Tian's properness conjectures and Finsler geometry of the space of Kähler metrics

Tamás Darvas and Yanir A. Rubinstein

Abstract

Well-known conjectures of Tian predict that existence of canonical Kähler metrics should be equivalent to various notions of properness of Mabuchi's K-energy functional. In some instances this has been verified, especially under restrictive assumptions on the automorphism group. We provide counterexamples to the original conjecture in the presence of continuous automorphisms. The construction hinges upon an alternative approach to properness that uses in an essential way the metric completion with respect to a *Finsler* metric and its quotients with respect to group actions. This approach also allows us to formulate and prove new optimal versions of Tian's conjecture in the setting of smooth and singular Kähler–Einstein metrics, with or without automorphisms, as well as for Kähler–Ricci solitons. Moreover, we reduce both Tian's original conjecture (in the absence of automorphisms) and our modification of it (in the presence of automorphisms) in the general case of constant scalar curvature metrics to a conjecture on regularity of minimizers of the K-energy in the Finsler metric completion. Finally, our results also resolve Tian's conjecture on the Moser–Trudinger inequality for Fano manifolds with Kähler–Einstein metrics.

1 Introduction

The main motivation for our work is Tian's properness conjecture. Consider the space

$$\mathcal{H} = \{\omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi \,:\, \varphi \in C^\infty(M),\, \omega_\varphi > 0\} \tag{1}$$

of all Kähler metrics representing a fixed cohomology class on a compact Kähler manifold (M, J, ω)

Motivated by results in conformal geometry and the direct method in the calculus of variations, in the 90's Tian introduced the notion of "properness on $\mathcal{H}$" [57, Definition 5.1] in terms of the Aubin nonlinear energy functional J [1] and the Mabuchi K-energy E [40] as follows.

Definition 1.1. The functional $E : \mathcal{H} \to \mathbb{R}$ is said to be proper if

$$\forall\, \omega_j \in \mathcal{H}, \quad \lim_j J(\omega_j) \to \infty \quad \Longrightarrow \quad \lim_j E(\omega_j) \to \infty. \tag{2}$$

Tian made the following influential conjecture [57, Remark 5.2], [59, Conjecture 7.12]. Denote by $\mathrm{Aut}(M, \mathrm{J})_0$ the identity component of the group of automorphisms of (M, J), and denote by $\mathrm{aut}(M, \mathrm{J})$ its Lie algebra, consisting of holomorphic vector fields.

Conjecture 1.2. *Let (M, J, ω) be a compact Kähler manifold.*
(i) If $\mathrm{aut}(M, \mathrm{J}) = 0$ then $\mathcal{H}$ contains a constant scalar curvature metric if and only if E is proper.

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(ii) Let K be a maximally compact subgroup of $\mathrm{Aut}(M,\mathrm{J})_0$. *Then $\mathcal{H}$ contains a constant scalar curvature metric if and only if E is proper on the subset $\mathcal{H}^K \subset \mathcal{H}$ consisting of K-invariant metrics.*

Tian's conjecture is central in Kähler geometry and has attracted much work over the past two decades including motivating much work on equivalence between algebro-geometric notions of stability and existence of canonical metrics, as well as on the interface of pluripotential theory and Monge–Ampère equations. We refer to the surveys [56, 44, 61, 42, 48].

We provide counterexamples to Conjecture 1.2 (ii)—see Example 2.2 below. Perhaps more interesting than the examples themselves is the realization that Tian's conjecture should be modified and phrased in terms of a Finsler structure on the space of Kähler metrics and properties of its metric completion. The metric completion approach turns out not only to be convenient but indispensable. In fact, our results show that properness with respect to the Finsler distance function *characterizes* existence of canonical Kähler metrics in many cases. It allows us to simultaneously unify, extend, and give new proofs of a number of instances of Tian's conjecture and our modification of it, as well as resolve some of the remaining open cases. Moreover, we reduce the properness conjecture in the most general case of constant scalar curvature metrics to a problem on minimizers of the K-energy in the Finsler metric completion. Finally, our results immediately imply Tian's conjecture on the Moser–Trudinger inequality for Kähler–Einstein Fano manifolds. This is the Kähler geometry analogue of Aubin's strong Moser–Trudinger inequality on S^2 in conformal geometry.

Approaching problems in Kähler geometry through an infinite-dimensional *Riemannian* perspective goes back to Calabi in 1953 [17] and later Mabuchi in 1986 [40]. These works proposed two different weak Riemannian metrics of L^2 type which have been studied extensively since. Historically, Calabi raised the question of computing the completion of his metric, which suggested a relation between the existence of canonical metrics on the finite-dimensional manifold M and the metric completion of the infinite-dimensional space $\mathcal{H}$. The first result in this spirit is due to Clarke–Rubinstein [22] who computed the Calabi metric completion, and proved the existence of Kähler–Einstein metrics on M is equivalent to the Ricci flow converging in the Calabi metric completion. Confirming a conjecture of Guedj [33], the Mabuchi metric completion was computed recently in [23] and in [24] a corresponding result for the Ricci flow in the Mabuchi metric completion was proved. Other results include the work of Streets [54], who shows that one gains new insight on the long time behavior of the Calabi flow by placing it in the context of the abstract Riemannian metric completion of the Mabuchi metric. In Darvas–He [25], the asymptotic behavior of the Kähler-Ricci flow in the metric completion is related to destabilizing geodesic rays. We refer the reader to the survey [48] for more references.

Perhaps surprisingly, a key observation of the present article is that not a Riemannian but rather a *Finsler* metric encodes the asymptotic behavior of essentially all energy functionals on $\mathcal{H}$ whose critical points are precisely various types of canonical metrics in Kähler geometry. In fact, as pointed out in Remark 7.3, the same kind of statement is in general false for the much-studied Riemannian metrics of Calabi and Mabuchi. The Finsler structure that we use was introduced in [24] where its metric completion was computed.

2 Results

Much of the progress on Conjecture 1.2 has focused on the case of Kähler–Einstein, Kähler–Einstein edge, or Kähler–Ricci soliton metrics. In the setting of Kähler–Einstein metrics, one direction of the conjecture (properness implies existence) follows from work of Ding–Tian

[28], while the converse for Conjecture 1.2 (i) was established by Tian [58] under a technical assumption that was removed by Tian–Zhu [62]. The result furnished the first 'stability' criterion equivalent to the existence of Kähler–Einstein metrics, in the absence of holomorphic vector fields.

Our first result disproves Conjecture 1.2 (ii) already in the setting of Kähler–Einstein metrics, and establishes an optimal replacement for it.

Theorem 2.1. *Suppose* (M, J, ω) *is Fano and that* K *is a maximal compact subgroup of* $\mathrm{Aut}(M, \mathrm{J})_0$ *with* $\omega \in \mathcal{H}^K$. *The following are equivalent:*

(i) There exists a Kähler–Einstein metric in $\mathcal{H}^K$ *and* $\mathrm{Aut}(M, \mathrm{J})_0$ *has finite center.*

(ii) There exists $C, D > 0$ *such that* $E(\eta) \geq CJ(\eta) - D, \ \eta \in \mathcal{H}^K$.

The estimate in (ii) gives a concrete version of the properness condition (2). The direction (i) $\Rightarrow$ (ii) is due to Phong et al. [43, Theorem 2], building on earlier work of Tian [58] and Tian–Zhu [62] in the case $\mathrm{aut}(M, \mathrm{J}) = 0$, who obtained a weaker inequality in (ii) with J replaced by J^δ for some $\delta \in (0, 1)$ (for more details see also the survey [61, p. 131]).

Example 2.2. Let M denote the blow-up of $\mathbb{P}^2$ at three non colinear points. It is well-known that it admits Kähler–Einstein metrics [51, 60]. According to [29, Theorem 8.4.2],

$$\mathrm{Aut}(M, \mathrm{J})_0 = (\mathbb{C}^\star)^2.$$

In particular, $\mathrm{Aut}(M, \mathrm{J})_0$ is equal to its center. Thus, Conjecture 1.2 (ii) fails for M by Theorem 2.1 (cf. [58, Theorem 4.4]).

These results motivate a reformulation of Tian's original conjecture. Albeit being a purely analytic criterion, properness should be morally equivalent to properness in a metric geometry sense, namely, that the Mabuchi functional should grow at least linearly relative to some metric on $\mathcal{H}$ precisely when a Kähler–Einstein metric exists in $\mathcal{H}$. Our goal in this article is to make this intuition rigorous.

To state our results we introduce some of the basic notions. Recall (3); the space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(M) \, : \, \omega_\varphi \in \mathcal{H}\} \tag{3}$$

can be identified with $\mathcal{H} \times \mathbb{R}$. Consider the following weak Finsler metric on $\mathcal{H}_\omega$ [24]:

$$\|\xi\|_\varphi := V^{-1} \int_M |\xi| \omega_\varphi^n, \quad \xi \in T_\varphi \mathcal{H}_\omega = C^\infty(M). \tag{4}$$

We denote by $d_1 : \mathcal{H}_\omega \times \mathcal{H}_\omega \to \mathbb{R}_+$ the associated path length pseudometric. According to [24] it is a bona fide metric. By looking at level sets of the Aubin–Mabuchi energy, it is possible to embed $\mathcal{H}$ into $\mathcal{H}_\omega$ (see (23)), giving a metric space $(\mathcal{H}, d_1)$.

Suppose G is a subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$. We will prove that G acts on $\mathcal{H}$ by d_1-isometries, hence induces a pseudometric on the orbit space $\mathcal{H}/G$,

$$d_{1,G}(Gu, Gv) := \inf_{f,g \in G} d_1(f.u, g.v).$$

Following Zhou–Zhu [69, Definition 2.1] and Tian [61, Definition 2.5],[64], we also define the descent of J to $\mathcal{H}/G$,

$$J_G(Gu) := \inf_{g \in G} J(g.u).$$

Definition 2.3. Let $F : \mathcal{H} \to \mathbb{R}$ be G-invariant.
• We say F is $d_{1,G}$-proper if for some $C, D > 0$,

$$F(u) \geq Cd_{1,G}(G0, Gu) - D.$$

• We say F is J_G-proper if for some $C, D > 0$,

$$F(u) \geq CJ_G(Gu) - D.$$

The following result complements Theorem 2.1 by giving yet another optimal replacement for Conjecture 1.2 (ii) in the setting of Kähler–Einstein metrics. The direction (iii) $\Rightarrow$ (i) is due to Tian [61, Theorem 2.6]. In the case of toric Fano manifolds, a variant of the direction (i) $\Rightarrow$ (iii) is due to Zhou–Zhu [69, Theorem 0.2].

Theorem 2.4. *Suppose (M, J, ω) is Fano. Set $G := \mathrm{Aut}(M, \mathrm{J})_0$. The following are equivalent:*
(i) There exists a Kähler–Einstein metric in $\mathcal{H}$.
(ii) E is G-invariant and its descent to the quotient space $\mathcal{H}/G$ is $d_{1,G}$-proper.
(iii) E is G-invariant and its descent to the quotient space $\mathcal{H}/G$ is J_G-proper.

Remark 2.5. In both Theorems 2.1 and 2.4 it is possible to replace E with the Ding functional [27]. See Theorems 7.2 and 7.1.

It is interesting to compare Theorem 2.4 with yet another—perhaps more familiar—notion of properness. Denote by Λ_1 the real eigenspace of the smallest positive eigenvalue of $-\Delta_\omega$, and set

$$\mathcal{H}_\omega^\perp := \{\varphi \in \mathcal{H} : \int \varphi\psi\omega^n = 0, \ \forall \psi \in \Lambda_1\}.$$

When ω is Kähler–Einstein, it is well-known that Λ_1 is in a one-to-one correspondence with holomorphic gradient vector fields [32]. Tian made the following conjecture in the 90's [58, Conjecture 5.5], [59, Conjecture 6.23],[61, Conjecture 2.15].

Conjecture 2.6. *Suppose (M, J, ω) is Fano Kähler-Einstein. Then for some $C, D > 0$,*

$$E(\varphi) \geq CJ(\varphi) - D, \qquad \varphi \in \mathcal{H}_\omega^\perp.$$

Conjecture 2.6 was originally motivated by results in conformal geometry related to the determination of the best constants in the borderline case of the Sobolev inequality. By restricting to functions orthogonal to the first eigenspace of the Laplacian, Aubin was able to improve the constant in the aforementioned inequality on spheres [2, p. 235]. This can be seen as the sort of coercivity of the Yamabe energy occuring in the Yamabe problem, and it clearly fails without the orthogonality assumption due to the presence of conformal maps. Conjecture 2.6 stands in clear analogy with the picture in conformal geometry, by stipulating that coercivity of the K-energy holds in 'directions perpendicular to holomorphic maps.' It can be thought of as a higher-dimensional fully nonlinear generalization of the classical Moser–Trudinger inequality.

It is a rather simple consequence of the work of Bando–Mabuchi [3] that when a Kähler–Einstein metric exists, J_G-properness implies J-properness on $\mathcal{H}_\omega^\perp$ [58, Corollary 5.4],[69, Lemma A.2],[61, Theorem 2.6]. Therefore, Theorem 2.4 resolves Tian's conjecture.

Corollary 2.7. *Conjecture 2.6 holds.*

In particular, this provides a new functional inequality on S^2. This inequality seems different from Aubin's well-known inequality [2, Theorem 6.70], especially in view of Sano's example [49, Remark 1.1]. It would be interesting to compare it to [47, Theorem 10.11].

Motivated by Theorem 2.4 we make the following conjecture. In the case of Kähler–Einstein metrics, Tian [61, p. 127] already conjectured the equivalence of (i) and (iii).

Conjecture 2.8. *Let (M, J, ω) be a compact Kähler manifold. Set $G := \mathrm{Aut}(M, \mathrm{J})_0$. The following are equivalent:*

(i) There exists a constant scalar curvature metric in $\mathcal{H}$.
(ii) E is G-invariant and its descent to the quotient space $\mathcal{H}/G$ is $d_{1,G}$-proper.
(iii) E is G-invariant and its descent to the quotient space $\mathcal{H}/G$ is J_G-proper.

It would be interesting to compare Conjecture 2.8 to [20, Conjecture 6.1]. We refer the reader to Remark 7.3.

We reduce Conjecture 2.8 to a purely PDE problem of regularity of minimizers. To phrase the result, we denote by $\mathcal{E}_1$ the d_1-metric completion of $\mathcal{H}$. We denote still by E the greatest d_1-lower semicontinuous extension of E to $\mathcal{E}_1$. We refer to Sections 4–5 for precise details.

Conjecture 2.9. *Minimizers of E over $\mathcal{E}_1$ are smooth.*

Conjecture 2.9 is inspired by previous work of many authors and, as recalled in §5.5, it is already known in the case of Kähler–Einstein (edge) metrics or Kähler–Ricci solitons by combining previously known results. Observe that a resolution of Conjecture 2.9 would also imply [20, Conjecture 6.3].

The modified properness conjecture follows from the previous one.

Theorem 2.10. *Conjecture 2.9 implies Conjecture 2.8.*

We also establish sharp versions of the properness conjecture in the setting of Kähler–Einstein edge metrics and Kähler–Ricci solitons. Denote by $\mathrm{Aut}^X(M, \mathrm{J})_0$ the subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$ defined in (85). For a precise statement of the following result see Theorem 8.1.

Theorem 2.11. *Suppose (M, J, ω) is Fano and let $X \in \mathrm{aut}(M, \mathrm{J})$. Set $G := \mathrm{Aut}^X(M, \mathrm{J})_0$. Then a version of Theorem 2.4 holds both for the Tian–Zhu modified K-energy E^X and for the modified Ding functional F^X.*

A version of Theorem 2.11 in the absence of holomorphic vector fields is due to [16].

Denote by $\mathrm{Aut}(M, D, \mathrm{J})_0$ the subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$ defined in (88). For a precise statement of the following result see Theorem 9.1.

Theorem 2.12. *Suppose (M, J, ω) is a compact Kähler manifold and a smooth divisor $D \subset M$ satisfying $c_1(M) - (1-\beta)[D] = [\omega]$ for some $\beta \in (0, 1)$. Set $G := \mathrm{Aut}(M, D, \mathrm{J})_0$. Then a version of Theorem 2.4 holds both for the twisted K-energy E^β and for the twisted Ding functional F^β.*

In the absence of holomorphic vector fields, i.e., when G is trivial, some versions of Theorem 2.12 exist in the literature. The direction (iii) $\Rightarrow$ (i) is due to [38, Theorem 2]. A version of the direction (i) $\Rightarrow$ (iii) in the special case of D plurianticanonical on a Fano manifold (which implies the triviality of G) is due to [21, Proposition 3.6], [65, Corollary 2.2],[66, Theorem 0.1].

Remark 2.13. Versions of Corollary 2.7 for Kähler–Einstein edge metrics and Kähler–Ricci solitons also follow from our work. We omit the statements for brevity.

2.1 Sketch of the proofs

As already noted, much work has gone into showing different versions of Tian's conjecture. These works are mostly based on the continuity method, the Ricci flow and the J-flow. To cite a few papers from a rapidly growing literature, we mention [16, 43, 52, 19, 53, 46, 69, 68] and references therein.

Perhaps one of the main thrusts of the present article is that it is considerably more powerful to use the Finsler geometry of $\mathcal{H}$ and various of its subspaces to treat in a unified manner essentially all instances of the properness conjectures. In this spirit, we state a completely general existence/properness principle (Theorem 3.4). This principle is stated in terms of four pieces of data: the space of regular candidates $\mathcal{R}$, a metric d on it, a functional F defined on $\mathcal{R}$, and a group G acting on $\mathcal{R}$. The data is assumed to satisfy four axioms, or conditions, that we denote by (A1)–(A4); we refer the reader to Notation 3.1. Roughly stated, the existence principle guarantees F is d_G-proper on $\mathcal{R}$ if and only if the data satisfies seven additional properties that we denote by (P1)–(P7); we refer the reader to Hypothesis 3.2.

Property (P1) guarantees F is convex along 'sufficiently many' d-geodesics, and is inspired by the work of Berndtsson [12]. Property (P2) is a compactness requirement for minimizing sequences, and is inspired by the work of Berman et al. [14, 8]. Property (P3) is a regularity requirement for minimizes, and is inspired by the work of Berman and Berman–Witt-Nyström [5, 11]. Property (P4) stipulates that G act by d-isometries. Property (P5) requires that G act transitively on the set of minimizes, and is inspired by the classical work of Bando–Mabuchi and its recent generalizations by Berndtsson and others. Property (P6) seems to be a new ingredient. It requires that in the presence of minimizers the pseudometric d_G is realized by elements in each orbit. Property (P7) is standard and requires the 'cocycle' functional associated to F to be G-invariant.

Section 3 is devoted to the proof of the existence principle. Section 4 recalls necessary preliminaries concerning the Finsler geometry from [24]. Section 5 is rather lengthy and studies basic properties of energy functionals relative to the Finsler metric completion. In particular we verify condition (A2) and properties (P3) and (P4) in the particular cases of interest. Section 6 is devoted to the proof of property (P6). A technical input here is a partial Cartan decomposition (Proposition 6.2). Finally, in the remaining sections we prove various instances of the properness conjecture. Section 7 contains the proof of results containing Theorems 2.1 and 2.4. Section 8 contains the proof of Theorem 2.11. Section 9 contains the proof of Theorem 2.12. Section 10 contains the proof of Theorem 2.10.

3 A general existence/properness principle

Notation 3.1. The data $(\mathcal{R}, d, F, G)$ is defined as follows.

(A1) $(\mathcal{R}, d)$ is a metric space with a special element $0 \in \mathcal{R}$, whose metric completion is denoted $(\overline{\mathcal{R}}, d)$.

(A2) $F : \mathcal{R} \to \mathbb{R}$ is lower semicontinuous (lsc). Let $F : \overline{\mathcal{R}} \to \mathbb{R} \cup \{+\infty\}$ be the largest lsc extension of $F : \mathcal{R} \to \mathbb{R}$:

$$F(u) = \sup_{\varepsilon>0} \Bigg(\inf_{\substack{v\in\mathcal{R} \\ d(u,v)\leq\varepsilon}} F(v) \Bigg), \quad u \in \overline{\mathcal{R}}.$$

For each $u, v \in \mathcal{R}$ define also

$$F(u, v) := F(v) - F(u).$$

(A3) The set of minimizers of F on $\overline{\mathcal{R}}$ is denoted

$$\mathcal{M} := \Big\{ u \in \overline{\mathcal{R}} \ : \ F(u) = \inf_{v \in \overline{\mathcal{R}}} F(v) \Big\}.$$

(A4) Let G be a group acting on $\mathcal{R}$ by $G \times \mathcal{R} \ni (g, u) \to g.u \in \mathcal{R}$. Denote by $\mathcal{R}/G$ the orbit space, by $Gu \in \mathcal{R}/G$ the orbit of $u \in \mathcal{R}$, and define $d_G : \mathcal{R}/G \times \mathcal{R}/G \to \mathbb{R}_+$ by

$$d_G(Gu, Gv) := \inf_{f,g \in G} d(f.u, g.v).$$

Hypothesis 3.2. The data $(\mathcal{R}, d, F, G)$ satisfies the following properties.

(P1) For any $u_0, u_1 \in \mathcal{R}$ there exists a d–geodesic segment $[0,1] \ni t \mapsto u_t \in \overline{\mathcal{R}}$ for which $t \mapsto F(u_t)$ is continuous and convex on $[0,1]$.

(P2) If $\{u_j\}_j \subset \overline{\mathcal{R}}$ satisfies $\lim_{j\to\infty} F(u_j) = \inf_{\overline{\mathcal{R}}} F$, and for some $C > 0$, $d(0, u_j) \le C$ for all j, then there exists a $u \in \mathcal{M}$ and a subsequence $\{u_{j_k}\}_k$ d-converging to u.

(P3) $\mathcal{M} \subset \mathcal{R}$.

(P4) G acts on $\mathcal{R}$ by d-isometries.

(P5) G acts on $\mathcal{M}$ transitively.

(P6) If $\mathcal{M} \neq \emptyset$, then for any $u, v \in \mathcal{R}$ there exists $g \in G$ such that $d_G(Gu, Gv) = d(u, g.v)$.

(P7) For all $u, v \in \mathcal{R}$ and $g \in G$, $F(u, v) = F(g.u, g.v)$.

We make two remarks. First, by (A2),

$$\inf_{v \in \overline{\mathcal{R}}} F(v) = \inf_{v \in \mathcal{R}} F(v). \tag{5}$$

Second, thanks to (P4) and the next lemma, the action of G, originally defined on $\mathcal{R}$ in (A4), extends to an action of G by d-isometries on the metric completion $\overline{\mathcal{R}}$.

Lemma 3.3. *Let (X, ρ) and (Y, δ) be two complete metric spaces, W a dense subset of X and $f : W \to Y$ a C-Lipschitz function, i.e.,*

$$\delta(f(a), f(b)) \le C\rho(a, b), \quad \forall\, a, b \in W. \tag{6}$$

Then f has a unique C-Lipschitz continuous extension to a map $\bar{f} : X \to Y$.

Proof. Let $w_k \in W$ be a Cauchy sequence converging to some $w \in X$. Lipschitz continuity gives

$$\delta(f(w_k), f(w_l)) \le C\rho(w_k, w_l),$$

hence $\bar{f}(w) := \lim_k f(w_k) \in Y$ is well defined and independent of the choice of approximating sequence w_k. Choose now another Cauchy sequence $z_k \in W$ with limit $z \in X$, plugging in w_k, z_k in (6) and taking the limit gives that $\bar{f} : X \to Y$ is C-Lipschitz continuous. □

The following result will provide the framework that relates existence of canonical Kähler metrics to properness of functionals with respect to the Finsler metric.

Theorem 3.4. *Let $(\mathcal{R}, d, F, G)$ be as in Notation 3.1 and satisfying Hypothesis 3.2. The following are equivalent:*

(i) $\mathcal{M}$ *is nonempty.*

(ii) $F : \mathcal{R} \to \mathbb{R}$ *is G-invariant, and for some* $C, D > 0$,

$$F(u) \geq C d_G(G0, Gu) - D, \quad \text{for all } u \in \mathcal{R}. \tag{7}$$

Remark 3.5. The G-invariance condition can be considered as a version of the Futaki obstruction [31].

Proof. (ii) $\Rightarrow$ (i). If condition (ii) holds, then F is bounded from below. By (5), (7), the G–invariance of F and the definition of d_G there exists $u_j \in \mathcal{R}$ such that $\lim_j F(u_j) = \inf_{\overline{\mathcal{R}}} F$ and $d(0, u_j) \leq d_G(G0, Gu_j) + 1 < C$ for C independent of j. By (P2), $\mathcal{M}$ is non-empty.

(i) $\Rightarrow$ (ii). We start with a standard lemma.

Lemma 3.6. *(i) If (P4) holds, $(\overline{\mathcal{R}}/G, d_G)$ and $(\mathcal{R}/G, d_G)$ are pseudo-metric spaces.*
(ii) If $\mathcal{M} \neq \emptyset$, (P4) and (P6) hold, $(\mathcal{R}/G, d_G)$ is a metric space.

Proof. (i) It is enugh to show that $(\overline{\mathcal{R}}/G, d_G)$ is a pseudo-metric space. Using (P4) and the fact that d is symmetric,

$$d_G(Gu, Gv) := \inf_{f,g \in G} d(f.u, g.v) = \inf_{h \in G} d(u, h.v) = \inf_{h \in G} d(h.v, u) = d_G(Gv, Gu).$$

Thus, d_G is symmetric.

Since d is nonnegative, given $u, v, w \in \overline{\mathcal{R}}$ and $\epsilon > 0$, there exist $f, g \in G$ such that $d_G(Gu, Gw) > d(f.u, w) - \epsilon$ and $d_G(Gv, Gw) > d(g.v, w) - \epsilon$. The triangle inequality for d and (P4) give

$$\begin{aligned} d_G(Gu, Gv) &\leq d(u, f^{-1}g.v) = d(f.u, g.v) \\ &\leq d(f.u, w) + d(g.v, w) < 2\varepsilon + d_G(Gu, Gv) + d_G(Gv, Gw). \end{aligned}$$

Letting ϵ tend to zero shows d_G satisfies the triangle inequality. Thus d_G is a pseudo-metric.

(ii) Since d is nonnegative so is d_G. Now, let $u, v \in \mathcal{R}$ satisfy $d_G(Gu, Gv) = 0$. By (P6), $d(u, f.v) = 0$ for some $f \in G$. Since d is a metric, $u = f.v$, hence $Gu = Gv$. ☐

A geodesic in $(\overline{\mathcal{R}}, d)$ need not descend to a geodesic in $(\overline{\mathcal{R}}/G, d_G)$. Even when a geodesic does descend, its speed may not be the same in the quotient space. A simple example in our cases of interest is a one-parameter subgroup of G acting on a fixed element $u \in \mathcal{R}$, whose descent is a trivial geodesic.

The next lemma gives a criterion for when a geodesic in $(\overline{\mathcal{R}}, d)$ descends to a geodesic in $(\overline{\mathcal{R}}/G, d_G)$ with the same speed.

Lemma 3.7. *Suppose that $\mathcal{M} \neq \emptyset$, (P4) and (P6) hold. Let $u_0, u_1 \in \mathcal{R}$ satisfy $d_G(Gu_0, Gu_1) = d(u_0, u_1)$, and let $\{u_t\}_{t \in [0,1]} \subset \overline{\mathcal{R}}$ be a d-geodesic connecting u_0 and u_1. Then, $\{Gu_t\}_{t \in [0,1]} \subset \overline{\mathcal{R}}/G$ is a d_G-geodesic satisfying*

$$d_G(Gu_a, Gu_b) = d(u_a, u_b) = |b - a| d(u_0, u_1), \quad \forall\, a, b \in [0, 1].$$

Proof. Since $d(u_0,u_a)+d(u_a,u_b)+d(u_b,u_1)=d(u_0,u_1)$ we can write

$$\begin{aligned} d_G([u_0],[u_1]) &\le d_G([u_0],[u_a])+d_G([u_a],[u_b])+d_G([u_b],[u_1]) \\ &\le d(u_0,u_a)+d(u_a,u_b)+d(u_b,u_1) \\ &= d(u_0,u_1)=d_G([u_0],[u_1]). \end{aligned}$$

Hence, there is equality everywhere, so $d_G([u_a],[u_b])=d(u_a,u_b)=|a-b|d(u_0,u_1)$. □

Lemma 3.8. *Suppose that (i) holds. Then $F:\mathcal{R}\to\mathbb{R}$ is G–invariant.*

Proof. By assumption, $\mathcal{M}$ is nonempty. Let $v\in\mathcal{M}$. By (P3), $v\in\mathcal{R}$. By (P4), $f.v\in\mathcal{M}$ for any $f\in G$. Thus, $F(v)=F(f.v)$. By (A3), subtracting $F(u)$ from both sides, $F(u,v)=F(u,f.v)$ for any $u\in\overline{\mathcal{R}}$. By (P7), $F(u,v)=F(f^{-1}.u,v)$, so adding $F(v)$ to both sides yields that $F(u)=F(f.u)$ for every $f\in G$. □

By the above lemma it makes sense to introduce $F_G:\mathcal{R}/G\to\mathbb{R}$, the descent of F to the quotient $\mathcal{R}/G$. Let $v\in\mathcal{M}\subset\mathcal{R}$. Define,

$$C:=\inf\left\{\frac{F_G(Gv,Gu)}{d_G(Gv,Gu)}\ :\ u\in\mathcal{R},\ d_G(Gv,Gu)\ge 1\right\}.$$

If $C>0$, then we are done. Suppose $C=0$. Then by (P4) there exists $u(k)\in\mathcal{R}$ such that $F_G(Gv,Gu(k))/d_G(Gv,Gu(k))\to 0$ and $d(v,u(k))=d_G(Gv,Gu(k))\ge 1$. By Lemma 3.8, in fact $F_G(Gv,Gu)=F(v,u)$. Thus,

$$\frac{F(v,u(k))}{d(v,u(k))}\to 0.$$

Using (P1), let $[0,d(v,u(k))]\ni t\mapsto u(k)_t\in\overline{\mathcal{R}}$ be a unit speed d-geodesic connecting $u(k)_0=v$ and $u(k)_{d(v,u(k))}=u(k)$ such that $t\mapsto F(u(k)_t)$ is convex. As v is a minimizer of F, by convexity we obtain

$$0\le F(u(k)_1)-F(v)\le\frac{F(v,u(k))}{d(v,u(k))}\to 0. \tag{8}$$

As $d(v,u(k)_1)=1$, (P2) and (8) imply that after perhaps passing to a subsequence of $u(k)_1$ we have $d(u(k)_1,\tilde v)\to 0$ for some $\tilde v\in\mathcal{M}$. By (P5), $\tilde v=f.v$ for some $f\in G$. Now,

$$0=d(f.v,\tilde v)\ge d(f.v,u(k)_1)-d(u(k)_1,\tilde v). \tag{9}$$

By Lemma 3.7, $d_G(Gv,Gu(k)_1)=d(v,u(k)_1)=1$. Thus, $d(f.v,u(k)_1)\ge d_G(Gv,Gu(k)_1)=1$. Since $d(u(k)_1,\tilde v)\to 0$, it follows that $d(f.v,\tilde v)\ge 1$, a contradiction with (9). Thus (ii) holds, concluding the proof of Theorem 3.4. □

Remark 3.9. (i) The first direction in the above proof only uses the compactness condition (P2).
(ii) By density, equation (7) is in fact equivalent to

$$F(u)\ge Cd_G(G0,Gu)-D,\quad \text{for all } u\in\overline{\mathcal{R}}. \tag{10}$$

(iii) In this article we will always verify a stronger condition than (P6), namely that for every $u,v\in\overline{\mathcal{R}}$ there exists $g\in G$ such that $d(u,g.v)=d_G(Gu,Gv)$.

Theorem 3.4 and the compactness condition (P2) have the following consequence.

Corollary 3.10. *Suppose $\mathcal{M}$ is nonempty. Then if $u_j\in\mathcal{R}$ satisfies $F(u_j)\to\inf_{\mathcal{R}}F$, then there exists $g_j\in G$ and $u\in\mathcal{M}$ such that $g._ju_j\to_d u$.*

In the search for canonical Kähler metrics one often studies geometric data that minimizes in the limit an appropriate energy functional F. Examples include the Ricci flow along with its twisted and modified versions, the Ricci iteration, or the (weak) Calabi flow. As stated, the above result partially generalizes [14, Theorem B (ii)] and [5, Theorem 1.2], each of which treats a particular case of interest in the presence of a trivial automorphism group.

To mention a concrete application, as it will be clear after the proof of Theorem 7.1, the above corollary gives d_1-convergence up to automorphisms of the Ricci iteration in the presence of a Kähler–Einstein metric. Smooth convergence up to automorphisms was conjectured in [47, Conjecture 3.2].

4 The Finsler geometry

Let (M, J, ω) denote a connected compact closed Kähler manifold. Recall (11); the space of smooth strictly ω-plurisubharmonic functions (Kähler potentials)

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(M) \,:\, \omega_\varphi \in \mathcal{H}\}, \tag{11}$$

can be identified with $\mathcal{H} \times \mathbb{R}$. For any $\varphi \in \mathcal{H}_\omega$ the total volume

$$V := \int_M \omega_\varphi^n. \tag{12}$$

is constant independent of φ. Consider the following weak Finsler metric on $\mathcal{H}_\omega$ [24]:

$$\|\xi\|_\varphi := V^{-1} \int_M |\xi| \omega_\varphi^n, \quad \xi \in T_\varphi \mathcal{H}_\omega = C^\infty(M). \tag{13}$$

Remark 4.1. More generally, one may consider L^p metrics on $\mathcal{H}_\omega$ [24]. The case $p = 2$ is the much-studied weak Riemannian metric of Mabuchi mentioned in the Introduction. Though it may seem surprising at first, in this article we only need an understanding of the case $p = 1$.

A curve $[0,1] \ni t \to \alpha_t \in \mathcal{H}$ is called smooth if $\alpha(t,z) = \alpha_t(z) \in C^\infty([0,1] \times M)$. Denote $\dot\alpha_t := \partial\alpha(t)/\partial t$. The length of a smooth curve $t \to \alpha_t$ is

$$\ell_1(\alpha) := \int_0^1 \|\dot\alpha_t\|_{\alpha_t} dt. \tag{14}$$

Definition 4.2. The path length distance of $(\mathcal{H}_\omega, ||\cdot||)$ is defined by

$$d_1(u_0, u_1) := \inf\{\ell_1(\alpha) \,:\, \alpha : [0,1] \to \mathcal{H}_\omega \text{ is a smooth curve with } \alpha(0) = u_0,\, \alpha(1) = u_1\}.$$

We call the pseudometric d_1 the *Finsler metric*.

It turns out d_1 is a bona fide metric [24, Theorem 3.5]. To state the result, consider $[0,1] \times \mathbb{R} \times M$ as a complex manifold of dimension $n+1$, and denote by $\pi_2 : [0,1] \times \mathbb{R} \times M \to M$ the natural projection.

Theorem 4.3. *$(\mathcal{H}_\omega, d_1)$ is a metric space. Moreover,*

$$d_1(u_0, u_1) = \|\dot u_0\|_{u_0} \geq 0, \tag{15}$$

with equality iff $u_0 = u_1$, where $\dot u_0$ is the image of $(u_0, u_1) \in \mathcal{H}_\omega \times \mathcal{H}_\omega$ under the Dirichlet-to-Neumann map for the Monge–Ampère equation,

$$\varphi \in \mathrm{PSH}(\pi_2^\star\omega, [0,1] \times \mathbb{R} \times M), \quad (\pi_2^\star\omega + \sqrt{-1}\partial\bar\partial\varphi)^{n+1} = 0, \quad \varphi|_{\{i\}\times\mathbb{R}} = u_i,\ i = 0, 1. \tag{16}$$

The Dirichlet-to-Neumann operator simply maps (u_0, u_1) to the initial tangent vector of the curve $t \mapsto u_t$ that solves (16). Since u is $\pi_2^*\omega$-psh and independent of the imaginary part of the first variable, it is convex in t. Thus,

$$\dot{u}_0(x) := \lim_{t\to 0^+} \frac{u(t,x) - u_0(x)}{t}, \tag{17}$$

with the limit well-defined since the difference quotient is decreasing in t. Let

$$\mathrm{PSH}(M,\omega) = \{\varphi \in L^1(M,\omega^n) \, : \, \varphi \text{ is upper semicontinuous and } \omega_\varphi \geq 0\}.$$

Following Guedj–Zeriahi [34, Definition 1.1] define,

$$\mathcal{E}(M,\omega) := \big\{\varphi \in \mathrm{PSH}(M,\omega) \, : \, \lim_{j\to-\infty} \int_{\{\varphi\leq j\}} (\omega + \sqrt{-1}\partial\bar{\partial}\max\{\varphi, j\})^n = 0\big\}.$$

For each $\varphi \in \mathcal{E}(M,\omega)$, define

$$\omega_\varphi^n := \lim_{j\to-\infty} \mathbf{1}_{\{\varphi>j\}}(\omega + \sqrt{-1}\partial\bar{\partial}\max\{\varphi, j\})^n.$$

By definition, $\mathbf{1}_{\{\varphi>j\}}(x)$ is equal to 1 if $\varphi(x) > j$ and zero otherwise, and the measure $(\omega + \sqrt{-1}\partial\bar{\partial}\max\{\varphi, j\})^n$ is defined by the work of Bedford–Taylor [4] since $\max\{\varphi, j\}$ is bounded. Define,

$$\mathcal{E}_1 := \big\{\varphi \in \mathcal{E}(M,\omega) \, : \, \int |\varphi|\omega_\varphi^n < \infty\big\}.$$

The next result characterizes the d_1-metric completion [24, Theorem 2].

Theorem 4.4. *The metric completion of* $(\mathcal{H}_\omega, d_1)$ *equals* $(\mathcal{E}_1, d_1)$*, where*

$$d_1(u_0, u_1) := \lim_{k\to\infty} d_1(u_0(k), u_1(k)),$$

for any smooth decreasing sequences $\{u_i(k)\}_{k\in\mathbb{N}} \subset \mathcal{H}_\omega$ *converging pointwise to* $u_i \in \mathcal{E}_1, i = 0,1$*. Moreover, for each* $t \in (0,1)$*, define*

$$u_t := \lim_{k\to\infty} u_t(k), \;\; t \in (0,1), \tag{18}$$

where $u_t(k)$ *is the solution of* (16) *with endpoints* $u_i(k), i = 0,1$*. Then* $u_t \in \mathcal{E}_1$*, and the curve* $t \to u_t$ *is well-defined independently of the choices of approximating sequences and is a* d_1*-geodesic.*

5 Action functionals and their Euler–Lagrange equations

In §5.1–5.3 we review certain energy functionals on $\mathcal{H}$ and $\mathcal{H}_\omega$. For an expository survey of this topic we refer to [48, §5].

5.1 Basic energy functionals

The most basic functional, introduced by Aubin [1], is defined by

$$J(\varphi) = J(\omega_\varphi) := V^{-1}\int_M \varphi\omega^n - \frac{V^{-1}}{n+1}\int_M \varphi \sum_{l=0}^{n} \omega^{n-l}\wedge\omega_\varphi^l. \tag{19}$$

The notation $J(\varphi) = J(\omega_\varphi)$ is justified by the fact that $J(\varphi) = J(\varphi + c)$ for any $c \in \mathbb{R}$. The Aubin–Mabuchi functional was introduced by Mabuchi [40, Theorem 2.3],

$$\mathrm{AM}(\varphi) := V^{-1} \int_M \varphi\omega^n - J(\varphi) = \frac{V^{-1}}{n+1} \sum_{j=0}^{n} \int_M \varphi\, \omega^j \wedge \omega_\varphi^{n-j}, \tag{20}$$

Note that

$$\mathrm{AM}(v) - \mathrm{AM}(u) = \frac{V^{-1}}{n+1} \int_M (v-u) \sum_{k=0}^{n} \omega_u^{n-k} \wedge \omega_v^k. \tag{21}$$

Among other things, this formula shows that

$$u \le v \quad \Rightarrow \quad \mathrm{AM}(u) \le \mathrm{AM}(v). \tag{22}$$

One can characterize d_1-convergence very concretely, as elaborated below. In addition, we note that monotone sequences with limit in $\mathcal{E}_1$ are always d_1-convergent [24, Proposition 5.9, Proposition 4.9]:

Lemma 5.1. *Suppose that* $\{u_k\} \subset \mathcal{E}_1$ *and* $u \in \mathcal{E}_1$. *Then the following hold:*
(i) $d_1(u_k, u) \to 0$ *if and only if* $\mathrm{AM}(u_k) \to \mathrm{AM}(u)$ *and* $u_k \to u$ *in* $L^1(M, \omega^n)$.
(ii) If $\{u_k\}$ *increases (or decreases) pointwise a.e. to* u *then* $d_1(u_k, u) \to 0$.

The subspace

$$\mathcal{H}_0 := \mathrm{AM}^{-1}(0) \cap \mathcal{H}_\omega \tag{23}$$

is isomorphic to $\mathcal{H}$ (1), the space of Kähler metrics. We use this isomorphism to endow $\mathcal{H}$ with a metric structure, by pulling back the Finsler metric defined on $\mathcal{H}_\omega$.

Lemma 5.2. *(i)* $\mathrm{AM}, J : \mathcal{H}_\omega \to \mathbb{R}$ *each admit a unique* d_1*-continuous extension to* $\mathcal{E}_1$ *using the same formula as* (20) *and* (19).
(ii) AM *is linear along the* d_1*-geodesic* $t \to u_t$ *defined in* (18).
(iii) The subspace $(\mathcal{E}_1 \cap \mathrm{AM}^{-1}(0), d_1)$ *is a complete geodesic metric space, coinciding with the metric completion of* $(\mathcal{H}_0, d_1)$ *(recall* (23)*).*

Proof. (i) In [9] it is shown that for $u_1, \ldots, u_k \in \mathcal{E}_1$, the positive currents

$$u_1 \omega_{u_2}^{j_2} \wedge \omega_{u_3}^{j_3} \wedge \ldots \wedge \omega_{u_k}^{j_k}$$

can be defined by approximating u_i by a decreasing sequence of smooth functions $\{u_i(k)\} \subset \mathcal{H}_\omega$ so that the limiting measure is independent of the choice of such sequences. Thus, formula (20) makes sense for all $u \in \mathcal{E}_1$. On the other hand, decreasing sequences converge in d_1 by Lemma 5.1 (ii). Thus, to prove (i) for AM it remains (by Lemma 3.3 and Theorem 4.4) to show that AM is d_1-Lipschitz continuous as a function from $\mathcal{H}_\omega$ to $\mathbb{R}$. This is proved below in Lemma 5.15 (take $X = 0$).

Next, Lemma 5.1 (i) gives that $u \to \int_X u\omega^n$ is also d_1-continuous. Thus, (20) implies that also J admits a unique d_1-continuous extension to $\mathcal{E}_1$, and (19) still holds for the extension.

(ii) It is well known that $t \to \mathrm{AM}(u_t)$ is linear for solutions of (16) when $u_0, u_1 \in \mathcal{E}_1 \cap L^\infty(M)$ [6, Remark 4.5].

When $u_0, u_1 \in \mathcal{E}_1$, let $u_t(k)$ be the sequence of curves in the definition of u_t (18). For each t, the sequence $u(k)_t$ decreases pointwise to u_t, hence by Lemma 5.1(ii) we have $d_1(u(k)_t, u_t) \to 0$. As AM is d_1-continuous, this gives $\mathrm{AM}(u(k)_t) \to \mathrm{AM}(u_t)$. By the above we also have that

$t \to \mathrm{AM}(u(k)_t)$ is linear. Taking the limit $k \to \infty$ we can conclude that $t \to \mathrm{AM}(u_t)$ is also linear.

(iii) As $\mathrm{AM} : \mathcal{E}_1 \to \mathbb{R}$ is d_1-continuous, it follows that $\mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)$ is d_1-closed. From (ii) it follows that $(\mathcal{E}_1 \cap \mathrm{AM}^{-1}(0), d_1)$ is a geodesic metric space. □

As proposed in [9], using monotonicity (22) it is possible to extend AM further to a functional on $\mathrm{PSH}(M, \omega)$ taking $-\infty$ as a possible value [9]:

$$\mathrm{AM}(\varphi) := \lim_{k \to -\infty} \mathrm{AM}(\max(\varphi, k)), \;\; \varphi \in \mathrm{PSH}(M, \omega). \tag{24}$$

By Lemma 5.1, $d_1(\max(\varphi, k), \varphi) \to 0$ when $\varphi \in \mathcal{E}_1$, hence this extension agrees with the one given the previous lemma. In fact the following result from [9]:

Lemma 5.3. *Suppose $\varphi \in \mathrm{PSH}(M, \omega)$. Then $\varphi \in \mathcal{E}_1$ if and only if $\mathrm{AM}(\varphi) > -\infty$.*

Proof. We can suppose that $\sup_M \varphi = 0$ and denote $\varphi_k = \max(\varphi, -k)$. By Bedford–Taylor [4], we can define a functional $I : \mathcal{E}_1 \cap L^\infty \to \mathbb{R}$ by

$$I(u) := V^{-1} \int_M u(\omega^n - \omega_u^n).$$

Recall that for $u \in \mathcal{E}_1 \cap L^\infty$ [59, 4]

$$\mathrm{AM}(u) = (I - J)(u) + V^{-1} \int u \omega_u^n, \tag{25}$$

and since $\int u \omega_u^j \wedge \omega^{n-j} \leq \int u \omega_u^{j-1} \wedge \omega^{n-j+1}$ (integration by parts is again justified by [4]) [1]

$$0 \leq (I - J)(u) \leq \frac{n}{n+1} I(u).$$

Thus,

$$V^{-1} \int_M \varphi_k \omega_{\varphi_k}^n \leq \mathrm{AM}(\varphi_k) \leq \frac{V^{-1}}{n+1} \int_M \varphi_k \omega_{\varphi_k}^n + \frac{nV^{-1}}{n+1} \int_M \varphi_k \omega^n.$$

For k big enough $\sup_M \varphi_k = 0$, hence by (27) below, the rightmost term in the above estimate is uniformly bounded. Lastly, [34, Proposition 1.4] gives that $-\int_M |\varphi| \omega_\varphi^n = \lim_k \int_M \varphi_k \omega_{\varphi_k}^n$, concluding the proof. □

Observe that,

$$J(\varphi) = V^{-1} \int_M \varphi \omega^n, \quad \varphi \in \mathcal{H}_0. \tag{26}$$

Recall that Green's formula implies that for all $u \in \mathrm{PSH}(M, \omega)$, there exists a constant $C > 0$ depending only on (M, ω) such that [59, p. 49]

$$\sup_M u \leq V^{-1} \int_M u \omega^n + C \leq \sup_M u + C. \tag{27}$$

Next, we recall a concrete formula for the d_1 metric relating it to the Aubin–Mabuchi energy and also give a concrete growth estimate for d_1. First we need to introduce the following rooftop type envelope for $u, v \in \mathcal{E}_1$:

$$P(u, v)(z) := \sup \{ w(z) \,:\, w \in \mathrm{PSH}(M, \omega), \, w \leq \min\{u, v\} \}.$$

Note that $P(u, v) \in \mathcal{E}_1$ [23, Theorem 2]. We recall the following properties of d_1 [24, Corollary 4.14, Theorem 3].

Proposition 5.4. *Let $u, v \in \mathcal{E}_1$. Then,*

$$d_1(u,v) = \mathrm{AM}(u) + \mathrm{AM}(v) - 2\mathrm{AM}(P(u,v)). \tag{28}$$

Also, there exists $C > 1$ such that for all $u, v \in \mathcal{E}_1$,

$$C^{-1} d_1(u,v) \leq \int_M |u-v|\omega_u^n + \int_M |u-v|\omega_v^n \leq C d_1(u,v). \tag{29}$$

The following result is stated in [24, Remark 6.3]. As it will be essential for us, we give a proof here.

Proposition 5.5. *There exists $C', C > 1$ such that for all $u \in \mathcal{H}_0$ (recall (23)):*

$$\frac{1}{C'} \sup_M u - C' \leq \frac{1}{C} J(u) - C \leq d_1(0,u) \leq C J(u) + C \leq C' \sup_M u + C'.$$

Proof. Let $u \in \mathcal{H}_0$. Equations (27) and (29) imply that $\frac{1}{C'} \sup_M u - C' \leq d_1(0,u)$. Now, $u - \sup_M u \leq \min\{0, u\}$, so $u - \sup_M u \leq P(0,u)$. Thus, $-\sup_M u = \mathrm{AM}(u - \sup_M u) \leq \mathrm{AM}(P(0,u))$. Combined with (28),

$$d_1(0,u) = -2\mathrm{AM}(P(0,u)) \leq 2 \sup_M u.$$

Finally, $J(u)$ and $\sup_M u$ are uniformly equivalent by (26) and (27). □

Finally, we recall two crucial compactness results. The first is a variant of a result of Berman et al [14].

Theorem 5.6. *Let $p > 1$ and suppose $\mu = f\omega^n$ is a probabilty measure with $f \in L^p(M)$. Suppose there exists $C > 0$ such that $\{u_k\}_k \subset \mathcal{E}_1$ satisfies*

$$|\sup_M u_k| < C, \qquad \int_M \log \frac{\omega_{u_k}^n}{\mu} \omega_{u_k}^n < C.$$

Then $\{u_k\}$ contains a d_1-convergent subsequence.

Proof. According to Berman et al. [14, Theorem 2.17], $\{u_k\}$ contains a subsequence u_{j_k} converging 'in energy' to some $u \in \mathcal{E}_1$, i.e., $||u - u_{j_k}||_{L^1(M,\omega^n)} \to 0$ and $\mathrm{AM}(u_{j_k}) \to \mathrm{AM}(u)$. According to Lemma 5.1, this latter convergence is equivalent to d_1-convergence. □

The second compactness result we recall is an often used version of Zeriahi's generalization of Skoda's uniform integrability theorem [67].

Theorem 5.7. *Consider the set*

$$\big\{u \in \mathcal{E}_1 \,:\, |\sup_M u|,\, |\mathrm{AM}(u)| \leq C\big\}. \tag{30}$$

For any $p > 0$ there exists $C'(C,p) > 0$ such that for all u belonging to (30),

$$\int_M e^{-pu} \omega^n \leq C'.$$

Proof. The map $\varphi \mapsto \mathrm{AM}(\varphi)$ is upper semicontinuous (usc) with respect to the $L^1(M,\omega^n)$-topology, while the map $\varphi \mapsto \sup_M u$ is continuous with respect to the $L^1(M,\omega^n)$-topology. Thus, the set (30) is compact with respect to the $L^1(M,\omega^n)$-topology. According to [34, Corollary 1.8] the elements of this set all have zero Lelong numbers. Hence, the requirements of [67, Corollary 3.2] are satisfied finishing the proof. □

5.2 Modified basic functionals arising from holomorphic vector fields

Let $\mathrm{Aut}_0(M,\mathrm{J})$ denote the connected component of the complex Lie group of automorphisms (biholomorphisms) of (M,J) and denote by $\mathrm{aut}(M,\mathrm{J})$ its Lie algebra of infinitesimal automorphisms composed of real vector fields X satisfying $\mathcal{L}_X\mathrm{J}=0$, equivalently,

$$\mathrm{J}[X,Y]=[X,\mathrm{J}Y],\quad \forall\, X\in \mathrm{aut}(M,\mathrm{J}),\ \forall\, Y\in \mathrm{diff}(M), \tag{31}$$

where $\mathrm{diff}(M)$ denotes all smooth vector fields on M. Thus $\mathrm{aut}(M,\mathrm{J})$ is a complex Lie algebra with complex structure J.

The automorphism group $\mathrm{Aut}(M,\mathrm{J})_0$ acts on $\mathcal{H}$ by pullback:

$$f.\eta := f^\star\eta,\qquad f\in \mathrm{Aut}(M,\mathrm{J})_0,\quad \eta\in\mathcal{H}. \tag{32}$$

Given the one-to-one correspondence between $\mathcal{H}$ and $\mathcal{H}_0$, the group $\mathrm{Aut}(M,\mathrm{J})_0$ also acts on $\mathcal{H}_0$. The action is described in the next lemma.

Lemma 5.8. *For $\varphi\in\mathcal{H}_0$ and $f\in\mathrm{Aut}(M,\mathrm{J})_0$ let $f.\varphi\in\mathcal{H}_0$ be the unique element such that $f.\omega_\varphi=\omega_{f.\varphi}$. Then,*

$$f.\varphi = f.0+\varphi\circ f,\qquad f\in \mathrm{Aut}(M,\mathrm{J})_0,\quad \varphi\in\mathcal{H}_0. \tag{33}$$

Proof. Note that (33) is a Kähler potential for $f^\star\omega_\varphi$. Indeed, $f\in\mathrm{Aut}(M,\mathrm{J})$ implies that $f^\star\sqrt{-1}\partial\bar\partial\varphi=\sqrt{-1}\partial\bar\partial\varphi\circ f$. That $\mathrm{AM}(f.0+\varphi\circ f)=0$ follows from (21) as we have:

$$\mathrm{AM}(f.\varphi)=\mathrm{AM}(f.\varphi)-\mathrm{AM}(f.0)=\int_M\varphi\circ f\sum_{j=0}^n f^\star\omega^{n-j}\wedge f^\star\omega_\varphi^j=\mathrm{AM}(\varphi)-\mathrm{AM}(0)=0.$$

□

Lemma 5.9. *The action of $\mathrm{Aut}(M,\mathrm{J})_0$ on $\mathcal{H}_0$ is a d_1-isometry.*

Proof. From (33),

$$\frac{d}{dt}f.\varphi_t=\dot\varphi_t\circ f,$$

for any smooth path $t\to\varphi_t$ in $\mathcal{H}_0$. Thus, the d_1-length of $t\to f.\varphi_t$ is

$$\int_0^1 V^{-1}\int_M|\dot\varphi_t\circ f|f^\star\omega_{\varphi_t}^n dt=\int_0^1 V^{-1}\int_M|\dot\varphi_t|\omega_{\varphi_t}^n dt,$$

equal to the d_1-length of φ_t. □

Lemma 5.10. *The action of $\mathrm{Aut}(M,\mathrm{J})_0$ on $\mathcal{H}_0$ has a unique d_1-isometric extension to the metric completion $\overline{(\mathcal{H}_0,d_1)}=(\mathcal{E}_1\cap\mathrm{AM}^{-1}(0),d_1)$.*

Proof. Because $\mathrm{Aut}(M,\mathrm{J})_0$ acts by d_1-isometries, each $f\in\mathrm{Aut}(M,\mathrm{J})_0$ induces a 1-Lipschitz continuous self-map of $\mathcal{H}_0$. By Lemma 3.3, such maps have a unique 1-Lipschitz extension to the completion $\mathcal{E}_1\cap\mathrm{AM}^{-1}(0)$ and the extension is additionally d_1-isometry. By density, the laws governing a group action have to be preserved as well. □

Let $G\subset\mathrm{Aut}(M,\mathrm{J})_0$ be a subgroup. Then the functional $J_G:\mathcal{E}_1\cap\mathrm{AM}^{-1}(0)/G\to\mathbb{R}$ is introduced as

$$J_G(Gu):=\inf_{f\in G}J(f.u). \tag{34}$$

We have the follwing estimates:

Lemma 5.11. *For $u \in \mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)$ we have*

$$\frac{1}{C} J_G(Gu) - C \leq d_{1,G}(G0, Gu) \leq C J_G(Gu) + C, \tag{35}$$

where $d_{1,G}$ is the pseudometric of the quotient $\mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)/G$.

Proof. By Lemma 5.9,

$$d_{1,G}(G0, Gu) = \inf_{f \in G} d_1(0, f.u).$$

The result now follows from Proposition 5.5. □

The infinitesimal action of the Lie algebra $\mathrm{aut}(M, \mathrm{J})$ on $\mathcal{H}$ associated to (32) naturally induces a vector field ψ^X on $\mathcal{H}$ given by

$$\eta \mapsto \psi^X_\eta \in C^\infty(M), \quad \text{where } \mathcal{L}_X \eta = \sqrt{-1}\partial\bar{\partial}\psi^X_\eta \text{ and } V^{-1} \textstyle\int_M e^{\psi^X_\eta} \eta^n = 1. \tag{36}$$

Denote the $L^2(M, e^{\psi^X_\eta} \eta^n)$ inner product by $\langle \,\cdot\, , \,\cdot\, \rangle_{\psi^X_\eta}$. The operator

$$L^X_\eta := \Delta_\eta + X \tag{37}$$

is self-adjoint with respect to $\langle \,\cdot\, , \,\cdot\, \rangle_{\psi^X_\eta}$ whenever X is the gradient vector field of ψ^X_η with respect to η, i.e.,

$$X = \nabla \psi^X_\eta, \tag{38}$$

where the gradient is with respect to g_η. Observe that

$$\psi^X_{\omega_\varphi} = \psi^X_\omega + X\varphi \tag{39}$$

since it follows from the definition (36) that (39) must hold up to an additive constant, say, $C(\varphi)$, but

$$\begin{aligned} \frac{d}{dt} \int_M e^{\psi^X_\omega + X(t\nu)} \omega^n_{t\nu} &= \int_M L^X_{\omega_{t\nu}} \nu e^{\psi^X_\omega + X(t\nu)} \omega^n_{t\nu} \\ &= e^{-C(t\varphi)} \langle L^X_{\omega_{t\nu}} \nu, 1 \rangle_{\psi^X_{\omega_{t\nu}}} = 0. \end{aligned}$$

Thus, $C(t\varphi) = 0$ for each t.

Let

$$g_\omega(\,\cdot, \,\cdot) := \omega(\,\cdot, \mathrm{J}\,\cdot). \tag{40}$$

Consider the following hypothesis on a vector field $X \in \mathrm{aut}(M, \mathrm{J})$:

- The closure $T = T(X)$ of the one-parameter subgroup generated by $\mathrm{J}X$ is a subgroup of of the isometry group of (M, g_ω). (41)

As the compact group T is the closure of a commutative subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$ it follows that T is in fact a torus. For any Lie subgroup K of the isometry group of (M, g_ω) define the subspace

$$\mathcal{H}^K_\omega := \{\varphi \in \mathcal{H}_\omega \,:\, \varphi \text{ is invariant under } K\}, \tag{42}$$

and similarly define $\mathcal{H}^K_0$. According to Theorem 4.4, the d_1-metric completion of $\mathcal{H}^K_\omega$ is

$$\mathcal{E}^K_1 := \{u \in \mathcal{E}_1 \,:\, u \text{ is invariant under } K\}.$$

The following lemma is well-known [70, 63]. We include a proof since our notation is somewhat different than in the original sources.

Lemma 5.12. *Suppose $X \in \mathrm{aut}(M, \mathrm{J})$ satisfies (38) for each $\eta \in \mathcal{H}^T$. Let $\gamma : [0,1] \to \mathcal{H}^T_\omega$ denote a smooth path with $\gamma(0) = 0, \gamma(1) = \varphi$. The functional*

$$\mathrm{AM}_X(\varphi) := V^{-1} \int_{[0,1]\times M} \dot\gamma(t) e^{\psi^X_{\omega_{\gamma(t)}}} \omega^n_{\gamma(t)} \wedge dt$$

is well-defined independently of the choice of γ.

Proof. Indeed, the 1-form $\alpha_X : \varphi \mapsto e^{\psi^X_{\omega_\varphi}} \omega^n_\varphi$ on $\mathcal{H}^T_\omega$ is closed since if $\nu, \mu \in T_\varphi \mathcal{H}^T_\omega$, and using (39),

$$\frac{d}{dt}\Big|_0 \alpha_X(\nu)|_{\varphi+t\mu} = \int_M \nu L^X_{\omega_\varphi} \mu e^{\psi^X_{\omega_\varphi}} \omega^n_\varphi = \frac{d}{dt}\Big|_0 \alpha_X(\mu)|_{\varphi+t\nu},$$

as by our assumption on X the operator $L^X_{\omega_\varphi}$ is self-adjoint with respect to $\langle\ \cdot\ ,\ \cdot\ \rangle_{\psi^X_{\omega_\varphi}}$ as observed after (37). □

Remark 5.13. (i) As verified in the proof of Lemma 5.15 (i) below, equation (38) follows from (41). (ii) We note in passing that from this lemma it follows that AM_X is monotone: $u \le v$ implies $\mathrm{AM}_X(u) \le \mathrm{AM}_X(v)$. This parallels (21).

The next result follows using the arguments in the proofs of Lemmas 5.2 and 5.10.

Lemma 5.14. *The metric completion of $(\mathcal{H}^K_0, d_1)$ is $\mathcal{E}^K_1 \cap \mathrm{AM}^{-1}(0)$.*

The Hodge decomposition implies that every $X \in \mathrm{aut}(M, \mathrm{J})$ can be uniquely written as [32]

$$X = X_H + \nabla \psi^X_\omega - \mathrm{J}\nabla \psi^{\mathrm{J}X}_\omega, \tag{43}$$

where ∇ is the gradient with respect to the Riemannian metric (40), and X_H is the g_ω-Riemannian dual of a g_ω-harmonic 1-form.

Lemma 5.15. *Assume that $T = T(X)$ satifies (41). Assume also that the first Betti number of M satisfies $b_1(M) = 0$.*
(i) $\mathrm{AM}_X : \mathcal{H}^T_\omega \to \mathbb{R}$ admits a unique d_1-continuous extension to $\mathcal{E}^T_1$.
(ii) AM_X is usc with respect to the $L^1(M, \omega^n)$ topology of $\mathcal{E}^T_1$.

Proof. We claim that AM_X is d_1-Lipschitz continuous. We also claim that X is a gradient field with respect to all elements of $\mathcal{H}^T_\omega$ so that AM_X is well-defined by Lemma 5.12. Lemma 3.3 then gives (i). To prove these claims observe first that as follows from (41), the torus T acts Hamiltonially on (M, ω). By (39) and (36),

$$\psi^{\mathrm{J}X}_{\omega_\varphi} = \psi^{\mathrm{J}X}_\omega = 0 \tag{44}$$

for every $\varphi \in \mathcal{H}^T_0$ since ω is $T_{\mathrm{J}X}$-invariant. Since $b_1(M) = 0$, $X_H = 0$ in (43); in particular, X is a gradient vector field with respect to all g_{ω_φ} with $\varphi \in \mathcal{H}^T_0$, i.e., (38) holds. Thus, $\iota_{\mathrm{J}X}\omega_\varphi = d\psi^X_{\omega_\varphi}$. In other words, $\psi^X_{\omega_\varphi}$ is the restriction of the moment map of the T-action to the vector field $\mathrm{J}X$ with respect to the symplectic form ω_φ. By a general result of Atiyah–Guillemin–Sternberg the image of a moment map of a toric action is independent of the choice of symplectic form in the cohomology class, and therefore

$$|\psi^X_{\omega_\varphi}| \le C, \tag{45}$$

for some uniform $C > 0$ independent of $\varphi \in \mathcal{H}_\omega^X$. Thus, if γ_t is any smooth path in $\mathcal{H}_0^X$ with endpoints u and v,

$$\begin{aligned} |\mathrm{AM}_X(u) - \mathrm{AM}_X(v)| &\le \int_0^1 V^{-1} \int_M |\dot\gamma_t| e^{\psi_{\gamma_t}^X} \omega_{\gamma_t}^n \wedge dt \\ &\le C \int_0^1 V^{-1} \int_M |\dot\gamma_t| \omega_{\gamma_t}^n dt \le C l_1(\gamma). \end{aligned} \tag{46}$$

Taking the infimum over all such γ we obtain

$$|\mathrm{AM}_X(u) - \mathrm{AM}_X(v)| \le C d_1(u, v),$$

as desired.
(ii) This is a consequence of the monotonicity of AM_X (Remark 5.13 (ii)) and is proved in [11, Proposition 2.15]. □

5.3 Condition (A2): action functionals and their lsc extensions

Each set of canonical Kähler metrics we consider in this article can be defined as the minimizers of an appropriate action functional over a space $\mathcal{R}$ of 'regular' potentials. It is crucial for us, however, to understand the greatest d_1-lsc extensions of these functionals to the corresponding d_1-metric completion $\overline{\mathcal{R}}$. This is the main goal of the present rather long subsection. We emphasize that it is crucial for us to obtain explicit formulas for the lsc extensions of our functionals in order to be able to apply existing results concerning regularity of minimizers over $\overline{\mathcal{R}}$. That is, if we only had an abstract, but not explicit, extension, this would not have allowed us to conclude property (P3).

5.3.1 Kähler–Einstein (edge) metrics

For an introduction to Kähler edge geometry we refer to the expository article [48]. Let $D \subset M$ denote a smooth divisor, and let $\beta \in (0, 1]$. Suppose that h is a smooth Hermitian metric on the line bundle L_D associated to D, and that s is a global holomorphic section of L_D, so that $D = s^{-1}(0)$. Let ω be a smooth Kähler metric on M and define

$$\omega_c := \omega + c\sqrt{-1}\partial\bar\partial(|s|_h^2)^\beta, \tag{47}$$

For $c > 0$ small enough, ω_c is a smooth Kähler metric away from D [38, Lemma 2.2]. Fix such a c once and for all.

Definition 5.16. A Kähler current η on M is called a *Kähler edge form of angle β* if it is a smooth Kähler form on $M \setminus D$ and satisfies $\omega_c / C \le \eta \le C\omega_c$ for some constant $C = C(\eta) > 0$.

The set of Kähler edge potentials (of angle β) is denoted by

$$\mathcal{H}_\omega^\beta := \{u \in \mathrm{PSH}(M, \omega) : \ \omega_u \text{ is a Kähler edge form of angle } \beta\}.$$

Set also,

$$\mathcal{H}_0^\beta := \mathcal{H}^\beta \cap \mathrm{AM}^{-1}(0).$$

There are plenty of Kähler edge metrics, according to the next result.

Lemma 5.17. *The d_1-metric completion of $\mathcal{H}_\omega^\beta$ is $\mathcal{E}_1$.*

Proof. Let $u \in \mathcal{H}_\omega$, and write
$$u_k := u + \frac{1}{k}(|s|_h^2)^\beta.$$
For k large enough $u_k \in \mathcal{H}_\omega^\beta$ [38, Lemma 2.2]. As u_k is decreasing pointwise to u, Lemma 5.1 implies that $d_1(u_k, u) \to 0$. As $\mathcal{H}_\omega$ is dense in $\mathcal{E}_1$, it follows that the metric completion of $\mathcal{H}_\omega^\beta$ is $\mathcal{E}_1$. □

We turn our attention to Kähler–Einstein edge metrics. Suppose that

$$c_1(M) - (1-\beta)[D] = \mu[\omega_0]/2\pi. \tag{48}$$

$$\sqrt{-1}\partial\bar\partial f_{\eta,\beta} = \operatorname{Ric}\eta - 2\pi(1-\beta)[D] - \mu\eta, \quad \int_M e^{f_{\eta,\beta}}\eta^n = \int_M \eta^n. \tag{49}$$

A Kähler edge form η is called Kähler–Einstein edge (KEE) when $f_{\eta,\beta} = 0$. We assume from now and on that $\mu > 0$ since the existence problem in the case $\mu \le 0$ is settled in [38]. In fact after rescaling the Kähler class we assume that $\mu = 1$.

We record the following estimate.

Lemma 5.18. *Let ω be a smooth Kähler form on M and let η be either a smooth Kähler form or a Kähler edge form of angle β. Then, $e^{f_{\eta,\beta}} \in L^p(M,\omega^n)$ for some $p > 1$.*

Proof. When η is smooth this is a direct consequence of the Poincaré–Lelong formula. When η is a Kähler edge form of angle β then $f_{\eta,\beta}$ is actually continuous [38, §4]. □

The Berger–Moser–Ding energy (or Ding functional for short) [27] is defined by

$$F^\beta(\varphi) = F^\beta(\omega_\varphi) := -\mathrm{AM}(\varphi) - \log\frac{1}{V}\int_M e^{f_{\omega,\beta}-\varphi}\omega^n, \quad \varphi \in \mathcal{H}^\beta. \tag{50}$$

The Mabuchi K-energy $E^\beta : \mathcal{H}_\omega^\beta \to \mathbb{R}$ is closely related and defined by [48, (5.27)],[40],

$$E^\beta(\varphi) := \operatorname{Ent}(e^{f_{\omega,\beta}}\omega^n, \omega_\varphi^n) - \mathrm{AM}(\varphi) + V^{-1}\int_M \varphi{\omega_\varphi}^n. \tag{51}$$

Observe that: (i) (51) differs from [48, (5.27)] by a constant equal to $V^{-1}\int f_{\omega,\beta}\omega^n$, (ii) the term $-(I-J)(\omega,\omega_\varphi)$ there equals the last two terms in (51) by (25). Here,

$$\operatorname{Ent}(\nu,\chi) = \frac{1}{V}\int_M \log\frac{\chi}{\nu}\chi, \tag{52}$$

is the entropy of the measure χ with respect to the measure ν. The following formula relating F^β and E^β was established by Ding–Tian [28]:

$$E^\beta(\varphi) = F^\beta(\varphi) - \frac{1}{V}\int f_{\omega_\varphi,\beta}\omega_\varphi^n. \tag{53}$$

Jensen's inequality gives that $V^{-1}\int_M f_{\omega_\varphi,\beta}\omega_\varphi^n \le \log V^{-1}\int_M e^{f_{\omega_\varphi,\beta}}\omega_\varphi^n = 0$, hence there exists $C > 0$ such that

$$E^\beta(\varphi) \ge F^\beta(\varphi) \tag{54}$$

The critical points in $\mathcal{H}_\omega^\beta$ of both F^β and E^β are precisely Kähler–Einstein (edge) potentials. The next result verifies condition (A2) for these functionals.

Proposition 5.19. *Formula* (50) *gives the unique* d_1*-continuous extension of* $F^\beta : \mathcal{H}^\beta \to \mathbb{R}$ *to a functional on* $\mathcal{E}_1$.

Proof. Lemma 5.1 (i) provides a d_1-continuous extension of AM. To deal with the other term we prove the following lemma.

Lemma 5.20. *Let* $e^a \in L^p(M, \omega^n)$ *for some* $p > 1$. *Then the functional defined on* $\mathcal{E}_1$ *by*

$$\tilde{F}(\varphi) = \int_M e^{a-\varphi}\omega^n,$$

is d_1*-continuous.*

Proof. Let $u_k, u \in \mathcal{E}_1$ be such that $d_1(u_k, u) \to 0$. We have to argue that $\tilde{F}(u_k) \to \tilde{F}(u)$. As $d_1(0, u_j)$ is bounded it follows from Lemma 5.1 (i) and (27) that for some $C > 0$ independent of j,

$$|\sup_M u_j|, \ |\mathrm{AM}(u_k)| \leq C. \tag{55}$$

Hence, Theorem 5.7 implies there exists $C'(s) > 0$ independent of j such that

$$\int_M e^{-su_j}\omega^n \leq C', \ j \in \mathbb{N}, \tag{56}$$

for all $s \geq 1$. Since $x \mapsto e^x$ is convex,

$$|e^a - e^b| \leq |a-b| \max\{e^a, e^b\} \leq |a-b|(e^a + e^b). \tag{57}$$

Suppose that $1/q + 1/p + 1/s = 1$. Then, (57) and Hölder's inequality imply that

$$\begin{aligned} \Big|\int_M \Big(e^{a-u_k} - e^{a-u}\Big)\omega^n\Big| &\leq \int_M |u_k - u|(e^{a-u_k} + e^{a-u})\omega^n \\ &\leq ||u_k - u||_{L^q(M,\omega^n)}||e^a||_{L^p(M,\omega^n)}(||e^{-u_k}||_{L^s(M,\omega^n)} + ||e^{-u_k}||_{L^s(M,\omega^n)}). \end{aligned} \tag{58}$$

The first term of this last expression converges to zero since by Lemma 5.1 (i), $\int_M |u_k - u|\omega^n \to 0$ while all L^s topologies are equivalent on $\mathrm{PSH}(M, \omega)$. The second term is bounded by hypothesis, while the third is bounded by (56). □

Thus, the proposition follows from the lemma by setting $a = f_{\omega,\beta}$, Lemma 5.18 and the d_1-density of $\mathcal{H}^\beta$ in $\mathcal{E}_1$ proved in Lemma 5.17. □

Proposition 5.21. *Formula* (51) *gives the greatest* d_1*-lsc extension of* $E^\beta : \mathcal{H}^\beta \to \mathbb{R}$ *to a functional on* $\mathcal{E}_1$.

A more precise version of this result is contained in the forthcoming paper [10], but for completeness we give a proof here:

Proof. The last two summands of (51) can be rewritten as

$$n\mathrm{AM}(\varphi) - (n+1)\Big[\mathrm{AM}(\varphi) - \frac{V^{-1}}{n+1}\int_M \varphi\omega_\varphi^n\Big] = n\mathrm{AM}(\varphi) - V^{-1}\sum_{j=0}^{n-1}\int_M \varphi \wedge {\omega_\varphi}^j \wedge \omega^{n-j}. \tag{59}$$

Combining Lemma 5.1 (i) and the following lemma (with $\alpha = \omega$) it follows that (59) is d_1-continuous.

Remark 5.22. At this stage, we could have also simply used (51) and proved that $\varphi \mapsto \int_M \varphi\omega_\varphi^n$ is d_1-continuous. We choose, however, to use the somewhat more complicated formula (59) since then essentially the same arguments allow us to deal with the K-energy on a general Kähler class (see Proposition 5.26).

Lemma 5.23. *Suppose α is a smooth closed $(1,1)$-form on M. The functional defined on $\mathcal{E}_1$ by*

$$\tilde{E}(\varphi) := V^{-1}\sum_{j=0}^{n-1}\int_M \varphi\alpha\wedge{\omega_\varphi}^j\wedge\omega^{n-1-j}$$

is d_1-continuous and bounded on d_1-bounded subsets of $\mathcal{E}_1$.

Proof. As in the proof of Lemma 5.2 (i), $\tilde{E}$ indeed makes sense on $\mathcal{E}_1$. Now, let $u_k, u \in \mathcal{E}_1$ be such that $d_1(u_k, u) \to 0$. An argument similar to that yielding (21) shows that

$$\tilde{E}(u) - \tilde{E}(u_k) = V^{-1}\int_M \sum_{j=0}^{n-1}(u-u_k)\alpha\wedge\omega_u^j\wedge\omega_{u_k}^{n-1-j}.$$

It is clear that for some $D > 0$ we have $-D\omega \leq \alpha \leq D\omega$. Thus, observing that $\omega_{(u+u_k)/4} = \omega/2 + \omega_v/4 + \omega_u/4$,

$$|\tilde{E}(u) - \tilde{E}(u_k)| \leq C\int_M |u-u_k|\omega^n_{(u+u_k)/4}. \tag{60}$$

By [24, Corollary 5.7] and its proof, for each $R > 0$ there exists $f_R : \mathbb{R} \to \mathbb{R}$ continuous with $f_R(0) = 0$ such that

$$\int_M |v-w|\omega_h^n \leq f_R(d_1(v,w)), \tag{61}$$

for any $v, w, h \in \mathcal{E}_1 \cap \{\varphi : d_1(0,\varphi) \leq R\}$. Using this, to show that (60) converges to 0, it is enough to argue that $d_1(0,(u+u_k)/4)$ is uniformly bounded. For this we use [24, Lemma 5.3] that says that there exists $C > 1$ such that $d_1(v,(w+v)/2) \leq Cd_1(v,w)$ for any $v, w \in \mathcal{E}_1$. Using this several times and the triangle inequality,

$$\begin{aligned} d_1(0,(u+u_k)/4) \leq& Cd_1(0,(u+u_k)/2) \leq C(d_1(0,u) + d_1(u,(u+u_k)/2)) \\ \leq& C^2(d_1(0,u) + d_1(u,u_k)). \end{aligned}$$

This last term is uniformly bounded, showing that $\tilde{E}$ is d_1-continuous. Also, by (61) it follows that $\tilde{E}$ is bounded on d_1-bounded subsets of $\mathcal{E}_1$. ☐

To prove Proposition 5.21 it thus remains to deal with the entropy term.

First, we claim this term can be extended to $\mathcal{E}_1$ in a d_1-lsc fashion. Indeed, d_1-convergence implies weak convergence of volume measures [24, Theorem 5 (i)]. By the display following (5.29) in [48], the map $\chi \to \mathrm{Ent}(\mu,\chi)$ is a supremum of continuous maps with respect to weak convergence of measures, hence is lsc with respect to this same convergence. These last two statements together imply that the map $u \to \mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n, \omega_u^n)$ is d_1-lsc, as claimed.

Second, the following result shows that thus extended to $\mathcal{E}_1$, the entropy is the greatest d_1-lsc extension of its restriction to $\mathcal{H}^\beta_\omega$. The result is due to a forthcoming paper where much more precise results are proved [10]:

Lemma 5.24. *Given $u \in \mathcal{E}_1$, there exists $u_k \in \mathcal{H}^\beta$ such that $d_1(u_k, u) \to 0$ and*

$$\mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n, \omega^n_{u_k}) \to \mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n, \omega^n_u).$$

Proof. If $\mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n,\omega_u^n)=\infty$, than any sequence $u_k\in\mathcal{H}^\beta$ with $d_1(u_k,u)\to 0$ satisfies the requirements, as follows from the d_1-lower semi-continuity of the entropy. We can suppose that $Ent(e^{f_{\omega,\beta}}\omega^n,\omega_u^n)$ is finite.

Let $g=\omega_u^n/\omega^n\geq 0$ be the density function of ω_u^n. We argue that there exists positive functions $g_k\in C^\infty(X)$ such that $|g-g_k|_{L^1}\to 0$ and

$$\int_M g_k\log\frac{g_k}{e^{f_{\omega,\beta}}}\omega^n\to\int_M g\log\frac{g}{e^{f_{\omega,\beta}}}\omega^n=\mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n,\omega_u^n).$$

First introduce $h_k=\min\{k,g\},\ k\in\mathbb{N}$. As $\phi(t)=t\log(t),\ t\geq 0$ is bounded from below by $-e^{-1}$ and increasing for $t\geq 1$, it follows that

$$-e^{-1}e^{f_{\omega,\beta}}\leq h_k\log\frac{h_k}{e^{f_{\omega,\beta}}}\leq\max\{0,g\log\frac{g}{e^{f_{\omega,\beta}}}\}.$$

Clearly $|h_k-g|_{L^1}\to 0$, and the dominated convergence theorem gives that

$$\int_M h_k\log\frac{h_k}{e^{f_{\omega,\beta}}}\omega^n\to\int_M g\log\frac{g}{e^{f_{\omega,\beta}}}\omega^n=\mathrm{Ent}(e^{f_{\omega,\beta}}\omega^n,\omega_u^n).\tag{62}$$

Using the density of $C^\infty(M)$ in $L^1(M)$, by another application of the dominated convergence theorem, we can find a positive sequence $g_k\in C^\infty(X)$ such that $|g_k-h_k|_{L^1}\leq 1/k$ and

$$\Big|\int_M h_k\log\frac{h_k}{e^{f_{\omega,\beta}}}\omega^n-\int_M g_k\log\frac{g_k}{e^{f_{\omega,\beta}}}\omega^n\Big|\leq\frac{1}{k}.\tag{63}$$

Using the Calabi-Yau theorem we find potentials $v_k\in\mathcal{H}_\omega$ with $\sup_M v_k=0$ and $\omega_{v_k}^n=g_k\omega^n/\int_M g_k\omega^n$. Proposition 5.28 now guarantees that after possibly passing to a subsequence $d_1(v_k,h)\to 0$ for some $h\in\mathcal{E}_1(X)$. But [24, Theorem 5 (i)] implies the equality of measures $\omega_h^n=\omega_u^n$. Finally the uniqueness theorem [34, Theorem B] gives that in fact h and u can differ by at most a constant. Hence, after possibly adding a constant, we can suppose that $d_1(v_k,u)\to 0$.

The last step is to perturb v_k slightly to obtain the conical metrics $u_k=v_k+\varepsilon_k|s|_h^{2\beta}\in\mathcal{H}^\beta$, where s is the section of L_D, with vanishing locus D. The argument of Lemma 5.17 gives that for small enough $\varepsilon_k>0$ one has

$$d_1(u_k,u)\to 0.$$

Using (62) and (63) a basic calculation gives that after possibly shrinking $\varepsilon_k>0$ further we obtain

$$\mathrm{Ent}(e^{f_{\omega_\beta}}\omega^n,\omega_{u_k}^n)\to\mathrm{Ent}(e^{f_{\omega_\beta}}\omega^n,\omega_u^n),$$

as desired. □

Since the last two terms in the formula for E^β are d_1-continuous, as already noted, the previous lemma implies that our extension is the largest d_1-lsc extension of its restriction to $\mathcal{H}_\omega^\beta$. This concludes the proof of Proposition 5.21. □

5.3.2 Kähler–Ricci solitons

Suppose that (M,J) is Fano and let $X\in\mathrm{aut}(M,\mathrm{J})$. Assume that X satisfies (41). Recall from the proof of Lemma (i) that this implies X is a gradient vector field with respect to metrics in $\mathcal{H}^T$. We say that $\eta\in\mathcal{H}^T$ is a Kähler–Ricci soliton associated to X if

$$\mathrm{Ric}\ \eta=\eta+\mathcal{L}_X\eta.\tag{64}$$

Following Tian-Zhu, we define the modified Ding functional and K-energy functional on $\mathcal{H}_\omega^T$ as follows:

$$F^X(\varphi) := -\mathrm{AM}_X(\varphi) - \log V^{-1} \int_M e^{f_\omega - \varphi} \omega^n, \tag{65}$$

$$E^X(\varphi) := F^X(\varphi) + V^{-1} \int_M (f_\omega - \psi_\omega^X) e^{\psi_\omega^X} \omega^n - V^{-1} \int_M (f_{\omega_\varphi} - \psi_{\omega_\varphi}^X) e^{\psi_{\omega_\varphi}^X} \omega_\varphi^n. \tag{66}$$

As in (54), it follows that

$$E^X(\varphi) \geq F^X(\varphi) - C. \tag{67}$$

The critical points of these functionals are Kähler-Ricci solitons. Both F^X and E^X are defined for smooth potentials, but as it turns out their definition can be extended to $\mathcal{E}_1^X$. In this article we will only need this for F^X.

Proposition 5.25. *Formula* (65) *gives a unique d_1-continuous extension of $F^X : \mathcal{H}_\omega^X \to \mathbb{R}$ to a functional on $\mathcal{E}_1^X$.*

Proof. Lemma 5.15 gives that the map $t \to \mathrm{AM}_X(u)$ is d_1-continuous. Lemma 5.20 with $a = f_\omega$ give that the map $u \to \log V^{-1} \int_M e^{f_\omega - u} \omega^n$ is also d_1-continuous. Uniqueness of the extension now follows from the density of $\mathcal{H}_\omega^X$ in $\mathcal{E}_1^X$. □

5.3.3 Constant scalar curvature metrics

On a general Kähler manifold (M, ω) Mabuchi's K-energy equals

$$E(\varphi) := \mathrm{Ent}(\omega^n, \omega_\varphi^n) + n\mathrm{AM}(\varphi) - \frac{1}{V} \sum_{j=0}^{n-1} \int_M \varphi \mathrm{Ric}\, \omega \wedge {\omega_\varphi}^j \wedge \omega^{n-1-j}. \tag{68}$$

Recall that by (59) this expression is equal to the one in (51) when $\beta = 1$. The advantage of this definition over the one in (51) is that this makes sense independently of the condition (48).

The critical points of E over $\mathcal{H}_0$ are metrics of constant scalar curvature. We have the following result, first obtained in [10], whose proof follows exactly the same line as that of Proposition 5.21, the only difference being that in Lemma 5.23 we now take $\alpha = \mathrm{Ric}\, \omega$.

Proposition 5.26. *Formula* (68) *gives the greatest d_1-lsc extension of $E : \mathcal{H}_\omega \to \mathbb{R} \cup \{\infty\}$ to a functional on $\mathcal{E}_1$.*

5.4 Property (P2): compactness of minimizing sequences

We turn to the compactness condition (P2). Most of the results that we present here have already been obtained in the works [8, 14, 11] in a different context. To fit well with the metric-geometric viewpoint on the space of Kähler metrics we employ here, we see it adequate to present a very detailed account of all theorems, often following the ideas of [8, 14, 11].

5.4.1 Kähler–Einstein (edge) metrics

Proposition 5.27. *Suppose $(M, \mathrm{J}, D, \omega)$ satisfies* (48) *with $\mu = 1$. Let $\mathcal{R} = \mathcal{H}_\omega^\beta$, so $\overline{(\mathcal{R}, d_1)} = \mathcal{E}_1$. Then F^β given by Proposition 5.19 satifies property (P2).*

Proof. Step 1. In this step we construct a candidate minimizer $u \in \mathcal{E}_1$. Suppose that $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{E}_1^X$ satisfies

$$\lim_j F^\beta(u_j) = \inf_{\mathcal{E}_1^X} F^\beta, \qquad d_1(0,u_j) \le C.$$

Thus, as in the proof of Lemma 5.20, the estimates (55) hold, in particular $|\sup_M u_j| \le C$, so there exists $j_k \to \infty$ and $u \in \mathrm{PSH}(M,\omega)$ such that $\int_M |u_{j_k} - u|\omega^n \to 0$ [26, Proposiion I.4.21].

Now, Lemma 5.15 (ii) (with $X = 0$) gives that AM is upper semicontinuous with respect to the weak $L^1(M,\omega^n)$ topology. This together with $|\mathrm{AM}(u_{j_k})| \le C$ yields,

$$-C \le \limsup_k \mathrm{AM}(u_{j_k}) \le \mathrm{AM}(u) \le \limsup_k \sup_M u_{j_k} \le C, \tag{69}$$

hence $u \in \mathcal{E}_1$ by Lemma 5.3.

Step 2. In this step we show u is actually a minimizer. Following the ideas giving estimate (58) we arrive at

$$\begin{aligned}\Big|\int_M \Big(e^{f_{\omega,\beta}-u_k} - e^{f_{\omega,\beta}-u}\Big)\omega^n\Big| &\le \int_M |u_k - u|(e^{f_{\omega,\beta}-u_k} + e^{f_{\omega,\beta}-u})\omega^n \\ &\le ||u_k - u||_{L^q(M,\omega^n)}||e^{f_{\omega,\beta}}||_{L^p(M,\omega^n)}(||e^{-u_k}||_{L^s(M,\omega^n)} + ||e^{-u_k}||_{L^s(M,\omega^n)}),\end{aligned}$$

where $1/q + 1/p + 1/s = 1$. The first term of this last expression converges to zero as all L^q topologies are equivalent on $\mathrm{PSH}(M,\omega)$. The second term is bounded by Lemma 5.18, while the third is bounded by (56). All this gives

$$\lim_{j_k} F^\beta(u_{j_k}) \ge -\limsup_{j_k} \mathrm{AM}(u_{j_k}) - \log\Big(V^{-1}\int_M e^{f_{\omega,\beta}-u}\omega^n\Big) \ge F^\beta(u). \tag{70}$$

As $j_k \to F^\beta(u_{j_k})$ minimizes F^β, it follows that the last inequality must be an equality. Thus, u minimizes F^β.

Step 3. Here, we show that there is a subsequence d_1-converging to u. Since equality holds in (70), $\limsup_k \mathrm{AM}(u_{j_k}) = \mathrm{AM}(u)$. Thus, after possibly passing to a further subsequence, $\lim_m \mathrm{AM}(u_{j_{k_m}}) = \mathrm{AM}(u)$. This together with $||u_{j_{k_m}} - u||_{L^1(M,\omega^n)} \to 0$ and Lemma 5.1 (i) gives that $d_1(j_{k_m}, u) \to 0$. □

Proposition 5.28. *Suppose $(M, \mathrm{J}, D, \omega)$ satisfies (48) with $\mu = 1$. Let $\mathcal{R} = \mathcal{H}_\omega^\beta$, so $\overline{(\mathcal{R}, d_1)} = \mathcal{E}_1$. Then E^β given by Proposition 5.21 satifies property (P2).*

Proof. According to Lemma 5.23 and Lemma 5.15 (i) (with $X = 0$), the second and third summands in (51) are controlled by the metric d_1. Thus, as $E^\beta(u_j)$ and $d(0,u_j)$ are uniformly bounded, this implies that both $\mathrm{Ent}(f_{\omega,\beta}\omega^n, \omega_{u_j}^n)$ and $|\sup_M u_j|$ remain uniformly bounded. Theorem 5.6 thus gives a subsequence u_{j_k} and $u \in \mathcal{E}_1$ such that $d_1(u_{j_k}, u) \to 0$. The d_1-lower semicontinuity of E^β (Proposition 5.21) then implies that u minimizes E^β and $\lim_k E^\beta(u_{j_k}) = E^\beta(u)$. □

5.4.2 Kähler–Ricci solitons

Proposition 5.29. *Suppose (M, J, ω) is Fano, $X \in \mathrm{aut}(M, \mathrm{J})$ satisfies (41). Let $\mathcal{R} = \mathcal{H}_\omega^X$, so $\overline{(\mathcal{R}, d_1)} = \mathcal{E}_1^X$. Then F^X given by Proposition 5.25 satifies property (P2).*

Proof. The proof is similar to that of Proposition 5.27, with a slight twist at the end.

Step 1. This is identical to Step 1 in the proof of Proposition 5.27, yielding a $u \in \mathcal{E}_1^X$ satisfying (69).

Step 2. This is again identical to Step 2 in the proof of Proposition 5.27, this time yielding

$$\lim_{j_k} F^X(u_{j_k}) \geq -\limsup_{j_k} \text{AM}_X(u_{j_k}) - \log\Big(V^{-1}\int_M e^{f-u}\omega^n\Big) \geq F^X(u). \tag{71}$$

As $j_k \to F^X(u_{j_k})$ minimizes F^X, it follows that the inequalities are actually equalities, so u minimizes F^X and $\limsup_k \text{AM}_X(u_{j_k}) = \text{AM}_X(u)$. Consequently, after possibly passing to a subsequence

$$\lim_k \text{AM}_X(u_{j_k}) = \text{AM}_X(u). \tag{72}$$

Step 3. We argue now that $d_1(u_{j_k}, u) \to 0$. By Lemma 5.1 we only need to show that $\text{AM}(u_{j_k}) \to \text{AM}(u)$. For this we introduce an auxiliary sequence

$$v_{j_k} := \max\{u, u_{j_k}\} \in \mathcal{E}_1^X.$$

Observe that

$$|\sup_M v_{j_k}| \leq C, \tag{73}$$

since this holds for u_j by (55) (this is part of Step 1). Similarly, since $v_{j_k} \geq u_{j_k}$, (22) and (55) imply that $\text{AM}(v_{j_k}) \geq C$. Since (73) and the definition of AM give that $\text{AM}(v_{j_k}) \leq C$, we have

$$|\text{AM}(v_{j_k})| \leq C.$$

Furthermore, $||v_{j_k} - u||_{L^1(M,\omega^n)} \to 0$. Thus, we may apply Step 2 to the sequence $\{v_{j_k}\}$ to obtain (71) for v_{j_k}. By monotonicity of AM_X we have $-\text{AM}_X(v_{j_k}) \leq -\text{AM}_X(u_{j_k})$, hence v_{j_k} is also F^X-minimizing. In the same manner (72) was derived one then has

$$\lim_k \text{AM}_X(v_{j_k}) = \text{AM}_X(u). \tag{74}$$

Examining (46) and recalling (45), one can show the existence of $C > 1$ such that for any $w, v \in \mathcal{E}_1$ *with* $w \leq v$ one has

$$0 \leq \frac{1}{C}(\text{AM}(v) - \text{AM}(w)) \leq \text{AM}_X(v) - \text{AM}_X(w) \leq C(\text{AM}(v) - \text{AM}(w)). \tag{75}$$

Using that $u \leq v_{j_k}$, $u_{j_k} \leq v_{j_k}$ equations (72), (74) and (75) we obtain that

$$\lim_k \text{AM}(u_{j_k}) = \lim_k \text{AM}(v_{j_k}) = \text{AM}(u),$$

as desired. □

5.4.3 Constant scalar curvature metrics

The proof of the following result is the same as that of Proposition 5.28 and it already appears in [25].

Proposition 5.30. *Let ω be an arbitrary Kähler metric on (M, J). Let $\mathcal{R} = \mathcal{H}_\omega$, so $\overline{(\mathcal{R}, d_1)} = \mathcal{E}_1$. Then E given by Proposition 5.26 satifies property (P2).*

5.5 Property (P3): regularity of minimizers

The regularity condition (P3) for F^β and E^β follows after combining together the result of Berman [5, Theorem 1.1] and the regularity theorems [39, Corollary 6.9],[38],[35] (see [48] for more details).

Theorem 5.31. *Suppose* $(M, \mathrm{J}, D, \omega)$ *satisfies* (48) *with* $\mu = 1$ *and* $\varphi \in \mathcal{E}_1$. *The following are equivalent: (i)* ω_φ *is a Kähler–Einstein edge metric, (ii)* φ *minimizes* F^β*, (iii)* φ *minimizes* E^β.

The regularity condition (P3) for F^X and E^X is due to the following result of Berman–Witt Nyström [11, Theorem 3.3].

Theorem 5.32. *Suppose* (M, J, ω) *is Fano, and that* $X \in \mathrm{aut}(M, \mathrm{J})$ *satisfies* (41). *Let* $\varphi \in \mathcal{E}_1^X$. *The following are equivalent: (i)* ω_φ *is a Kähler–Ricci soliton, (ii)* φ *minimizes* F^X*, (iii)* φ *minimizes* E^X.

6 Property (P6) via a partial Cartan decomposition

The purpose of this section is to give explicit criteria that imply condition (P6) in all cases of interest for us. The main result is Proposition 6.8, while the main technical ingredient is Proposition 6.2.

6.1 Partial Cartan decomposition

Recall the following form of the classical Cartan decomposition [15, Proposition 32.1, Remark 31.1].

Theorem 6.1. *Let* S *be a compact connected semisimple Lie group. Denote by* $(S^{\mathbb{C}}, \mathrm{J})$ *the complexification of* S*, namely the unique connected complex Lie group whose Lie algebra is the complexification of that of* $\mathfrak{s}$*, the Lie algebra of* S. *Then the map* C *from* $S \times \mathfrak{s}$ *to* $S^{\mathbb{C}}$ *given by*

$$(s, X) \mapsto C(s, X) := s \exp_I \mathrm{J}X \tag{76}$$

is a diffeomorphism.

The following result can be thought of as an extension of Theorem 6.1 to the setting of a compact (but not necessary semisimple) Lie group. We state a result in a form that will be most useful for us in applications, albeit it not being quite optimal, perhaps.

Proposition 6.2. *Let* K *be a compact connected subgroup of a connected complex Lie group* (G, J). *Denote by* $\mathfrak{k}$ *and* $\mathfrak{g}$ *their Lie algebras. Suppose that*

$$\begin{aligned} \mathfrak{k} &= \mathfrak{a} \oplus \tilde{\mathfrak{k}}, \\ \mathfrak{g} &= \mathfrak{a} \oplus \tilde{\mathfrak{k}} \oplus \mathrm{J}\tilde{\mathfrak{k}}, \end{aligned} \tag{77}$$

where $\mathfrak{a}$ *is a complex Lie subalgebra of* $\mathfrak{g}$ *contained in the center* $\mathfrak{z}(\mathfrak{k})$ *of* $\mathfrak{k}$*, and* $\tilde{\mathfrak{k}}$ *is a Lie subalgebra of* $\mathfrak{k}$. *Then the map* $C : K \times \tilde{\mathfrak{k}} \to G$ *given by* $C(k, X) = k \exp_I \mathrm{J}X$ *is surjective.*

Proof. We start with several simple claims.

Claim 6.3. $\mathfrak{z}(\mathfrak{k}) = \mathfrak{a} \oplus \mathfrak{z}(\tilde{\mathfrak{k}})$.

Proof. The inclusion $\mathfrak{z}(\mathfrak{k}) \supset \mathfrak{a} \oplus \mathfrak{z}(\tilde{\mathfrak{k}})$ follows from (77). For the converse, let $X \in \mathfrak{z}(\mathfrak{k})$. Since $X \in \mathfrak{k}$, by (77), $X = X_1 + X_2$ for $X_1 \in \mathfrak{a}, X_2 \in \tilde{\mathfrak{k}}$. Since $X_2 \in \mathfrak{z}(\mathfrak{k})$, and $\tilde{\mathfrak{k}} \cap \mathfrak{z}(\mathfrak{k}) \subset \mathfrak{z}(\tilde{\mathfrak{k}})$, we are done. □

Claim 6.4. *(i)* $[\mathfrak{k}, \mathfrak{k}] = [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}]$. *(ii)* $\tilde{\mathfrak{k}} = \mathfrak{z}(\tilde{\mathfrak{k}}) \oplus [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}]$.

Proof. (i) This follows from (77) since $[\mathfrak{a}, \mathfrak{k}] = 0$.
(ii) Since K is compact [37, Proposition 6.6 (ii), p. 132] gives

$$\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus [\mathfrak{k}, \mathfrak{k}]. \tag{78}$$

From (i) and Claim 6.3,

$$\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}] = \mathfrak{a} \oplus \mathfrak{z}(\tilde{\mathfrak{k}}) \oplus [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}].$$

The conclusion now follows from (77). □

Claim 6.5. $\mathfrak{z}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{z}(\tilde{\mathfrak{k}}) \oplus \mathrm{J}\mathfrak{z}(\tilde{\mathfrak{k}})$.

Proof. First, $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{k}) \subset \mathfrak{z}(\mathfrak{g})$ by (77). By Claim 6.3, $\mathfrak{z}(\tilde{\mathfrak{k}}) \subset \mathfrak{z}(\mathfrak{k}) \subset \mathfrak{z}(\mathfrak{g})$. Since $\mathfrak{z}(\mathfrak{g})$ is complex, we obtain $\mathfrak{z}(\mathfrak{g}) \supset \mathfrak{a} \oplus \mathfrak{z}(\tilde{\mathfrak{k}}) \oplus \mathrm{J}\mathfrak{z}(\tilde{\mathfrak{k}})$. Conversely, let $X \in \mathfrak{z}(\mathfrak{g})$. By (77), $X = X_1 + X_2 + X_3$, with $X_1 \in \mathfrak{a}, X_2 \in \tilde{\mathfrak{k}} \cap \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\tilde{\mathfrak{k}})$, and $X_3 \in J\tilde{\mathfrak{k}} \cap \mathfrak{z}(\mathfrak{g}) = \mathrm{J}(\tilde{\mathfrak{k}} \cap \mathfrak{z}(\mathfrak{g})) = J\mathfrak{z}(\tilde{\mathfrak{k}})$, since $\mathfrak{z}(\mathfrak{g})$ is complex. □

Let $Z(K)$ and $Z(G)$ denote the connected closed Lie groups whose Lie algebras are $\mathfrak{z}(\mathfrak{k})$ and $\mathfrak{z}(\mathfrak{g})$.

Claim 6.6. *The map* $\Theta_1 : Z(K) \times \mathfrak{z}(\tilde{\mathfrak{k}}) \to Z(G)$ *given by* $(z, X) \mapsto z \exp_I \mathrm{J}X$ *is surjective.*

Proof. Claims 6.3 and 6.5 imply that $\dim Z(K) + \dim \tilde{\mathfrak{k}} = \dim Z(G)$ and the differential of Θ_1 at $(I, 0)$ is invertible by [37, Proposition 1.6, p. 104]. But, considering $\mathfrak{z}(\tilde{\mathfrak{k}})$ as an abelian Lie group with respect to the additive structure, it follows that Θ_1 is a Lie group homomorphism, thus it must be surjective as its image is a connected subgroup of the same dimension as that of $Z(G)$. □

We now conclude the proof of Proposition 6.2. Let L denote the connected compact Lie subgroup of K whose Lie algebra is $[\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}]$ (since the Killing form is negative on $[\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}] = [\mathfrak{k}, \mathfrak{k}]$, L is indeed compact). By Claim 6.4 and [37, Proposition 6.6 (i), p. 132], L is semisimple. By Theorem 6.1, the map $\Theta_2 : L \times [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}] \to L^{\mathbb{C}}$ given by

$$\Theta_2(l, X) = l \exp_I \mathrm{J}X,$$

is a diffeomorphism.

Claim 6.7. *The multiplication maps* $Z(K) \times L \to K$*,* $Z(G) \times L^{\mathbb{C}} \to G$ *are surjective.*

Proof. By Claim 6.4 (i) and (78), the multiplication map $Z(K) \times L \to K$ is a local isomorphism near (I, I) by dimension count. The map is also a group homomorphism since elements of $Z(K)$ commute with elements of $K \supset L$. Thus, it is surjective.

The same argument works for the multiplication map $Z(G) \times L^{\mathbb{C}} \to G$. Here the dimension count is provided by Claim 6.4 (ii), Claim 6.5 and (77), as together they give $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}] \oplus \mathrm{J}[\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}]$. □

Given $k \in K$ and $X \in \tilde{\mathfrak{k}}$, observe that

$$C(k, X) = zl \exp_I \mathrm{J}X,$$

where $z \in Z(K)$ and $l \in L$ are such that $k = zl$ (these exist by Claim 6.7). Now let X_1 and X_2 are the unique elements such that $X_1 \in \mathfrak{z}(\tilde{\mathfrak{k}})$, $X_2 \in [\tilde{\mathfrak{k}}, \tilde{\mathfrak{k}}]$, and $X = X_1 + X_2$, given by Claim 6.4 (ii). Since $\exp_I \mathrm{J}X_1 \in Z(G)$,

$$C(k, X) = z \exp_I \mathrm{J}X_1 l \exp_I \mathrm{J}X_2 = \Theta_1(z, X_1)\Theta_2(l, X_2).$$

By Claim 6.6 and the fact that Θ_2 is a diffeomorphism, it follows that C surjects onto $Z(G) \times L^{\mathbb{C}}$. However, the multiplication map $Z(G) \times L^{\mathbb{C}} \to G$ is surjective by Claim 6.7. Thus, C is surjective, concluding the proof of Proposition 6.2. □

6.2 Properness of the distance function on orbits

Given data $(\mathcal{R}, d, F, G)$ satisfying (A1)-(A4), the following result gives a criteria that implies a stronger version of condition (P6). Though it may seem artificial at first glance, all the assumptions are verified naturally in the presence of canonical metrics.

Proposition 6.8. *Suppose K and G satisfies the assumptions of Proposition 6.2, $(\mathcal{R}, d, F, G)$ satisfies (A1)-(A4), (P4), and $w \in \mathcal{R}$. We additionally assume the following:*
(i) $K.w = w$.
(ii) For each $X \in \tilde{\mathfrak{k}}$, $t \mapsto \exp_I t\mathrm{J}X.w$ is a d-geodesic whose speed depends continuously on X.
(iii) $G \times G \ni (f, g) \mapsto d(f.u, g.v)$ is a continuous map for every $u, v \in \mathcal{R}$.
Then, for any $u, v \in \overline{\mathcal{R}}$ there exists $g \in G$ such that $d(u, g.v) = d_G(Gu, Gv)$.

Proof. Let $u, v \in \mathcal{R}$. By (P4) and Proposition 6.2,

$$d_G(Gu, Gv) = \inf_{g \in G} d(u, g.v) = \inf_{k \in K, X \in \tilde{\mathfrak{k}}} d(u, C(k, X).v). \tag{79}$$

By (P4),

$$\begin{aligned} d(u, C(k, tX).v) &\geq d(w, C(k, tX).w) - d(w, u) - d(C(k, tX).w, C(k, tX).v) \\ &= c_X t - d(w, u) - d(w, v), \end{aligned}$$

since using (P4), (i) and (ii) we have

$$d(w, C(k, tX).w) = d(k^{-1}.w, \exp_I t\mathrm{J}X.w) = d(w, \exp_I t\mathrm{J}X.w) = c_X t,$$

with c_X depending continuously on $X \in \tilde{\mathfrak{k}}$. Since $\tilde{\mathfrak{k}}$ is finite-dimensional it follows that $(k, X) \mapsto d(u, C(k, X).v)$ is proper. Hence the infimum in (79) is attained, because by (iii) $(k, X) \mapsto d(u, C(k, X).v)$ is continuous. This finishes the proof for $u, v \in \mathcal{R}$.

Finally, when $u, v \in \overline{\mathcal{R}}$, let $\{u_j\}, \{v_j\} \subset \mathcal{R}$ denote sequences that d-converge to u and v. Then,

$$\begin{aligned} |d(u, C(k, X).v) - d(u_j, C(k, X).v_j)| &\leq d(u, u_j) + |d(u, C(k, X).v) - d(u, C(k, X).v_j)| \\ &\leq d(u, u_j) + d(C(k, X).v, C(k, X).v_j) \\ &= d(u, u_j) + d(v, v_j), \end{aligned}$$

hence the continuous proper maps $(k, X) \to d(u_j, C(k, X).v_j)$ converge uniformly to $(k, X) \to d(u, C(k, X).v)$, making this latter map also continuous and proper. This gives $d_G(Gu, Gv) = d(u, g.v)$ for some $g \in G$, finishing the proof. □

6.3 Automorphism groups of canonical Kähler manifolds

In this section we recall some classical theorems about the automorphism group of a Kähler manifold (M, J, ω) when the metric ω is canonical, following Gauduchon [32] to which we refer for more details. These results will be very helpful once we try to verify the conditions of Proposition 6.8 in concrete situations.

Let $g(\cdot\,,\cdot\,) = \omega(\cdot\,, \mathrm{J}\,\cdot\,)$ denote the Riemannian metric associated to (M, J, ω). Denote by $\mathrm{Isom}(M, g)_0$ the identity component of the isometry group of (M, g). Since M is compact so is $\mathrm{Isom}(M, g)_0$ [45, Proposition 29.4]. Denote by $\mathrm{isom}(M, g)$ the Lie algebra of $\mathrm{Isom}(M, g)_0$. Consider the Lie subalgebra of $\mathrm{aut}(M, \mathrm{J})$ of harmonic fields,

$$\mathfrak{a} := \{X \in \mathrm{aut}(M, \mathrm{J}) \,:\, g(X, \cdot\,) \text{ is a } g\text{-harmonic 1-form}\}, \tag{80}$$

and the Lie subalgebra of $\mathrm{isom}(M, g)$ of Hamiltonian fields,

$$\mathfrak{h} := \{X \in \mathrm{isom}(M, g) \,:\, \iota_X \omega \text{ is an exact 1-form}\}. \tag{81}$$

The following theorem is due to Matsushima and Lichnerowitz [32, Theorem 3.6.1].

Proposition 6.9. *Let $(M, \mathrm{J}, \omega, g)$ be as above. Suppose g has constant scalar curvature. Then,*

$$\mathrm{isom}(M, g) = \mathfrak{a} \oplus \mathfrak{h}, \tag{82}$$

$$\mathrm{aut}(M, \mathrm{J}) = \mathfrak{a} \oplus \mathfrak{h} \oplus \mathrm{J}\mathfrak{h}. \tag{83}$$

In particular, Proposition 6.9 implies that $\mathrm{Isom}(M, g)_0 \subset \mathrm{Aut}(M, \mathrm{J})_0$.

Analogues of this result have been established in several settings. First, consider the Lie subalgebra [63, Lemma A.2]

$$\mathrm{aut}^X(M, \mathrm{J}) := \{Y \in \mathrm{aut}(M, \mathrm{J}) \,:\, [X, Y] = 0\}, \tag{84}$$

and denote the associated connected complex Lie group by

$$\mathrm{Aut}^X(M, \mathrm{J})_0 \subset \mathrm{Aut}(M, \mathrm{J})_0. \tag{85}$$

Tian–Zhu proved the following result [63, Appendix A].

Proposition 6.10. *Let $(M, \mathrm{J}, \omega, g)$ be as above, and let $X \in \mathrm{aut}(M, \mathrm{J})$. Suppose g is a Kähler–Ricci soliton. Then,*

$$\mathrm{aut}^X(M, \mathrm{J}) = \mathrm{isom}(M, g) \oplus \mathrm{J}\,\mathrm{isom}(M, g). \tag{86}$$

Next, let $D \subset M$ be a smooth divisor and consider

$$\mathrm{aut}(M, D, \mathrm{J}) := \{Y \in \mathrm{aut}(M, \mathrm{J}) \,:\, (1 - \beta)Y \text{ is tangent to } D\}, \tag{87}$$

and denote the associated connected complex Lie group by

$$\mathrm{Aut}(M, D, \mathrm{J})_0 \subset \mathrm{Aut}(M, \mathrm{J})_0. \tag{88}$$

Cheltsov–Rubinstein proved the following [18, Theorem 1.12].

Proposition 6.11. *Let $(M, D, \mathrm{J}, \omega, g)$ be as above and let $\beta \in (0, 1]$. Suppose g is a Kähler–Einstein edge metric. Then,*

$$\mathrm{aut}(M, D, \mathrm{J}) = \mathrm{isom}(M, g) \oplus \mathrm{J}\,\mathrm{isom}(M, g). \tag{89}$$

The following result is classical, and we only state its Kähler–Einstein version, whose proof we sketch.

Theorem 6.12. *Let* $(M, \mathrm{J}, \omega, g)$ *be Kähler–Einstein. Then any maximally compact subgroup of* $\mathrm{Aut}(M, \mathrm{J})_0$ *is conjugate to* $\mathrm{Isom}(M, g)_0$.

Proof. By a Theorem of Iwasawa–Malcev [55, Theorem 32.5], if G is a connected Lie group then its maximal compact subgroup must be connected and any two maximal compact subgroups are conjugate. But then by Proposition 6.11 ($\beta = 1$) $\mathrm{Isom}(M, g)_0$ has to be a maximal compact subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$. □

7 Kähler–Einstein metrics

In this section we prove two results about existence of Kähler–Einstein metrics. Recall that $d_{1,G}$ and J_G are defined in (A4) and (34), and F^1 and E^1 are defined in (50) and (53). Our first result gives characterizes Kähler classes admitting Kähler–Einstein metrics. This theorem will be generalized in two different directions in the next two sections.

Theorem 7.1. *Suppose* (M, J, ω) *is Fano. Set* $G := \mathrm{Aut}(M, \mathrm{J})_0$ *and let* $F \in \{F^1, E^1\}$*. The following are equivalent:*

(i) There exists a Kähler–Einstein metric in $\mathcal{H}$*.*

(ii) F *is* G*-invariant and for some* $C, D > 0$,

$$F(u) \geq C d_{1,G}(G0, Gu) - D, \qquad u \in \mathcal{H}_0.$$

(iii) F *is* G*-invariant and for some* $C, D > 0$,

$$F(u) \geq C J_G(Gu) - D, \qquad u \in \mathcal{H}_0.$$

The purpose of our second theorem is to indicate what modifications are necessary in Tian's original conjecture (Conjecture 1.2 (ii)). Let $K \subset \mathrm{Aut}(M, \mathrm{J})_0$ be a compact Lie subgroup. Denote,

$$\mathcal{H}^K := \{\eta \in \mathcal{H} \,:\, g.\eta = \eta \text{ for any } g \in K\}.$$

The isomorphic space of potentials, denoted $\mathcal{H}_0^K$, is defined after (42).

Theorem 7.2. *Suppose* (M, J, ω) *is Fano and that* K *is a maximal compact subgroup of* $\mathrm{Aut}(M, \mathrm{J})_0$*. Assume that* $\omega \in \mathcal{H}^K$*. Finally, let* $F \in \{E^1, F^1\}$*. The following are equivalent:*

(i) There exists a Kähler–Einstein metric in $\mathcal{H}^K$ *and* $\mathrm{Aut}(M, \mathrm{J})_0$ *has finite center.*

(ii) For some $C, D > 0$ *and all* $u \in \mathcal{H}_0^K$,

$$F(u) \geq C d_1(0, u) - D.$$

(iii) For some $C, D > 0$ *and all* $u \in \mathcal{H}_0^K$,

$$F(u) \geq C J(u) - D.$$

Remark 7.3. In Theorems 7.1 and 7.2 when $F = F^1$ one *cannot* replace d_1 by the Mabuchi metric d_2. Indeed, according to (50), Jensen's inequality, and Proposition 5.5, for $\varphi \in \mathcal{H}_0$,

$$F^1(\varphi) = -\log V^{-1} \int_M e^{f_\omega - \varphi} \omega^n \leq V^{-1} \int_M (\varphi - f_\omega)\omega^n \leq C(\sup_M \varphi + 1) \leq C(d_1(0, \varphi) + 1). \quad (90)$$

If F^1 were d_2-proper it would follow that $d_2(0,\varphi) \leq C'(d_1(0,\varphi)+1)$. However, this is impossible. E.g., when M is toric, each of the d_p metrics are equivalent (via the Legendre transform) to the L^p metric on the space of convex functions on the Delzant polytope P of M [33, Proposition 4.5], and one can construct d_2-unbounded sequences contained in a d_1-unit ball. More generally, for arbitrary M, the results of [24] can be readily used to construct such sequences. Finally, properness of the Calabi metric is not the correct notion either, since J is unbounded on $\mathcal{H}$ (Proposition 5.5) while Calabi's metric has finite diameter [17].

In connection with [20, Conjecture 6.1], it would be interesting to see if similar facts also hold the for the K-energy E^1.

Remark 7.4. It would be interesting to extend Theorem 7.2 to the settings of Kähler–Ricci solitons or Kähler–Einstein edge metrics. Also, it is possible to modify the proof of Theorem 7.1 to other settings, e.g., Sasaki–Einstein metrics, twisted KE metrics [68], or multiplier Hermitian structures [41]. For brevity, we do not pursue these here and leave this and related extensions to the reader.

7.1 Proof of Theorem 7.1

The equivalence of (ii) and (iii) is the content of Lemma 5.11.

For the equivalene between (i) and (ii) we wish to apply Theorem 3.4 to the data

$$\mathcal{R} = \mathcal{H}_0, \quad d = d_1, \quad F \in \{E^1, F^1\}, \quad G := \mathrm{Aut}_0(M, \mathrm{J}).$$

First, we go over Notation 3.1. First, in (A1), $\overline{\mathcal{R}} = \mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)$ by Theorem 4.4 and Lemma 5.2. Observe that (A2) holds by Propositions 5.19 and 5.21 (with $\beta = 1$). In (A3), the minimizers of F are denoted by $\mathcal{M}$. Finally, (A4) holds since $G \subset \mathrm{Aut}(M, \mathrm{J})_0$ implies that if $g \in G$ and $\eta \in \mathcal{H}$ then $g.\eta$ is both Kähler and cohomologous to η, i.e., $g.\eta \in \mathcal{H}$. Thus, it remains to verify Hypothesis 3.2.

(P1) This is due to Berndtsson [12, Theorem 1.1] for F^1 and to Berman–Berndtsson [7, Theorem 1.1] for E^1.

(P2) For E^1 this is Proposition 5.28 with $\beta = 1$. For F^1, this follows from Proposition 5.27 with $\beta = 1$.

(P3) This is Theorem 5.31 with $\beta = 1$.

(P4) This is Lemma 5.9.

(P5) This follows from (P3) and the Bando–Mabuchi uniqueness theorem [3, Theorem A (ii)].

(P6) Suppose $u \in \mathcal{H}_0$ is a Kähler-Einstein metric. We wish to apply Proposition 6.8 with $K = \mathrm{Isom}(M, g_u)_0$ and $G = \mathrm{Aut}(M, \mathrm{J})_0$. There are several points to check. First, we verify the assumptions of Proposition 6.2 (used in Proposition 6.8): (i) K is a compact connected subgroup of G by Proposition 6.12; (ii) According to Proposition 6.11 (with $\beta = 1$), Equation (77) holds with $\mathfrak{a} = 0$ and $\tilde{\mathfrak{k}} = \mathrm{isom}(M, g_{\omega_u})$. Second, we verify the

assumptions of Proposition 6.8:
(i) $K.u = u$ by definition;
(ii) for each $X \in \tilde{\mathfrak{k}}$, $t \mapsto \exp_I t\mathrm{J}X.u$ is a d_1-geodesic. This is classical since by (43) and the fact that $X \in \mathrm{isom}(M, g_{\omega_u})$ it follows that

$$\mathrm{J}X = \nabla \psi_{\omega_u}^{\mathrm{J}X} \tag{91}$$

is a gradient (with respect to g_{ω_u}) vector field [40, Theorem 3.5]. Indeed, first remark that Set $\omega_{\varphi(t)} := \omega(t) = \exp_I t\mathrm{J}X.\omega_u$. Thus,

$$\dot{\omega}(t) = \frac{d}{dt} \exp_I t\mathrm{J}X.\omega_u = \mathcal{L}_{\mathrm{J}X}\omega \circ \exp_I t\mathrm{J}X = \sqrt{-1}\partial\bar{\partial}\psi_{\omega_u}^{\mathrm{J}X} \circ \exp_I t\mathrm{J}X, \tag{92}$$

and

$$\ddot{\omega}(t) = \sqrt{-1}\partial\bar{\partial}|\nabla\psi_{\omega_u}^{\mathrm{J}X}|^2 \circ \exp_I t\mathrm{J}X,$$

i.e., $\ddot{\varphi}(t) - |\nabla\dot{\varphi}(t)|^2_{\omega_{\varphi(t)}} = 0$, which by an observation of Semmes and Donaldson [50, 30] means that $\varphi(t)$ solves (16). Thus, Theorem 4.4 implies $t \mapsto \varphi(t)$ is a d_1-geodesic. The speed of this geodesic depends continuously on X by (92) and (15);
(iii) $G \times G \ni (f, g) \mapsto d_1(f.u, g.v) = d_1(u, f^{-1} \circ g.v)$ by Lemma 5.9, and this is a continuous map in $G \times G$ whenever $u, v \in \mathcal{H}_0$ are fixed. Indeed, if $h_k \in G$ converges to $h \in G$ then $h_k.\omega_v$ converges smoothly to $h.\omega_v$ and using the Green kernel of ω we see that also $h_k.v$ converges smoothly to $h.v$. Thus, $d_1(u, h_k.v)$ converges to $d_1(u, h.v)$ by (29).

(P7) Both F^1 and E^1 are the path-integrals of G-invariant closed 1-forms on $\mathcal{H}_\omega$.

7.2 Proof of Theorem 7.2

By Proposition 5.5, it suffices to verify the equivalence between (i) and (ii). We apply Theorem 3.4, but this time only in the direction (i) ⇒ (ii). To do so, we set

$$\mathcal{R} = \mathcal{H}_0^K, \quad d = d_1, \quad F \in \{E^1, F^1\}, \quad G = \{I\}.$$

By Lemma 5.14, $\overline{\mathcal{R}} = \mathcal{E}_1^K \cap \mathrm{AM}^{-1}(0)$. All the properties (A1)-(A4), (P1)-(P7) are inherited from Therem 7.1, with the exception of (P3) and (P5). We verify these then and then Theorem 3.4 will automatically yield (ii).

Claim 7.5. *Assume that (i) holds. Property (P3) holds.*

Proof. As $\mathcal{H}_0^K$ contains a Kähler–Einstein metric u, Theorem 5.31 ($\beta = 1$) gives that u minimizes F globally on $\mathcal{E}_1$, so in particular also on $\mathcal{E}_1^K$, giving $u \in \mathcal{M}$. If $v \in \mathcal{M}$ arbitrary, then $F(u) = F(v)$, hence another application of Theorem 5.31 gives that v is also smooth Kähler-Einstein, concluding that $\mathcal{M} \subset \mathcal{H}_0^K$. □

As we chose the group G to be trivial, to verify (P5), we have to show that $\mathcal{M}$ is a singleton. Before proving this we need to understand properties of the group K.

Claim 7.6. *Suppose $(M, \mathrm{J}, \omega, g_{\omega_u})$ is Kähler–Einstein with $u \in \mathcal{H}_0^K$. Then $K = \mathrm{Isom}(M, g_{\omega_u})_0$.*

Proof. By Theorem 6.12 $\mathrm{Isom}(M, g_{\omega_u})_0$ is a maximal compact subgroup of $\mathrm{Aut}(M, \mathrm{J})_0$. By assumption $K \subset \mathrm{Aut}(M, \mathrm{J})_0$ is also maximal and trivially $K \subset \mathrm{Isom}(M, g_{\omega_u})_0$, hence in fact $K = \mathrm{Isom}(M, g_{\omega_u})_0$. □

Let L be a group. Recall that the centralizer and normalizer of a subgroup H are defined as follows:

$$N_H(L) := \{g \in L : ghg^{-1} \in H, \ \forall h \in H\}.$$
$$C_H(L) := \{g \in L : ghg^{-1} = h, \ \forall h \in H\} \subset N_H(L).$$

Note that $C_L = C_L(L)$ is just the center of L. The following result is due to Hazod et al. [36, Theorem A] and we will make us of it shortly.

Theorem 7.7. *Suppose H be a compact subgroup of a connected Lie group L. Then the group $N_H(L)/(HC_H(L))$ is a finite.*

Lemma 7.8. *Suppose $(M, \mathrm{J}, \omega, g_{\omega_u})$ is Kähler–Einstein and that $\mathrm{Aut}(M, \mathrm{J})_0$ has finite center. Then, $N_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0) = \mathrm{Isom}(M, g_{\omega_u})_0$.*

Proof. We claim that

$$C_{\mathrm{Aut}(M,\mathrm{J})_0}(\mathrm{Aut}(M, \mathrm{J})_0) = C_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0). \tag{93}$$

One inclusion is by definition; for the converse suppose that $h \in C_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0)$. The map $\mathrm{Aut}(M, \mathrm{J})_0 \ni g \mapsto C_h(g) := hgh^{-1} \in \mathrm{Aut}(M, \mathrm{J})_0$ is biholomorphic. Thus, $dC_h(\mathrm{J}X) = \mathrm{J}dC_h(X)$. Since $C_h(g) = g$ whenever $g \in \mathrm{Isom}(M, g_{\omega_u})_0$, it follows that $dC_h(X) = X$ for each $X \in \mathrm{isom}(M, g_{\omega_u})$. It follows from Proposition 6.9 that in fact $dC_h = \mathrm{Id}$ identically. Since $\mathrm{Aut}(M, \mathrm{J})_0$ is connected (i.e., the union of all of its 1-parameter subgroups passing through the identity), it follows that C_h is the identity map. Thus, $h \in C_{\mathrm{Aut}(M,\mathrm{J})_0}(\mathrm{Aut}(M, \mathrm{J})_0)$.

Hence, By Theorem 7.7,

$$N_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0)/\mathrm{Isom}(M, g_{\omega_u})_0$$

is finite, implying that $N_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0)$ is compact. By Theorem 6.12, it follows that $N_{\mathrm{Isom}(M,g_{\omega_u})_0}(\mathrm{Aut}(M, \mathrm{J})_0) = \mathrm{Isom}(M, g_{\omega_u})_0$. □

Claim 7.9. *Assume that (i) holds. Then $\mathcal{M}$ is a singleton, hence (P5) holds.*

Proof. Suppose $u, v \in \mathcal{M}$. By Claims 7.5 and 7.6 we have $\mathrm{Isom}(M, g_{\omega_v})_0 = \mathrm{Isom}(M, g_{\omega_u})_0 = K$. Thus,

$$f^{-1}Kf \subset \mathrm{Isom}(M, f^\star g_{\omega_u})_0 = \mathrm{Isom}(M, g_{\omega_v})_0 = K.$$

Thus, $f \in N_K(\mathrm{Aut}(M, \mathrm{J})_0)$. By Lemma 7.8, $N_K(Aut_0(M, \mathrm{J})) = K$, so $u = f.u = v$. □

We now turn to proving the direction (ii) ⇒ (i).

Claim 7.10. *If (ii) holds, $\mathcal{H}_0^K$ contains a Kähler–Einstein potential.*

Proof. As remarked earlier, (iii) holds by Proposition 5.5. The classical continuity method with a K-invariant reference metric (observe $\mathcal{H}^K$ is nonempty by averaging an arbitrary element of $\mathcal{H}$ with respect to the Haar measure of K) produces a Kähler–Einstein potential when $F = F^1$ [58, Proposition] or $F = E^1$ [46, pp. 2651–2653]. Since solutions $\varphi(t)$ to $\omega^n_{\varphi(t)} = \omega^n e^{f_\omega - t\varphi(t)}$ are unique for each $t \in (0, 1)$ [12, Theorem 7.1], it follows that the Kähler–Einstein potential must be K-invariant. □

For the remainder of the proof we fix a Kähler–Einstein potential $u \in \mathcal{H}_0^K$ provided by the last claim. By Claim 7.6 we have $K = \mathrm{Isom}_0(M, g_{\omega_u})$. It remains to show that the center of $\mathrm{Aut}(M, \mathrm{J})_0$ is finite. For this we take a closer look at the normalizer of K:

Claim 7.11. *Suppose (ii) holds. Then* $N_K(\mathrm{Aut}(M,\mathrm{J})_0)$ *is compact.*

Proof. It is trivial to verify that $N_K(\mathrm{Aut}(M,\mathrm{J})_0)$ acts on $\mathcal{H}_0^K$, hence $g.u \in \mathcal{H}_0^K$ for any $g \in N_K(\mathrm{Aut}(M,\mathrm{J})_0)$. As $g.u$ is Kähler–Einstein it follows that $F(u) = F(g.u)$.

By (ii), $d_1(u, g.u)$ is uniformly bounded for $g \in N_K(\mathrm{Aut}(M,\mathrm{J})_0)$. Let C be the map given by (76) (recall Proposition 6.11) and consider

$$S := C^{-1}(N_K(\mathrm{Aut}(M,\mathrm{J})_0)).$$

Since C is continuous, S is closed in $K \times \mathrm{isom}(M, g_{\omega_u})$. To show that $N_K(\mathrm{Aut}(M,\mathrm{J})_0)$ is compact we only need to show that S is bounded. For any $(k, X) \in S$ we can write

$$d_1(u, C(k, \mathrm{J}X).u) = d_1(u, k \exp_I \mathrm{J}X.u) = d_1(u, \exp_I \mathrm{J}X.u) \geq \rho|X|,$$

where, using Theorem 4.3,

$$\rho := \inf_{Y \in \mathrm{isom}(M, g_{\omega_u}), |Y|=1} d_1(u, \exp_I \mathrm{J}Y.u) > 0,$$

since $[0,\infty) \ni t \to \exp_I t\mathrm{J}X.u \in \mathcal{H}_0$ is a d_1-geodesic ray initiating from u by the proof of (P6) in §7.1. Thus, $|X|$ is uniformly bounded giving that S is a bounded set. □

As $K = \mathrm{Isom}(M, g_{\omega_u})_0$ is maximally compact in $\mathrm{Aut}(M,\mathrm{J})_0$,

$$N_K(\mathrm{Aut}(M,\mathrm{J})_0) = K.$$

Thus, by (93) (which uses only Proposition 6.9),

$$C_{\mathrm{Aut}(M,\mathrm{J})_0} = C_K(\mathrm{Aut}(M,\mathrm{J})_0) \subset N_K(\mathrm{Aut}(M,\mathrm{J})_0) = K. \tag{94}$$

But the Lie algebra of $C_{\mathrm{Aut}(M,\mathrm{J})_0}$ must be trivial, since it is complex and is a Lie subalgebra of $\mathrm{isom}(M, g_{\omega_u})$. Indeed, by Proposition 6.9, the only complex Lie subalgebra of $\mathrm{isom}(M, g_{\omega_u})$ is the trivial one. Since $C_{\mathrm{Aut}(M,\mathrm{J})_0}$ is compact by (94), it must be finite. This concludes the proof of Theorem 7.2.

8 Kähler–Ricci solitons

We now state our main result concerning existence of Kähler–Ricci solitons. Recall the definition of $\mathrm{aut}^X(M,\mathrm{J})$ and $\mathrm{Aut}^X(M,\mathrm{J})$ from (85). In this section we set

$$G := \mathrm{Aut}^X(M,\mathrm{J}). \tag{95}$$

Theorem 8.1. *Let* (M,J,ω) *be a Fano manifold with* $[\omega] = c_1(X)$*, let* $X \in \mathrm{aut}(M,\mathrm{J})$*, let* T *be the group determined by* X *given in* (41)*, and let* $F \in \{F^X, E^X\}$*. The following are equivalent:*
(i) There exists a Kähler–Ricci soliton associated to X *in* $\mathcal{H}_0^T$*.*
(ii) F *is* G*-invariant and for some* $C, D > 0$*,*

$$F(u) \geq C d_{1,G}(G0, Gu) - D, \qquad u \in \mathcal{H}_0^T.$$

(iii) F *is* G*-invariant and for some* $C, D > 0$*,*

$$F(u) \geq C J_G(G0, Gu) - D, \qquad u \in \mathcal{H}_0^T.$$

Proof. By Lemma 5.11, it suffices to verify the equivalence between (i) and (ii). We apply Theorem 3.4, but this time only for $F = F^X$. Recall G is defined in (95). For the remaining data in Notation 3.1 we set

$$\mathcal{R} = \mathcal{H}_0^T, \quad d = d_1.$$

Thus, in (A1), $\overline{\mathcal{R}} = \mathcal{E}_1^T \cap \mathrm{AM}^{-1}(0)$, by Lemma 5.14. Observe that (A2) holds by Proposition 5.25. In (A3), the minimizers of F^X are denoted by $\mathcal{M}$. Finally, (A4) holds by Lemma 5.10. Thus, it remains to verify Hypothesis 3.2.

(P1) This is due to [12, Theorem 1.1, Proposition 10.4] .

(P2) This is Proposition 5.29.

(P3) This is Theorem 5.32.

(P4) This is Lemma 5.9.

(P5) This follows from (P3) and the Tian–Zhu uniqueness theorem [63].

(P6) Suppose $u \in \mathcal{M}$ is a Kähler–Ricci solition. We may apply Proposition 6.8 with $K = \mathrm{Isom}(M, g_{\omega_u})_0$ and G, since the assumptions of Proposition 6.2 are verified by Proposition 6.10 and an argument identical to the proof of (P6) in §7.1.

(P7) F^X is the path-integrals of G-invariant closed 1-forms on $\mathcal{H}_\omega^X$.

This concludes the proof of Theorem 8.1 for $F = F^X$.

We now let $F = E^X$. By (67) and the previous paragraph it suffices to verify the direction (ii) ⇒ (i). This is done in [63],[11, Theorem 1.6]. □

9 Kähler–Einstein edge metrics.

We now state our main result concerning existence of Kähler–Einstein edge metrics. Recalling (88), in this section we set

$$G := \mathrm{Aut}(M, D, \mathrm{J})_0. \tag{96}$$

Theorem 9.1. *Suppose (M, J, ω) is compact Kähler, $D \subset M$, and $\beta \in (0, 1)$ satisfy (48). Let $F \in \{F^\beta, E^\beta\}$. The following are equivalent:*

(i) There exists a Kähler–Einstein edge metric in $\mathcal{H}^\beta$.

(ii) F is G-invariant and for some $C, D > 0$,

$$F(u) \geq C d_{1,G}(G0, Gu) - D, \qquad u \in \mathcal{H}_0^\beta.$$

(iii) F is G-invariant and for some $C, D > 0$,

$$F(u) \geq C J_G(Gu) - D, \qquad u \in \mathcal{H}_0^\beta.$$

Proof. By Lemma 5.11, it suffices to verify the equivalence between (i) and (ii). We apply Theorem 3.4, but as in §8 only for $F = F^\beta$. Recall G is defined in (96). For the remaining data in Notation 3.1 we set

$$\mathcal{R} = \mathcal{H}_0^\beta, \quad d = d_1.$$

Thus, in (A1), $\overline{\mathcal{R}} = \mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)$, by Lemmas 5.2 and 5.17. Observe that (A2) holds by Proposition 5.19. In (A3), the minimizers of F are denoted by $\mathcal{M}$. Finally, (A4) holds by Definition 5.16. Thus, it remains to verify Hypothesis 3.2.

(P1) This is [12, Theorem 6.4].

(P2) This is Proposition 5.27.

(P3) This is Theorem 5.31.

(P4) This is Lemma 5.10.

(P5) This follows from (P3) and the Berndtsson's uniqueness theorem [12, Theorem 6.4].

(P6) Let $u \in \mathcal{M}$ be a Kähler–Einstein edge metric. We may apply Proposition 6.8 with $K = \mathrm{Isom}(M, D, g_{\omega_u})_0$ and G, since the assumptions of Proposition 6.2 are verified by Proposition 6.11 and the proof of (P6) in §7.1.

(P7) F^β is the path-integrals of G-invariant closed 1-forms on $\mathcal{H}_\omega^\beta$.

This concludes the proof of Theorem 9.1 for $F = F^\beta$.

We now let $F = E^\beta$. By (67) and the previous paragraph it suffices to verify the direction (ii) $\Rightarrow$ (i). For this, Theorem 3.4, thanks to Remark 3.9 (i) and Proposition 5.28, implies that $\mathcal{M} \neq \emptyset$. Finally, Theorem 5.31 gives that $\mathcal{M} \subset \mathcal{R}$, as desired. □

10 Constant scalar curvature metrics

We now state our main result concerning existence of constant scalar curvature metrics. In this section we set

$$G := \mathrm{Aut}(M, \mathrm{J})_0. \tag{97}$$

Theorem 10.1. *Let (M, ω) be a Kähler manifold and suppose that minimizers of the K-energy E on $\mathcal{E}_1$ are smooth. Then the following are equivalent: (i) There exists a constant scalar curvature metric in $\mathcal{H}$.*
(ii) E is G-invariant and for some $C, D > 0$,

$$E(u) \geq C d_{1,G}(G0, Gu) - D, \qquad u \in \mathcal{H}_0.$$

(iii) E is G-invariant and for some $C, D > 0$,

$$F(u) \geq C J_G(Gu) - D, \qquad u \in \mathcal{H}_0.$$

Proof. The proof is the same as that of Theorem 7.1 except for the following points. For the equivalence between (i) and (ii) we apply Theorem 3.4 to the data

$$\mathcal{R} = \mathcal{H}_0, \quad d = d_1, \quad F = E.$$

Again, in (A1), $\overline{\mathcal{R}} = \mathcal{E}_1 \cap \mathrm{AM}^{-1}(0)$. Observe that condition (A2) holds by Proposition 5.26. Property (P1) holds by [7, Theorem 1.1]. Property (P2) holds by Proposition 5.28. Property (P3) holds by assumption. Property (P5) holds by the work of Berman–Berndtsson [7, Theorem 1.3]. Suppose $u \in \mathcal{H}_0$ is a constant scalar curvature metric. By the same argument as the one in Theorem 7.1, due to Propositions 6.8 and 6.9, property (P6) holds with $K = \mathrm{Isom}(M, g_{\omega_u})_0$ and $G = \mathrm{Aut}(M, \mathrm{J})_0$. □

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University of Maryland
tdarvas@math.umd.edu, yanir@umd.edu