Donaldson-Tian-Yau's Conjectuture for general polarization

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1. *Introduction*

Setting-up: (X,L) : a polarized algebraic manifold, i.e.,

X : non-singular irreducible algebraic variety

L : very ample line bundle on X

In this talk, Donaldson-Tian-Yau's Conjecture for general polarization will be considered:

Conjecture: ***If*** (X,L) ***is strongly K-stable***, ***then the polarization class*** $c_1(L)$ ***admits a CSC Kähler metric.***

2. *Background Materials*

A test configuration $(\mathcal{X},\mathcal{L},\varphi)$ *for* (X,L)

Let $\mathcal{X} \subset \mathbf{C}\times\mathbf{P}^*(V)$ be a $\mathbf{C}^*$-invariant subset for the $\mathbf{C}^*$-action

$$\mathbf{C}^*\times(\mathbf{C}\times\mathbf{P}^*(V)) \ni (t, (z,p)) \to (tz,\varphi(t)p),$$

for a 1-PS φ: $\mathbf{C}^* \to \mathrm{SL}(V)$, where $\mathrm{SL}(V)$ acts naturally on the set $\mathbf{P}^*(V)$ of all hyperplanes in V passing through the origin, and we usually take $V = V_l = \Gamma(X,L^l)$ assuming that V has a natural metric structure such that $S^1 \subset \mathbf{C}^*$ acts isometrically on V.
A triple $(\mathcal{X},\mathcal{L},\varphi)$ is called a ***test configuration*** for (X,L) if
(1) $\mathcal{L}$ is the restriction to $\mathcal{X}$ of the pullback $\mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^*(V)}(1)$ of the hyperplane bundle on $\mathbf{P}^*(V)$ on which $\mathbf{C}^*$ acts naturally;
(2) $(\mathcal{X}_t, \mathcal{L}_t) \cong (X, L^l)$, $t\neq 0$,
for some positive integer l independent of the choice of t.
This l is called the ***exponent*** of the test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$.

The Chow norm on W_k by Zhang

$d(k)$:= degree of X in $\mathbf{P}^*(V_k)$ embedded by $|L^m|$, where $V_k := \Gamma(X, L^m)$, $m = kl$, $n = \dim X$, $W_k := \{S^{d(k)}(V_k)^*\}^{\otimes n+1}$. Let $0 \neq \mathrm{CH}_k(X) \in W_k$ be the Chow form for the irreducible reduced algebraic cycle X on $\mathbf{P}^*(V_k)$, so that $[\mathrm{CH}_k(X)] \in \mathbf{P}(W_k)$ is the Chow point for the cycle X. Consider the ***Chow norm*** $\|\ \|: W_k \to \mathbf{R}$,

$$0 \leq \|w\| \in \mathbf{R}, \qquad w \in W_k.$$

The Donaldson-Futaki invariant
$F_1 = F_1(\mu)$ *for* $\mu = (\mathcal{X},\mathcal{L},\varphi)$

- N_k := dim $\Gamma(X,L^k)$ = dim $\Gamma(\mathcal{X}_0,\mathcal{L}_0{}^m)$, where $k = l\,m$.
- w_k := weight of the $\mathbf{C}^*$-action on det $\Gamma(\mathcal{X}_0,\mathcal{L}_0{}^m)$

$$w_k / (kN_k) \quad (k >> 1)$$
$$= F_0(\mu) + F_1(\mu)\,k^{-1} + F_2(\mu)\,k^{-2} + \ldots$$

$F_1 = F_1(\mu)$ is called the ***Donaldson-Futaki invariant*** for test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$ of (X,L).

K-stability and Li-Xu's pathology

- A test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$ for (X,L) is called ***trivial*** if φ is trivial.
- (X,L) is called ***K-stable*** if the following conditions are satisfied:

(1) $F_1(\mu) \leq 0$ for all test configurations $\mu = (\mathcal{X},\mathcal{L},\varphi)$ for (X,L).

(2) If $F_1(\mu) = 0$, then the normalization test configuration of μ is trivial.

- In the original definition of K-stability by Donaldson, (2) is stated as "If $F_1(\mu) = 0$, then μ is trivial." However, Li-Xu gave an example of a nontrivial test configuraton $\mu = (\mathcal{X},\mathcal{L},\varphi)$ for $(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}1}(3))$ such that $F_1(\mu)$ vanishes, and that the normalization of μ is trivial. Hence the definition of K-stability is reformulated as above.

Characterization of F_1 and the Chow weight in terms of the Chow norm

For a test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$, we consider the homogeneous ideal $I = \oplus_m I_m$ for $\mathcal{X}_0$ in $\mathbf{P}^*(V)$, where $V := V_l$. For $k = l\,m$, we put

$$V_k = S^m(V) \,/\, I_m, \quad m = 1,2, \dots ,$$

where $S^k(V)$ denotes the k-th symmetric tensor product of V.
Then φ: $\mathbf{C}^* \to \mathrm{SL}(V)$ induces a represemntation $\varphi_k : \mathbf{C}^* \to \mathrm{GL}(V_k)$.
For its special linear form $\varphi_k^{\mathrm{SL}} : \mathbf{C}^* \to \mathrm{SL}(V_k)$ (modulo finite group), by using the Chow norm || ||, we set

$$\nu_k(s) := \log \| \varphi_k^{\mathrm{SL}}(\exp(s)) \cdot \mathrm{CH}_k(X) \|,$$

Let q_k be the (possibly rational) Chow weight of the $\mathbf{C}^*$-action by φ_k^{SL} on the line $\mathbf{C} \cdot \mathrm{CH}_k(\mathcal{X}_0)$. Then by writing $F_i(\mu)$ as F_i shortly, we obtain

$$q_k = \lim_{s \to -\infty} d\nu_k(s)/ds = (n+1)!\, c_1(L)^n[X]\, (F_1 k^n + F_2 k^{n-1} + F_3 k^{n-2} + \dots).$$

3. *The Donaldson-Futaki invariant* F_1 *is generalized to* f_1

Norms for the the infinitesimal generator u

For a test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$, we consider the infinitesimal generator u of φ satisfying $exp\,(2\pi u\sqrt{-1}) = \mathrm{id}_V$, so that

$$\varphi(exp\ s) = exp(su), \qquad s \in \mathbf{C}.$$

Then φ is called the 1-PS generated by u, and is written as φ_u. Let $b_1, b_2, \dots , b_N$ be the weights of the $\mathbf{C^*}$-action for φ, so that each $\varphi(t)$, $t \in \mathbf{C^*}$, is written as a diagonal matrix with the γ-th diagonal elements $t^{b\gamma}$, $\gamma=1,2,\dots, N$. Let n be the dimension of X, and l be the exponent of the test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$. Define

$$|u|_1 := l^{-n-1}(\ |b_1| + |b_2| + \dots + |b_N|\ ),$$
$$|u|_\infty := l^{-1} \max\{|b_1|, |b_2|, \dots , |b_N|\}.$$

Definition of f_1: $\mathcal{M} \to \mathbf{R} \cup \{-\infty\}$

Consider a sequence $\{\mu_j\}$ of test configurations $\mu_j = (\mathcal{X}_j, \mathcal{L}_j, \varphi_j)$ such that the exponent l_j of μ_j satisfies $l_j \to +\infty$ as $j \to \infty$.
Let $\mathcal{M}$ be the set of all such sequences $\{\mu_j\}$. For $s \in \mathbf{R}$, we define

$$\nu_j(s) := (|u_j|_\infty / |u_j|_1)\, l_j^{-n} \log \|\varphi_j(t) \cdot \mathrm{CH}_j(X)\|$$
$$= (|u_j|_\infty / |u_j|_1)\, l_j^{-n} \log \|exp(su_j / |u_j|_\infty) \cdot \mathrm{CH}_j(X)\|,$$

where $t = exp(s/|u_j|_\infty)$. We can then define

$$f_1(\{\mu_j\}) := \lim_{s \to -\infty} \underline{\lim}_{j \to \infty} d\nu_j / ds.$$

If the double limit commutes and if $\lim_{j \to \infty} |u_j|_1$ exists as a positive real number $r > 0$, then by characterization of F_1 in terms of the Chow norm, we obtain (compare this with Szekelyhidi's approach)

$$f_1(\{\mu_j\}) = r^{-1}(n+1)!\, c_1(L)^n [X]\, \underline{\lim}_{j \to \infty} F(\mu_j).$$

Some remark on f_1

For instance, for a test configuration $\mu = (\mathcal{X},\mathcal{L},\varphi)$ for (X,L) of exponent 1, let $\mu_j = (\mathcal{X}_j,\mathcal{L}_j,\varphi_j)$, j =1,2, … , be the test configurations such that

$$\mathcal{X}_j = \mathcal{X}, \quad \mathcal{L}_j = \mathcal{L}^j$$

and that $\varphi_j : \mathbf{C}^* \to \mathrm{SL}(V_j)$ is induced by $\varphi: \mathbf{C}^* \to \mathrm{SL}(V)$, where $V_j := \Gamma(\mathcal{X}_0,\mathcal{L}_0^{\,j})$ and $V := \Gamma(\mathcal{X}_0,\mathcal{L}_0)$. Replacing $\{\mu_j\}$ by its subsequence if necessary, we may assume that $r = \lim_{j\to\infty} |u_j|_1 \geq 0$ exists.

For this sequence, if $r > 0$, then the double limit commutes, so that

$$f_1(\{\mu_j\}) = r^{-1}(n+1)!\, c_1(L)^n [X]\, F_1(\mu).$$

Note that, for the test configuration in Li-Xu's pathology, $r = 0$.

4. *Stabilities*

Asymptotic Chow-stability

Let G_k := SL(V_k) which naturally acts on V_k and also on W_k.
For the Chow form CH$_k$(X) for $X \subset \mathbf{P}^*(V_k)$, we consider its orbit $G_k\cdot$ CH$_k$(X) in W_k.

Definition: (1) (X,L^k) is called ***Chow-stable***, if $G_k\cdot$ CH$_k$(X) is closed in W_k and the isotropy subgroup of G_k at CH$_k$(X) is finite.
(2) (X,L) is called ***asymptotically Chow-stable***, if for all $k >> 1$, (X,L^k) is Chow-stable.

Hilbert-Mumford stability criterion

Definition: (1) Fix a Hermitian metric ρ_k on V_k. We define $(\mathbf{sl}_k)_{\mathbf{Z}}$ as the set of all $u \in \mathbf{sl}(V_k)$ such that $exp(2\pi\sqrt{-1}u) = 1_{Vk}$ and that the circle group $exp(2\pi\sqrt{-1}su)$, $s \in \mathbf{R}$, acts isometrically on (V_k, ρ_k).
(2) For each $u \in \mathbf{sl}(V_k)$, let $\mathbf{G}_u$ denote the 1-dimensional algebraic torus in $\mathrm{SL}(V_k)$ generated by u.

Then by the Hilbert-Mumford stability criterion, in order to show the closedness of $G_k \cdot \mathrm{CH}_k(X)$ in W_k, it suffices to show the closedness of the orbit $\mathbf{G}_u \cdot \mathrm{CH}_k(X)$ in W_k for all $0 \neq u \in (\mathbf{sl}_k)_{\mathbf{Z}}$, i.e., suffices to show that the Chow weight $q(u)$ at $\lim_{s \to -\infty} exp(2\pi\sqrt{-1}su) \cdot \mathrm{CH}_k(X)$ is negative for all $0 \neq u \in (\mathbf{sl}_k)_{\mathbf{Z}}$.

Strong K-stability

For each $u \in (\mathbf{sl}_k)_{\mathbf{Z}}$, we consider the 1-PS φ_u: $\mathbf{C}^* \to \mathrm{SL}(V_k)$ generated by u, and let $(\mathcal{X}^u, \mathcal{L}^u, \varphi_u)$ be the associated test configuration obtained as the DeContini Procesi family.

Let $\mathcal{M}$ be the set of all sequences $\mu_j = (\mathcal{X}^{u_j}, \mathcal{L}^{u_j}, \varphi_{u_j})$, $j = 1, 2, \dots$, of test configurations for (X, L) such that $u_j \in (\mathbf{sl}_k)_{\mathbf{Z}}$, and that the exponent l_j of μ_j satisfies $l_j \to +\infty$ as $j \to \infty$.

Definition: (1) (X,L) is ***strongly K-semistable***, if $f_1(\{\mu_j\}) \leq 0$ for all $\{\mu_j\} \in \mathcal{M}$.

(2) Let (X,L) be strongly K-semistable. Then (X,L) is called ***strongly K-stable***, if the equality $f_1(\{\mu_j\}) = 0$ for $\{\mu_j\} \in \mathcal{M}$ implies that there exists a j_0 such that μ_j is trivial for all j satisfying $j \geq j_0$.

Strong K-stability and Li-Xu's pathology

Li-Xu's pathology doesn't occur in our new definition of f_1.
Actually, for their example of a test configuration, we have $f_1 = -\infty$ (see arXiv: 1305.6411). Hence the following conjecture in the introduction is proposed:

C***onjecture***: ***If*** **(*X,L*)** ***is strongly K-stable, then the polarization class*** **c_1(*L*)** ***admits a CSC Kähler metric.***

Strong K-stability implies asymptotic Chow-stability

Our strong K-stability concept seems to be natural in the sense that we have the following result (“*Strong K-stability and asymptotic Chow-stability*”, joint work with Y. Nitta, arXiv: 1307.1959):

Theorem*: *If* (X,L) *is strongly K-stable, then* (X,L) *is asymptotically Chow-stable.

Outline of proof for the Theorem

We here explain how, for $l >> 1$, the Chow weight $q(u)$ is shown to satisfy: $q(u) < 0$ for all $0 \neq u \in (\mathtt{sl}_l)_{\mathbf{Z}}$. Assume, for contradiction, that there exists a sequence

$$l_1 < l_2 < \ldots < l_j < \ldots$$

with $0 \neq u_j \in (\mathtt{sl}_{l_j})_{\mathbf{Z}}$ such that $q(u_j) \geq 0$ for all positive integers j.

Now we consider the test configurations

$$(\mathcal{X}_j, \mathcal{L}_j, \varphi_j), \qquad j = 1,2, \ldots ,$$

associated to u_j above. By the characterization of F_1 and the Chow weight in terms of the Chow norm, we obtain

$$0 \leq l_j^{-n} |u_j|_1^{-1} q(u_j) = \lim_{s\to-\infty} d\nu_j(s)/ds$$

By convexity of the function ν_j, we have $d\nu_j(s)/ds \geq 0$ on $-\infty<s<-\infty$.

Then by taking $\lim_{s\to-\infty}\underline{\lim}_{j\to\infty}$, we obtain $f_1(\{\mu_j\}) \geq 0$. Hence by strong K-stability, it follows that $f_1(\{\mu_j\}) = 0$ and μ_j is trivial for $j >> 1$.

This is a contradiction to the fact that $u_j \neq 0$ for all j.

5. *Existence of CSC Kähler metrics*

Geometry of Hermitian metrics on a complex vector space V

Let ρ_1, ρ_2 be Hermitian metrics on a vector space V. Then for a suitable orthonormal basis $(e_1, e_2, \ldots, e_N)$ for (V, ρ_1), we can write

$$\rho_2(e_\alpha, e_\beta) = 0, \qquad \alpha \neq \beta,$$
$$\rho_2(e_\alpha, e_\alpha) = \lambda_\alpha, \qquad \alpha = 1,2, \ldots, N,$$

where λ_α are positive real numbers. Replacing ρ_2 by its positive constant multiple, we may assume that $\prod_\alpha \lambda_\alpha = 1$. Put $b_\alpha = \log \sqrt{\lambda_\alpha}$. Then the 1-PS $\varphi(exp\ s) = \sum_\alpha exp(b_\alpha s)\, e_\alpha \otimes e_\alpha^*$ from $\mathbf{R}_+$ to $\mathrm{SL}(V)$ interpolates ρ_1 and ρ_2 in the sense that $\varphi(exp\ s) \cdot \rho_1$ is ρ_1 or ρ_2 accoding as $s = 0$ or $s = 1$.

The (multiplicative) ***C^0-distance*** $d(\rho_1, \rho_2)$ between ρ_1 and ρ_2 is defined as

$$d(\rho_1, \rho_2) = C \max\{\, |b_\alpha| \,;\, \alpha = 1,2, \ldots, N \},$$

where C is a positive real constant depending only on V. We often reparametrize $\varphi(exp\ s)$ by replacing s by $d(\rho_1, \rho_2)^{-1} s$. Namely, $\varphi(exp\ s) = \sum_\alpha exp(b_\alpha d(\rho_1, \rho_2)^{-1} s)\, e_\alpha \otimes e_\alpha^*$ is ***C^0-distance parametrized*** in the sense that $\varphi(exp\ s) \cdot \rho_1$ is ρ_1 or ρ_2 accoding as $s = 0$ or $s = d(\rho_1, \rho_2)$.

An outline of our approach (I)

Assume that (X,L) is strongly K-stable relative to T. By the joint work with Nitta, (X,L) is asymptotically Chow-stable. Then replacing L by its positive integral multiple, we may assume that (X,L^j) are Chow-stable for all positive integers j. Hence we have a sequence of balanced metrics ω_j in the sense that it is a critical point of the Chow norm satisfying $B_j(\omega_j)$ = constant.

Fix a Hermitian metric h on L such that $\omega = c_1(L,h)$ is Kähler.
Let $\omega_j = c_1(L,h_j)$ be a balanced metric. Then $V_j = H^0(X, L^j)$ has Hermitian metrics ρ_1 and ρ_j defined by

$$\rho_1(\tau,\tau') = \int_X (\tau,\tau')_h\, \omega^n \quad \text{and} \quad \rho_j(\tau,\tau') = \int_X (\tau,\tau')_{hj}\, \omega_j^{\,n},$$

for $\tau, \tau' \in V_j$. We then have the interpolation of ρ_1 and ρ_2 by a 1-PS φ_j of $SL(V_j)$, where replacing s by $d(\rho_1,\rho_2)^{-1}s$, we may assume that φ_j is C^0-distance parametrized.

An outline of our approach (II)

- If $\beta_j = (b_1, b_2, \dots, b_{Nj})$ is a rational vector, then 1-PS φ_j comes from a $\mathbf{C}^*$-action. If not, approximating β_j by a sequence of rational vectors, we may assume that β_j is a rational vector.
- Since the Chow norm is critical at balanced metrics, we have $d\nu_j(s)/ds = 0$ at s corresponding to the Chow norm, i.e., at $s = - d(\rho_1,\rho_j)$.
- Then we consider the sequence of test configurations $\mu_j = (\mathcal{X}_j,\mathcal{L}_j,\varphi_j)$ associated to φ_j.

Put $d_\infty = \sup_j d(\rho_1,\rho_j)$

Then the following two cases are possible:

(Case 1) $d_\infty < +\infty$.

(Case 2) $d_\infty = +\infty$.

An outline of our aproach (III)

・If Case 1 occurs, then the C^0-distances $d(\rho_1,\rho_j)$ are uniformly bounded from above, and hence it is not difficult to show that a suitable subsequence of the balanced Kähler metrics ω_j, j =1,2, … , converges to a CSC Kähler metric.

・If Case 2 occurs, replacing $\{\omega_j\}$ by its subsequence if necessary, we may assume that the sequence $d(\rho_1,\rho_j)$, j =1,2, … , is monotone-increasing (to + ∞), as $j \to \infty$.

Note that $d\nu_j(s)/ds = 0$ at $s = -\, d(\rho_1,\rho_j)$. Then for every j' with $j' \geq j$, since $d\nu_{j'}(s)/ds$ is non-decreasing in s,

$$0 \;=\; d\nu_{j'}(s)/ds|_{s=-d(\rho_1,\rho_{j'})} \;\leq\; d\nu_{j'}(s)/ds|_{s=-d(\rho_1,\rho_j)}$$

Hence the non-decreasing function $\underline{\lim}_{j'\to\infty}\, d\nu_{j'}(s)/ds$ satisfies ≥ 0 on the interval $-d(\rho_1,\rho_j) \leq s$. Since $d(\rho_1,\rho_j) \to +\infty$ as $j \to \infty$, it now follows that $\underline{\lim}_{j'\to\infty}\, d\nu_{j'}(s)/ds \geq 0$ on the whole real line $-\infty < s < +\infty$. Then

$f_1(\{\mu_j\}) = \lim_{s\to-\infty} \underline{\lim}_{j'\to\infty}\, d\nu_{j'}(s)/ds \geq 0$. Now by strong K-stability of (X,L), $f_1(\{\mu_j\}) = 0$ and μ_j are trivial for $j >>1$.

Thus $\{\omega_j\}$ converges to a CSC Kähler metric.

6. *Concluding remarks*

Why strong K-stability ?

E : irreducible holomorphic vector bundle over a
compact Kähler manifold (M,ω)

$\mu(\mathcal{S}) := \deg(\mathcal{S})/rk(\mathcal{S})$ for $\mathcal{S}$ below.

E is *stable* (*in the sense of Mumford-Takemoto*)

$\Leftrightarrow$ $\mu(\mathcal{S}) < \mu(E)$ $\forall$ coherent subsheaf $\mathcal{S}$ of $\mathcal{O}(E)$
with $0 < rk(\mathcal{S}) < rk(E)$

Remark: If $\dim_{\mathbf{C}} M = 1$, then $\mathcal{S}$ can be chosen as a vector subbundle of E.

Hitchin-Kobayashi correspondence (Narasimhan-Seshadri, Kobayashi, Lübke, Donaldson, Uhlenbeck-Yau):

E is stable $\Leftrightarrow$ $\exists$ *Hermitian-Einstein metric on E*

Manifolds case versus vector bundles case in the Hitchin-Kobayashi correspondence

vector bundles	*manifolds*
holomorphic vector bundles	polarized algebraic manifolds
vector subbundles	test configurations
rank of a vector subbundle	exponent of a test configuration
coherent subsheaf	?
slope $\nu(\mathcal{S}) = \deg(\mathcal{S})/\mathrm{rk}(\mathcal{S})$	the Donaldson-Futaki invariant
$\nu(\mathcal{S}) \leq \nu(E)$	$F_1(\mu) \leq 0$
the Mumford-Takemoto stability	(strong) K-stability
compact Riemann surfaces case	Kähler-Einstein case

This indicates the necessity of considering a suitable compactification $\mathcal{M}$ of the moduli space of test configurations for (X,L).

K-stability versus strong K-stability

Recent results by Tian and Chen-Donaldson-Sun show that K-stability is equivalent (up to Li-Xu's pathology) to strong K-stability for Kähler-Einstein case, i.e., for anti-canonical polarization. Donaldson's result for toric case shows also that these two stability concepts coincide for toric case.

However, for general polarization L on a general X, the graded **C**-*algebra associated to the limit* μ_∞ *in the compactified moduli space* $\mathcal{M}$ *is not finitely generated in general. Hence we proposed the concept of strong K-stability.*

Extremal Kähler case

By taking a maximal algebraic torus T in Aut(X), we see that the arguments above is valid not only for non-discrete automorphisms cases but also for extremal Kähler cases.

For extremal Kähler cases,
(1) *balanced metrics, the Chow norm, asymptotic Chow stability, CSC Kähler metrics, strong K-stability, the Kodaira embedding have to be replaced by*
(2) *polybalanced metrics, the weighted Chow norm, asymptotic relative Chow stability, extremal Kähler metrics, strong relative K-stability, the weighted Kodaira embedding, respectively.*

Thank you.