Monge-Ampère functionals for the curvature tensor of a holomorphic vector bundle

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Dedicated to Professor László Lempert on the occasion of his 70^{th} birthday

Abstract. Let E be a holomorphic vector bundle on a projective manifold X such that $\det E$ is ample. We introduce three functionals Φ_P related to Griffiths, Nakano and dual Nakano positivity respectively. They can be used to define new concepts of volume for the vector bundle E, by means of generalized Monge-Ampère integrals of $\Phi_P(\Theta_{E,h})$, where $\Theta_{E,h}$ is the Chern curvature tensor of (E,h). These volumes are shown to satisfy optimal Chern class inequalities. We also prove that the functionals Φ_P give rise in a natural way to elliptic differential systems of Hermitian-Yang-Mills type for the curvature, in such a way that the related P-positivity threshold of $E\otimes(\det E)^t$, where $t>-1/\operatorname{rank}E$, can possibly be investigated by studying the infimum of exponents t for which the Yang-Mills differential system has a solution.

Keywords. Holomorphic vector bundle, hermitian metric, curvature tensor, Griffiths positivity, Nakano positivity, dual Nakano positivity, Hermitian-Yang-Mills equation, Monge-Ampère equation, elliptic operator.

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1. Introduction

Let X be a projective n-dimensional manifold, and $E\to X$ a holomorphic vector bundle equipped with a smooth hermitian metric h. Putting $\operatorname{rank}E=r$, the Chern curvature tensor $\Theta_{E,h}=i\nabla^2_{E,h}$ can be written

$$\Theta_{E,h}=i\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r}c_{jk\lambda\mu}dz_j\wedge d\overline{z}_k\otimes e^*_\lambda\otimes e_\mu \tag{1.1}$$

in terms of holomorphic coordinates $(z_1,\ldots,z_n)$ on X and of an orthonormal frame $(e_\lambda)_{1\le\lambda\le r}$ of E. There is an associated quadratic form $\widetilde\Theta_{E,h}$ on $T_X\otimes E$ defined by

$$\widetilde\Theta_{E,h}(\gamma):=\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r}c_{jk\lambda\mu}\gamma_{j\lambda}\overline{\gamma}_{k\mu},\quad \gamma=\sum_{j,\lambda}\gamma_{j\lambda}\frac{\partial}{\partial z_j}\otimes e_\lambda\in T_X\otimes E, \tag{1.2}$$

so that we have in particular

$$\widetilde\Theta_{E,h}(\xi\otimes v):=\langle\Theta_{E,h}(\xi,\overline{\xi})\cdot v,v\rangle_h=\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r}c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda\overline{v}_\mu.$$

As is well known, the dual hermitian bundle (E^*, h^*) has a curvature tensor that is the opposite of the transpose of $\Theta_{E,h}$, and for $\gamma = \sum_{j,\lambda} \gamma_{j\lambda} \frac{\partial}{\partial z_j} \otimes e^*_\lambda \in T_X \otimes E^*$ we have

$$-\Theta_{E^*,h^*} = {}^T\Theta_{E,h} = \sum_{1\le j,k\le n,\, 1\le \lambda,\mu\le r} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes (e^*_\lambda)^* \otimes e^*_\mu, \tag{1.3}$$

$$-\widetilde{\Theta}_{E^*,h^*}(\gamma) = {}^T\widetilde{\Theta}_{E,h}(\gamma) = \sum_{1\le j,k\le n,\, 1\le \lambda,\mu\le r} c_{jk\mu\lambda} \gamma_{j\lambda} \overline{\gamma}_{k\mu} > 0. \tag{1.4}$$

Let us recall the following standard positivity concepts.

1.5. Definition. *The hermitian bundle (E, h) is said to be*

(a) *Griffiths positive if $\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0$ for all decomposable nonzero tensors $\xi \otimes v \in T_X \otimes E$,*

(b) *Nakano positive if $\widetilde{\Theta}_{E,h}(\gamma) > 0$ for all nonzero tensors $\gamma \in T_X \otimes E$,*

(c) *dual Nakano positive if ${}^T\widetilde{\Theta}_{E,h}(\gamma) > 0$ for all nonzero tensors $\gamma \in T_X \otimes E^*$.*

One says that E itself has one of these three positivity properties if it possesses a smooth hermitian metric h satisfying the corresponding positivity assumption.

This definition gives rise to well known implications

$$\begin{array}{l} E \text{ Nakano positive} \searrow \\ \qquad\qquad\qquad\qquad E \text{ Griffiths positive} \Longrightarrow E \text{ ample.} \\ E \text{ dual Nakano positive} \nearrow \end{array}$$

The last implication comes from the Kodaira embedding theorem [Kod54] and the easy verification that the Griffiths positivity of $\Theta_{E,h}$ implies the positivity of the curvature of the induced metric on the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, where $\mathbb{P}(E)$ is the projectivized bundle of hyperplanes of E. A basic problem raised by [Gri69, Problem (0.9)] is

1.6. Griffiths problem. *Does it hold that E ample $\Rightarrow$ E Griffiths positive ?*

One might wonder whether the ampleness of E would even imply the Nakano or dual Nakano positivity of E, but it turns out that none of these implications holds true. In fact the tangent bundle $E = T_X$ of the complex projective space $X = \mathbb{P}^n$ is ample but not Nakano positive (the fact that $H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(X, K_X \otimes T_X) \neq 0$ would contradict the Nakano vanishing theorem [Nak55]), and the cotangent bundle $E = T^*_X$ of a ball quotient $X = \mathbb{B}^n/\Gamma$ is ample but not dual Nakano positive, since $\mathrm{Id}_E \in H^0(X, \Omega^1_X \otimes E^*) \neq 0$ contradicts the dual version of the Nakano vanishing theorem ([Nak55], [DemB]). The latter fact, that was very briefly mentioned in [LSY13, p. 304], had been overlooked in [Dem21], where we proposed an approach to investigate the dual Nakano positivity of an ample vector bundle. However, as we will see here, the above counterexample raises new interesting problems and does not invalidate the approach of [Dem21]. We thank Dr Junsheng Zhang for pointing out to us the observation made in [LSY13].

In section 2, we introduce three functionals $\Phi_N(\Theta_{E,h})$, $\Phi_{N^*}(\Theta_{E,h})$, $\Phi_G(\Theta_{E,h})$ and corresponding integrated Monge-Ampère volumes $\mathrm{MAVol}_N(E)$, $\mathrm{MAVol}_{N^*}(E)$, $\mathrm{MAVol}_G(E)$ that are related respectively to Nakano, dual Nakano and Griffiths positivity. One can check that these Monge-Ampère volumes reach their maximum value if and only if the bundle E is projectively flat – see Corollary 2.7 for a detailed statement. The corresponding densities are determinants of the curvature tensor that can be used to define global scalar equations for the curvature. In section 3, extending the approach proposed in [Dem21], we show that it suffices to add a trace free Hermite-Einstein condition to a scalar determinantal equation to yield families of elliptic systems of Yang-Mills type, denoted respectively $\mathrm{YM}_{N,\beta}(t)$, $\mathrm{YM}_{N^*,\beta}(t)$, $\mathrm{YM}_{G,\beta}(t)$, depending on a time parameter t and a suitable positive constant β, where the unknown is a time dependent hermitian metric h_t on E. These solutions could hopefully help in the study of the Griffiths' problem if one could obtain an appropriate existence theorem. On a more differential geometric side, if (E,h) is a hermitian vector bundle and $t\in\mathbb{R}$ a real number, we consider formally the curvature tensor of $E\otimes(\det E)^t$, namely

$$\Theta_{E,h}+t\,\Theta_{\det E,\det h}\otimes \mathrm{Id}_E=\Theta_{E,h}+t\,\mathrm{tr}_E\,\Theta_{E,h}\otimes\mathrm{Id}_E, \tag{1.7}$$

where $\Theta_{\det E,\det h}$ is the $(1,1)$-curvature form of the determinant bundle $\det E=\Lambda^r E$ and tr_E the trace operator on $\mathrm{Hom}(E,E)$. We introduce the following threshold values, defined for any vector bundle possessing an ample determinant.

1.8. Definition. *Let $E\to X$ be a holomorphic vector bundle such that* $\det E$ *is ample. We define the Nakano, dual Nakano, Griffiths and ample thresholds, denoted respectively*

$$\tau_N(E),\quad \tau_{N^*}(E),\quad \tau_G(E),\quad \tau_A(E),$$

to be the infimum of values $t\in\mathbb{R}$ such that there exists a smooth hermitian metric h for which $\Theta_{E,h}+t\,\Theta_{\det E,\det h}\otimes\mathrm{Id}_E$ is Nakano, dual Nakano, Griffiths positive, respectively the infimum of $t\in\mathbb{Q}$ such that $E\otimes(\det E)^t$ is ample (i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)\otimes\pi^*(\det E)^t$ is a $\mathbb{Q}$-ample line bundle on the total space of the projectivized bundle $\pi:\mathbb{P}(E)\to X$).

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E)\ge\tau_G(E)\ge\tau_A(E),\quad \tau_{N^*}(E)\ge\tau_G(E)\ge\tau_A(E). \tag{1.9}$$

Since $E\otimes(\det E)^{-1/r}$ has trivial determinant, no positivity property can hold for it, and we conclude that $\tau_A(E)\ge -1/r$. The equality may however occur, e.g. when $E=L^{\oplus r}$ is the direct sum of r copies of an ample line bundle L. In fact, the following fact is easy to prove.

1.10. Proposition. *Let $E\to X$ be a holomorphic vector bundle such that* $\det E$ *is ample. If $\tau_A(E)=-1/r$, then $F=E\otimes(\det E)^{-1/r}$ is numerically flat, in other words, as a $\mathbb{Q}$-vector bundle, we have $E=F\otimes L$ where F is numerically flat of rank r, and $L=(\det E)^{1/r}$ is an ample $\mathbb{Q}$-line bundle (one will refer to this situation by saying that E is projectively numerically flat, see Definition* 2.8*). Then we have*

$$\tau_N(E)=\tau_{N^*}(E)=\tau_G(E)=\tau_A(E)=-\frac{1}{r}.$$

In this setting, the Griffiths problem translates into the conjectural implication

$$E \text{ ample}\Rightarrow \tau_G(E)<0\ ? \tag{1.11}$$

On the other hand, in the counterexamples $E = T_{\mathbb{P}^n}$ (resp. $E = T^*_{\mathbb{B}^n/\Gamma}$) just mentioned for Nakano (resp. dual Nakano) positivity, one can check that E is in fact Griffiths positive and Nakano (resp. dual Nakano) semipositive, hence we have $\tau_G(E) < 0$, while $\tau_N(E) = 0$ (resp. $\tau_{N^*}(E) = 0$). By investigating the curvature of direct images of adjoint positive line bundles, Berndtsson [Ber09] (see also Mourougane-Takayama [MoT07]) has proved that

$$(1.12)\qquad E \text{ ample} \Rightarrow S^m E \otimes \det E = \pi_*(\mathcal{O}_{\mathbb{P}(E)}(m) \otimes K_{\mathbb{P}(E)}) \text{ Nakano positive for all } m \in \mathbb{N}.$$

We infer from this that E ample implies $\tau_N(E) < 1$ and $\tau_N(S^m E) < \frac{r}{r_m}$ where r_m is the rank of $S^m E$, namely $r_m = \binom{m+r-1}{r-1}$. Furthermore, we know that $S^m E$ generates its jets for $m \geq m_0$ large, hence $\tau_{N^*}(S^m E) < 0$ for $m \geq m_0$. In [LSY13, Cor. 4.12], it is further proved that

$$(1.13)\qquad E \text{ ample} \Rightarrow S^m E \otimes \det E \text{ dual Nakano positive for all } m \in \mathbb{N},$$

hence we have as well $\tau_{N^*}(E) < 1$ and $\tau_{N^*}(S^m E) < \frac{r}{r_m}$ when E is ample. The only counterexamples we know about still leave room for the following question.

1.14. Question. *Assume that E is an ample vector bundle. Are there examples for which $\tau_N(E) > 0$, $\tau_{N^*}(E) > 0$ or $\tau_G(E) \geq 0$?*

Of course finding an example with $\tau_G(E) \geq 0$ would be equivalent to answer negatively Griffiths' problem 1.6. On the PDE side, our main result is as follows (see section § 3).

1.15. Theorem. *Given any value t_0 such that $E \otimes (\det E)^{t_0} >_P 0$, one can always arrange the corresponding differential systems $\mathrm{YM}_{N,\beta}(t)$, $\mathrm{YM}_{N^*,\beta}(t)$, $\mathrm{YM}_{G,\beta}(t)$ to be elliptic invertible and to have unique solutions that depend continuously (and even differentiably) on t on a small interval $[t_0 - \delta_0, t_0 + \delta_0]$, $\delta_0 > 0$.*

The proof depends only on the theory of elliptic equations and on the implicit function theorem. In the end, checking the ellipticity is just a sophisticated exercise of linear algebra. A natural problem is whether such Yang-Mills type equations can be used to compute the positivity thresholds, by trying to get solutions for $t \in\,]-1/r, t_0]$ as small as possible.

1.16. Question. *Can one design the Yang-Mills systems $\mathrm{YM}_{N,\beta}(t)$, $\mathrm{YM}_{N^*,\beta}(t)$, $\mathrm{YM}_{G,\beta}(t)$ so that the infimum of times $t_{\inf}$ for which a smooth solution exists on $]t_{\inf}, t_0]$ coincides respectively with the positivity thresholds $\tau_N(E)$, $\tau_{N^*}(E)$, $\tau_G(E)$, for suitably chosen initial data at $t = t_0$ (or whatever they are) ?*

Getting $t_{\inf} = t_{\inf}(\beta)$ to converge to the positivity threshold $\tau_P(E)$ as $\beta \to +\infty$ instead of being equal to $\tau_P(E)$ would be good as well. In the above question, we somehow expect that the differential systems can be made invertible elliptic throughout an almost maximal interval $[t_P(E) + \delta, t_0]$, $0 < \delta \ll 1$, and not just on a small interval $[t_0 - \delta_0, t_0]$.

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2. Monge-Ampère functionals for vector bundles

Let $E \to X$ be a holomorphic vector bundle equipped with a smooth hermitian metric h. If the Chern curvature tensor $\Theta_{E,h}$ is Nakano positive, then the $\frac{1}{r}$-power of the $(n \times r)$-dimensional determinant of the corresponding hermitian quadratic form on $T_X \otimes E$ can be seen as a positive (n,n)-form

$$(2.1) \quad \Phi_N(\Theta_{E,h}) = \det{}_{T_X\otimes E}(\Theta_{E,h})^{1/r} = \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

Moreover, this (n,n)-form does not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E (but (e_λ) must be orthornormal). Similarly, if the Chern curvature tensor $\Theta_{E,h}$ is dual Nakano positive, we can consider the $(n\times r)$-dimensional determinant of the hermitian quadratic form on $T_X \otimes E^*$, namely

$$(2.2) \quad \Phi_{N^*}(\Theta_{E,h}) = \det{}_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r} = \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n,$$

and view it as a positive (n,n)-form. Finally, if $\Theta_{E,h}$ is Griffiths positive, the most natural substitute for (2.1) and (2.2) is

$$(2.3) \quad \Phi_G(\Theta_{E,h}) = \inf_{\substack{v\in E,\\ |v|_h=1}} \frac{1}{n!}\big(\langle\Theta_{E,h}\cdot v, v\rangle_h\big)^n = \inf_{|v|=1} \frac{1}{n!}\Bigg(\sum_{1\le\lambda,\mu\le r} c_{jk\lambda\mu}\, v_\lambda \overline{v}_\mu\, idz_j \wedge d\overline{z}_k\Bigg)^n.$$

It is easy to see that the three volume forms coincide when (E,h) is projectively flat, namely when $\Theta_{E,h} = \alpha \otimes \mathrm{Id}_E$ where α is a positive $(1,1)$-form on X (which is then equal to $\frac{1}{r}\operatorname{tr}_E \Theta_{E,h} = \frac{1}{r}\Theta_{\det E,\det h}$). In this case, we clearly have

$$(2.4) \qquad \Phi_N(\Theta_{E,h}) = \Phi_{N^*}(\Theta_{E,h}) = \Phi_G(\Theta_{E,h}) = \frac{1}{n!}\,\alpha^n = \frac{1}{n!\,r^n}\,(\Theta_{\det E,\det h})^n.$$

In general, we have the following inequalities.

2.5. Proposition. *Let (E,h) be a hermitian vector bundle.*

(a) *If $\Theta_{E,h}$ is Nakano positive, then* $\Phi_N(\Theta_{E,h}) \le \dfrac{1}{n!\,r^n}\,(\Theta_{\det E,\det h})^n$.

(b) *If $\Theta_{E,h}$ is dual Nakano positive, then* $\Phi_{N^*}(\Theta_{E,h}) \le \dfrac{1}{n!\,r^n}\,(\Theta_{\det E,\det h})^n$.

(c) *If $\Theta_{E,h}$ is Griffiths positive, then* $\Phi_G(\Theta_{E,h}) \le \dfrac{1}{n!\,r^n}\,(\Theta_{\det E,\det h})^n$.

In all three cases, the equality of volume forms occurs if and only if (E,h) is projectively flat and $\Theta_{\det E,\det h} > 0$.

Proof. (a) We take h to be a hermitian metric on E such that $\Theta_{E,h}$ is Nakano positive, and consider the Kähler metric

$$\omega = \Theta_{\det E,\det h} = \operatorname{tr}_E \Theta_{E,h}.$$

If $(\alpha_j)_{1\le j\le nr}$ are the eigenvalues of the associated hermitian form $\widetilde{\Theta}_{E,h}$ with respect to $\omega\otimes h$, we have

$$\det{}_{T_X\otimes E^*}\big(\Theta_{E,h}\big)^{1/r} = \Big(\prod_j \alpha_j\Big)^{1/r}\frac{\omega^n}{n!}$$

and $\big(\prod_j \alpha_j\big)^{1/nr} \le \frac{1}{nr}\sum_j \alpha_j$ by the inequality between the geometric and arithmetic means. Since

$$\sum_j \alpha_j = \mathrm{tr}_\omega\,\big(\mathrm{tr}_E\,\Theta_{E,h}\big) = \mathrm{tr}_\omega\,\omega = n,$$

we obtained the asserted inequality

$$\det{}_{T_X\otimes E}\,\big(\Theta_{E,h}\big)^{1/r} \le \bigg(\frac{1}{nr}\sum_j \alpha_j\bigg)^n \frac{\omega^n}{n!} = \frac{1}{n!\,r^n}\,\omega^n.$$

(b) In the case of dual Nakano positivity, the proof is almost identical, except that we take the α_j's to be the eigenvalues of ${}^T\widetilde{\Theta}_{E,h}$ with respect to $\omega\otimes h^*$ on $T_X\otimes E^*$. In both cases, the equality of volume forms occurs if only if all eigenvalues α_j are equal at all points, and then we must have $\alpha_j = \frac{1}{r}$, hence $\Theta_{E,h} = \frac{1}{r}\,\omega\otimes h$.

(c) When (E,h) is Griffiths positive, we pick an h-orthonormal frame $(e_\lambda)_{1\le\lambda\le r}$ of E and observe that we have by definition

$$(*)\qquad \Phi_G(\Theta_{E,h}) \le \frac{1}{n!}\,\big(\langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle_h\big)^n,\quad 1\le\lambda\le r.$$

We view this inequality as a comparison between positive real numbers by referring to the volume form $dV = \frac{1}{n!}\omega^n$ of the metric $\omega = \Theta_{\det E,\det h}$. Let us consider the $(1,1)$-form $A_\lambda = \langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle_h$ on $(T_X,\omega)\simeq(\mathbb{C}^n,\mathrm{std})$ as a hermitian form (or matrix) on $\mathbb{C}^n$. Then we have $\frac{1}{n!}\,\langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle_h^n = \det A_\lambda\,dV$. We use the well-known fact that the function $A\mapsto(\det A)^{1/n}$ is concave on the cone of positive hermitian $(n\times n)$-matrices. This implies

$$\frac{1}{r}\sum_{\lambda=1}^r(\det A_\lambda)^{1/n} \le \Bigg(\det\bigg(\frac{1}{r}\sum_{\lambda=1}^r A_\lambda\bigg)\Bigg)^{1/n},$$

in other words

$$\frac{1}{r}\sum_{\lambda=1}^r\bigg(\frac{1}{n!}\,\langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle_h^n\bigg)^{1/n} \le \Bigg(\frac{1}{n!}\bigg(\frac{1}{r}\sum_{\lambda=1}^r\langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle_h\bigg)^n\Bigg)^{1/n} = \bigg(\frac{1}{n!}\Big(\frac{1}{r}\omega\Big)^n\bigg)^{1/n}.$$

By $(*)$, the left hand side is greater or equal to $\Phi_G(\Theta_{E,h})^{1/n}$, and by taking the n-th power of the above inequality, we find $\Phi_G(\Theta_{E,h}) \le \frac{1}{n!\,r^n}\,\omega^n$, as desired. The only line segments that lie in the graph of $A\mapsto(\det A)^{1/n}$ project into rays of the cone of positive hermitian matrices. Therefore, the equality case may occur only when the hermitian forms A_λ are proportional and we have $\Phi_G(\Theta_{E,h}) = \frac{1}{n!}\,\langle\Theta_{E,h}\cdot e_\lambda, e_\lambda\rangle^n$ for each λ. This forces the $(1,1)$-forms A_λ to be equal, and therefore equal to $\frac{1}{r}\omega$, for any choice of h-orthonormal frame (e_λ). It follows that (E,h) must be projectively flat. □

By considering their integrals over X, the above functionals give rise to interesting concepts of volume for vector bundles.

2.6. Definition. *Let $E\to X$ be a holomorphic vector bundle. If E is P-positive, where P is any of the symbols N, N^* or G, we define the related Monge-Ampère volume of E to be*

$$\mathrm{MAVol}_P(E) = \sup_h \frac{1}{(2\pi)^n}\int_X \Phi_P(\Theta_{E,h}),$$

where the supremum is taken over all smooth metrics h on E such that $\Theta_{E,h}$ is P-positive.

By Proposition 2.5, the supremum is always finite, and in fact we immediately get the following upper bound from the fact that $\frac{1}{2\pi}\Theta_{\det E,\det h}$ is a $(1,1)$-form representing the first Chern class $c_1(\det E) = c_1(E)$.

2.7. Corollary. *For any P-positive vector bundle E, we have*

$$\mathrm{MAVol}_P(E) \le \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Moreover, the equality occurs, with the supremum being a maximum, if and only if E is projectively flat.

It may happen that the equality occurs for the supremum, without E being projectively flat. In fact, one has to take account the following more general situation.

2.8. Definition. *We say that a rank r vector bundle E is numerically projectively flat if $F = S^rE\otimes(\det E)^{-1}$ is numerically flat, i.e; both F and F^* are nef vector bundles. An equivalent condition is that the $\mathbb{Q}$-line bundles $\mathcal{O}_{\mathbb{P}(E)}(1)\otimes\pi^*(\det E)^{-1/r}$ over $\mathbb{P}(E)$ and $\mathcal{O}_{\mathbb{P}(E^*)}(1)\otimes\pi^*(\det E^*)^{-1/r}$ over $\mathbb{P}(E^*)$ are both nef.*

If $\det E$ happens to admit an r-th root $(\det E)^{-1/r}$ that is a genuine line bundle on X, then the numeric projective flatness of E is equivalent to $F = E\otimes(\det E)^{-1/r}$ being numerically flat. In that case, we know by [DPS94, Theorem 1.18] (assuming X to be projective), that this is equivalent to the existence of a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_k = F$ by vector bundles F_j such that the graded pieces F_j/F_{j-1} are hermitian flat for $1 \le j \le k$, i.e. given by representatons of $\pi_1(X)$ into the a unitary group $U(r_j)$. Since there always exists a finite morphism $Y \to X$ such that the pull-back of $\det E$ to Y admits an r-th root on Y, we can always obtain such a filtration by pulling back E itself. The above considerations lead to the following fact.

2.9. Proposition. *Assume that E is numerically projectively flat, that $\det E$ is ample and admits an r-th root on X. Then*

$$\mathrm{MAVol}_N(E) = \mathrm{MAVol}_{N^*}(E) = \mathrm{MAVol}_G(E) = \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Proof. The proof proceeds by showing that there exist smooth metrics $\tilde h_\varepsilon$ on the numerically flat bundle $F = E\otimes(\det E)^{-1/r}$, such that the curvature tensor $\Theta_{F,\tilde h_\varepsilon}$ is arbitrarily small in L^∞ norm. This is a standard fact, resulting from the property that the filtered bundle F deforms to its graded bundle $G = \bigoplus_{1\le j\le k} F_j/F_{j-1}$. In fact, it is enough to fix a C^∞ splitting of the filtration (F_j) and hermitian flat metrics h_j on the graded pieces F_j/F_{j-1}. We can then use $\tilde h_\varepsilon = \bigoplus_{1\le j\le k}\varepsilon^{k-j}h_j$ on $F \simeq G$ (as a C^∞ vector bundle). An easy check shows that the second fundamental forms of the Chern connections become arbitrarily small in L^∞ norms. We take a metric of positive curvature η on $\det E$, and consider the metrics $h_\varepsilon = \tilde h_\varepsilon\otimes\eta^{1/r}$ on E. One can then see that the supremum of the Monge-Ampère integrals over the family (h_ε) reaches the equality in 2.9. When the filtration is non split, the supremum is never a maximum, as this would imply E to be projectively flat by Proposition 2.5. □

2.10. Complements. (a) The argument used in the proof of Proposition 2.9 also implies Proposition 1.10, even without assuming that the n-th root $(\det E)^{1/r}$ exists on X. In fact, we can extract the n-th root of $\det E$ by pulling back via a finite morphism $\mu : Y \to X$. We then get a family of metrics $\tilde h$ on $\mu^* E$ achieving the desired threshold $-1/r$ over Y. We define a metric$h$ on E (or rather h^* on E^*) by putting

$$|\xi|^2_{h^*(x)} = \sum_{y\in\mu^{-1}(x)} |\xi|^2_{\tilde h^*(y)}, \quad \xi \in E^*_x, \quad x \in X, \quad y \in Y,$$

where the sum is counted with multiplicity at branched points. Since Griffiths sempositivity is equivalent to the plurisubharmonicity of $|\xi|^2_{h^*}$ on E^*, this process preserves Griffiths (semi)-positivity; in general the metric h^* is just continuous, but we can apply a Richberg regularization process to make it smooth. The armument is complete for the Griffiths threshold τ_G. For the Nakano and dual Nakano positivity, we use the fact that E is a subbundle of $\mu_*\mu^*E$ (and likewise for E^*), Nakano seminegativity being preserved by going to subbundles.

(b) In the case of a completely split bundle $E = \bigoplus_{j=1}^r E_j$ with ample factors E_j of rank 1, equipped with a split metric $h = \bigoplus_{j=1}^r h_j$, Yau's theorem [Yau78] allows us to normalize the metrics h_j to have proportional volume forms $(\frac{1}{2\pi}\Theta_{E_j,h_j})^n = \beta_j\omega^n$ for any Kähler metric $\omega \in c_1(E)$, $\beta_j > 0$ being a suitable constant. We then get $\beta_j = c_1(E_j)^n/c_1(E)^n$, and find

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \frac{1}{(2\pi)^n}\int_X \Phi_{N^*}(\Theta_{E,h}) = \left(\prod_{j=1}^r \beta_j\right)^{1/r}\int_X \frac{\omega^n}{n!} = \frac{1}{n!}\left(\prod_{j=1}^r c_1(E_j)^n\right)^{1/r}.$$

For $P = N, N^*$, the inequality of Corollary 2.7 then reads

$$\left(\prod_{j=1}^r c_1(E_j)^n\right)^{\frac{1}{r}} \le \frac{1}{r^n}\,c_1(E)^n.$$

It is an equality when $E_1 = \cdots = E_r$, thus Corollary 2.7 is optimal as far as the constant $\frac{1}{n!\,r^n}$ is concerned. For a completely split bundle $E = \bigoplus_{1\le j\le r} E_j$ with arbitrary ample factors, it seems natural to conjecture that

$$\mathrm{MAVol}_N(E) = \mathrm{MAVol}_{N^*}(E) = \frac{1}{n!}\left(\prod_{j=1}^r c_1(E_j)^n\right)^{\frac{1}{r}},$$

i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$. In the case of the Griffiths functional, it is easy to see that

$$\mathrm{MAVol}_G(E) = \frac{1}{(2\pi)^n}\int_X \Phi_G(\Theta_{E,h}) = \min_{1\le j\le r}\beta_j\int_X\frac{\omega^n}{n!} = \frac{1}{n!}\min_{1\le j\le r} c_1(E_j)^n. \tag{2.11}$$

In fact, $\Phi_G(\Theta_{E,h})$ is obtained by picking vectors v in the component E_j for which β_j is minimum. Moreover, for any G-positive metric h on E, even a non split one, (b1) is proved by arguing with the induced metric $h_{|E_j}$ on E_j, which is again G-positive as a quotient of the metric of E by the projection $E \to E_j$.

(c) It would be interesting to characterize the "extremal metrics" h achieving the supremum in $\mathrm{MAVol}_N(E)$, $\mathrm{MAVol}_{N^*}(E)$, when a maximum exists (we have seen in the proof of Proposition 2.9 that this is not always the case). Suitable calculations (see §3 for this) would show that they satisfy a certain Euler-Lagrange equation

$$\int_X (\det\theta)^{1/r}\cdot \mathrm{tr}_{T_X\otimes E^*}\Big(\theta^{-1}\cdot {}^T\big(i\partial_{h^*\otimes h}\overline\partial u\big)\Big)=0\qquad \forall u\in C^\infty(X,\mathrm{Herm}(E)),\tag{2.12}$$

where θ is the $(n\times r)$-matrix representing ${}^T\Theta_{E,h}$. After integrating by parts twice, freeing u from any differentiation, we get a fourth order nonlinear differential system that h has to satisfy. Such a system is somewhat akin to the equation for cscK metrics, in the special case $E=T_X$.

(d) When $r=1$, we clearly have

$$\Phi_N(\Theta_{E,h})=\Phi_{N^*}(\Theta_{E,h})=\Phi_G(\Theta_{E,h})=\frac{1}{n!}\,(\Theta_{E,h})^n,\tag{2.13}$$

and we infer that the integrals $\frac{1}{(2\pi)^n}\int_X\Phi_P(\Theta_{E,h})=\frac{1}{n!}\,c_1(E)^n$ do not depend on h. In the case of ranks $r>1$, it is natural to ask what is the infimum

$$\inf_h\frac{1}{(2\pi)^n}\int_X\Phi_P(\Theta_{E,h})$$

for the three types of functionals. Let us consider the split case $(E,h)=\bigoplus(E_j,h_j)$. By [Yau78] again, we can renormalize Θ_{E_j,h_j} to get volume form equalities $(2\pi)^{-n}\Theta^n_{E_j,h_j}=f_j\omega^n$ with arbitrary functions $f_j>0$ such that $\int_X f_j\omega^n=c_1(E_j)^n$. Then

$$\frac{1}{(2\pi)^n}\int_X\Phi_N(\Theta_{E,h})=\frac{1}{(2\pi)^n}\int_X\Phi_{N^*}(\Theta_{E,h})=\int_X(f_1\cdots f_r)^{1/r}\,\frac{\omega^n}{n!},\tag{2.14}$$

$$\frac{1}{(2\pi)^n}\int_X\Phi_G(\Theta_{E,h})=\int_X\min_{1\le j\le r}f_j\,\frac{\omega^n}{n!},\tag{2.15}$$

and these integrals become arbitrarily small if we take the f_j's to be large on disjoint open sets, and very small elsewhere. This example leads us to suspect that for $r>1$ and any $P=N,N^*,G$, one always have

$$\inf_h\frac{1}{(2\pi)^n}\int_X\Phi_P(\Theta_{E,h})=0.\tag{2.16}$$

(e) In example (2.15), we have $\Phi_G(\Theta_{E,h})=\min_{1\le j\le r}f_j$. This shows that the functional Φ_G fails to be differentiable in general, even though it is clearly Lipschitz continuous. In order to mitigate this difficulty, it suffices to take a large parameter $s>0$ and to consider the family of functionals

$$\Phi_{G,s}(\Theta_{E,h})=\left(\int_{v\in E,\,|v|_h=1}\frac{1}{\big((\langle\Theta_{E,h}\cdot v,v\rangle_h)^n\big)^s}\,d\sigma(v)\right)^{-1/s}\tag{2.17}$$

where $d\sigma$ is the unitary invariant probability measure on the unit sphere. Then $\Phi_{G,s}\ge\Phi_G$ and $\Phi_{G,\infty}=\lim_{s\to+\infty}\Phi_{G,s}=\Phi_G$. A differentiation under the integral sign shows that $\Phi_{G,s}$ is a differentiable functional whenever $s<\infty$.

(f) One desirable property for our functionals $\Phi_P(\Theta_{E,h})$ is that the conditions $\Theta_{E,h} \geq_P 0$ and $\Phi_P(\Theta_{E,h}) > 0$ should enforce $\Theta_{E,h} >_P 0$, thus preventing the P-positivity of $\Theta_{E,h}$ to degenerate. This is clearly the case for Φ_N, Φ_{N^*}, Φ_G. This will be also the case for $\Phi_{G,s}$ for $s \geq r-1:$ in fact, if $\langle\Theta_{E,h}\cdot v, v\rangle_h \geq 0$ and $(\langle\Theta_{E,h}\cdot v, v\rangle_h)^n$ vanishes at some point $(x_0, v_0) \in E$, $|v_0| = 1$, then the differentiability of the non negative polynomial expressing the volume form along the $(r-1)$-dimensional projectivized fiber $P(E_{x_0})$ implies $(\langle\Theta_{E,h}\cdot v, v\rangle_h)^n/|v|^{2n} = O(|v-v_0|^2)$ near $[v_0] \in P(E_{x_0})$; we conclude that the integral (2.17) is divergent on $P(E_{x_0})$ as well as on the unit sphere of E_{x_0}, and this shows that $\Phi_{G,s}(\Theta_{E,h})(x_0) = 0$ for $s \geq r-1$.

3. Hermitian-Yang-Mills equations and positivity thresholds

Following the strategy suggested in [Dem21], we propose here to study certain differential systems of Yang-Mills type, that could be useful to obtain information on the positivity thresholds of a holomorphic vector bundle. Throughout this section, we assume that X is a complex projective manifold of dimension n, and that $E \to X$ is a rank r holomorphic vector bundle such that $\det E$ is ample. Then there exists $t_0 > 0$ such that $E \otimes (\det E)^{t_0}$ has all positivity properties $P = N, N^*, G$ we may desire. If E itself is assumed to be ample, we know by [Ber09] and [LSY13] that one can take $t_0 = 1$. We consider time dependent smooth metrics $(h_t)_{t\in]t_1,t_0]}$ on E, such that

$$\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t} \otimes \mathrm{Id}_E >_P 0. \tag{3.1}$$

We start at $t = t_0$ and try to decrease t as much as we can, eventually going down to the positivity threshold $t_{\inf} = \tau_P(E)$. In any case, we can find a P-positive metric on $E \otimes (\det E)^{t_0}$, and since $\det(E \otimes (\det E)^{t_0}) = (\det E)^{1+rt_0}$, we can derive from it a metric h_{t_0} on E satisfying (3.1). In the sequel, we also set

$$\omega_t = \Theta_{\det E,\det h_t}. \tag{3.2}$$

By our assumption (3.1), ω_t is a Kähler metric that lies in the Kähler class $2\pi\, c_1(E)$. We wish to enforce suitable differential equations on (h_t) so that the family (h_t) is uniquely determined, running t backwards as long as possible. One natural condition is to require

$$\Phi_P\big(\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t} \otimes \mathrm{Id}_E\big) = \text{some positive volume form on } X,$$

in the hope of enforcing the P-positivity of the tensor $\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t} \otimes \mathrm{Id}_E$. Let $\Omega \in C^\infty(X, \Lambda^{n,n}T^*_X)$ be a fixed positive volume form on X. For reasons that will become apparent later, we introduce a new parameter $\beta \in \mathbb{R}_+$ and the differential equation

$$\Phi_P\big(\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t} \otimes \mathrm{Id}_E\big) = f_t\left(\frac{\Omega}{\omega_t^n}\right)^{\beta}\Omega, \quad f_t > 0, \quad f_t \in C^\infty(X,\mathbb{R}). \tag{3.3}$$

In the case of Griffiths positivity, we take $\Phi_P = \Phi_{G,s}$ with p large, according to the discussion conducted in 2.10 (e,f). As was pointed out in [Dem21], equality (3.3) yields only one scalar differential equation, whereas h_t is represented by r^2 unknown real coefficients. Therefore we need to couple (3.3) with an additional matrix equation of real rank r^2-1 to achieve exact determinacy. It turns out that r^2-1 is precisely the real dimension of trace free hermitian endomorphisms of E. It is therefore natural to consider trace free Hermite-Einstein equations of the form

$$\omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = g_t\,\omega_t^n, \quad g_t \in C^\infty(X, \mathrm{Herm}^\circ_{h_t}(E,E)), \tag{3.3°}$$

expressed in terms of the direct sum decomposition

(3.4) $\mathrm{Herm}^\circ_h(E,E) = \big\{u \in \mathrm{Herm}_h(E,E)\,;\; \mathrm{tr}(u) = 0\big\}$,

(3.4′) $\mathrm{Herm}_h(E,E) = \mathrm{Herm}^\circ_h(E,E) \oplus \mathbb{R}\,\mathrm{Id}_E, \quad u = u^\circ + \frac{1}{r}\,\mathrm{tr}(u)\otimes \mathrm{Id}_E, \quad \mathrm{tr}(u^\circ) = 0.$

In the above notation, $\Theta^\circ_{E,h}$ is the curvature tensor of $E\otimes(\det E)^{-1/r}$, namely

$$\Theta^\circ_{E,h} = \Theta_{E,h} - \frac{1}{r}\Theta_{\det E,\det h}\otimes \mathrm{Id}_E \in C^\infty(X,\Lambda^{1,1}_{\mathbb{R}}T^*_X\otimes \mathrm{Herm}^\circ_h(E,E)). \tag{3.5}$$

By the fundamental work of [Don85] and [UhY86], we know that (3.3°) can be solved with $g_t = 0$ if E is $c_1(E)$-polystable, and with a suitable choice of the right hand side $g_t = G(h_t)$ otherwise; as shown by [UhY86], it suffices to take for G an appropriate matrix functional, for instance $G(h) = -\varepsilon\log h$ in suitable coordinates, with $\varepsilon > 0$ arbitrary. The “friction term” $g_t\,\omega_t^n = -\varepsilon\log h_t\,\omega_t^n$ helps in getting a priori bounds for the solutions, and in our case, we will possibly need to take ε large. The following simple observation is essential.

3.6. Observation. *As long as $t\mapsto h_t$ is continuous with values in $C^2(X,\mathrm{Hom}(E,E))$ and we start with an initial value h_{t_0} such that $\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t}\otimes \mathrm{Id}_E >_P 0$ at time $t = t_0$, complement* (2.10 f) *shows that the positivity property P is preserved on the whole interval $]t_{\inf}, t_0]$ where the solution exists. One would therefore need to show that the solution persists to times $t < 0$ to conclude that $0\in\,]t_{\inf}, t_0]$ and $(E,h_0) >_P 0$.*

Now, the differential system (3.2, 3.3, 3.3°) can be considered with arbitrary right hand sides f_t, g_t of order at most 1 in h_t, i.e. of the form

$$f_t(z) = F(t,z,h_t(z),D_zh_t(z)) > 0, \tag{3.7}$$

$$g_t(z) = G(t,z,h_t(z),D_zh_t(z)), \quad g_t\in C^\infty(X,\mathrm{Herm}^\circ_{h_t}(E,E)). \tag{3.7°}$$

These right hand sides do not affect the principal symbol of the system, which is or order 2, as we will see very soon. At this stage, our first concern is whether the above (fully non linear) differential system is actually elliptic. It is a priori exactly determined in the sense that there are exactly as many equations as unknowns, namely r^2 scalar coefficients for $h_t(z)$.

3.8. Theorem. *Let $E\to X$ be a holomorphic vector bundle such that* $\det E$ *is ample and $t\in\mathbb{R}$ such that $E\otimes(\det E)^t >_P 0$. Then there exist explicit distortion functions $\beta_{P,h,t}$ in $C^0(X,\mathbb{R}_+)$ such that for any metric h_t on E satisfying $\Theta_{E,h_t} + t\,\Theta_{\det E,\det h_t}\otimes\mathrm{Id}_E >_P 0$ and any $\beta > \sup_X \beta_{P,h_t,t}$, the system of differential equations* $(3.2, 3.3, 3.3^\circ)$ *possesses an elliptic linearization in a C^2 neighborhood of h_t, whatever is the choice of right hand sides $f_t = F(t,z,h_t,D_zh_t) > 0$, $g_t = G(t,z,h_t,D_zh_t)\in \mathrm{Herm}^\circ_{h_t(z)}(E_z,E_z)$.*

Proof. The proof is similar to the one given in [Dem21], although we have somewhat extended our perspective and allowed more flexible equations. For simplicity of notation, we put $h = h_t$ and, in general, we set

$M := \mathrm{Herm}(E) =$ hermitian forms $E\times E\to\mathbb{C}$, $\quad M_+ =$ positive ones in M,

$M_h := \mathrm{Herm}_h(E,E) =$ hermitian endomorphisms $E\to E$ with respect to $h\in M_+$,

$M^\circ_h := \mathrm{Herm}^\circ_h(E,E) =$ trace free hermitian endomorphisms $E\to E$.

The system of equations $(3.2, 3.3, 3.3^\circ)$ is associated with the nonlinear differential operator

$$Q : C^\infty(X, M_+) \to C^\infty(X, \mathbb{R} \oplus M_h^\circ), \qquad h \mapsto Q(h)$$

defined by $Q = Q_{\mathbb{R}} \oplus Q^\circ$ where

$$\tag{3.9} \begin{cases} \omega_h := \Theta_{\det E, \det h} > 0, \\ Q_{\mathbb{R}}(h) := \big(\omega_h^n/\Omega\big)^\beta \, \Omega^{-1} \, \Phi_P\big(\Theta_{E,h} + t\,\omega_h \otimes \mathrm{Id}_E\big), \\ Q^\circ(h) := (\omega_h^n)^{-1}\big(\omega_h^{n-1} \wedge \Theta_{E,h}^\circ\big). \end{cases}$$

By definition, it is elliptic at h if its linearization $dQ(h)$ is an elliptic linear operator, the exact determinacy being reflected in the fact that M and $\mathbb{R} \oplus M_h^\circ$ have the same rank r^2 over the field $\mathbb{R}$ of real numbers. Our goal is to compute the principal symbol

$$\sigma_2(dQ(h)) \in C^\infty(X, S^2 T_X^{\mathbb{R}} \otimes \mathrm{Hom}(M, \mathbb{R} \oplus M_h^\circ))$$

of the linearized operator $dQ(h)$, and to check that $\sigma_2(dQ(h))(\xi) \in \mathrm{Hom}(M, \mathbb{R} \oplus M_h^\circ)$ is invertible for every non zero vector cotangent vector $\xi \in T_X^*$. For the calculation in coordinates, we fix locally on X a holomorphic frame $(\varepsilon_\lambda^0)_{1 \le \lambda \le r}$ of E, and denote by H_0 the trivial hermitian metric for which (ε_λ^0) is orthonormal. Any hermitian metric h is then represented by a hermitian matrix, denoted again $h = (h_{\lambda\mu})$, such that the corresponding inner product is $\langle h\bullet, \bullet\rangle_{H_0}$. It is well known that the Chern curvature tensor $\Theta_{E,h}$ is given locally by the matrix of $(1,1)$-forms

$$\Theta_{E,h} = i\overline\partial(h^{-1}\partial h).$$

Next, we pick an infinitesimal variation δh of h in $C^\infty(X, M)$. It is convenient to write it under the form $\delta h = \langle u\,\bullet, \bullet\rangle_h = \langle hu\,\bullet, \bullet\rangle_{H_0}$ with $u \in M_h = \mathrm{Herm}_h(E, E)$. In terms of matrices, we have $\delta h = hu$, that is, $u = (u_{\lambda\mu}) = h^{-1}\delta h$ is some sort of "logarithmic variation of h". In this setting, we first evaluate $(d\Theta_{E,h})(u)$. We have $h + \delta h = h(I + u)$ and $(h + \delta h)^{-1} = (\mathrm{Id} - u)h^{-1}$ modulo $O(u^2)$, thus

$$d\Theta_{E,h}(u) = i\overline\partial\big((I-u)h^{-1}\partial(h(I+u)\big) - i\overline\partial(h^{-1}\partial h) \mod O(|u|^2 + |du|^2),$$

that is,

$$\tag{3.10} d\Theta_{E,h}(u) = \overline\partial(h^{-1}\partial(hu)) - i\overline\partial(uh^{-1}\partial h) = -\partial\overline\partial u + \overline\partial((h^{-1}\partial h)u) - i\overline\partial(u(h^{-1}\partial h)).$$

As a consequence, the order 2 part $(\bullet)^{[2]}$ of the linearized operator $d(\Theta_{E,h})$, in other words its principal symbol, is simply given by

$$\tag{3.10'} (d\Theta_{E,h})^{[2]}(u) = -i\partial\overline\partial u.$$

Since $d\omega_h(u) = \mathrm{tr}(d\Theta_{E,h}(u))$, we find

$$\tag{3.11} (d\omega_h)^{[2]}(u) = -i\,\mathrm{tr}\,\partial\overline\partial u = -i\partial\overline\partial\,\mathrm{tr}(u), \qquad (d\Theta_{E,h}^\circ)^{[2]}(u) = -i\partial\overline\partial u^\circ,$$

$$\tag{3.11'} (d\log\omega_h^n)^{[2]}(u) = n\,(\omega_h^n)^{-1}\omega_h^{n-1} \wedge (d\omega_h)^{[2]}(u).$$

In order to compute dQ, we need the differential of the functional Φ_P. In the case $P=N,N^*$, we have to consider the $(nr\times nr)$-matrix $\theta_t(h)$ of the hermitian form on $T_X\otimes E$ defined by

$$\theta_t(h)\simeq\Theta_{E,h}+t\,\omega_h\otimes\mathrm{Id}_E>_P 0$$

(or its transpose), and the logarithmic differential of $\det(\theta_t(h))^{1/r}$ is $\frac{1}{r}\,\mathrm{tr}\left(\theta_t(h)^{-1}d\theta_t(h)\right)$ where $\theta_t(h)^{-1}=(\det\theta_t(h))^{-1}\,{}^T(\theta_t(h)^{\mathrm{cof}})$ and $\theta_t(h)^{\mathrm{cof}}$ is the $\mathrm{Hom}(T_X\otimes E,T_X\otimes E)$-cofactor matrix of $\theta_t(h)$, ${}^T(\bullet)$ the corresponding transposition operator. We pursue our calculations with respect to ω_h-orthonormal coordinates $(z_j)_{1\le j\le n}$ on X at a given point $z^0\in X$, and also use later on an h-orthonormal frame $(e_\lambda)_{1\le\lambda\le r}$ of E_{z_0}. We then get respectively

$$d\theta_t(h)^{[2]}(u)=-\bigg(\frac{\partial^2u_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}+t\,\delta_{\lambda\mu}\sum_\nu\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_k}\bigg)_{(j,\lambda),(k,\mu)},\tag{3.12}$$

$$(d\log\omega_h^n)^{[2]}(u)=-\sum_{j,\nu}\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_j},\tag{3.13}$$

$$d\log\Phi_N(\theta_t)_h^{[2]}(u)=\frac{-1}{r\,\det\theta_t(h)}\sum_{j,k,\lambda,\mu}\theta_t(h)^{\mathrm{cof}}_{jk\lambda\mu}\bigg(\frac{\partial^2u_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}+t\,\delta_{\lambda\mu}\sum_\nu\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_k}\bigg),\tag{3.14$_N$}$$

$$d\log\Phi_{N^*}(\theta_t)_h^{[2]}(u)=\frac{-1}{r\,\det{}^T\theta_t(h)}\sum_{j,k,\lambda,\mu}\big({}^T\theta_t(h)\big)^{\mathrm{cof}}_{jk\mu\lambda}\bigg(\frac{\partial^2u_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}+t\,\delta_{\lambda\mu}\sum_\nu\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_k}\bigg)\tag{3.14$_{N^*}$}$$

where $\big({}^T\theta_t(h)\big)^{\mathrm{cof}}$ is the $\mathrm{Hom}(T_X\otimes E^*,T_X\otimes E^*)$-cofactor matrix of ${}^T\theta_t(h)$. The calculation for the functional $\Phi_{G,s}$ requires a differentiation of (2.17) and is more involved. If we notice that the differentiation of $\langle\bullet,\bullet\rangle_h$ in h does not contribute to the order 2 terms, we find

$$\text{(3.14$_G$)}\quad d\log\Phi_{G,s}(\theta_t)_h^{[2]}(u)=$$
$$\bigg(\int_{\substack{v\in E\\|v|_h=1}}\frac{d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^s}\bigg)^{-1}\int_{\substack{v\in E\\|v|_h=1}}\frac{n\,(\langle\theta_t(h)\cdot v,v\rangle_h)^{n-1}\wedge\langle d\theta_t(h)^{[2]}(u)\cdot v,v\rangle_h\,d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^{s+1}}.$$

Notice that the s-th power of an (n,n)-form in the first integral and the quotient of an (n,n)-form by the $(s+1)$-st power of an (n,n)-form in the second integral actually combine into a dimensionless value. In normal coordinates, $\langle d\theta_t(h)^{[2]}(u)\cdot v,v\rangle_h$ is the $(1,1)$-form

$$\langle d\theta_t(h)^{[2]}(u)\cdot v,v\rangle_h=-\sum_{j,k,\lambda,\mu}\bigg(\frac{\partial^2u_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}\,v_\lambda\overline{v}_\mu+t\,\frac{\partial^2u_{\lambda\lambda}}{\partial z_j\partial\overline{z}_k}\,|v_\mu|^2\bigg)\,dz_j\wedge d\overline{z}_k.\tag{3.15}$$

Let us begin with the case of Nakano positivity $P=N$. By the above identities, the logarithmic differential of the first scalar component $Q_{\mathbb{R}}(h)$ of $Q(h)$ has order 2 terms

$$\begin{aligned}Q_{\mathbb{R}}(h)^{-1}\,(dQ_{\mathbb{R},h})^{[2]}(u)&=d\log\Phi_N(\theta_t)_h^{[2]}(u)+\beta\,(d\log\omega_h^n)^{[2]}(u)\\&=\frac{-1}{r\,\det\theta_t(h)}\sum_{j,k,\lambda,\mu}\theta_t(h)^{\mathrm{cof}}_{jk\lambda\mu}\bigg(\frac{\partial^2u_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}+t\,\delta_{\lambda\mu}\sum_\nu\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_k}\bigg)-\beta\sum_{j,\nu}\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_j},\end{aligned}$$

$$=\frac{-1}{r\,\det\theta_t(h)}\sum_{j,k,\lambda,\mu}\theta_t(h)^{\mathrm{cof}}_{jk\lambda\mu}\bigg(\frac{\partial^2u^\circ_{\lambda\mu}}{\partial z_j\partial\overline{z}_k}+\Big(t+\frac{1}{r}\Big)\,\delta_{\lambda\mu}\sum_\nu\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_k}\bigg)-\beta\sum_{j,\nu}\frac{\partial^2u_{\nu\nu}}{\partial z_j\partial\overline{z}_j},\tag{3.16$_N$}$$

and we get a similar expression for Φ_{N^*} by (3.14_{N^*}). Finally, we compute the order 2 terms in the differential of the second component

$$h \mapsto Q^\circ(h) = (\omega_h^n)^{-1}\,(\omega_h^{n-1} \wedge \Theta^\circ_{E,h}) \in M_h^\circ.$$

The above calculations imply

$$\begin{aligned} dQ^\circ(h)^{[2]}(u) = & -n(\omega_h^n)^{-2}\left(\omega_h^{n-1} \wedge (d\omega_h)^{[2]}(u)\right) \cdot \left(\omega_h^{n-1} \wedge \Theta^\circ_{E,h}\right) \\ & + (n-1)\,(\omega_h^n)^{-1}\left(\omega_h^{n-2} \wedge (d\omega_h)^{[2]}(u) \wedge \Theta^\circ_{E,h}\right) \\ & + (\omega_h^n)^{-1}\left(\omega_h^{n-1} \wedge (d\Theta^\circ_{E,h})^{[2]}(u)\right). \end{aligned}$$

If we denote $\Theta^\circ_{E,h} = \sum_{j,k,\lambda,\mu} c^\circ_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$ at z^0, this yields

$$\begin{aligned} dQ^\circ(h)^{[2]}(u) = & +\frac{1}{n}\sum_{j,\nu}\frac{\partial^2 u_{\nu\nu}}{\partial z_j \partial\overline{z}_j} \cdot \sum_{k,\lambda,\mu} c^\circ_{kk\lambda\mu}\, e^*_\lambda \otimes e_\mu \\ & -\frac{1}{n}\sum_{j,k,\lambda,\mu,\nu}\left(\frac{\partial^2 u_{\nu\nu}}{\partial z_j \partial\overline{z}_j}\, c^\circ_{kk\lambda\mu}\, e^*_\lambda \otimes e_\mu - \frac{\partial^2 u_{\nu\nu}}{\partial z_k \partial\overline{z}_j}\, c^\circ_{jk\lambda\mu}\, e^*_\lambda \otimes e_\mu\right) \\ & -\frac{1}{n}\sum_{j,\lambda,\mu}\frac{\partial^2 u^\circ_{\lambda\mu}}{\partial z_j \partial\overline{z}_j}\, e^*_\lambda \otimes e_\mu \\ = & \;\frac{1}{n}\sum_{j,k,\lambda,\mu,\nu}\frac{\partial^2 u_{\nu\nu}}{\partial z_k \partial\overline{z}_j}\, c^\circ_{jk\lambda\mu}\, e^*_\lambda \otimes e_\mu - \frac{1}{n}\sum_{j,\lambda,\mu}\frac{\partial^2 u^\circ_{\lambda\mu}}{\partial z_j \partial\overline{z}_j}\, e^*_\lambda \otimes e_\mu. \end{aligned} \tag{3.16°}$$

The principal symbol $\sigma_2(dQ(h))$ at h, taken on a cotangent vector $\xi \in T^*_X$, is thus given by the two components $\sigma_2(dQ_{\mathbb{R}}(h))$ and $\sigma_2(dQ^\circ(h))$ such that

$$\frac{\sigma_2(dQ_{\mathbb{R}}(h))(\xi)\cdot u}{dQ_{\mathbb{R}}(h)} = \frac{-1}{r\,\det\theta_t(h)}\sum_{j,k,\lambda,\mu}\theta_t(h)^{\rm cof}_{jk\lambda\mu}\,\xi_j\overline{\xi}_k\left(u^\circ_{\lambda\mu} + \left(t+\frac{1}{r}\right)\delta_{\lambda\mu}\,{\rm tr}(u)\right) - \beta\,|\xi|^2\,{\rm tr}(u),$$

(3.17, 3.17°)

$$\sigma_2(dQ^\circ(h))(\xi)\cdot u = -\frac{1}{n}\,\frac{\omega_h^n}{\Omega}\sum_{\lambda,\mu}\left(\sum_{j\neq k}\left(c^\circ_{kk\lambda\mu}\,|\xi_j|^2 - c^\circ_{jk\lambda\mu}\xi_j\overline{\xi}_k\right){\rm tr}(u) + |\xi|^2\,u^\circ_{\lambda\mu}\right)e^*_\lambda \otimes e_\mu.$$

By definition $dQ(h)$ is elliptic if and only if $\sigma_2(dQ(h))(\xi) \in {\rm Hom}(M_h, \mathbb{R}\oplus M_h^\circ)$ is injective for all cotangent vectors $\xi \neq 0$. Now, since $t + \frac{1}{r} > 0$ and since the cofactor matrix is hermitian positive by the Nakano positivity assumption, we see that the vanishing of $\sigma_2(dQ_{\mathbb{R}}(h))(\xi)\cdot u$ implies by a simple Cauchy-Schwarz argument that

$$|\,{\rm tr}(u)| \le \frac{1}{\beta r}\,\frac{|\theta_t(h)^{\rm cof}|}{\det\theta_t(h)}\,|u^\circ|, \tag{3.18}$$

where the norms of tensors are taken with respect to (E,h) and (T_X,ω_h). By plugging this inequality into $\sigma_2(dQ^\circ(h))$, taking the inner product with $u^\circ = \sum_{\lambda,\mu} u^\circ_{\lambda\mu}\, e^*_\lambda \otimes e_\mu$ and using again Cauchy-Schwarz, we see that $\sigma_2(dQ_{\mathbb{R}}(h))(\xi)\cdot u = 0$ entails

$$\langle -\sigma_2(dQ^\circ(h))(\xi)\cdot u, u^\circ\rangle \ge \frac{1}{n}\,\frac{\omega_h^n}{\Omega}\left(|\xi|^2\,|u^\circ|^2 - (\sqrt{n-1}+1)\,|\Theta^\circ_{E,h}|\,|\xi|^2\,|u^\circ|\,|\,{\rm tr}(u)|\right),$$

hence

$$(3.18^\circ)\qquad \big|\sigma_2(dQ^\circ(h))(\xi)\cdot u\big| \ge \frac{1}{n}\,\frac{\omega_h^n}{\Omega}\,|\xi|^2\,|u^\circ|\left(1-\frac{\sqrt{n-1}+1}{\beta r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\rm cof}|}{\det\theta_t(h)}\right).$$

Let us introduce the "distortion function" $\beta_{N,h,t}\in C^0(X,\mathbb{R}_+)$

$$(3.19_N)\qquad \beta_{N,h,t}=\frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\rm cof}|}{\det\theta_t(h)}$$

computed at each point $z\in X$ and, in a similar manner for dual Nakano positivity,

$$(3.19_{N^*})\qquad \beta_{N^*,h,t}=\frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|({}^T\theta_t(h))^{\rm cof}|}{\det({}^T\theta_t(h))}.$$

Then, for $\beta>\sup_X\beta_{N,h,t}$ (resp. $\beta>\sup_X\beta_{N^*,h,t}$), inequalities (3.18) and (3.18°) imply the ellipticity of our differential system. In the case of $\Phi_{G,s}$, the identities (3.14_G) and (3.15) yield

$$\begin{aligned}Q_{\mathbb{R}}(h)^{-1}\,\sigma_2(dQ_{\mathbb{R},h})(u)\cdot\xi&=\sigma_2(d\log\Phi_{G,s}(\theta_t)_h)(u)\cdot\xi+\beta\,\sigma_2(d\log\omega_h^n)(u)\cdot\xi\\&=-\beta\,|\xi|^2\,{\rm tr}(u)-\left(\int_{\substack{v\in E\\|v|_h=1}}\frac{d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^s}\right)^{-1}\\&\qquad\times\int_{\substack{v\in E\\|v|_h=1}}\frac{n\,(\langle\theta_t(h)\cdot v,v\rangle_h)^{n-1}\wedge\big(\langle u^\circ(v),v\rangle+(t+\frac{1}{r})\,{\rm tr}(u)\,|v|^2\big)\,i\xi\wedge\overline{\xi}\,d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^{s+1}}.\end{aligned}$$

By easy estimates left to the reader, this leads in the case of the Griffiths functional to the distortion function

$$\begin{aligned}&\beta_{G,s,h,t}=(\sqrt{n-1}+1)\,|\Theta^\circ_{E,h}|\\(3.19_G)\qquad&\quad\times\left(\int_{\substack{v\in E\\|v|_h=1}}\frac{d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^s}\right)^{-1}\int_{\substack{v\in E\\|v|_h=1}}\frac{n\,(\langle\theta_t(h)\cdot v,v\rangle_h)^{n-1}\wedge\omega_h\,d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^{s+1}},\end{aligned}$$

where $\beta_{G,s,h,t}(z)$ is obtained by computing the integrals fiberwise on E_z. The proof of Theorem 3.8 is complete. □

3.20. Remark. It the curvature tensor $\Theta_{E,h}(z)$ happens to be just rescaled by a positive multiplication factor at some point $z\in X$, the value of the above distortion function $\beta_{P,h,t}(z)$ can be seen to remain invariant. In some sense, $\beta_{P,h,t}(z)$ measures the ratio of "eigenvalues" along directions of maximum and minimum P-positivity for $\Theta_{E,h}+t\,\Theta_{\det E,\det h}\otimes{\rm Id}_E$, at each point $z\in X$. Hopefully, there might be a way of relating these distortion functions to simpler geometric invariants, such as the slopes in the Harder-Narasimhan filtration of E with respect to $c_1(E)$.

Our next concern is to ensure that the existence and uniqueness of solutions hold, at least on suitable subsets of $\mathbb{R}\times C^\infty(X,M_+)$, consisting of pairs (t,h) such that $\theta_{E,h}+t\,\Theta_{\det E,\det h}\otimes{\rm Id}_E>_P 0$. We fix such a pair (t_0,h_0) and use $H_0=h_{t_0}$ and $\omega_{t_0}=\Theta_{\det E,\det h_{t_0}}$

as the reference metrics on E and T_X respectively. For $K \geq K_0 \gg 1$, we consider the subset $S_K \subset]-1/r, t_0] \times C^\infty(X, M_+)$ of pairs (t, h) such that

$$|h|_{C^2} \leq K, \quad |h^{-1}|_{C^2} \leq K, \quad \theta_{E,h} + t\,\Theta_{\det E, \det h} \otimes \mathrm{Id}_E \geq_P K^{-1}\,\omega_0 \otimes \mathrm{Id}_E \tag{3.21}$$

with respect to (H_0, ω_{t_0}). In the case of a rank one metric $h = e^{-\varphi}$, it is well-known that the Kähler-Einstein equation $(\omega_0 + i\partial\overline{\partial}\varphi_t)^n = e^{tf+\lambda\varphi_t}\omega_0^n$ yields easily the openness and closedness of solutions when λ is positive, as a consequence of the fact that the linearized operator $\psi \mapsto \Delta_{\omega_{\varphi_t}}\psi - \lambda\psi$ is always invertible. Here we can still play the game of adjusting the right hand sides f_t, g_t in $(3,3,3.3^\circ)$ to achieve the invertibility of the related elliptic operator $\widehat{Q}$, at least for $(t, h) \in S_K$. Before doing so, we introduce some notation. If $h \in \mathrm{Herm}(E)$ is a hermitian form, we have an isomorphism

$$\mathrm{Herm}(E) \to \mathrm{Herm}_{h_{t_0}}(E, E), \quad h \mapsto \tilde{h} \ \text{ such that } \ h(v, w) = \langle v, w\rangle_h = \langle \tilde{h}(v), w\rangle_{h_{t_0}}, \tag{3.22}$$

and for $h \in \mathrm{Herm}_+(E)$, we let $\log\tilde{h} \in \mathrm{Herm}_{h_{t_0}}(E, E)$ be its logarithm as a hermitian endomorphism. Finally, we define $\tilde{h}^{(1)} = (\det\tilde{h})^{-1/r}\tilde{h}$, so that $\det(\tilde{h}^{(1)}) = 1$ and

$$\log\tilde{h}^{(1)} = (\log\tilde{h})^\circ \ \in \ \mathrm{Herm}^\circ_{h_{t_0}}(E, E) \tag{3.23}$$

is the trace free part of $\log\tilde{h}$. One way to generalize the Kähler-Einstein condition to the case of arbitrary ranks $r \geq 1$ is to consider pairs (t, h) satisfying a differential equation of the form (3.3) with a factor $f_t(z) = (\det h_{t_0}(z)/\det h(z))^\lambda$, namely

$$\Phi_P\big(\Theta_{E,h} + t\,\omega_h \otimes \mathrm{Id}_E\big) = \left(\frac{\det h_{t_0}}{\det h}\right)^\lambda \left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega, \quad \text{where } \omega_h = \Theta_{\det E, \det h},\ \lambda, \beta > 0, \tag{3.24}$$

and the volume form $\Omega > 0$ is chosen so that equation (3.24) is satisfied by (t_0, h_{t_0}). The choice $\lambda > 0$ has the advantage that the right hand side gets automatically rescaled when multiplying h by a constant (while the left hand side remains untouched), thus avoiding a trivial non invertibility issue. When $r = 1$, one easily sees that equation (3.24) actually reduces to the usual Kähler-Einstein equation. By Uhlenbeck-Yau [UhY86], if one chooses for the right hand side of (3.3°) a "friction term" g_t of the type $g_t(z) = -\varepsilon\,a(t, z)\log\tilde{h}^{(1)}(z)$, $a(t, z) > 0$, then the Hermite-Einstein equation always has a solution, although it usually blows up as $\varepsilon \to 0$ when E is unstable. This leads to couple (3.24) with a trace free Hermite-Einstein equation of the form

$$\omega_h^{-n}\left(\omega_h^{n-1} \wedge \Theta^\circ_{E,h}\right) = -\varepsilon\,A(\det h)\,\log\tilde{h}^{(1)}, \tag{3.24°}$$

where $A \in C^\infty(]0, +\infty[\,, \mathbb{R}_+)$ is a positive function; one could also use more generally a factor $A(\det u(z), z)$ where $A \in C^\infty(\,]0, +\infty[\, \times X, \mathbb{R}_+)$, such as $A(y, z) = (\det h_{t_0}(z)/y)^\mu$, $\mu \in \mathbb{R}$. The precise form of A is irrelevant here, provided that $A > 0$; one could just take $A \equiv 1$. The right hand sides used in $(3.24, 3.24^\circ)$ do not depend on higher derivatives of h, thus Theorem 3.8 ensures the ellipticity of the differential system as soon $\beta > \sup_X \beta_{P,t,h}$ (see (3.19_P)).

3.25. Theorem. *Consider the differential operator* $\widehat{Q} : C^\infty(X, M_+) \to C^\infty(X, \mathbb{R} \oplus M_h^\circ)$ *defined by*

$$\widehat{Q}_{\mathbb{R}}(h) = \left(\frac{\det h}{\det h_{t_0}}\right)^\lambda Q_{\mathbb{R}}(h) = \left(\frac{\det h}{\det h_{t_0}}\right)^\lambda \left(\frac{\omega_h^n}{\Omega}\right)^\beta \Omega^{-1}\,\Phi_P\big(\Theta_{E,h} + t\,\omega_h \otimes \mathrm{Id}_E\big), \tag{YM}$$

$$\widehat{Q}^\circ(h) = Q^\circ(h) + \varepsilon\,A(h)\,\log\tilde{h}^{(1)} = \omega_h^{-n}\big(\omega_h^{n-1} \wedge \Theta^\circ_{E,h}\big) + \varepsilon\,A(\det h)\,\log\tilde{h}^{(1)}. \tag{YM°}$$

where $Q = Q_{\mathbb{R}} \oplus Q^\circ$ is the operator introduced in the proof of Theorem 3.8*, and A is any smooth positive function. There exist bounds $\beta_0(K) := \sup_{(t,h)\in S_K} \sup_X \beta_{P,t,h}$, $\varepsilon_0(A,K,\beta)$ and $\lambda_0(A,K,\beta)$ such that for any choice of constants $\beta > \beta_0(K)$, $\varepsilon > \varepsilon_0(A,K,\beta)$ and $\lambda > \lambda_0(A,K,\beta)$, the elliptic operator defined by* (YM, YM°) *possesses an invertible elliptic linearization $d\widehat{Q}(h)$ for all $(t,h) \in S_K$. As a consequence, there exists an open interval $[t_0-\delta_0, t_0]$, $\delta_0 > 0$, such that the solution h_t of the system $(\widehat{Q}_{\mathbb{R}}(h), \widehat{Q}^\circ(h)) = (1,0)$ exists and is unique for $t \in [t_0-\delta_0, t_0]$. This solution h_t depends differentiably on t.*

Proof. Here, we have to keep an eye on the linearized operator $d\widehat{Q}$ itself, and not just its principal symbol. We let again $u = h^{-1}\delta h \in \mathrm{Herm}_h(E,E)$ and use the formulas established for $dQ(h)$ in the proof of Theorem 3.8. The logarithmic derivative of $\widehat{Q}_{\mathbb{R}}(h)$ is

$$\widehat{Q}_{\mathbb{R}}(h)^{-1}\, d\widehat{Q}_{\mathbb{R}}(h)(u) = Q_{\mathbb{R}}(h)^{-1}\, dQ_{\mathbb{R}}(h)(u) + \lambda\ \mathrm{tr}(u). \tag{3.26}$$

For $\widehat{Q}^\circ$, we need the fact that, when viewed as a hermitian endomorphism, $h^\circ = h\cdot(\det h)^{-1/r}$ possesses a logarithmic variation

$$(\tilde{h}^{(1)})^{-1}\delta\tilde{h}^{(1)} = u^\circ = u - \frac{1}{r}\,\mathrm{tr}(u)\cdot \mathrm{Id}_E.$$

By the classical formula expressing the differential of the logarithm of a matrix, we have

$$(d\log g)(\delta g) = \int_0^1 \big((1-s)\mathrm{Id} + sg\big)^{-1}\delta g\,\big((1-s)\mathrm{Id} + sg\big)^{-1}\,ds$$

(g and δg need not commute here!), which implies

$$d\log\tilde{h}^{(1)}(\delta h) = \int_0^1 \big((1-s)\mathrm{Id} + s\,\tilde{h}^{(1)}\big)^{-1}\,\tilde{h}^{(1)}u^\circ\,\big((1-s)\mathrm{Id} + s\,\tilde{h}^{(1)}\big)^{-1}\,ds.$$

If $(\alpha_\lambda)_{1\le\lambda\le r}$ are the eigenvalues of $h^{(1)}$ with respect to h_{t_0} and we use an orthonormal basis of eigenvectors, we obtain in coordinates

$$d\log\tilde{h}^{(1)} : \delta h \longmapsto L_h(u^\circ) = \big(\gamma_{\lambda\mu}u^\circ_{\lambda\mu}\big)_{1\le\lambda,\mu\le r}, \qquad \gamma_{\lambda\mu} = \frac{\alpha_\mu}{\alpha_\mu - \alpha_\lambda}\log\frac{\alpha_\mu}{\alpha_\lambda},$$

where $u = (u_{\lambda\mu})_{1\le\lambda,\mu\le r} = h^{-1}\delta h$ and the coefficient $\gamma_{\lambda\mu} > 0$ is to be interpreted as 1 if $\alpha_\lambda = \alpha_\mu$. In the end, we obtain

$$d\widehat{Q}^\circ(h)(u) = dQ^\circ(h)(u) + \varepsilon\,\big(A(h)\,L_h(u^\circ) + A'(\det h)\,\det h\ \mathrm{tr}(u)\,\log\tilde{h}^{(1)}\big). \tag{3.26°}$$

In order to check the invertibility, we compare the operators

$$d\log Q_{\mathbb{R}}(h) \oplus dQ^\circ(h) \quad \text{and} \quad d\log\widehat{Q}_{\mathbb{R}}(h) \oplus d\widehat{Q}^\circ(h).$$

The principal symbol calculations (3.17, 3.17°) show that for

$$\beta > \beta_0(K) = \sup_{(t,h)\in S_K}\ \sup_X \beta_{P,h,t},$$

the linearized operator $d\log Q_{\mathbb{R}}(h) \oplus dQ^\circ(h)$ is elliptic and essentially positive for all $(t,h) \in S_K$. We consider the natural L^2 metric on $L^2(X,M) \simeq L^2(X, \mathbb{R} \oplus M_h^\circ)$ defined by

$$\|u\|^2 = \|\operatorname{tr}(u)\|^2 + \|u^\circ\|^2,$$

using the hermitian metric $h^* \otimes h$ on $\mathrm{Herm}_h(E,E)$ and the volume element $\omega_h^n/n!$ on X. By the ellipticity of $dQ(h)$ and an elementary case of Gårding's inequality, there exist constants $C, C' > 0$ such that

$$\langle\!\langle d\log Q_{\mathbb{R}}(h)(u) \oplus dQ^\circ(h)(u), \operatorname{tr}(u) \oplus u^\circ \rangle\!\rangle_{\mathbb{R}\oplus M_h^\circ} \geq C\,\|\nabla_h u\|^2 - C'\|u\|^2. \tag{3.27}$$

Moreover, the estimate is valid with constants $C = C(K,\beta)$, $C' = C'(K,\beta)$, uniformly for all $(t,h) \in S_K$. In such a C^2 bounded set, we also have bounds

$$\begin{cases} \langle A(h)\, L_h(u^\circ), u^\circ\rangle = A(h) \sum_{1\leq\lambda,\mu} \gamma_{\lambda\mu}|u^\circ_{\lambda\mu}|^2 \geq C''\,|u^\circ|^2, \\ |A'(\det h)\ \det h\ \log \tilde h^{(1)}| \leq C''' \end{cases} \tag{3.28}$$

with $C'' = C''(A,K), C''' = C'''(A,K) > 0$. Estimates $(3.26, 3.26^\circ)$, $(3.27, 3.28)$ and the Cauchy-Schwarz inequality imply

$$\begin{aligned} &\langle\!\langle d\log \widehat{Q}_{\mathbb{R}}(h)(u) \oplus d\widehat{Q}^\circ(h)(u), \operatorname{tr}(u) \oplus u^\circ \rangle\!\rangle_{\mathbb{R}\oplus M_h^\circ} \\ &\qquad \geq C\,\|\nabla_h u\|^2 - C'\|u\|^2 + \lambda\|\operatorname{tr}(u)\|^2 + \varepsilon\big(C''\,\|u^\circ\|^2 - C'''\,\|\operatorname{tr}(u)\|\,\|u^\circ\|\big) \\ &\qquad \geq C\,\|\nabla_h u\|^2 + \Big(\lambda - C' - \frac{2(C''')^2}{2C''}\Big)\|\operatorname{tr}(u)\|^2 + \Big(\frac{1}{2}\,\varepsilon\, C'' - C'\Big)\|u^\circ\|^2 \end{aligned} \tag{3.29}$$

by the inequality $\|\operatorname{tr}(u)\|\,\|u^\circ\| \leq \frac{C'''}{2C''}\|\operatorname{tr}(u)\|^2 + \frac{C''}{2C'''}\|u^\circ\|^2$. If we take

$$\varepsilon > \varepsilon_0(A,K,\beta) = \frac{2C'}{C''}, \quad \lambda > \lambda_0(A,K,\beta) = C' + \frac{2(C''')^2}{2C''}, \tag{3.30}$$

we conclude from (3.29) that $d\log \widehat{Q}_{\mathbb{R}}(h) \oplus d\widehat{Q}^\circ(h)$ is an invertible elliptic operator $W^{s+2} \to W^s$ for all Sobolev spaces W^s, $s \geq 0$. The proof of Theorem 3.25 is achieved by applying standard results in the theory of elliptic operators and the implicit function theorem. □

3.31. Remark. Theorem 3.25 is somehow purely local. The main point would be to obtain more uniform estimates with respect to the metric h, especially in terms of the distortion fonctions, so that one could keep control on the solution throughout the expected maximal interval of time. This obviously requires a finer analysis than the one we conducted here. If $\beta_{P,t,h}$ could be better understood, explicit expressions of the constants C, C', C'', C''' and thus of $\varepsilon_0(A,K,\beta)$ and $\lambda_0(A,K,\beta)$ would perhaps become accessible by looking more in depth at the Bochner formula.

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