, K_X + L)$. Over V , it can be written as

$$u_s(dt)|_V = \tau_{sV} dz_1 \wedge \dots \wedge dz_{n+m} \text{ with } \tau_{sV} \in H^0(V, \mathcal{O}_X),$$

and hence $\tau_{sV} dz_1 \wedge \dots \wedge dz_{n+m} = (\sigma_{sV} dz_1 \wedge \dots \wedge dz_n) \wedge f^* dt$. As in 3.2.2, every $u_s \in H^0(Y, E) = H^0(Y, f_*(K_{X/Y} + L))$ defines $\sigma_{st} \in H^0(X_t, K_{X_t} + L_t)$ for every $t \in Y_0$. Remark 3.2.2 also shows $u_s|_{V \cap X_t} = (\sigma_{sV} dz_1 \wedge \dots \wedge dz_n)|_{V \cap X_t}$ on $V \cap X_t$.

At the point $s_0 \in S^{2r-1}$, since $u_{s_0}(dt)$ does not vanish identically along $B \cap f^{-1}(y)$ and since $x_0 \in B \cap f^{-1}(y)$ is general, we have $\sigma_{s_0V}(x_0) \neq 0$.

(4) By the continuity of $s \mapsto u_s \mapsto u_s(dt)$, we can take an ε -polydisk $V(\varepsilon) = \{z = (z_1, \dots, z_{n+m}) \in V; |z_i| < \varepsilon \text{ for any } 1 \leq i \leq n+m\}$ centered at x_0 for some $\varepsilon > 0$ and a neighborhood $S(s_0)$ of s_0 in S^{2r-1} , such that

$$A := \inf\{|\sigma_{sV}(z)|; s \in S(s_0), z \in V(\varepsilon)\} > 0.$$

We set $W_y := f(V(\varepsilon))$, which is an open neighborhood of $y \in Y$, since f is flat (in particular it is open). Then for any $s \in S(s_0)$ and any $t \in W_y \setminus \Delta$, we have

$$\begin{aligned} \int_{X_t} (-1)^{n^2/2} \sigma_s \wedge \bar{\sigma}_s h|_{X_t} &\geq a \int_{X_t \cap V} (-1)^{n^2/2} \sigma_s \wedge \bar{\sigma}_s|_{X_t \cap V} \\ &\geq a \int_{z \in X_t \cap V(\varepsilon)} A^2 dV_n = aA^2(2\pi\varepsilon^2)^n, \end{aligned}$$

where $dV_n = \bigwedge_{i=1}^n \sqrt{-1} dz_i \wedge d\bar{z}_i$ is an euclidean volume form on \mathbb{C}^n . □

Remark 3.3.13. (1) It may be possible to show $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h))$ is Griffiths semi-positive in 3.3.5 without assuming that $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is generically isomorphic. However there are some technical difficulties. To explain this, suppose Y is a disk, and $X_0 = \sum m_i X_i$ is the singular fiber over $0 \in Y$. There are difficulties to compare the ideal $\mathcal{I}_{X_0}, \mathcal{I}(h)$, and $\text{div}(u)$ for $u \in H^0(X, (K_{X/Y} + L) \otimes \mathcal{I}(h))$. Embedded components of $\mathcal{O}_X/\mathcal{I}(h)$ are difficult to handle in general. More subtle thing is, because h may have non-algebraic/analytic singularities, we cannot reduce to the normal crossing situation even after a modification of the manifold.

(2) In the proof of 3.3.5, it is not enough to show that, on each line ℓ on Y which is not contained in Δ , $g_{1NS, X/Y}|_{\ell \setminus \Delta}$ is bounded as $g_{1NS, X/Y}|_{\ell \setminus \Delta} \geq c_\ell$ by a constant $c_\ell > 0$. We need a uniform bound $c_\ell \geq c > 0$ independent of ℓ . We can not assume $\dim Y = 1$ without special attentions.

4. POSITIVITY OF RELATIVE PLURICANONICAL BUNDLES

In this part of our article we will provide the necessary tools needed in order to generalize Theorem 3.3.5 in the context of pluricanonical bundles. Part of our motivation is that the metric associated to the bundle $f_*(mK_{X/Y} + L)$ for $m \geq 2$ has better regularity properties than the one on $f_*(K_{X/Y} + L)$, and this is important for applications. The path we will follow is to write $mK_{X/Y} + L = K_{X/Y} + L_{m-1}$. We endow the bundle L_{m-1} with the metric $h_{m-1} := (B_{m,X/Y}^{-1})^{(m-1)/m} h^{1/m}$, and then we apply the results of the preceding section with the data f and (L_{m-1}, h_{m-1}) . Therefore, it is fundamental to understand the relative Bergman type metric $B_{m,X/Y}$ (as the metric h on L is given).

To start with, we will consider the case $m = 1$. In the first part of this section we will revisit the construction of the Bergman-type metric on $K_{X/Y} + L$. The proof we present here is different from the original argument in [3]. It relies on the version of the Ohsawa-Takegoshi extension theorem obtained by Blocki [6] and Guan-Zhou [21], combined with an approach due to Tsuji. The construction of the metric $B_{m,X/Y}^{-1}$ on $mK_{X/Y} + L$ for $m \geq 2$ is treated afterwards; to a large extent, it is a consequence of the case $m = 1$. In both cases, the metric is constructed first on a Zariski open set which will be explicitly described here. We show that the curvature current corresponding to $B_{m,X/Y}^{-1}$ is semi-positive definite. The extension of this current to the manifold X will be a consequence of the version of the Ohsawa-Takegoshi extension theorem we establish here.

The set-up in this section is as follows: let $f : X \rightarrow Y$ be a projective surjective morphism of complex manifolds with connected fibers, and let $L \rightarrow X$ be a line bundle, endowed with a singular Hermitian metric $h_L = e^{-\varphi_L}$ whose curvature current is positive.

4.1. The log-plurisubharmonicity of the Bergman kernel metric revisited. “Ideally”, the relative Bergman kernel metric $B_{1,X/Y}$ on $-(K_{X/Y} + L)$ would be defined as a collection of fiberwise Bergman kernel metrics $B_{1,y}$ with respect to $h_y := h_L|_{X_y}$ for X_y smooth (notations as in 3.2.1). However, it is not a-priori clear that the resulting metric $B_{1,X/Y}$ is even semi-continuous, mainly because the dimension of the space of sections of the bundle $K_{X_y} + L_y$ may not be constant as a function of y .

In order to overcome this phenomena, we first define explicitly $B_{1,X/Y}$ on the set

$$Y_{\text{ext}} = Y_{1,\text{ext}} := \{y \in Y_0; h^0(X_y, K_{X_y} + L_y) \text{ equals to the rank of } f_*(K_{X/Y} + L)\},$$

which is the Zariski open set of regular values $y \in Y$ such that every section of $K_{X_y} + L_y$ extends to the f -inverse image of an open set containing y (as we defined in 3.3.1). Let

$$X_{\text{ext}} = f^{-1}(Y_{\text{ext}})$$

so that in particular the map $f : X_{\text{ext}} \rightarrow Y_{\text{ext}}$ is a submersion. In [3, pp. 346–367] (or 3.2.1 here) we have defined the metric $B_{1,X/Y}^{-1}$ on $(K_{X/Y} + L)|_{f^{-1}(Y_{\text{ext}} \cap Y_h)}$. We assign the value $+\infty$ on the set $Y_{\text{ext}} \setminus Y_h$, and in this way $B_{1,X/Y}^{-1}$ is well-defined on X_{ext} .

The following result was basically established in [3]; we will nevertheless provide here an alternative argument which is based on Blocki’s and Guan-Zhou’s version of the Ohsawa-Takegoshi extension theorem with optimal constant.

Theorem 4.1.1. [3, 0.1] *We assume that there exists a point $y \in Y_h$ such that*

$$H^0(X_y, (K_{X_y} + L_y) \otimes \mathcal{I}(h_y)) \neq 0.$$

Then the curvature current corresponding to the metric $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ defined above is positive. Moreover, the local weights of this metric are upper semi-continuous, and uniformly bounded on $X \setminus X_{\text{ext}}$. Hence, $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ extends to X in a unique manner, and it endows the bundle $K_{X/Y} + L$ with a metric whose curvature current is positive definite.

Proof. A first observation is that it suffices to show that the weights $\varphi_{X/Y}^{(1)}$ of $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ are psh. Indeed, by the original argument in [3], the resulting metric extends to X , and it has the positivity property stated in the last part of Theorem 4.1.1. We remark that the extension property amounts to prove that $\sup_{X_{\text{ext}}} \varphi_{X/Y}^{(1)} < \infty$. We refer to [3] for a complete argument, based on the classical version of the Ohsawa-Takegoshi extension theorem.

(i) We first show that *the weights $\varphi_{X/Y}^{(1)}$ of metric $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ are upper semi-continuous.* This claim will be a consequence of the fact that $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ can be written as increasing limit of continuous metrics on the bundle $(K_{X/Y} + L)|_{X_{\text{ext}}}$. Let $(h_{L,\varepsilon} = e^{-\varphi_{L,\varepsilon}})_{\varepsilon>0}$ be an increasing sequence of smooth metrics on L , converging pointwise and in L^1 to the metric h_L . We recall that this can be constructed by a global convolution kernel, cf. [13]. Actually, the negative part of the curvature form associated to $(L, h_{L,\varepsilon})$ is uniformly bounded and it converges almost everywhere to zero, but we will not use this important information here.

Let $B_{1,\varepsilon}^{-1}|_{X_{\text{ext}}}$ be the metric on $(K_{X/Y} + L)|_{X_{\text{ext}}}$ whose local weights $\varphi_{\alpha,\varepsilon}$ are defined as follows ([3, pp.346–347]). Given a point $x_0 \in \Omega_\alpha \subset X_{\text{ext}}$ in a coordinate set Ω_α we consider the expression

$$B_{1,\varepsilon}(x_0) = \exp(\varphi_{\alpha,\varepsilon}(x_0)) := \sup_u \frac{|u'_\alpha(x_0)|^2}{\|u\|_{y_0,\varepsilon}^2},$$

where the notations are as follows: $y_0 := f(x_0)$ is the projection of x_0 , u is a global section of the bundle $K_{X_{y_0}} + L_{y_0}$, and $\|\cdot\|_{\cdot,\varepsilon}$ means that we are using the metric $h_{L,\varepsilon}|_{X_{y_0}}$ in defining the L^2 norm of u . We fix $z = (z_1, \dots, z_k)$ and $t = (t_1, \dots, t_\ell)$ coordinates on Ω_α and $f(\Omega_\alpha)$ respectively. The local holomorphic function u'_α is defined as $u'_\alpha dz = u \wedge dt$, where $dz = dz_1 \wedge \dots \wedge dz_k$ and $dt = dt_1 \wedge \dots \wedge dt_\ell$. In other words, $B_{1,\varepsilon}^{-1}|_{X_{\text{ext}}}$ is defined exactly as $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$, except that we are using the metric $h_{L,\varepsilon}$ for the normalization of the sections on the fibers of f .

The fact that $h_{L,\varepsilon}$ is non-singular for each $\varepsilon > 0$, together with the choice of $y_0 \in Y_{\text{ext}}$ shows that the weights $\varphi_{\alpha,\varepsilon}$ defined on Ω_α are continuous. Given the monotonicity property of the family of metrics $h_{L,\varepsilon}$ with respect to ε , we infer that we have

$$(4.1.1) \quad \varphi_{\alpha,\varepsilon} \geq \varphi_{\alpha,\varepsilon'} \geq \varphi_{\alpha,X/Y}^{(1)}$$

provided that $\varepsilon \geq \varepsilon'$; we denote by $\varphi_{\alpha,X/Y}^{(1)}$ the local weight of the metric $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ on Ω_α with respect to the chosen trivializations.

We show next that we have

$$(4.1.2) \quad \lim_{\varepsilon \rightarrow 0} \varphi_{\alpha,\varepsilon} = \varphi_{\alpha,X/Y}^{(1)}.$$

To this end, we will analyze the following two cases.

- We have $\varphi_{\alpha, X/Y}^{(1)}(x_0) = -\infty$ (indeed this may happen, e.g. if the point y_0 belongs to the set $Y_{\text{ext}} \setminus Y_h$). Let u_ε be a holomorphic section of the bundle $K_{X_{y_0}} + L_{y_0}$, such that the L^2 norm $\|u_\varepsilon\|_{y_0, \varepsilon}^2 := \int_{X_{y_0}} |u_\varepsilon|^2 e^{-\varphi_{L, \varepsilon}}$ is equal to 1, and such that

$$\varphi_{\alpha, \varepsilon}(x_0) = \log |u'_{\varepsilon, \alpha}(x_0)|^2.$$

We claim that $\lim_{\varepsilon \rightarrow 0} |u'_{\varepsilon, \alpha}(x_0)|^2 = 0$. If this is not true, then let u_∞ be the limit of u_ε ; it is not identically zero, given our assumption. Let e^{m_ε} be the maximum of the function $(\tilde{h}/h_{L, \varepsilon})|_{X_{y_0}}$, where \tilde{h} is a smooth, fixed metric on L . If $m_\varepsilon \rightarrow -\infty$, then we obtain a contradiction as follows. We have

$$\int_{X_{y_0}} |u_\varepsilon|^2 e^{-\varphi_{L, \varepsilon} + m_\varepsilon} = e^{m_\varepsilon}$$

so that by the definition of m_ε , we infer

$$\int_{X_{y_0}} |u_\varepsilon|^2 e^{-\phi_L} \leq e^{m_\varepsilon},$$

where $e^{-\phi_L}$ denotes the smooth metric \tilde{h} . This cannot happen if ε is small enough, because (u_ε) is converging towards a non-identically zero limit.

Thus, we can assume that $\limsup_{\varepsilon \rightarrow 0} m_\varepsilon := m_\infty > -\infty$; in other words, $y_0 \in Y_h$. Then we have

$$(4.1.3) \quad \int_{X_{y_0}} |u_\infty|^2 e^{-\varphi_L} \leq 1$$

by a standard argument (we apply the dominate convergence theorem in the complement of the support of the multiplier ideal sheaf corresponding to $h_L|_{X_{y_0}}$), and this contradicts the fact that $\varphi_{\alpha, X/Y}^{(1)}(x_0) = -\infty$.

- If $\varphi_{\alpha, X/Y}^{(1)}(x_0) > -\infty$, then we know a-priori that $y_0 \in Y_h$, and by the arguments already used in the preceding case we have $u_\infty := \lim_\varepsilon u_\varepsilon$, such that (4.1.3) holds. We easily see that

$$\varphi_{\alpha, X/Y}^{(1)}(x_0) = \log |u'_{\infty, \alpha}(x_0)|^2,$$

as if not, the inequality (4.1.1) will be contradicted.

This concludes the proof of the upper semi-continuity of $\varphi_{X/Y}^{(1)}$.

(ii) Next we show that $\varphi_{X/Y}^{(1)}$ verifies the mean value inequality; at this point we follow the approach outlined by H. Tsuji in [45], which appears in the article by Guan-Zhou [21, §3.5].

A first observation is that it is enough to treat the case $\dim Y = 1$. Given a smooth projective family of varieties $f : X \rightarrow \mathbb{D}$ over the disc of radius 1 in \mathbb{C} (so that y_0 will correspond to 0), the minimal norm of the extension of twisted canonical forms can be controlled in an optimal way, thanks to the important version of Ohsawa-Takegoshi theorem, recently obtained by Błocki [6] and [21]. The result we need states as follows.

Theorem 4.1.2. [21, Thm 2.2] *Given a section u of the bundle $(K_{X_{y_0}} + L_{y_0}) \otimes \mathcal{I}(h_L|_{X_{y_0}})$ and a positive real number $r < 1$, there exists a section U_r of $K_X + L$ extending u , defined over the pre-image $f^{-1}(\mathbb{D}_r)$ of the disk of radius r such that*

$$(4.1.4) \quad \frac{1}{\pi r^2} \int_{t \in \mathbb{D}_r} \left\| \frac{U_r}{dt} \right\|_t^2 d\lambda(t) \leq \|u\|_{y_0}^2.$$

Actually, this result is established in [21] only in the case of a smooth metric h_L ; the singular version we need here can be derived from it by the usual regularization procedure.

We consider a particular section $u \in H^0(X_{y_0}, K_{X_{y_0}} + L_{y_0})$ so that we have

$$B_{1,X/Y}(x_0) = \exp(\varphi_{X/Y}^{(1)}(x_0)) = \frac{|u'(x_0)|^2}{\|u\|_{y_0}^2}.$$

We extend u as in Theorem 4.1.2, and thus we obtain the section U_r defined on $f^{-1}(\mathbb{D}_r)$ whose L^2 norm is satisfying the inequality (4.1.4). Let $t \rightarrow \gamma(t)$ be a local section of f , such that $x_0 = \gamma(0)$. We infer that we have

$$\varphi_{X/Y}^{(1)}(x_0) \leq \frac{1}{\pi r^2} \int_{t \in \mathbb{D}_r} \varphi_{X/Y}^{(1)}(\gamma(t)) d\lambda(t)$$

thanks to the estimate (4.1.4), combined with the concavity of \log and the fact that the absolute value of the holomorphic functions is log-psh.

In conclusion, we have showed that the weights of the metric $B_{1,X/Y}^{-1}|_{X_{\text{ext}}}$ are psh functions. \square

4.2. Log-plurisubharmonicity of the m -Bergman kernel metric for $m \geq 2$. We still keep the notation $f : X \rightarrow Y$ and $(L, h_L = e^{-\varphi_L})$. We will analyze here some of the properties of the metric $(B_{m,X/Y})^{-1}$ on the bundle

$$mK_{X/Y} + L$$

whose fiberwise construction was recalled in 3.1.1. As in the case $m = 1$ treated in the previous subsection, we first consider the following non-empty Zariski open subset of Y

$$Y_{m,\text{ext}} := \{y \in Y_0; h^0(X_y, mK_{X_y} + L_y) \text{ equals to the rank of } f_*(mK_{X/Y} + L)\},$$

where $Y_0 \subset Y$ is the set of regular values of f . We denote by $Y_h = \{y \in Y_0; h_y \neq +\infty\}$.

Let φ be a psh function on an open set in \mathbb{C}^n . The following ideal sheaf $\mathcal{J}_m(\varphi)$ will appear in our considerations

$$\mathcal{J}_{m,x}(\varphi) := \left\{ f \in \mathcal{O}_{\mathbb{C}^n,x}; \int_{(\mathbb{C}^n,x)} |f|^{\frac{2}{m}} e^{-\varphi} d\lambda < \infty \right\},$$

where $d\lambda$ is the Euclidean volume form. Although we will not use it here, we mention the result of [8, 3.4], showing that there exists a psh function with analytic singularities ψ such that we have $\mathcal{J}_m(\varphi) = \mathcal{J}_m(\psi)$. In particular, the ideal $\mathcal{J}_m(\varphi)$ is coherent. Moreover, as in the case of the usual multiplier ideal sheaf, the notion above can be formulated in the case of a line bundle endowed with a metric with semi-positive curvature current.

We extend the definition of $(B_{m,X/Y})^{-1}$ to the Zariski open set

$$X_{m,\text{ext}} = f^{-1}(Y_{m,\text{ext}})$$

by assigning the value $+\infty$ on the set $f^{-1}(Y_{m,\text{ext}} \setminus Y_h)$; formally, we have the following.

Definition 4.2.1. *Let b_m be a smooth Hermitian metric on $-(mK_{X/Y} + L)$. For every $y \in Y_{m,\text{ext}}$, we can write as $B_{m,y} = (b_m|_{X_y}) \exp(\varphi_y^{(m)})$, where either $\varphi_y^{(m)} \in L_{\text{loc}}^1(X_y, \mathbb{R})$ or $\varphi_y^{(m)} \equiv -\infty$ on X_y (we are using here the notations in 3.1.1). The latter holds if and only if $V_{m,y} =$*

$H^0(X_y, (mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})) = 0$ (including the case $y \in Y_{m,\text{ext}} \setminus Y_h$ by convention). We then define a function

$$\varphi_{X/Y}^{(m)o} : X \rightarrow [-\infty, +\infty) \quad \text{by} \quad \varphi_{X/Y}^{(m)o}(x) = \varphi_y^{(m)}(x) \quad \text{if } x \in X_y,$$

and define

$$B_{m,X/Y}^o = b_m \exp(\varphi_{X/Y}^{(m)o}).$$

We remark that the metric $B_{m,X/Y}^o$ is so far only defined on $X_{m,\text{ext}} \subset X$.

By definition, $B_{m,X/Y}^o \not\equiv 0$ (non-trivial) if and only if there exists $y \in Y_h \cap Y_{m,\text{ext}}$ such that $H^0(X_y, (mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})) \neq 0$. In the expression above, we have used the identification $K_{X/Y}|_{X_y} \cong K_{X_y}$ so that $b_m|_{X_y}$ becomes a smooth Hermitian metric on $-(mK_{X_y} + L_y)$. It is not difficult to see $B_{m,X/Y}^o$ is independent of the choices of these identifications, even though $\varphi_{X/Y}^{(m)o}$ depends on them formally. The choice of a reference metric b_m is also not essential. We may regard b_m as a part of local trivializations, and then we may simply write as $B_{m,X/Y}^o = \exp(\varphi_{X/Y}^{(m)o})$ and $B_{m,X/Y}^o|_{X_y} = B_{m,y}$ for every $y \in Y_{m,\text{ext}}$.

We recall a result in [3] (compare with [42] and the references therein for related results).

Theorem 4.2.2. [3, 4.2] *We assume that the map $f : X \rightarrow Y$ is smooth, $Y_{m,\text{ext}} = Y$, and that there exists a point $y \in Y_h$ such that*

$$H^0(X_y, (mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})) \neq 0.$$

Then the dual of the relative m -Bergman kernel metric: $(B_{m,X/Y}^o)^{-1}$ in 4.2.1 defines a singular Hermitian metric on $mK_{X/Y} + L$ whose associated curvature current is semi-positive. Moreover, the local weights $\varphi_{X/Y}^{(m)o}$ of this metric are upper semi-continuous.

Remark 4.2.3. We take this opportunity to point out two *imprecisions* in the formulation [3, 4.2]. In the first place, it must be of course assumed that the metric $(B_{m,X/Y}^o)^{-1}$ (called the relative NS-metric in [3]) is not identically $+\infty$. Secondly, the assumption that $f_*(mK_{X/Y} + L)$ is locally free should be replaced by $Y_{m,\text{ext}} = Y$. Just before [3, 4.2], it is stated that these two conditions are equivalent, but this is not the case. In fact, the property of being locally free concerns the direct image $f_*(mK_{X/Y} + L)$ as a sheaf: the stalks of this sheaf consists of sections over the fibers of f that *do extend* to neighboring fibers, and local freeness means that this space of extendable sections has everywhere the same rank. We remark that the direct image $f_*(mK_{X/Y} + L)$ is torsion free, hence locally free if the dimension of the base is equal to one. For the proof of [3, 4.2] we need the stronger property $Y_{m,\text{ext}} = Y$, saying that all sections extend locally to neighboring fibers.

This distinction between local freeness and the condition $Y_{m,\text{ext}} = Y$ is precisely the heart of the question of invariance of plurigenera for smooth projective families over one dimensional base, where the local freeness is automatic, and $Y_{m,\text{ext}} = Y$ is exactly what is to be proved. \square

Remark 4.2.4. We highlight here some of the new difficulties in the proof of 4.2.2 (along the line given in 4.1.1 for $m = 1$) specific to the case $m \geq 2$; as for the complete argument, we refer to [3, 4.2].

In the first place, the upper semi-continuity of the metric $(B_{m,X/Y}^o)^{-1}$ follows *mutatis mutandis* from the arguments in §4.1, by using a monotone approximation of the metric h_L with non-singular metrics. Hence we provide no further details about this point here.

In order to check the mean value inequality, we cannot argue as in the case $m = 1$, i.e. by using an extension with minimal $L^{2/m}$ norm. The reason is very simple: it may happen that some section u of the sheaf $(mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})$ does not admits any extension locally near X_y which moreover is $L^{2/m}$ -integrable with respect to the metric $h_L^{1/m}$. We would have such an extension e.g. if the variety of zeros corresponding to the multiplier ideal of $h_L^{1/m}$ is f -vertical, but we do not want/need to impose this additional requirement. Also, we remark that even a-posteriori the existence of the metric $(B_{m,X/Y}^o)^{-1}$ *does not* implies the existence of an extension belonging to the ideal $\mathcal{J}_m(h_L^{1/m})$, despite of the fact that we know (by the choice of the set $Y_{m,\text{ext}}$) that some extension exists. \square

We now show that the metric $(B_{m,X/Y}^o)^{-1}$ extends across the set $X \setminus X_{m,\text{ext}}$; it is our main result in this section. Prior to this, we give a proof of Proposition 1.2, as it plays an important role in our arguments.

4.2.5. Proof of 1.2: The $L^{2/m}$ version of the Ohsawa-Takegoshi theorem. We recall briefly the set-up in Proposition 1.2. Let $\Omega \subset \mathbb{C}^n$ be a ball of radius r and let $\sigma : \Omega \rightarrow \mathbb{C}$ be a holomorphic function, such that $\sup_{\Omega} |\sigma| \leq 1$; moreover, we assume that the gradient $\partial\sigma$ of σ is nowhere zero on the set $V := \{\sigma = 0\}$. We denote by φ a plurisubharmonic function on Ω , such that its restriction to V is well-defined (i.e., $\varphi|_V \not\equiv -\infty$).

Let u be a holomorphic function on V with the property that

$$\int_V |u|^{2/m} \exp(-\varphi) \frac{d\lambda_V}{|\partial\sigma|^2} = 1.$$

The question is to find an extension of u to Ω , whose $L^{2/m}$ norm is bounded by the quantity above, modulo a universal constant. We may suppose $m > 1$.

We begin with some reductions. In the first place we can assume that the function φ is smooth, and that the functions σ (respectively u) can be extended in a neighbourhood of Ω (of V inside $V \cap \overline{\Omega}$, respectively). Once the result is established under these additional assumptions, the general case follows by approximations and standard normal families arguments.

We can then clearly find some holomorphic F_1 in Ω that extends u and satisfies

$$\int_{\Omega} |F_1|^{2/m} \exp(-\varphi) d\lambda \leq A < \infty.$$

We then apply the Ohsawa-Takegoshi theorem with weight

$$\varphi_1 = \varphi + (1 - 1/m) \log |F_1|^2$$

and obtain a new extension F_2 of f satisfying

$$\int_{\Omega} \frac{|F_2|^2}{|F_1|^{2-2/m}} \exp(-\varphi) d\lambda \leq C_0 \int_V \frac{|u|^2}{|F_1|^{2-2/m}} \exp(-\varphi) \frac{d\lambda_V}{|\partial\sigma|^2} = C_0.$$

Hölder's inequality gives that

$$\int_{\Omega} |F_2|^{2/m} \exp(-\varphi) d\lambda \int_{\Omega} \frac{|F_2|^{2/m}}{|F_1|^{(2-2/m)/m}} |F_1|^{(2-2/m)/m} \exp(-\varphi) d\lambda \leq$$

$$\leq \left(\int_{\Omega} \frac{|F_2|^2}{|F_1|^{2-2/m}} \exp(-\varphi) d\lambda \right)^{1/m} \left(\int_{\Omega} |F_1|^{2/m} \exp(-\varphi) d\lambda \right)^{(m-1)/m}$$

which is smaller than

$$C_0^{1/m} A^{(m-1)/m} = A(C_0/A)^{1/m} =: A_1.$$

If $A > C_0$, then $A_1 < A$. We can then repeat the same argument with F_1 replaced by F_2 , etc, and get a decreasing sequence of constants A_k , such that

$$A_{k+1} = A_k(C_0/A_k)^{1/m}$$

for $k \geq 0$. It is easy to see that A_k tends to C_0 . Indeed, if $A_k > rC_0$ for some $r > 1$, then A_k would tend to zero by the relation above. This completes the proof. \square

For further use, we reformulate Proposition 1.2 in terms of differential forms. Since this is absolutely immediate, we provide no further explanations. As in 3.1.1(1), we will denote by $|u|_{\frac{2}{m}} = (cu \wedge \bar{u})^{1/m}$ for $u \in H^0(V, mK_V)$, with an absolute constant c .

Lemma 4.2.6. *Let m be a positive integer, and let u a holomorphic m -canonical form on V such that*

$$\int_V |u|_{\frac{2}{m}} e^{-\varphi} < \infty.$$

Then there exists a holomorphic m -canonical form U on Ω such that:

- (i) $U = u \wedge (d\sigma)^{\otimes m}$ on V i.e. the form U is an extension of u .
- (ii) The next $L^{2/m}$ bound holds true

$$\int_{\Omega} |U|_{\frac{2}{m}} e^{-\varphi} \leq C_0 \int_V |u|_{\frac{2}{m}} e^{-\varphi},$$

where C_0 is the constant in the Ohsawa-Takegoshi extension theorem.

After this preparation, we can complete the proof of the following result.

Theorem 4.2.7. *We assume that there exists a point $y \in Y_h \cap Y_{m,\text{ext}}$ such that*

$$H^0(X_y, (mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})) \neq 0.$$

Then the metric $(B_{m,X/Y}^o)^{-1}$ on $(mK_{X/Y} + L)|_{f^{-1}(Y_{m,\text{ext}})}$ obtained in 4.2.2 is not identically $+\infty$, and it extends as a singular Hermitian metric $(B_{m,X/Y})^{-1}$ on $mK_{X/Y} + L$ with semi-positive curvature current.

Remark 4.2.8. We ignore whether the equality $B_{m,X/Y}|_{X_y} = B_{m,y}$ holds for $y \in Y_0 \setminus Y_{m,\text{ext}}$. This issue will be discussed later in 4.3.5.

Proof. We recall the expression of $B_{m,X/Y}^o$ on a fiber of a point $y \in Y_{m,\text{ext}}$. Let $x \in X_y$ be an arbitrary point, and denote by $z = (z_1, \dots, z_k)$ respectively $t = (t_1, \dots, t_\ell)$ some local coordinates centered at x on an open set Ω , respectively y , and assume L is also trivialized on Ω . We consider a section $u \in H^0(X_y, mK_{X_y} + L_y)$. Then we have $\tilde{u} = u \wedge (f^*dt)^{\otimes m} \in H^0(X_y, (mK_X + L)|_{X_y})$ via the standard identification. We write as $\tilde{u} = u'(dz)^{\otimes m}$ with $u' \in H^0(\Omega_y, \mathcal{O}_{X_y})$, where $\Omega_y = \Omega \cap X_y$, and then we have

$$B_{m,X/Y}^o(x) = \exp(\varphi_{X/Y}^{(m)o}(x)) = \sup_{\|u\|_{m,y} \leq 1} |u'(x)|^2,$$

where $\|u\|_{m,y} = (\int_{X_y} |u|^{2/m} h_y^{1/m})^{m/2}$. Thus, our task is to show that the above quantity is bounded from above on the open set $X_{m,\text{ext}} \subset X$. Let u be a global section of the bundle $mK_{X_y} + L_y$, normalized by the condition

$$\int_{X_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\varphi_L} \leq 1.$$

By Lemma 4.2.6 we obtain a local m -canonical form \tilde{U} on Ω whose restriction to Ω_y is equal to \tilde{u} , and such that the next inequality holds

$$\int_{\Omega} |\tilde{U}|^{\frac{2}{m}} e^{-\frac{1}{m}\varphi_L} \leq C_0 \int_{\Omega_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m}\varphi_L} \leq C_0,$$

where C_0 is an absolute constant. We write as $\tilde{U} = U'(dz)^{\otimes m}$ with $U' \in H^0(\Omega, \mathcal{O}_X)$. Let $\Omega' \subset \Omega$ be the polydisc of poly-radius (r, \dots, r) centered at x . Then by the mean value inequality, we have

$$|U'(x)|^{\frac{2}{m}} \leq \frac{1}{(\pi r^2)^{\dim X}} \int_{\Omega'} |U'|^{\frac{2}{m}} d\lambda_z \leq \frac{1}{(\pi r^2)^{\dim X}} e^{\frac{1}{m} \sup_{\Omega'} \varphi_L} \int_{\Omega'} |\tilde{U}|^{\frac{2}{m}} e^{-\frac{1}{m}\varphi_L}.$$

As $U'|_{\Omega_y} = u'$, we thus have

$$(\sharp) \quad |u'(x)|^{\frac{2}{m}} \leq C,$$

where moreover the bound C above does not depend at all on the geometry of the fiber X_y , but on the ambient manifold X . (A direct application of the mean value inequality for $|u'|^{2/m}$ on Ω_y would encode the geometry of the fiber X_y .) In particular, the metric $(B_{m,X/Y}^o)^{-1}$ admits an extension to the whole manifold X and its curvature current is semi-positive, so 4.2.7 is proved. \square

4.3. A few consequences. We further investigate the positivity of $mK_{X/Y} + L$ and give some variants and refinements of 4.2.7. We still keep the notation $f : X \rightarrow Y$ and $(L, h_L = e^{-\varphi_L})$. The crucial observation is as follows.

Remark 4.3.1. The constant “ C ” in the relation (\sharp) of the proof of 4.2.7 above only depends on the sup of the m^{th} -root of the metric of the bundle L and the geometry of the ambient manifold X (and *not at all* on the geometry of the fiber X_y). As a consequence we see that if we apply the above arguments to a sequence $mK_{X/Y} + L$ when m varies, then the constant in question will be uniformly bounded with respect to m , provided that $(1/m)\varphi_L$ is bounded from above.

As a consequence of 4.3.1, we obtain the following corollaries (see also [12] and [42] for related statements).

Corollary 4.3.2. *Suppose that there exists a very general point $w \in Y$ such that we have*

$$H^0(X_w, (mK_{X_w} + L_w) \otimes \mathcal{J}_m(h_w^{1/m})) \neq 0.$$

Then the bundle $mK_{X/Y} + L$ admits a singular Hermitian metric $h_{X/Y}^{(\infty)} = \exp(-\varphi_{X/Y}^{(\infty)})$ with semi-positive curvature current, whose restriction to any very general fiber X_w of f has the

following (minimality) property: for any integer $k > 0$ and any section $v \in H^0(X_w, kmK_{X_w} + kL_w)$, we have

$$|v|_{\frac{2}{km}} e^{-\frac{1}{m}\varphi_{X/Y}^{(\infty)}} \leq \int_{X_w} |v|_{\frac{2}{km}} e^{-\frac{1}{m}\varphi_L}$$

up to the identification of $K_{X/Y}|_{X_w}$ with K_{X_w} .

In the statement of 4.3.2, we call a point $w \in Y$ *very general* if it is not a critical point of f and any section of the bundle $k(mK_{X/Y} + L)|_{X_w}$ extends over the nearby fibers, for any integer $k > 0$. By the usual semi-continuity arguments we infer that the set of very general points is the complement of a countable union of Zariski closed sets of codimension at least one in Y . If h_L has a better regularity, every regular value $y \in Y_0$ of f can be very general. For example, if h is continuous, or more generally the multiplier ideal $\mathcal{I}(h_y^k) = \mathcal{O}_{X_y}$ on X_y with $y \in Y_0$ for any integer $k > 0$, the invariance of plurigena [41] and [38] verify this.

Proof of 4.3.2. The proof is an immediate consequence of the arguments of 4.2.7. Indeed, we consider the dual of the km -Bergman metric (twisted by kL) on the bundle $k(mK_{X/Y} + L)$ and we know that the weights of its k^{th} -root are *uniformly* bounded, independently of k (see 4.3.1). We obtain the metric $h_{X/Y}^{(\infty)} = \exp(-\varphi_{X/Y}^{(\infty)})$ by the usual upper envelope construction, namely

$$\varphi_{X/Y}^{(\infty)} := \sup_{k \geq 1}^* \left(\frac{1}{k} \varphi_{X/Y}^{(km)} \right).$$

□

Corollary 4.3.2 admits a metric version which we discuss next.

Corollary 4.3.3. *Suppose that X is projective, and that there exists a very general point $w \in Y$ and a singular Hermitian metric $h_w = e^{-\varphi_w}$ of the bundle $mK_{X_w} + L_w$ with semi-positive curvature such that an integrability condition*

$$e^{\varphi_w - \varphi_L}|_{X_w} \in L^{\frac{1+\varepsilon}{m}}$$

holds locally at each point of X_w , where $\varepsilon > 0$ is a real number. (Here h_w does not mean $h_L|_{X_w}$.)

Then the bundle $mK_{X/Y} + L$ admits a singular Hermitian metric with semi-positive curvature, whose restriction to X_w is less singular than the metric h_w .

In the statement of 4.3.3, we call a point $w \in Y$ *very general*, if it is not a critical value of f and if any section of the bundle $(kmK_{X/Y} + kL + A)|_{X_w}$ extends over the nearby fibers, for any integer $k > 0$. Here A is an auxiliary ample line bundle on X , for example, we take $A = 2(\dim X)H$ and H is a very ample line bundle on X .

We remark that 4.3.3 is more *coherent* than 4.3.2, in the sense that we start with a metric on a very general fiber X_w of f , and we produce a metric defined on X . Even if we give ourselves a section on a very general fiber, the object we are able to produce (via 4.3.2) will be in general a metric, which is not necessarily algebraic (i.e., it is constructed by sections of the bundle), as one would hope or guess.

We first write $e^{-\varphi_w}$ as limit of a sequence of algebraic metrics $e^{-\varphi_w^{(k)}}$ of $(kmK_{X/Y} + kL + A)|_{X_w}$, and then we apply 4.2.7 to construct a sequence of global metrics on

$$kmK_{X/Y} + kL + A,$$

whose restriction to X_w is comparable with $e^{-\varphi_w^{(k)}}$. The metric we seek is obtained by a limit process. In order to complete this program we have to control several constants which are involved in the arguments, and this is possible thanks to the version of the Ohsawa-Takegoshi theorem 1.2. It would be very interesting to give a more direct, *sections-less* proof of 4.3.3.

Proof of 4.3.3. The choice of the bundle $A \rightarrow X$ as above is explained by the following fact we borrow from the approximation theorem in [11], or [13, §13].

We take a smooth Hermitian metric $e^{-\varphi_A}$ of A with strictly positive curvature. Associated to the metric $e^{-\varphi_w}$ on the bundle $mK_{X_w} + L_w$ given by hypothesis, we consider the following space of sections

$$(4.3.1) \quad \mathcal{V}_k \subset H^0(X_w, kmK_{X_w} + kL_w + A_w)$$

defined by $u \in \mathcal{V}_k$ if and only if

$$(4.3.2) \quad \|u\|_w^2 := \int_{X_w} |u|^2 \exp(-k\varphi_w - \varphi_A) dV_w < \infty.$$

We denote by $e^{-\varphi_w^{(k)}}$ the metric on the bundle $kmK_{X_w} + kL_w + A_w$ induced by an orthonormal basis $\{u_j^{(k)}\}_j$ of \mathcal{V}_k endowed with the scalar product corresponding to $\|u\|_w^2$. Then it is proved in [13, 13.21] that there exists a constant C_1 independent of k such that we have

$$(4.3.3) \quad k\varphi_w + \varphi_A \leq \varphi_w^{(k)} + C_1.$$

We remark that the ample bundle A is independent of the particular point $w \in Y$ as well as on the metric $e^{-\varphi_w}$. This fact is very important for the following arguments.

The Hölder inequality shows that the $(km)^{\text{th}}$ -root of the sections $v \in \mathcal{V}_k$ are integrable with respect to the metric $h_L^{1/m}$, as soon as k is large enough. Indeed we have:

$$(4.3.4) \quad \int_{X_w} |v|^{\frac{2}{km}} e^{-\frac{k\varphi_L + \varphi_A}{km}} \leq c_k \left(\int_{X_w} |v|^2 e^{-k\varphi_w - \varphi_A} dV_w \right)^{\frac{1}{km}},$$

where we use the following notation

$$c_k^{\frac{km}{km-1}} := \int_{X_w} e^{\frac{k}{km-1}(\varphi_w - \varphi_L)} dV_w < \infty.$$

We remark that the last integral is indeed convergent, by the integrability hypothesis $e^{\varphi_w - \varphi_L}|_{X_w} \in L^{\frac{1+\varepsilon}{m}}$, provided that $k \gg 0$. In conclusion, we get

$$(4.3.5) \quad \int_{X_w} |v|^{\frac{2}{km}} e^{-\frac{k\varphi_L + \varphi_A}{km}} \leq C_2$$

for every section $v := u_j^{(k)} \in \mathcal{V}_k$. By definition, the expression of the constant in the relation (4.3.5) is the following

$$C_2 := \sup_{k \gg 0} c_k < \infty,$$

therefore it is independent of k . By Theorem 4.2.7, for each k large enough we can construct the dual of the km -Bergman metric $h_{X/Y}^{(km)}$ on the bundle $kmK_{X/Y} + kL + A$ with semi-positive

curvature current. Thanks to the relation (4.3.5), its restriction to X_w is less singular than the metric $\varphi_w^{(k)}$, in the following precise way:

$$(4.3.6) \quad \varphi_{X/Y}^{(km)}|_{X_w} \geq \varphi_w^{(k)} - km \log C_2;$$

(we identify $K_{X/Y}$ with K_{X_w} , which is harmless in this context, since the point w is “far” from the critical loci of f). The inequality (4.3.6) combined with (4.3.3) shows furthermore that we have

$$(4.3.7) \quad \frac{1}{k} \varphi_{X/Y}^{(km)}|_{X_w} \geq \varphi_w + C_4,$$

where $C_4 := -m \log C_2$.

On the other hand, by 4.3.1 we have an a-priori upper bound

$$(4.3.8) \quad \frac{1}{k} \varphi_{X/Y}^{(km)} \leq C_5,$$

where C_5 is *uniform with respect to k* . Then the metric we seek is obtained by the usual upper envelope construction, namely

$$(4.3.9) \quad \varphi_{X/Y}^{(\infty)} := \limsup_{k \rightarrow \infty}^* \left(\frac{1}{k} \varphi_{X/Y}^{(km)} \right).$$

The local weights $(\varphi_{X/Y}^{(\infty)})$ glue together to give a metric for the bundle $mK_{X/Y} + L$, since the auxiliary bundle A used in the approximation process is removed by the normalization factor $1/k$. Moreover, the relation (4.3.7) shows that we have

$$(4.3.10) \quad \varphi_{X/Y}^{(\infty)}|_{X_w} \geq \varphi_w + \mathcal{O}(1);$$

hence, the local weight $\varphi_{X/Y}^{(\infty)}$ restricted to the fiber X_w is less singular than the metric $e^{-\varphi_w}$ and 4.3.3 is proved. \square

Remark 4.3.4. The Corollary 4.3.3 can be immediately adapted to the case where X is only assumed to be quasi-projective. Another by-product of the proof of this result is the case where $f : X \rightarrow Y$ is a projective family over a Stein manifold. The verification of these statements is left to the interested reader.

Remark 4.3.5. It is certainly worthwhile to understand the behavior of the metric constructed in 4.2.7 over the *exceptional fibers*, where it is just defined as the (unique) extension of $(B_{m,X/Y}^o)^{-1}$ (defined over $X_{m,\text{ext}} = f^{-1}(Y_{m,\text{ext}})$) as a singular Hermitian metric of $mK_{X/Y} + L$.

Let us here look at the fibers over a point $y \in Y$ where the a-priori extension of the sections is not known to be satisfied, but such that y is still a regular value of f (i.e., $y \in Y_0 \setminus Y_{m,\text{ext}}$). On the bundle $mK_{X_y} + L_y$ we can consider (at least) two natural extremal metrics: the one induced by *all* its global sections satisfying the $L^{2/m}$ integrability condition with respect to h_L which we denote by h_1 , and metric corresponding to the subspace of sections which extend locally near y , denoted by h_2 . It is clear that we have $h_1 \preceq h_2$ (h_1 is less singular than h_2 , refer [13, 6.3]), as one can see by the comparison between the corresponding m -Bergman kernels. Our claim is

Claim. *The metric $h_{X/Y}^{(m)} = B_{m,X/Y}^{-1}$ extends over X_y to a metric which is less singular than h_2 i.e., $h_{X/Y}^{(m)}|_{X_y} \preceq h_2$, at least if the singularities of the weight φ_L are “mild” enough.*

Indeed, as a consequence of 4.2.7 we have

$$\varphi_{X/Y}^{(m)}(x) = \limsup_{x' \rightarrow x} \varphi_{X/Y}^{(m)}(x'),$$

where $x' \in X_{m,\text{ext}}$ in the limit above. In order to establish the result we claim, let us consider a section $u \in H^0(X_y, mK_{X_y} + L_y)$ which computes the local weight of h_2 at x , i.e. $\|u\|_{m,y} = 1$ and

$$\sup_{\|v\|_{m,y}=1} |v'(x)|^2 = |u'(x)|^2$$

as in the proof of 4.2.7 ([3, pp. 346–347]). If $x' \in X_0$ is close enough to x and $y' := f(x') \in Y_{m,\text{ext}}$, then we certainly have

$$\sup_{\|v\|_{m,y'}=1} |v'(x')|^2 \geq \frac{|u'(x')|^2}{\|u\|_{m,y'}^2}$$

(here we use the fact that the section u extends over the fibers near X_y). Hence in order to establish the claim above, all that we need is the relation

$$\liminf_{y' \rightarrow y} \|u\|_{m,y'} \leq \|u\|_{m,y}.$$

This last inequality clearly holds at least if φ_L is continuous, and we actually believe that *it always holds*. \square

5. POSITIVITY OF DIRECT IMAGE SHEAVES OF PLURICANONICAL TYPE

Our aim here is to show 5.1.2; the positivity of direct image sheaves of pluricanonical type: $f_*(mK_{X/Y} + L)$.

5.1. Positivity of direct image sheaves of pluricanonical type. We shall discuss in the setup 3.2.1 throughout this subsection. We further introduce the following subset of Y to make statements clearer. For every integer $m > 0$, we set

$$Y_{m,h} := \{y \in Y_h; \mathcal{I}(h_y^{1/m}) = \mathcal{O}_{X_y}\}.$$

We can see $Y_{m,h} \subset Y_{\ell,h}$ if $m < \ell$ in general. Let us see further

Lemma 5.1.1. (1) *If $Y_{m,h}$ is not empty, $Y_0 \setminus Y_{m,h}$ has measure zero.*

(2) $Y_{1,h} = \{y \in Y_h; \mathcal{I}(h_y) = \mathcal{O}_{X_y}\} \subset (Y_{1,h,\text{ext}} \cap Y_{m,\text{ext}})$ for every integer $m > 0$.

(3) For $y \in Y_h$, $y \in Y_{m,h}$ if and only if $\mathcal{J}_m(h_y^{1/m}) = \mathcal{O}_{X_y}$.

Proof. (1) By Ohsawa-Takegoshi, we have $\mathcal{I}(h_y^{1/m}) \subset \mathcal{I}(h^{1/m}) \cdot \mathcal{O}_{X_y}$ for every $y \in Y_0$. If $Y_{m,h}$ is not empty, then $f(V\mathcal{I}(h^{1/m})) \subset Y$ is a proper analytic subset, where $V\mathcal{I}(h^{1/m}) \subset X$ is the complex subspace defined by $\mathcal{I}(h^{1/m})$. Thus by Fubini, $h_y^{1/m} \in L_{loc}^1$ on X_y for a.a. $y \in Y_0 \setminus f(V\mathcal{I}(h^{1/m}))$.

(2) It is clear that $Y_{1,h} \subset Y_{1,h,\text{ext}}$. Moreover by Siu [41] (see also [38]), if $y \in Y_{1,h}$, the natural homomorphism $f_*(mK_{X/Y} + L) \rightarrow H^0(X_y, mK_{X_y} + L_y)$ is surjective, and hence $Y_{1,h} \subset Y_{m,\text{ext}}$.

(3) We see $\mathcal{J}_m(h_y^{1/m}) \subset \mathcal{I}(h_y^{1/m})$, as $|u|^2 h_y^{1/m} \leq C_u |u|^{2/m} h_y^{1/m}$ for a constant $C_u > 0$ depends only on any given local holomorphic function u . In particular $\mathcal{I}(h_y^{1/m}) = \mathcal{O}_{X_y}$ if $\mathcal{J}_m(h_y^{1/m}) = \mathcal{O}_{X_y}$. While if $\mathcal{I}(h_y^{1/m}) = \mathcal{O}_{X_y}$, i.e., $h_y^{1/m} \in L_{loc}^1$, then $\mathcal{J}_m(h_y^{1/m}) = \mathcal{O}_{X_y}$. \square

We shall convert the positivity of $mK_{X/Y} + L$ to that of $f_*(mK_{X/Y} + L)$ via 3.3.5. As a consequence of 5.1.1, as soon as $f_*(mK_{X/Y} + L)$ is non-zero and $Y_{m,h} \neq \emptyset$, the assumption in 4.2.7: $H^0(X_y, (mK_{X_y} + L_y) \otimes \mathcal{J}_m(h_y^{1/m})) \neq 0$ for some $y \in Y_h \cap Y_{m,\text{ext}}$ is satisfied. We then can apply results in the previous section. The main result in this section is the following statement.

Theorem 5.1.2. *Suppose that $Y_{m,h} = \{y \in Y_h; \mathcal{I}(h_y^{1/m}) = \mathcal{O}_{X_y}\}$ is not empty. Then $f_*(mK_{X/Y} + L)$ (if it is non-zero) admits a singular Hermitian metric $g_{mNS,X/Y}$ with positive curvature and satisfying a base change property on $Y_{m,\text{ext}}$, which means, $g_{mNS,X/Y,y} = g_{mNS,y}$ holds on every fiber $f_*(mK_{X/Y} + L)_y = H^0(X_y, mK_{X_y} + L_y)$ at $y \in Y_{m,\text{ext}}$, where $g_{mNS,y}$ is the m -th Narashimhan-Simha Hermitian form on $H^0(X_y, mK_{X_y} + L_y)$ with respect to h_y . In particular, if Y is projective, $f_*(mK_{X/Y} + L)$ is weakly positive at every $y \in Y_{m,\text{ext}} \cap Y_{m,h}$ in the sense of Nakayama.*

Proof. As $Y_{m,h} \neq \emptyset$, the assumption in 4.2.7 is satisfied, and then we can put the fiberwise m -th Narashimhan-Simha Hermitian form $g_{mNS,y}$ on $f_*(mK_{X/Y} + L)_y = H^0(X_y, mK_{X_y} + L_y)$ for every $y \in Y_{m,\text{ext}}$. Moreover, by 3.1.1(4), it satisfies $0 < \det g_{mNS,y} < +\infty$ for every $y \in Y_{m,\text{ext}} \cap Y_{m,h}$. We apply 4.2.7 and have a singular Hermitian metric $h_{m-1} = (h^{(m)})^{(m-1)/m} h^{1/m}$ on $L_{m-1} := (m-1)K_{X/Y} + L$ with semi-positive curvature, and $h_{m-1}|_{X_y} = h_{m-1,y} := (h_y^{(m)})^{(m-1)/m} h_y^{1/m}$ on $(m-1)K_{X_y} + L_y$ for every $y \in Y_{m,\text{ext}}$.

As we remarked in 3.1.1, we have $H^0(X_y, (K_{X_y} + L_{m-1}|_{X_y}) \otimes \mathcal{I}(h_{m-1,y})) = H^0(X_y, K_{X_y} + L_{m-1}|_{X_y})$ for any $y \in Y_{m,h}$, in particular $Y_{m,h} \subset Y_{1,h_{m-1},\text{ext}}$ holds, where $Y_{1,h_{m-1},\text{ext}} := \{y \in$

$Y_{h_{m-1}}$; $H^0(X_y, (K_{X_y} + L_{m-1}|_{X_y}) \otimes \mathcal{I}(h_{m-1}|_{X_y})) = H^0(X_y, K_{X_y} + L_{m-1}|_{X_y})$. Thus, by 3.3.2 for (L_{m-1}, h_{m-1}) , $f_*(K_{X/Y} + L_{m-1})$ is locally free and the natural inclusion $f_*((K_{X/Y} + L_{m-1}) \otimes \mathcal{I}(h_{m-1})) \subset f_*(K_{X/Y} + L_{m-1})$ is isomorphic both on a Zariski open subset of Y containing $Y_{1, h_{m-1}, \text{ext}} (\supset Y_{m, h})$.

Then by 3.3.5 for (L_{m-1}, h_{m-1}) instead of (L, h) , $f_*(K_{X/Y} + L_{m-1})$ is positively curved with the canonical L^2 -metric with respect to h_{m-1} . Here the canonical L^2 -metric is nothing but the singular Hermitian metric g_m on $f_*(mK_{X/Y} + L)$, more precisely $g_{mNS, X/Y, y} = g_{mNS, y}$ for $y \in Y_{m, \text{ext}}$.

The pointwise weak positivity follows from a general result 2.5.2. \square

5.2. Variants and refinements. We will state some corollaries in the rest, and still keep the set up 3.2.1. We first give a statement of MMP flavor.

Corollary 5.2.1. *Let Δ be an effective \mathbb{Q} -divisor on X such that a pair (X, Δ) is klt. Then $f_*(\mathcal{O}_X(\lfloor m(K_{X/Y} + \Delta) \rfloor))$ has a singular Hermitian metric with positive curvature.*

Here $\lfloor \bullet \rfloor$ is the round down. In this algebraic setting, we can set up a singular Hermitian metric h so that h has the so-called algebraic/analytic singularities; (L, h) corresponds to a line bundle $\mathcal{O}_X(\lfloor m\Delta \rfloor)$ with a singular Hermitian metric obtained by a defining section of $\lfloor m\Delta \rfloor$. In such a case, $Y_h = \{y \in Y_0; X_y \not\subset \text{Supp } \Delta\}$ and $Y_{m, h} = \{y \in Y_h; (X_y, \frac{1}{m}\lfloor m\Delta_y \rfloor) \text{ is klt}\}$ are Zariski open (where $\Delta_y = \Delta|_{X_y}$ is the restriction as a divisor), and hence $g_{mNS, X/Y, y} = g_{mNS, y}$ on some Zariski open subset.

If we pose a stronger regularity assumption on the singular Hermitian metric h , we can obtain a stronger positivity conclusion on the direct image sheaves. For example,

Corollary 5.2.2. (1) *Suppose that $Y_{1, h} := \{y \in Y_h; \mathcal{I}(h_y) = \mathcal{O}_{X_y}\}$ is not empty. Then for every $m > 0$, the m -th Narashimhan-Simha singular Hermitian metric $g_m = g_{mNS, X/Y}$ on $f_*(mK_{X/Y} + L)$ in 5.1.2 satisfies $0 < \det g_m < \infty$ on $Y_{1, h}$ (which is contained in $Y_{m, \text{ext}} \cap Y_{m, h}$ by 5.1.1).*

(2) *Suppose h is continuous on $f^{-1}(Y_0) \subset X$. Then the m -th Narashimhan-Simha singular Hermitian metric g_m on $f_*(mK_{X/Y} + L)$ in 5.1.2 is continuous and $0 < \det g_m < \infty$ both on Y_0 . In particular, if Y is projective, $f_*(mK_{X/Y} + L)$ is weakly positive at every $y \in Y_0$ in the sense of Nakayama.*

Various weak positivities are known for direct images of pluricanonical type $f_*(mK_{X/Y})$ after Viehweg. However, according to Nakayama, Viehweg's method (and also Nakayama's method in [35]) can only show the weak positivity at some point $y \in Y$, and do not imply weak positivity at all points $y \in Y_0$.

It can be possible to draw several variants from the basic results above. For example, we have a result with variable m . The following statement (1) is a variant of 4.3.2.

Corollary 5.2.3. *Let $f : X \rightarrow Y$ be a projective surjective morphism of complex manifolds with connected fibers, and let $Y_0 \subset Y$ be the maximum Zariski open subset where f is smooth. Suppose that the Kodaira dimension of general fibers are non-negative: $\kappa(X_y) \geq 0$. Then*

(1) $K_{X/Y}$ is pseudo-effective and admits a singular Hermitian metric $h_{X/Y}$ with semi-positive curvature and satisfying, for any $y \in Y_0$, any integer $m > 0$, and any $u \in H^0(X_y, mK_{X_y})$,

$$(|u|^2 h_{X/Y}^m|_{X_y})^{1/m} \leq \int_{X_y} |u|^{2/m}$$

(up to the identification of $K_{X/Y}|_{X_y}$ with K_{X_y}).

(2) For any integer $m > 0$ with $f_*(mK_{X/Y}) \neq 0$, $f_*(K_{X/Y} + (m-1)K_{X/Y})$ endows with a canonical L^2 -metric g'_m with respect to $h_{X/Y}^{m-1}$ (of $(m-1)K_{X/Y}$ with $h_{X/Y}$ in (1)), is positively curved, and g'_m is continuous and $0 < \det g'_m < \infty$ both on Y_0 .

When X is further compact, we can use the notion (from [15, 1.5]) of singular Hermitian metric with minimal singularities, say h_{min} , of $K_{X/Y}$ with semi-positive curvature, and then $h_{min} \preceq h_{X/Y}$, i.e, h_{min} is less singular than $h_{X/Y}$. Then the same conclusion in 5.2.3 holds for h_{min} instead of $h_{X/Y}$.

Proof of 5.2.3. We are going to use 4.3.2 with $(L, h_L) = (\mathcal{O}_X, 1)$. Noting Siu's invariance of plurigenera [41], every point in Y_0 is a very general point in 4.3.2.

(1) In 4.3.2, m is fixed. We vary m here. For every integer m with $f_*(mK_{X/Y}) \neq 0$, $mK_{X/Y}$ admits a singular Hermitian metric $h^{(m)}$ with semi-positive curvature and satisfying, for any $y \in Y_0$, any integer $k > 0$, and any $v \in H^0(X_y, kmK_{X_y})$,

$$(|v|^2 (h^{(m)}|_{X_y})^k)^{1/(km)} \leq \int_{X_y} |v|^{2/(km)}$$

holds (up to the identification of $K_{X/Y}|_{X_y}$ with K_{X_y}). Here $|v|^2 (h^{(m)}|_{X_y})^k$ is a function on X_y , and $|v|^{2/(km)}$ is a semi-positive volume form on X_y .

We consider $h_{X/Y} := \inf_m^* (h^{(m)})^{1/m}$ a lower regularized limit (or writing as $h^{(m)} = e^{-\varphi^{(m)}}$ on each local coordinate with a weight function, we consider $\varphi := \sup_m^* (\frac{1}{m}\varphi^{(m)})$ an upper regularized limit), where m runs through all integers $m > 0$ with $f_*(mK_{X/Y}) \neq 0$. As $(L, h_L) = (\mathcal{O}_X, 1)$ in 4.3.1, an upper bound of $\varphi^{(m)}$ depends only on the geometry of X , and hence for every relatively compact set $U \subset X$, there exists a constant $C_U > 0$ such that $\varphi^{(m)} \leq \frac{1}{m}C_U \leq C_U$. As a consequence, $h_{X/Y} \not\equiv +\infty$, namely $h_{X/Y}$ defines a singular Hermitian metric, and then by definition, $h_{X/Y}$ is less singular than $(h^{(m)})^{1/m}$ for any integer $m > 0$ with $f_*(mK_{X/Y}) \neq 0$.

(2) We let $L = (m-1)K_{X/Y}$ with a singular Hermitian metric $h = h_{X/Y}^{m-1}$. By (1), we have $H^0(X_y, (K_{X_y} + L_y) \otimes \mathcal{I}(h_y)) = H^0(X_y, K_{X_y} + L_y)$ for any $y \in Y_0$, namely $Y_{1,h,\text{ext}} = Y_0$. In particular the natural inclusion $f_*((K_{X/Y} + L) \otimes \mathcal{I}(h)) \subset f_*(K_{X/Y} + L)$ is isomorphic over Y_0 by 3.3.2. We can further apply 3.3.5. \square

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