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Académie des sciences

Hyperbolic algebraic varieties and holomorphic differential equations

Jean-Pierre Demailly

Institut Fourier, Université de Grenoble I, France
& Académie des Sciences de Paris

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- A complex torus $X = \mathbb{C}^n/\Lambda$ (Λ lattice) has a lot of entire curves. As $\mathbb{C}$ simply connected, every $f : \mathbb{C} \to X = \mathbb{C}^n/\Lambda$ lifts as $\tilde{f} : \mathbb{C} \to \mathbb{C}^n$, $\tilde{f}(t) = (\tilde{f}_1(t), \ldots, \tilde{f}_n(t))$, and $\tilde{f}_j : \mathbb{C} \to \mathbb{C}$ can be arbitrary entire functions.

Projective algebraic varieties

- Consider now the complex projective n-space

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \smallsetminus \{0\})/\mathbb{C}^*, \qquad [z] = [z_0 : z_1 : \ldots : z_n].$$

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- An entire curve $f : \mathbb{C} \to \mathbb{P}^n$ is given by a map

$$t \longmapsto [f_0(t) : f_1(t) : \ldots : f_n(t)]$$

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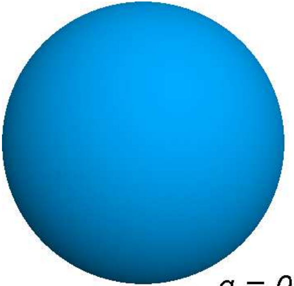
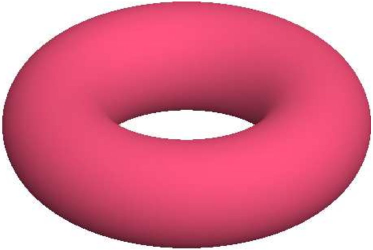
- More generally, look at a (complex) projective manifold, i.e.

$$X^n \subset \mathbb{P}^N, \qquad X = \{[z]\,;\ P_1(z) = \ldots = P_k(z) = 0\}$$

where $P_j(z) = P_j(z_0, z_1, \ldots, z_N)$ are homogeneous polynomials (of some degree d_j), such that X is non singular.

Complex curves (genus 0 and 1)





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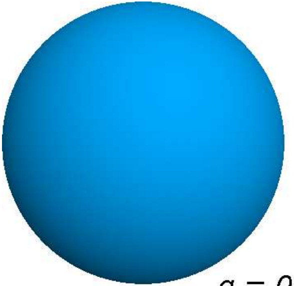

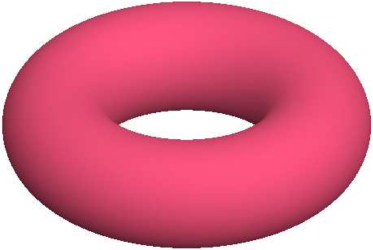

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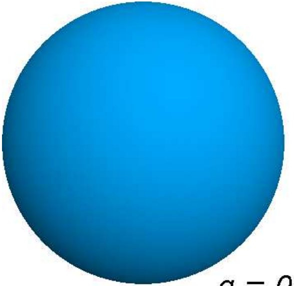

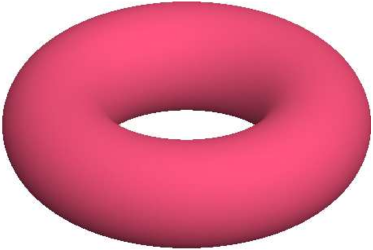

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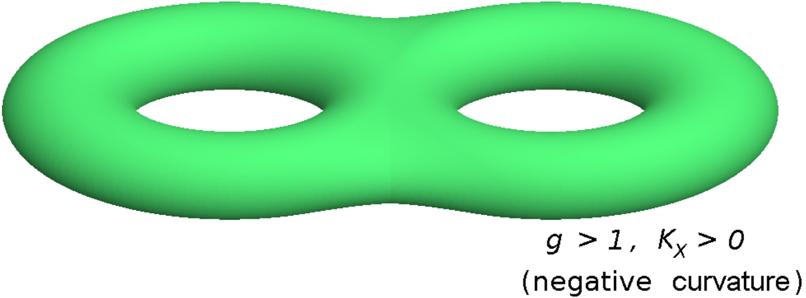
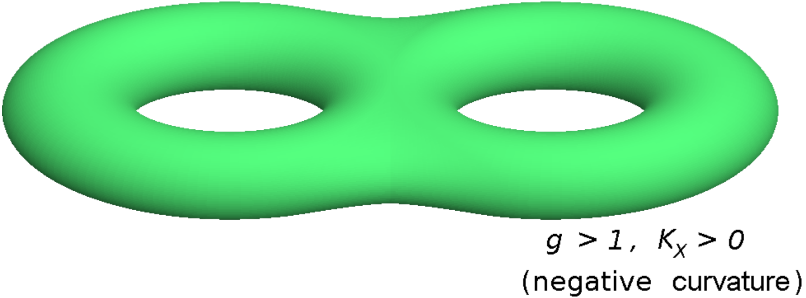
- $g = 1$: $X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ courbure $T_X = 0$ not hyperbolic

Complex curves (genus $g \geq 2$)



$\deg K_X = 2g - 2$

If $g \geq 2$, $X \simeq \mathbb{D}/\Gamma$ $(T_X < 0)$ $\Rightarrow$ X is hyperbolic.



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If $g \geq 2$, $\quad X \simeq \mathbb{D}/\Gamma \quad (T_X < 0) \quad \Rightarrow \quad X$ is hyperbolic.

In fact every curve $f : \mathbb{C} \to X \simeq \mathbb{D}/\Gamma$ lifts to $\widetilde{f} : \mathbb{C} \to \mathbb{D}$, and so must be constant by Liouville.

- For a complex manifold, $n = \dim_{\mathbb{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X$, $\xi \in T_X$
$\kappa_x(\xi) = \inf\{\lambda > 0\,;\ \exists f : \mathbb{D} \to X,\ f(0) = x,\ \lambda f_*(0) = \xi\}$
On $\mathbb{C}^n$, $\mathbb{P}^n$ or complex tori $X = \mathbb{C}^n/\Lambda$, one has $\kappa_X \equiv 0$.

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- **Examples.** $*$ $X = \Omega/\Gamma$, Ω bounded symmetric domain.
  $*$ any product $X = X_1 \times \ldots \times X_s$ where X_j hyperbolic.
- **Theorem (dimension n arbitrary)** (Kobayashi, 1970)
  T_X negatively curved ($T_X^* > 0$, i.e. ample) $\Rightarrow$ X hyperbolic.
  Recall that a holomorphic vector bundle E is ample iff its symmetric powers $S^m E$ have global sections which generate 1-jets of (germs of) sections at any point $x \in X$.

Ahlfors-Schwarz lemma

The proof of the above Kobayashi result depends crucially on:

Ahlfors-Schwarz lemma. Let $\gamma = i\sum \gamma_{jk} dt_j \wedge d\overline{t}_k$ be an almost everywhere positive hermitian form on the ball $B(0,R) \subset \mathbb{C}^p$, such that $-\mathrm{Ricci}(\gamma) := i\,\partial\overline{\partial}\log\det\gamma \geq A\gamma$ in the sense of currents, for some constant $A > 0$ (this means in particular that $\det\gamma = \det(\gamma_{jk})$ is such that $\log\det\gamma$ is plurisubharmonic). Then the γ-volume form is controlled by the Poincaré volume form :

$$\det(\gamma) \leq \left(\frac{p+1}{AR^2}\right)^p \frac{1}{(1-|t|^2/R^2)^{p+1}}.$$

Brody reparametrization Lemma. *Assume that X is compact, let ω be a hermitian metric on X and $f : \mathbb{D} \to X$ a holomorphic map. For every $\varepsilon > 0$, there exists a radius $R \geq (1-\varepsilon)\|f'(0)\|_\omega$ and a homographic transformation ψ of the disk $D(0,R)$ onto $(1-\varepsilon)\mathbb{D}$ such that $\|(f\circ\psi)'(0)\|_\omega = 1$ and $\|(f\circ\psi)'(t)\|_\omega \leq (1-|t|^2/R^2)^{-1}$ for every $t \in D(0,R)$.*

$\Rightarrow$ *if f' unbounded, $\exists g = \lim f\circ\psi_\nu : \mathbb{C} \to X$ with $\|g'\|_\omega \leq 1$.*

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Brody theorem (1978). *If X is compact then X is Kobayashi hyperbolic if and only if there are no entire holomorphic curves $f : \mathbb{C} \to X$ (Brody hyperbolicity).*

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Hyperbolic varieties are especially interesting for their expected diophantine properties :

Conjecture (S. Lang, 1986) *An arithmetic projective variety X is hyperbolic iff $X(\mathbb{K})$ is finite for every number field $\mathbb{K}$.*

- **Definition** *A non singular projective variety X is said to be of general type if the growth of pluricanonical sections*

$$\dim H^0(X, K_X^{\otimes m}) \sim cm^n, \qquad K_X = \Lambda^n T_X^*$$

is maximal.

(sections locally of the form $f(z)\,(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}$)

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- **Conjecture CGT.** *If a compact variety X is hyperbolic, then it should be of general type, and if X is non singular, then $K_X = \Lambda^n T_X^*$ should be ample, i.e. $K_X > 0$ (Kodaira) (equivalently $\exists$ Kähler metric ω such that $Ricci(\omega) < 0$).*

- **Theorem.** *Let X be projective algebraic. Consider the following properties :*
  (GT) *Every subvariety Y of X is of general type.*
  (AH) $\exists\varepsilon > 0$, $\forall C \subset X$ *algebraic curve*

  $$2g(\bar{C}) - 2 \geq \varepsilon \deg(C).$$

  (X *"algebraically hyperbolic"*)
  (HY) X *is hyperbolic*
  (JC) X *possesses a jet-metric with negative curvature on its k-jet bundle X_k [to be defined later], for $k \geq k_0 \gg 1$.*

  Then $(\mathrm{JC}) \Rightarrow (\mathrm{GT}),\ (\mathrm{AH}),\ (\mathrm{HY})$,
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  Then (JC) $\Rightarrow$ (GT), (AH), (HY),
  (HY) $\Rightarrow$ (AH),
  and if Conjecture CGT holds, (HY) $\Rightarrow$ (GT).
- It is expected that all 4 properties are in fact equivalent for projective varieties.

Green-Griffiths-Lang conjecture

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- Combining the above conjectures, we get :
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  are equivalent if CGT + GGL *hold.*
- **Arithmetic counterpart** (Lang 1987). *If X is a variety of general type defined over a number field and Y is the Green-Griffiths locus (Zariski closure of $\bigcup f(\mathbb{C})$), then $X(\mathbb{K}) \smallsetminus Y$ is finite for every number field $\mathbb{K}$.*

- Using "jet technology" and deep results of McQuillan for curve foliations on surfaces, D. – El Goul proved
  Theorem (solution of Kobayashi conjecture, 1998).
  A very generic surface $X \subset \mathbb{P}^3$ of degree ≥ 21 is hyperbolic.
  Independently McQuillan got degree ≥ 35.
  Recently improved to degree ≥ 18 (Păun, 2008).

  For $X \subset \mathbb{P}^{n+1}$, the optimal bound should be degree $\geq 2n+1$ for $n \geq 2$ **(Zaidenberg)**.

- **Generic GGL conjecture for** $\dim_{\mathbb{C}} X = n$
  (S. Diverio, J. Merker, E. Rousseau, 2009).
  If $X \subset \mathbb{P}^{n+1}$ is a generic n-fold of degree $d \geq d_n := 2^{n^5}$,
  [also $d_3 = 593$, $d_4 = 3203$, $d_5 = 35355$, $d_6 = 172925$]
  then $\exists Y \subsetneq X$ s.t. $\forall$ non const. $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$

Results obtained so far

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  then $\exists Y \subsetneq X$ s.t. $\forall$ non const. $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Y$
  Moreover (S. Diverio, S. Trapani, 2009) $\operatorname{codim}_{\mathbb{C}} Y \geq 2 \Rightarrow$ generic hypersurface $X \subset \mathbb{P}^4$ of degree ≥ 593 is hyperbolic.

The main idea in order to attack GGL is to use differential equations. Let

$$\mathbb{C} \to X, \quad t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$$

be a curve written in some local holomorphic coordinates $(z_1, \ldots, z_n)$ on X.

Definition of algebraic differential operators

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Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

where $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are holomorphic coefficients on X and $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet.

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Obvious $\mathbb{C}^*$-action :

$$\lambda \cdot f(t) = f(\lambda t), \quad (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

Vanishing theorem for differential operators

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- **Fundamental vanishing theorem**
  [Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]
  Let $P \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then $\forall f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.

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  Let $P \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ *be a global algebraic differential operator whose coefficients vanish on some ample divisor A. Then* $\forall f : \mathbb{C} \to X$, $P(f_{[k]}) \equiv 0$.
- *Proof*. One can assume that A is very ample and intersects $f(\mathbb{C})$. Also assume f' bounded (this is not so restrictive by Brody !). Then all $f^{(k)}$ are bounded by Cauchy inequality. Hence

$$\mathbb{C} \ni t \mapsto P(f', f'', \ldots, f^{(k)})(t)$$

is a bounded holomorphic function on $\mathbb{C}$ which vanishes at some point. Apply Liouville's theorem ! $\square$

- Let $X_k^{\mathrm{GG}} = J_k(X)^*/\mathbb{C}^*$ be the projectivized k-jet bundle of X = quotient of non constant k-jets by $\mathbb{C}^*$-action. Fibers are weighted projective spaces.
  Observation. *If $\pi_k : X_k^{\mathrm{GG}} \to X$ is canonical projection and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ is the tautological line bundle, then*

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  $$E_{k,m}^{\mathrm{GG}} = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m)$$
- Saying that $f : \mathbb{C} \to X$ satisfies the differential equation $P(f_{[k]}) = 0$ means that
  $$f_{[k]}(\mathbb{C}) \subset Z_P$$
  where Z_P is the zero divisor of the section
  $$\sigma_P \in H^0(X_k^{\mathrm{GG}}, \mathcal{O}_{X_k^{\mathrm{GG}}}(m) \otimes \pi_k^* \mathcal{O}(-A))$$
  associated with P.

- **Consequence of fundamental vanishing theorem.**
If $P_j \in H^0(X, E_{k,m}^{\mathrm{GG}} \otimes \mathcal{O}(-A))$ *is a basis of sections then the image* $f(\mathbb{C})$ *lies in* $Y = \pi_k(\bigcap Z_{P_j})$*, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that*

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

- **Consequence of fundamental vanishing theorem.**
If $P_j \in H^0(X, E^{\mathrm{GG}}_{k,m} \otimes \mathcal{O}(-A))$ is a basis of sections then the image $f(\mathbb{C})$ lies in $Y = \pi_k(\bigcap Z_{P_j})$, hence property asserted by the GGL conjecture holds true if there are "enough independent differential equations" so that

$$Y = \pi_k(\bigcap_j Z_{P_j}) \subsetneq X.$$

- However, some differential equations are not very useful. On a surface with coordinates (z_1, z_2), a Wronskian equation $f_1' f_2'' - f_2' f_1'' = 0$ tells us that $f(\mathbb{C})$ sits on a line, but $f_2''(t) = 0$ says that the second component is linear affine in time, an essentially meaningless information which is lost by a change of parameter $t \mapsto \varphi(t)$.

- The k-th order Wronskian operator

$$W_k(f) = f' \wedge f'' \wedge \ldots \wedge f^{(k)}$$

(locally defined in coordinates) has degree $m = \frac{k(k+1)}{2}$ and

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- **Definition.** *A differential operator P of order k and degree m is said to be invariant by reparametrization if*

$$P(f \circ \varphi) = \varphi'^m P(f) \circ \varphi$$

for any parameter change $t \mapsto \varphi(t)$. Consider their set

$$E_{k,m} \subset E_{k,m}^{\mathrm{GG}} \quad (\textit{a subbundle})$$

(Any polynomial $Q(W_1, W_2, \ldots W_k)$ is invariant, but for $k \geq 3$ there are other invariant operators.)

Category of directed manifolds

- **Goal.** We are interested in curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$ where V is a subbundle (or subsheaf) of T_X.

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  - Objects : pairs (X, V), X manifold/$\mathbb{C}$ and $V \subset \mathcal{O}(T_X)$
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- **Fonctor "1-jet" :** $(X,V)\mapsto(\tilde X,\tilde V)$ where :

$$\tilde X = P(V) = \text{bundle of projective spaces of lines in } V$$
$$\pi:\tilde X = P(V)\to X,\quad (x,[v])\mapsto x,\quad v\in V_x$$
$$\tilde V_{(x,[v])} = \big\{\xi\in T_{\tilde X,(x,[v])}\,;\ \pi_*\xi\in\mathbb{C}v\subset T_{X,x}\big\}$$

- For every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ tangent to V

$$f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X}$$

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- **Basic exact sequences**

$$0 \to T_{\tilde{X}/X} \to \tilde{V} \xrightarrow{\pi_\star} \mathcal{O}_{\tilde{X}}(-1) \to 0 \qquad \Rightarrow \mathrm{rk}\, \tilde{V} = r = \mathrm{rk}\, V$$

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- For $n = \dim X$ and $r = \mathrm{rk}\, V$, get a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\mathrm{rk}\, V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

Direct image formula

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- **Theorem.** *X_k is a smooth compactification of*

$$X_k^{\mathrm{GG,reg}}/G_k = J_k^{\mathrm{GG,reg}}/G_k$$

  where G_k is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$, acting on the right by reparametrization: $(f,\varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

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- **Direct image formula.** $(\pi_{k,0})_* \mathcal{O}_{X_k}(m) = E_{k,m} V^* =$ *invariant algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$.*

Algebraic structure of differential rings

- Although very interesting, results are currently limited by lack of knowledge on jet bundles and differential operators
- Theorem (Bérczi-Kirwan, 2009). *The ring of germs of invariant differential operators on* $(\mathbb{C}^n, T_{\mathbb{C}^n})$ *at the origin*
$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{is finitely generated.}$$

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$$\mathcal{A}_{k,n} = \bigoplus_m E_{k,m} T^*_{\mathbb{C}^n} \quad \textit{is finitely generated.}$$
- Checked by direct calculations $\forall n$, $k \le 2$ and $n = 2$, $k \le 4$:

$$\begin{aligned}
\mathcal{A}_{1,n} &= \mathcal{O}[f_1', \ldots, f_n'] \\
\mathcal{A}_{2,n} &= \mathcal{O}[f_1', \ldots, f_n', W^{[ij]}], \quad W^{[ij]} = f_i' f_j'' - f_j' f_i'' \\
\mathcal{A}_{3,2} &= \mathcal{O}[f_1', f_2', W_1, W_2][W]^2, \quad W_i = f_i' DW - 3 f_i'' W \\
\mathcal{A}_{4,2} &= \mathcal{O}[f_1', f_2', W_{11}, W_{22}, S][W]^6, \quad W_{ii} = f_i' DW_i - 5 f_i'' W_i
\end{aligned}$$

where $W = f_1' f_2'' - f_2' f_1''$, $S = (W_1 DW_2 - W_2 DW_1)/W$.

- **Generalized GGL conjecture.** *If* (X, V) *is directed manifold of general type, i.e.* $\det V^*$ *big, then* $\exists Y \subsetneq X$ *such that* $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ *non const.,* $f(\mathbb{C}) \subset Y$.

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- **Strategy.** Try some sort of induction on $r = \mathrm{rk}\, V$. First try to get differential equations $f_{[k]}(\mathbb{C}) \subset Z \subsetneq X_k$. Take minimal such k. If $k = 0$, we are done! Otherwise $k \geq 1$ and $\pi_{k,k-1}(Z) = X_{k-1}$, thus $V' = V_k \cap T_Z$ has rank $< \mathrm{rk}\, V_k = r$ and should have again $\det V'^*$ big (unless some unprobable geometry situation occurs ?).

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- **Needed induction step.** *If (X,V) has* $\det V^*$ *big and* $Z \subset X_k$ *irreducible with* $\pi_{k,k-1}(Z) = X_{k-1}$*, then* (Z, V'), $V' = V_k \cap T_Z$ *has* $\mathcal{O}_{Z_\ell}(1)$ *big on* (Z_ℓ, V'_ℓ), $\ell \gg 0$.

Holomorphic Morse inequalities (D-, 1985) Let $L \to X$ be a holomorphic line bundle on a compact complex manifold X, h a smooth hermitian metric on L and

$$\Theta_{L,h} = \frac{i}{2\pi}\nabla^2_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$$

its curvature form. Then $\forall q = 0, 1, \ldots, n = \dim_{\mathbb{C}} X$

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X, L^{\otimes k}) \le \frac{k^n}{n!}\int_{X(L,h,\le q)}(-1)^q\Theta^n_{L,h} + o(k^n).$$

where

$$X(L,h,q) = \{x \in X\,;\ \Theta_{L,h}(x) \text{ has signature } (n-q,q)\}$$

(q-index set), and

$$X(L,h,\le q) = \bigcup_{0\le j\le q} X(L,h,\le j)$$

As a consequence, one gets an upper bound

$$h^0(X, L^{\otimes k}) \le \frac{k^n}{n!} \int_{X(L,h,0)} \Theta^n_{L,h} + o(k^n)$$

and a lower bound

$$h^0(X, L^{\otimes k}) \ge h^0(X, L^{\otimes k}) - h^1(X, L^{\otimes k}) \ge$$
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Holomorphic Morse inequalities (continued)

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and similar bounds for the higher cohomology groups H^q:

$$h^q(X, L^{\otimes k}) \le \frac{k^n}{n!} \int_{X(L,h,q)} |\Theta_{L,h}^n| + o(k^n)$$

$$h^q(X, L^{\otimes k}) \ge \frac{k^n}{n!} \Big(\int_{X(L,h,q)} - \int_{X(L,h,q-1)} - \int_{X(L,h,q+1)} |\Theta_{L,h}^n| \Big) - o(k^n)$$

Finsler metric on the k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$

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Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1 \le s \le k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \Big)^{1/p}, \;\; 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

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Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\, dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$. The expression gets simpler by using polar coordinates $x_s = |\xi_s|_h^{2p/s}$, $u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|$.

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$. The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s}\gamma(u_s)$ where u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in SV} \gamma(u)\, du.$$

As γ is quadratic here, $\int_{u\in SV} \gamma(u)\, du = \frac{1}{r}\,\mathrm{Tr}(\gamma)$.

It follows that the leading term in the estimate only involves the trace of Θ_{V^*,h^*}, i.e. the curvature of $(\det V^*, \det h^*)$, which can be taken to be > 0 if $\det V^*$ is big.

Corollary (D-, 2010) Let (X, V) be a directed manifold, $F \to X$ a $\mathbb{Q}$-line bundle, (V, h) and (F, h_F) hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathcal{O}\Big(\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) F\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F, h_F}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r} \Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k} \Big)$$

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Partial solution of the GGL conjecture

Using the above cohomological estimate, we obtain:

Theorem (D-, 2010) Let (X, V) be of general type, i.e. $K_V = (\det V)^*$ is a big line bundle. Then there exists $k \geq 1$ and an algebraic hypersurface $Z \subsetneq X_k$ such that every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \mapsto (X, V)$ satisfies $f_{[k]}(\mathbb{C}) \subset Z$ (in other words, f satisfies an algebraic differential equation of order k).

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Another important consequence is:

Theorem (D-, 2012) A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n$ with

$$d_2 = 286, \quad d_3 = 7316, \quad d_n = \left\lfloor \frac{n^4}{3}\big(n\log(n\log(24n))\big)^n \right\rfloor$$

(for $n \geq 4$) satisfies the Green-Griffiths conjecture.

The proof of the last result uses an important idea due to Yum-Tong Siu, itself based on ideas of Claire Voisin and Herb Clemens, and then refined by M. Păun [Pau08], E. Rousseau [Rou06b] and J. Merker [Mer09].

The idea consists of studying vector fields on the relative jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$.

A differentiation technique by Yum-Tong Siu

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The idea consists of studying vector fields on the relative jet space of the universal family of hypersurfaces of $\mathbb{P}^{n+1}$.

Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d}$ be the universal hypersurface, i.e.

$$\mathcal{X} = \{(z,a)\,;\; a = (a_\alpha) \text{ s.t. } P_a(z) = \sum a_\alpha z^\alpha = 0\},$$

let $\Omega \subset \mathbb{P}^{N_d}$ be the open subset of a's for which $X_a = \{P_a(z) = 0\}$ is smooth, and let

$$p : \mathcal{X} \to \mathbb{P}^{n+1}, \quad \pi : \mathcal{X}_{|\Omega} \to \Omega \subset \mathbb{P}^{N_d}$$

be the natural projections.

Let

$$p_k : \mathcal{X}_k \to \mathcal{X} \to \mathbb{P}^{n+1}, \quad \pi_k : \mathcal{X}_k \to \Omega \subset \mathbb{P}^{N_d}$$

be the relative Green-Griffiths k-jet space of $\mathcal{X} \to \Omega$. Then J. Merker [Mer09] has shown that global sections η_j of

$$\mathcal{O}(T_{\mathcal{X}_k}) \otimes p_k^*\mathcal{O}_{\mathbb{P}^{n+1}}(k^2+2k) \otimes \pi_k^*\mathcal{O}_{\mathbb{P}^{N_d}}(1)$$

generate the bundle at all points of $\mathcal{X}_k^{\mathrm{reg}}$ for $k = n = \dim X_a$. From this, it follows that if P is a non zero global section over Ω of $E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes p_k^*\mathcal{O}_{\mathbb{P}^{n+1}}(-s)$ for some s, then for a suitable collection of $\eta = (\eta_1, \ldots, \eta_m)$, the m-th derivatives

$$D_{\eta_1} \ldots D_{\eta_m} P$$

yield sections of $H^0(\mathcal{X}, E_{k,m}^{\mathrm{GG}} T_{\mathcal{X}}^* \otimes p_k^*\mathcal{O}_{\mathbb{P}^{n+1}}(m(k^2+2k)-s))$ whose joint base locus is contained in $\mathcal{X}_k^{\mathrm{sing}}$, whence the result.

References

[Dem85] Demailly, J.-P.: *Champs Magnétiques et Inégalités de Morse pour la d''-cohomologie*. Ann. Inst. Fourier (Grenoble) **35** (1985), no. 4, 189–229.

[Dem95] Demailly, J.-P.: *Algebraic Criteria for Kobayashi Hyperbolic Projective Varieties and Jet Differentials*. Algebraic geometry – Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[Dem10] Demailly, J.-P.: *Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture*, November 2010, arXiv: math.AG/1011.3636, dedicated to the memory of Eckart Viehweg; Pure and Applied Mathematics Quarterly **7** (2011) 1165–1208

[Dem12] Demailly, J.-P.: *Hyperbolic algebraic varieties and holomorphic differential equations*, lecture given at the VIASM Annual Meeting 2012, Hanoi 26 August 2012, 63p, to appear

in Acta Vietnamica

[D-EG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of Generic Surfaces of High Degree in Projective 3-Space*. Amer. J. Math. **122** (2000), no. 3, 515–546.

[Div09] Diverio, S.: *Existence of global invariant jet differentials on projective hypersurfaces of high degree*. Math. Ann. **344** (2009) 293-315.

[DMR09] Diverio, S., Merker, J., Rousseau, E.: *Effective algebraic degeneracy*. e-print arXiv:0811.2346v5.

[DT9] Diverio, S., Trapani, T.: *A remark on the codimension of the Green-Griffiths locus of generic projective hypersurfaces of high degree*. e-print arXiv:0902.3741v2.

[F-H91] Fulton, W., Harris, J.: *Representation Theory: A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991, xvi+551 pp.

[G-G79] Green, M., Griffiths, P.: *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*. The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 41–74, Springer, New York-Berlin, 1980.

[Kobayashi70] Kobayashi S.: *Hyperbolic Manifolds and Holomorphic Mappings*. Marcel Dekker, Inc., New York 1970 ix+148 pp.

[Lang86] Lang S.: *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. **14** (1986), no. 2, 159–205.

[Mer08] An algorithm to generate all polynomials in the k-jet of a holomorphic disc $D \to \mathbb{C}^n$ that are invariant under source reparametrization, arxiv.org:math.CV/0808.3547.

[Mer09] Merker, J.: *Low pole order frames on vertical jets of the universal hypersurface*. Ann. Inst. Fourier (Grenoble) **59** (2009) 1077-1104.

[Pau08] Păun, M.: *Vector fields on the total space of*

hypersurfaces in the projective space and hyperbolicity. Math. Ann. **340** (2008), 875-892.

[Rou05] Rousseau, E: *Weak Analytic Hyperbolicity of Generic Hypersurfaces of High Degree in the Complex Projective Space of Dimension* 4. arXiv:math/0510285v1 [math.AG].

[Rou06a] Rousseau, E.: *Étude des Jets de Demailly-Semple en Dimension* 3. Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 397–421.

[Rou06b] Rousseau, E: *Équations Différentielles sur les Hypersurfaces de* $\mathbb{P}^4$. J. Math. Pures Appl. (9) **86** (2006), no. 4, 322–341.

[Siu04] Siu, Y.-T.: *Hyperbolicity in Complex Geometry*. The legacy of Niel Henrik Abel, 543–566, Springer, Berlin, 2004.

[SY96] Siu, Y.T., Yeung, S.K.: *Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane*. Invent. Math. **124** (1996), 573–618

[SY97] Siu, Y.T., Yeung, S.K.: *Defects for ample divisors of Abelian varieties, Schwarz lemma and hyperbolic surfaces of low degree*. Amer. J. Math. **119** (1997), 1139–1172

[Tra95] Trapani, S.: *Numerical criteria for the positivity of the difference of ample divisors*, Math. Z. **219** (1995), no. 3, 387–401.

[Voj87] Vojta, P.: *Diophantine Approximations and Value Distribution Theory*, Springer-Verlag, Lecture Notes in Mathematics no. 1239, 1987.