Analytic techniques in algebraic geometry

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Lectures given at the School held in Mahdia, Tunisia, July 14 – July 31, 2004
Analyse Complexe et Géométrie

The purpose of this series of lectures is to explain some advanced techniques of Complex Analysis which can be applied to obtain fundamental results in algebraic geometry: vanishing of cohomology groups, embedding theorems, description of the geometric structure of projective algebraic varieties.

Contents

0. Preliminary material

Let X be a complex manifold and $n = \dim_{\mathbb{C}} X$. The bundle of differential forms of typr (p,q) is denoted by $\Lambda^{p,q}T^*_X$. We are especially interested in *closed positive currents* of type (p,p)

$$T = i^{p^2} \sum_{|J|=|K|=p} T_{JK}(z) dz_J \wedge d\overline{z}_J, \qquad dz_J = dz_{j_1} \wedge \ldots \wedge dz_{j_p}, \quad dT = 0.$$

Recall that a current is a differential form with distribution coefficients, and that such a (p,p) current is said to be positive (in the "medium positivity" sense) if the distribution $\sum \lambda_J \overline{\lambda}_K T_{JK}$ is a positive measure for all complex numbers λ_J. The coefficients T_{JK} are then complex measures. Important examples of closed positive (p,p)-currents are currents of integration over analytic cycles of codimension p :

$$Z = \sum c_j Z_j, \qquad [Z] = \sum c_j [Z_j]$$

where the current $[Z_j]$ is defined by duality as

$$\langle [Z_j], u\rangle = \int_{Z_j} u_{|Z_j}$$

for every $(n-p, n-p)$ test form u on X. Another important example of positive $(1,1)$-current is the Hessian form $T = i\partial\overline{\partial}\varphi$ of a plurisubharmonic function on an open set $\Omega \subset X$. A *Kähler metric* on X is a positive definite hermitian $(1,1)$-form

$$\omega(z) = i \sum_{1\le j,k\le n} \omega_{jk}(z) dz_j \wedge d\overline{z}_k \qquad \text{such that } d\omega = 0,$$

with smooth coefficients. To every closed real $(1,1)$-form (or current) α is associated its De Rham cohomology class

$$\{\alpha\} \in H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R}).$$

We denote here by $H^k(X,\mathbb{C})$ (resp. $H^k(X,\mathbb{R})$) the complex (real) De Rham cohomology group of degree k, and by $H^{p,q}(X,\mathbb{C})$ the subspace of classes which can be represented as closed (p,q)-forms, $p+q=k$.

We will rely on the nontrivial fact that all cohomology groups involved (De Rham, Dolbeault, ...) can be defined either in terms of smooth forms or in terms of currents. In fact, if we consider the associated complexes of sheaves, forms and currents both provide acyclic resolutions of the same sheaf (locally constant functions, resp. holomorphic sections), hence define the same cohomology groups.

In the sequel, we are mostly interested in the geometry of *compact complex manifolds*. The compactness assumption brings many interesting features such as finitess results for the cohomology or the topology, Stokes theorem, intersection formulas of Bezout type, etc. A *projective algebraic manifold* is a closed submanifold X of some complex projective space $\mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$ defined by a finite collection of homogeneous polynomial equations

$$P_j(z_0, z_1, \ldots, z_N) = 0, \qquad 1 \le j \le k$$

(in such a way that X is non singular). An important theorem due to Chow states that every complex analytic submanifold of $\mathbb{P}^N$ is in fact automatically algebraic, i.e. defined as above by a finite collection of polynomials. We will prove this in section 4.

However, we will sometimes need to study local situations, and in that case it is also useful to consider the case of (pseudoconvex) open sets in $\mathbb{C}^n$.

(0.1) Definition.

a) *A hermitian manifold is a pair (X,ω) where ω is a C^∞ positive definite $(1,1)$-form on X.*

b) *X is said to be a Kähler manifold if X carries at least one Kähler metric ω.*

Since ω is real, the conditions $d\omega = 0$, $d'\omega = 0$, $d''\omega = 0$ are all equivalent. In local coordinates we see that $d'\omega = 0$ if and only if

$$\frac{\partial \omega_{jk}}{\partial z_l} = \frac{\partial \omega_{lk}}{\partial z_j} \quad , \quad 1 \le j, k, l \le n.$$

A simple computation gives

$$\frac{\omega^n}{n!} = \det(\omega_{jk}) \bigwedge_{1 \le j \le n} \left(\mathrm{i} dz_j \wedge d\overline{z}_j\right) = 2^n \det(\omega_{jk})\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

where $z_n = x_n + \mathrm{i} y_n$. Therefore the (n, n)-form

$$dV = \frac{1}{n!}\omega^n \tag{0.2}$$

is positive and coincides with the hermitian volume element of X. If X is compact, then $\int_X \omega^n = n!\, \mathrm{Vol}_\omega(X) > 0$. This simple remark already implies that compact Kähler manifolds must satisfy some restrictive topological conditions:

(0.3) Consequence.

a) *If (X, ω) is compact Kähler and if $\{\omega\}$ denotes the cohomology class of ω in $H^2(X, \mathbb{R})$, then $\{\omega\}^n \neq 0$.*

b) *If X is compact Kähler, then $H^{2k}(X, \mathbb{R}) \neq 0$ for $0 \le k \le n$. In fact, $\{\omega\}^k$ is a non zero class in $H^{2k}(X, \mathbb{R})$.*

(0.4) Example. The complex projective space $\mathbb{P}^n$ is Kähler. A natural Kähler metric ω_{FS} on $\mathbb{P}^n$, called the *Fubini-Study metric*, is defined by

$$p^\star \omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi} d'd'' \log\left(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2\right)$$

where $\zeta_0, \zeta_1, \ldots, \zeta_n$ are coordinates of $\mathbb{C}^{n+1}$ and where $p : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the projection. Let $z = (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0)$ be non homogeneous coordinates on $\mathbb{C}^n \subset \mathbb{P}^n$. Then a calculation shows that

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi} d'd'' \log(1 + |z|^2), \qquad \int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = 1.$$

It is also well-known from topology that $\{\omega_{\mathrm{FS}}\} \in H^2(\mathbb{P}^n, \mathbb{Z})$ is a generator of the cohomology algebra $H^\bullet(\mathbb{P}^n, \mathbb{Z})$.

(0.5) Example. A *complex torus* is a quotient $X = \mathbb{C}^n/\Gamma$ by a lattice Γ of rank $2n$. Then X is a compact complex manifold. Any positive definite hermitian form $\omega = \mathrm{i} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$ with constant coefficients defines a Kähler metric on X.

(0.6) Example. Every (complex) submanifold Y of a Kähler manifold (X, β) is Kähler with metric $\omega = \beta_{\restriction Y}$. Especially, all complex submanifolds of $X \subset \mathbb{P}^N$ are Kähler with Kähler metric $\omega = \omega_{\mathrm{FS}\restriction X}$. Since ω_{FS} is in $H^2(\mathbb{P}, \mathbb{Z})$, the restriction ω is an integral class in $H^2(X, \mathbb{Z})$. Conversely, the Kodaira embedding theorem [Kod54] states that every compact Kähler manifold X possessing a Kähler metric ω with an integral cohomology class $\{\omega\} \in H^2(X, \mathbb{Z})$ can be embedded in projective space as a projective algebraic subvariety. We will prove this in section 4.

(0.7) Example. Consider the complex surface

$$X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$$

where $\Gamma = \{\lambda^n \ ; \ n \in \mathbb{Z}\}$, $\lambda < 1$, acts as a group of homotheties. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}_+^\star \times S^3$, we have $X \simeq S^1 \times S^3$. Therefore $H^2(X, \mathbb{R}) = 0$ by Künneth's formula, and property 0.3 b) shows that X is not Kähler. More generally, one can take Γ to be an infinite cyclic group generated by a holomorphic contraction of $\mathbb{C}^2$, of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \end{pmatrix}, \qquad \text{resp.} \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda z_1 \\ \lambda z_2 + z_1^p \end{pmatrix},$$

where $\lambda, \lambda_1, \lambda_2$ are complex numbers such that $0 < |\lambda_1| \le |\lambda_2| < 1$, $0 < |\lambda| < 1$, and p a positive integer. These non Kähler surfaces are called *Hopf surfaces*.

1. Hermitian Vector Bundles, Connections and Curvature

The goal of this section is to recall the most basic definitions of hemitian differential geometry related to the concepts of connection, curvature and first Chern class of a line bundle.

Let F be a complex vector bundle of rank r over a smooth differentiable manifold M. A *connection* D on F is a linear differential operator of order 1

$$D : C^\infty(M, \Lambda^q T_M^\star \otimes F) \to C^\infty(M, \Lambda^{q+1} T_M^\star \otimes F)$$

such that

$$D(f \wedge u) = df \wedge u + (-1)^{\deg f} f \wedge Du \tag{1.1}$$

for all forms $f \in C^\infty(M, \Lambda^p T_M^\star)$, $u \in C^\infty(X, \Lambda^q T_M^\star \otimes F)$. On an open set $\Omega \subset M$ where F admits a trivialization $\theta : F_{\restriction \Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}^r$, a connection D can be written

$$Du \simeq_\theta du + \Gamma \wedge u$$

where $\Gamma \in C^\infty(\Omega, \Lambda^1 T_M^\star \otimes \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ is an arbitrary matrix of 1-forms and d acts componentwise. It is then easy to check that

$$D^2 u \simeq_\theta (d\Gamma + \Gamma \wedge \Gamma) \wedge u \quad \text{on } \Omega.$$

Since D^2 is a globally defined operator, there is a global 2-form

$$\Theta(D) \in C^\infty(M, \Lambda^2 T_M^\star \otimes \mathrm{Hom}(F, F)) \tag{1.2}$$

such that $D^2u = \Theta(D) \wedge u$ for every form u with values in F.

Assume now that F is endowed with a C^∞ hermitian metric along the fibers and that the isomorphism $F_{|\Omega} \simeq \Omega \times \mathbb{C}^r$ is given by a C^∞ frame (e_λ). We then have a canonical sesquilinear pairing

$$(1.3) \quad \begin{aligned} C^\infty(M, \Lambda^p T_M^\star \otimes F) \times C^\infty(M, \Lambda^q T_M^\star \otimes F) &\longrightarrow C^\infty(M, \Lambda^{p+q} T_M^\star \otimes \mathbb{C}) \\ (u, v) &\longmapsto \{u, v\} \end{aligned}$$

given by

$$\{u, v\} = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \langle e_\lambda, e_\mu \rangle, \qquad u = \sum u_\lambda \otimes e_\lambda, \quad v = \sum v_\mu \otimes e_\mu.$$

The connection D is said to be *hermitian* if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{\deg u}\{u, Dv\}.$$

Assuming that (e_λ) is orthonormal, one easily checks that D is hermitian if and only if $\Gamma^\star = -\Gamma$. In this case $\Theta(D)^\star = -\Theta(D)$, thus

$$\mathrm{i}\Theta(D) \in C^\infty(M, \Lambda^2 T_M^\star \otimes \mathrm{Herm}(F, F)).$$

(1.4) Special case. For a bundle F of rank 1, the connection form Γ of a hermitian connection D can be seen as a 1-form with purely imaginary coefficients $\Gamma = \mathrm{i}A$ (A real). Then we have $\Theta(D) = d\Gamma = \mathrm{i}dA$. In particular $\mathrm{i}\Theta(F)$ is a closed 2-form. The *first Chern class* of F is defined to be the cohomology class

$$c_1(F)_\mathbb{R} = \Big\{\frac{\mathrm{i}}{2\pi}\Theta(D)\Big\} \in H^2_{\mathrm{DR}}(M, \mathbb{R}).$$

The cohomology class is actually independent of the connection, since any other connection D_1 differs by a global 1-form, $D_1u = Du + B \wedge u$, so that $\Theta(D_1) = \Theta(D) + dB$. It is well-known that $c_1(F)_\mathbb{R}$ is the image in $H^2(M, \mathbb{R})$ of an integral class $c_1(F) \in H^2(M, \mathbb{Z})$; by using the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^\star \to 0,$$

$c_1(F)$ can be defined in Čech cohomology theory as the image by the coboundary map $H^1(M, \mathcal{E}^\star) \to H^2(M, \mathbb{Z})$ of the cocycle $\{g_{jk}\} \in H^1(M, \mathcal{E}^\star)$ defining F; see e.g. [GrH78] for details. □

We now concentrate ourselves on the complex analytic case. If $M = X$ is a complex manifold X, every connection D on a complex C^∞ vector bundle F can be splitted in a unique way as a sum of a $(1,0)$ and of a $(0,1)$-connection, $D = D' + D''$. In a local trivialization θ given by a C^∞ frame, one can write

$$(1.5') \qquad D'u \simeq_\theta d'u + \Gamma' \wedge u,$$

$$(1.5'') \qquad D''u \simeq_\theta d''u + \Gamma'' \wedge u,$$

with $\Gamma = \Gamma' + \Gamma''$. The connection is hermitian if and only if $\Gamma' = -(\Gamma'')^\star$ in any orthonormal frame. Thus there exists a unique hermitian connection D corresponding to a prescribed $(0,1)$ part D''.

Assume now that the bundle F itself has a *holomorphic* structure. The unique hermitian connection for which D'' is the d'' operator defined in § 1 is called the *Chern connection* of F. In a local holomorphic frame (e_λ) of $E_{|\Omega}$, the metric is given by the hermitian matrix $H = (h_{\lambda\mu})$, $h_{\lambda\mu} = \langle e_\lambda, e_\mu\rangle$. We have

$$\{u,v\} = \sum_{\lambda,\mu} h_{\lambda\mu} u_\lambda \wedge \overline{v}_\mu = u^\dagger \wedge H\overline{v},$$

where $u^\dagger$ is the transposed matrix of u, and easy computations yield

$$\begin{aligned} d\{u,v\} &= (du)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge (dH \wedge \overline{v} + H\overline{dv}) \\ &= \big(du + \overline{H}^{-1} d'\overline{H} \wedge u\big)^\dagger \wedge H\overline{v} + (-1)^{\deg u} u^\dagger \wedge \overline{(dv + \overline{H}^{-1} d'\overline{H} \wedge v)} \end{aligned}$$

using the fact that $dH = d'H + \overline{d'\overline{H}}$ and $\overline{H}^\dagger = H$. Therefore the Chern connection D coincides with the hermitian connection defined by

$$(1.6)\qquad \begin{cases} Du \simeq_\theta du + \overline{H}^{-1} d'\overline{H} \wedge u, \\ D' \simeq_\theta d' + \overline{H}^{-1} d'\overline{H} \wedge \bullet = \overline{H}^{-1} d'(\overline{H}\bullet), \qquad D'' = d''. \end{cases}$$

It is clear from this relations that $D'^2 = D''^2 = 0$. Consequently D^2 is given by to $D^2 = D'D'' + D''D'$, and the curvature tensor $\Theta(D)$ is of type $(1,1)$. Since $d'd'' + d''d' = 0$, we get

$$\begin{aligned} (D'D'' + D''D')u &\simeq_\theta \overline{H}^{-1} d'\overline{H} \wedge d''u + d''(\overline{H}^{-1} d'\overline{H} \wedge u) \\ &= d''(\overline{H}^{-1} d'\overline{H}) \wedge u. \end{aligned}$$

(1.7) Proposition. *The Chern curvature tensor $\Theta(F) := \Theta(D)$ is such that*

$$\mathrm{i}\,\Theta(F) \in C^\infty(X, \Lambda^{1,1}T^\star_X \otimes \mathrm{Herm}(F,F)).$$

If $\theta : E_{\restriction\Omega} \to \Omega \times \mathbb{C}^r$ is a holomorphic trivialization and if H is the hermitian matrix representing the metric along the fibers of $F_{\restriction\Omega}$, then

$$\mathrm{i}\,\Theta(F) \simeq_\theta \mathrm{i}\, d''(\overline{H}^{-1} d'\overline{H}) \quad \text{on } \Omega. \qquad \square$$

Let $(z_1, \ldots, z_n)$ be holomorphic coordinates on X and let $(e_\lambda)_{1\le\lambda\le r}$ be an orthonormal frame of F. Writing

$$\mathrm{i}\Theta(F) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu} dz_j \wedge dz_k \otimes e^\star_\lambda \otimes e_\mu,$$

we can identify the curvature tensor to a hermitian form

$$(1.8)\qquad \widetilde{\Theta}(F)(\xi\otimes v) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu$$

on $T_X \otimes F$. This leads in a natural way to positivity concepts, following definitions introduced by Kodaira [Kod53], Nakano [Nak55] and Griffiths [Gri69].

(1.9) Definition. *The hermitian vector bundle F is said to be*

a) *positive in the sense of Nakano if $\widetilde{\Theta}(F)(\tau) > 0$ for all non zero tensors $\tau = \sum \tau_{j\lambda} \partial/\partial z_j \otimes e_\lambda \in T_X \otimes F$.*

b) *positive in the sense of Griffiths if $\widetilde{\Theta}(F)(\xi \otimes v) > 0$ for all non zero decomposable tensors $\xi \otimes v \in T_X \otimes F$;*

Corresponding semipositivity concepts are defined by relaxing the strict inequalities.

(1.10) Special case of rank 1 bundles. Assume that F is a line bundle. The hermitian matrix $H = (h_{11})$ associated to a trivialization $\theta : F_{\restriction\Omega} \simeq \Omega \times \mathbb{C}$ is simply a positive function which we find convenient to denote by $e^{-\varphi}$, $\varphi \in C^\infty(\Omega, \mathbb{R})$. In this case the curvature form $\Theta(F)$ can be identified to the $(1,1)$-form $2d'd''\varphi$, and

$$\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{\mathrm{i}}{\pi} d'd''\varphi = dd^c\varphi$$

is a real $(1,1)$-form. Hence F is semipositive (in either the Nakano or Griffiths sense) if and only if φ is psh, resp. positive if and only if φ is *strictly psh*. In this setting, the Lelong-Poincaré equation can be generalized as follows: let $\sigma \in H^0(X, F)$ be a non zero holomorphic section. Then

$$dd^c \log \|\sigma\| = [Z_\sigma] - \frac{\mathrm{i}}{2\pi}\Theta(F). \tag{1.11}$$

Formula (1.11) is immediate if we write $\|\sigma\| = |\theta(\sigma)| e^{-\varphi}$ and if we apply (1.20) to the holomorphic function $f = \theta(\sigma)$. As we shall see later, it is very important for the applications to consider also singular hermitian metrics.

(1.12) Definition. *A singular (hermitian) metric on a line bundle F is a metric which is given in any trivialization $\theta : F_{\restriction\Omega} \xrightarrow{\simeq} \Omega \times \mathbb{C}$ by*

$$\|\xi\| = |\theta(\xi)|\, e^{-\varphi(x)}, \qquad x \in \Omega,\ \xi \in F_x$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the weight of the metric with respect to the trivialization θ.

If $\theta' : F_{\restriction\Omega'} \longrightarrow \Omega' \times \mathbb{C}$ is another trivialization, φ' the associated weight and $g \in \mathcal{O}^\star(\Omega \cap \Omega')$ the transition function, then $\theta'(\xi) = g(x)\,\theta(\xi)$ for $\xi \in F_x$, and so $\varphi' = \varphi + \log|g|$ on $\Omega \cap \Omega'$. The curvature form of F is then given formally by the closed $(1,1)$-current $\frac{\mathrm{i}}{2\pi}\Theta(F) = dd^c\varphi$ on Ω; our assumption $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ guarantees that $\Theta(F)$ exists in the sense of distribution theory. As in the smooth case, $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is globally defined on X and independent of the choice of trivializations, and its De Rham cohomology class is the image of the first Chern class $c_1(F) \in H^2(X, \mathbb{Z})$ in $H^2_{DR}(X, \mathbb{R})$. Before going further, we discuss two basic examples.

(1.13) Example. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{Z}$ and let $F = \mathcal{O}(D)$ be the associated invertible sheaf of meromorphic functions u such that $\mathrm{div}(u) + D \geq 0$; the corresponding line bundle can be equipped with the singular metric defined by $\|u\| = |u|$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$ then $\theta(u) = u \prod g_j^{\alpha_j}$ defines a trivialization of $\mathcal{O}(D)$ over Ω, thus our singular metric is associated to the weight $\varphi = \sum \alpha_j \log |g_j|$. By the Lelong-Poincaré equation, we find

$$\frac{\mathrm{i}}{2\pi} \Theta\big(\mathcal{O}(D)\big) = dd^c\varphi = [D],$$

where $[D] = \sum \alpha_j [D_j]$ denotes the current of integration over D. □

(1.14) Example. Assume that $\sigma_1, \ldots, \sigma_N$ are non zero holomorphic sections of F. Then we can define a natural (possibly singular) hermitian metric on $F^\star$ by

$$\|\xi^\star\|^2 = \sum_{1 \leq j \leq n} \big|\xi^\star.\sigma_j(x)\big|^2 \quad \text{for } \ \xi^\star \in F_x^\star.$$

The dual metric on F is given by

$$\|\xi\|^2 = \frac{|\theta(\xi)|^2}{|\theta(\sigma_1(x))|^2 + \ldots + |\theta(\sigma_N(x))|^2}$$

with respect to any trivialization θ. The associated weight function is thus given by $\varphi(x) = \log \big(\sum_{1 \leq j \leq N} |\theta(\sigma_j(x))|^2\big)^{1/2}$. In this case φ is a psh function, thus $\mathrm{i}\Theta(F)$ is a closed positive current. Let us denote by Σ the linear system defined by $\sigma_1, \ldots, \sigma_N$ and by $B_\Sigma = \bigcap \sigma_j^{-1}(0)$ its base locus. We have a meromorphic map

$$\Phi_\Sigma : X \smallsetminus B_\Sigma \to \mathbb{P}^{N-1}, \qquad x \mapsto (\sigma_1(x) : \sigma_2(x) : \ldots : \sigma_N(x)).$$

Then $\frac{\mathrm{i}}{2\pi}\Theta(F)$ is equal to the pull-back over $X \smallsetminus B_\Sigma$ of the Fubini-Study metric $\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2\pi} \log(|z_1|^2 + \ldots + |z_N|^2)$ of $\mathbb{P}^{N-1}$ by Φ_Σ. □

(1.15) Ample and very ample line bundles. *A holomorphic line bundle F over a compact complex manifold X is said to be*

a) *very ample if the map $\Phi_{|F|} : X \to \mathbb{P}^{N-1}$ associated to the complete linear system $|F| = P(H^0(X, F))$ is a regular embedding (by this we mean in particular that the base locus is empty, i.e. $B_{|F|} = \emptyset$).*

b) *ample if some multiple mF, $m > 0$, is very ample.*

Here we use an additive notation for $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\star)$, hence the symbol mF denotes the line bundle $F^{\otimes m}$. By Example 1.14, every ample line bundle F has a smooth hermitian metric with positive definite curvature form; indeed, if the linear system $|mF|$ gives an embedding in projective space, then we get a smooth hermitian metric on $F^{\otimes m}$, and the m-th root yields a metric on F such that $\frac{\mathrm{i}}{2\pi}\Theta(F) = \frac{1}{m}\Phi_{|mF|}^\star \omega_{\mathrm{FS}}$. Conversely, the Kodaira embedding theorem [Kod54] tells us that every positive line bundle F is ample (see (4.14) for a straightforward analytic proof of the Kodaira embedding theorem).

2. Bochner Technique and Vanishing Theorems

We first recall briefly a few basic facts of Hodge theory. Assume for the moment that M is a differentiable manifold equipped with a Riemannian metric $g = \sum g_{ij} dx_i \otimes dx_j$. Given a q-form u on M with values in F, we consider the global L^2 norm

$$\|u\|^2 = \int_M |u(x)|^2 dV_g(x)$$

where $|u|$ is the pointwise hermitian norm and dV_g is the Riemannian volume form. The Laplace-Beltrami operator associated to the connection D is

$$\Delta = DD^\star + D^\star D$$

where

$$D^\star : C^\infty(M, \Lambda^q T_M^\star \otimes F) \to C^\infty(M, \Lambda^{q-1} T_M^\star \otimes F)$$

is the (formal) adjoint of D with respect to the L^2 inner product. Assume that M is *compact*. Since

$$\Delta : C^\infty(M, \Lambda^q T_M^\star \otimes F) \to C^\infty(M, \Lambda^q T_M^\star \otimes F)$$

is a self-adjoint elliptic operator in each degree, standard results of PDE theory show that there is an orthogonal decomposition

$$C^\infty(M, \Lambda^q T_M^\star \otimes F) = \mathcal{H}^q(M, F) \oplus \operatorname{Im} \Delta$$

where $\mathcal{H}^q(M, F) = \operatorname{Ker} \Delta$ is the space of harmonic forms of degree q; $\mathcal{H}^q(M, F)$ is a finite dimensional space. Assume moreover that the connection D is *integrable*, i.e. that $D^2 = 0$. It is then easy to check that there is an orthogonal direct sum

$$\operatorname{Im} \Delta = \operatorname{Im} D \oplus \operatorname{Im} D^\star,$$

indeed $\langle Du, D^\star v \rangle = \langle D^2 u, v \rangle = 0$ for all u, v. Hence we get an orthogonal decomposition

$$C^\infty(M, \Lambda^q T_M^\star \otimes F) = \mathcal{H}^q(M, F) \oplus \operatorname{Im} D \oplus \operatorname{Im} D^\star,$$

and $\operatorname{Ker} D = (\operatorname{Im} D^*)^\perp$ is precisely equal to $\mathcal{H}^q(M, F) \oplus \operatorname{Im} D$. Especially, the q-th cohomology group $H^q_{\mathrm{DR}}(M, F) := \operatorname{Ker} D / \operatorname{Im} D$ is isomorphic to $\mathcal{H}^q(M, F)$. In general, a nontrivial vector bundle F does not admit an integrable connection, but this is certainly the case for the trivial bundle $F = M \times \mathbb{C}$. This implies that the De Rham cohomology groups $H^q_{\mathrm{DR}}(M, \mathbb{C})$ can be computed in terms of harmonic forms:

(2.1) Hodge Fundamental Theorem. *If M is a compact Riemannian manifold, there is an isomorphism*

$$H^q_{\mathrm{DR}}(M, \mathbb{C}) \simeq \mathcal{H}^q(M, \mathbb{C})$$

from De Rham cohomology groups onto spaces of harmonic forms. □

A rather important consequence of the Hodge fundamental theorem is a proof of the *Poincaré duality theorem*. Assume that the Riemannian manifold (M, g) is oriented. Then there is a (conjugate linear) Hodge star operator

$$\star : \Lambda^q T^\star_M \otimes \mathbb{C} \to \Lambda^{m-q} T^\star_M \otimes \mathbb{C}, \qquad m = \dim_{\mathbb{R}} M$$

defined by $u \wedge \star v = \langle u, v\rangle dV_g$ for any two complex valued q-forms u, v. A standard computation shows that $\star$ commutes with Δ, hence $\star u$ is harmonic if and only if u is. This implies that the natural pairing

$$(2.2) \qquad H^q_{\mathrm{DR}}(M,\mathbb{C}) \times H^{m-q}_{\mathrm{DR}}(M,\mathbb{C}), \qquad (\{u\},\{v\}) \mapsto \int_M u \wedge v$$

is a nondegenerate duality, the dual of a class $\{u\}$ represented by a harmonic form being $\{\star u\}$.

Let us now suppose that X is a compact complex manifold equipped with a hermitian metric $\omega = \sum \omega_{jk} dz_j \wedge d\overline{z}_k$. Let F be a holomorphic vector bundle on X equipped with a hermitian metric, and let $D = D' + D''$ be its Chern curvature form. All that we said above for the Laplace-Beltrami operator Δ still applies to the complex Laplace operators

$$\Delta' = D'D'^\star + D'^\star D', \qquad \Delta'' = D''D''^\star + D''^\star D'',$$

with the great advantage that we always have $D'^2 = D''^2 = 0$. Especially, if X is a compact complex manifold, there are isomorphisms

$$(2.3) \qquad H^{p,q}(X,F) \simeq \mathcal{H}^{p,q}(X,F)$$

between Dolbeault cohomology groups $H^{p,q}(X,F) := \operatorname{Ker} D''/\operatorname{Im} D''$ and spaces $\mathcal{H}^{p,q}(X,F)$ of Δ''-harmonic forms of bidegree (p,q) with values in F; indeed, as above, we have an orthogonal direct sum

$$C^\infty(X, \Lambda^{p,q} T^*_X \otimes F) = \operatorname{Ker} \Delta'' \oplus \operatorname{Im} \Delta'' = \mathcal{H}^{p,q}(X,F) \oplus \operatorname{Im} D'' \oplus \operatorname{Im} D''^*$$

and $\operatorname{Ker} D'' = (\operatorname{Im} D''^*)^\perp = \mathcal{H}^{p,q}(X,F) \oplus \operatorname{Im} D''$. Now, there is a generalized Hodge star operator

$$\star : \Lambda^{p,q} T^\star_X \otimes F \to \Lambda^{n-p,n-q} T^\star_X \otimes F^\star, \qquad n = \dim_{\mathbb{C}} X,$$

such that $u \wedge \star v = \langle u, v\rangle dV_g$, for any two F-valued (p,q)-forms, when the wedge product $u \wedge \star v$ is combined with the pairing $F \times F^\star \to \mathbb{C}$. This leads to the *Serre duality theorem* [Ser55]: the bilinear pairing

$$(2.4) \qquad H^{p,q}(X,F) \times H^{n-p,n-q}(X,F^\star), \qquad (\{u\},\{v\}) \mapsto \int_X u \wedge v$$

is a nondegenerate duality. Combining this with the Dolbeault isomorphism, we may restate the result in the form of the duality formula

$$(2.4') \qquad H^q(X, \Omega^p_X \otimes \mathcal{O}(F))^\star \simeq H^{n-q}(X, \Omega^{n-p}_X \otimes \mathcal{O}(F^\star)).$$

We now proceed to explain the basic ideas of the Bochner technique used to prove vanishing theorems. Great simplifications occur in the computations if the hermitian metric on X is supposed to be *Kähler*, i.e. if the associated *fundamental* $(1,1)$*-form*

$$\omega = \mathrm{i} \sum \omega_{jk} dz_j \wedge d\overline{z}_k$$

satisfies $d\omega = 0$. It can be easily shown that ω is Kähler if and only if there are holomorphic coordinates $(z_1, \ldots, z_n)$ centered at any point $x_0 \in X$ such that the matrix of coefficients (ω_{jk}) is tangent to identity at order 2, i.e.

$$\omega_{jk}(z) = \delta_{jk} + O(|z|^2) \quad \text{at } x_0.$$

It follows that all order 1 operators D, D', D'' and their adjoints $D^\star$, $D'^\star$, $D''^\star$ admit at x_0 the same expansion as the analogous operators obtained when all hermitian metrics on X or F are constant. From this, the basic commutation relations of Kähler geometry can be checked. If A, B are differential operators acting on the algebra $C^\infty(X, \Lambda^{\bullet,\bullet} T_X^\star \otimes F)$, their graded commutator (or graded Lie bracket) is defined by

$$[A, B] = AB - (-1)^{ab} BA$$

where a, b are the degrees of A and B respectively. If C is another endomorphism of degree c, the following purely formal *Jacobi identity* holds:

$$(-1)^{ca}\big[A, [B, C]\big] + (-1)^{ab}\big[B, [C, A]\big] + (-1)^{bc}\big[C, [A, B]\big] = 0.$$

(2.5) Basic commutation relations. *Let (X, ω) be a Kähler manifold and let L be the operators defined by $Lu = \omega \wedge u$ and $\Lambda = L^\star$. Then*

$$\begin{aligned} [D''^\star, L] &= \mathrm{i}D', & [D'^\star, L] &= -\mathrm{i}D'', \\ [\Lambda, D''] &= -\mathrm{i}D'^\star, & [\Lambda, D'] &= \mathrm{i}D''^\star. \end{aligned}$$

Proof (sketch). The first step is to check the identity $[d''^\star, L] = \mathrm{i}d'$ for constant metrics on $X = \mathbb{C}^n$ and $F = X \times \mathbb{C}$, by a brute force calculation. All three other identities follow by taking conjugates or adjoints. The case of variable metrics follows by looking at Taylor expansions up to order 1; essentially nothing changes since the Kähler metric ω just introduces additional $O(|z|^2)$ terms which have zero derivative at the center of the coordinate chart. □

(2.6) Bochner-Kodaira-Nakano identity. *If (X, ω) is Kähler, the complex Laplace operators Δ' and Δ'' acting on F-valued forms satisfy the identity*

$$\Delta'' = \Delta' + [\mathrm{i}\Theta(F), \Lambda].$$

Proof. The last equality in (2.5) yields $D''^\star = -\mathrm{i}[\Lambda, D']$, hence

$$\Delta'' = [D'', \delta''] = -\mathrm{i}[D'', [\Lambda, D']].$$

By the Jacobi identity we get

$$\big[D'', [\Lambda, D']\big] = \big[\Lambda, [D', D'']\big] + \big[D', [D'', \Lambda]\big] = [\Lambda, \Theta(F)] + \mathrm{i}[D', D'^\star],$$

taking into account that $[D', D''] = D^2 = \Theta(F)$. The formula follows. □

(2.7) Corollary (Hodge decomposition)**.** *If (X, ω) is a compact Kähler manifold, there is a canonical decomposition*

$$H^k(X,\mathbb{R}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$$

of the De Rham cohomology groups in terms of the Dolbeault cohomology groups.

Proof. If we apply the Bochner-Kodaira-Nakano identity to the trivial bundle $F = X \times \mathbb{C}$, we find $\Delta'' = \Delta'$. Morever

$$\Delta = [d' + d'', d'^\star + d''^\star] = \Delta' + \Delta'' + [d', d''^\star] + [d'', d'^\star].$$

We claim that $[d', d''^\star] = [d'', d'^\star] = 0$. Indeed, we have $[d', d''^\star] = -\mathrm{i}\big[d', [\Lambda, d']\big]$ by (2.5), and the Jacobi identity implies

$$-\big[d', [\Lambda, d']\big] + \big[\Lambda, [d', d']\big] + \big[d', [d', \Lambda]\big] = 0,$$

hence $-2\big[d', [\Lambda, d']\big] = 0$ and $[d', d''^\star] = 0$. The second identity is similar. As a consequence

$$\Delta' = \Delta'' = \frac{1}{2}\Delta.$$

We infer that Δ preserves the bidegree of forms and operates “separately” on each term $C^\infty(X, \Lambda^{p,q}T^*_X)$. Hence, on the level of harmonic forms we have

$$\mathcal{H}^k(X,\mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X,\mathbb{C}).$$

The decomposition theorem (2.7) now follows from the Hodge isomorphisms for De Rham and Dolbeault groups. The decomposition is canonical since $H^{p,q}(X)$ coincides with the set of classes in $H^k(X,\mathbb{C})$ which can be represented by d-closed (p,q)-forms. □

Now, assume that X is compact Kähler and that $u \in C^\infty(X, \Lambda^{p,q}T^\star X \otimes F)$ is an arbitrary (p,q)-form. An integration by parts yields

$$\langle \Delta' u, u\rangle = \|D'u\|^2 + \|D'^\star u\|^2 \geq 0$$

and similarly for Δ'', hence we get the basic a priori inequality

$$\|D''u\|^2 + \|D''^\star u\|^2 \geq \int_X \langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle dV_\omega. \tag{2.8}$$

This inequality is known as the *Bochner-Kodaira-Nakano* inequality (see [Boc48], [Kod53], [Nak55]). When u is Δ''-harmonic, we get

$$\int_X \big(\langle [\mathrm{i}\Theta(F), \Lambda]u, u\rangle + \langle T_\omega u, u\rangle\big) dV \leq 0.$$

If the hermitian operator $[\mathrm{i}\Theta(F), \Lambda]$ acting on $\Lambda^{p,q}T^\star_X \otimes F$ is positive on each fiber, we infer that u must be zero, hence

$$H^{p,q}(X,F) = \mathcal{H}^{p,q}(X,F) = 0$$

by Hodge theory. The main point is thus to compute the curvature form $\Theta(F)$ and find sufficient conditions under which the operator $[\mathrm{i}\Theta(F), \Lambda]$ is positive definite.

Elementary (but somewhat tedious) calculations yield the following formulae: if the curvature of F is written as in (1.8) and $u = \sum u_{J,K,\lambda} dz_I \wedge d\overline{z}_J \otimes e_\lambda$, $|J| = p$, $|K| = q$, $1 \le \lambda \le r$ is a (p,q)-form with values in F, then

$$\begin{aligned}\langle[\mathrm{i}\Theta(F),\Lambda]u,u\rangle = &\sum_{j,k,\lambda,\mu,J,S} c_{jk\lambda\mu}\, u_{J,jS,\lambda}\, \overline{u_{J,kS,\mu}} \\ &+ \sum_{j,k,\lambda,\mu,R,K} c_{jk\lambda\mu}\, u_{kR,K,\lambda}\, \overline{u_{jR,K,\mu}} \\ &- \sum_{j,\lambda,\mu,J,K} c_{jj\lambda\mu}\, u_{J,K,\lambda}\, \overline{u_{J,K,\mu}},\end{aligned} \tag{2.9}$$

where the sum is extended to all indices $1 \le j,k \le n$, $1 \le \lambda,\mu \le r$ and multiindices $|R| = p-1$, $|S| = q-1$ (here the notation $u_{JK\lambda}$ is extended to non necessarily increasing multiindices by making it alternate with respect to permutations). It is usually hard to decide the sign of the curvature term (2.9), except in some special cases.

The easiest case is when $p = n$. Then all terms in the second summation of (2.9) must have $j = k$ and $R = \{1,\ldots,n\} \smallsetminus \{j\}$, therefore the second and third summations are equal. It follows that $[\mathrm{i}\Theta(F),\Lambda]$ is positive on (n,q)-forms under the assumption that F is positive in the sense of Nakano. In this case X is automatically Kähler since

$$\omega = \mathrm{Tr}_F(\mathrm{i}\Theta(F)) = \mathrm{i}\sum_{j,k,\lambda} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k = \mathrm{i}\Theta(\det F)$$

is a Kähler metric.

(2.10) Nakano vanishing theorem ([Nak55])**.** *Let X be a compact complex manifold and let F be a Nakano positive vector bundle on X. Then*

$$H^{n,q}(X,F) = H^q(X, K_X \otimes F) = 0 \qquad \textit{for every } q \ge 1. \qquad \square$$

Another tractable case is the case where F is a line bundle ($r = 1$). Indeed, at each point $x \in X$, we may then choose a coordinate system which diagonalizes simultaneously the hermitians forms $\omega(x)$ and $\mathrm{i}\Theta(F)(x)$, in such a way that

$$\omega(x) = \mathrm{i}\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j, \qquad \mathrm{i}\Theta(F)(x) = \mathrm{i}\sum_{1\le j\le n} \gamma_j dz_j \wedge d\overline{z}_j$$

with $\gamma_1 \le \ldots \le \gamma_n$. The curvature eigenvalues $\gamma_j = \gamma_j(x)$ are then uniquely defined and depend continuously on x. With our previous notation, we have $\gamma_j = c_{jj11}$ and all other coefficients $c_{jk\lambda\mu}$ are zero. For any (p,q)-form $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \otimes e_1$, this gives

$$\begin{aligned}\langle[\mathrm{i}\Theta(F),\Lambda]u,u\rangle &= \sum_{|J|=p,\,|K|=q} \Big(\sum_{j\in J}\gamma_j + \sum_{j\in K}\gamma_j - \sum_{1\le j\le n}\gamma_j\Big)|u_{JK}|^2 \\ &\ge (\gamma_1 + \ldots + \gamma_q - \gamma_{n-p+1} - \ldots - \gamma_n)|u|^2.\end{aligned} \tag{2.11}$$

Assume that $\mathrm{i}\Theta(F)$ is positive. It is then natural to make the special choice $\omega = \mathrm{i}\Theta(F)$ for the Kähler metric. Then $\gamma_j = 1$ for $j = 1, 2, \ldots, n$ and we obtain $\langle[\mathrm{i}\Theta(F), \Lambda]u, u\rangle = (p + q - n)|u|^2$. As a consequence:

(2.12) Akizuki-Kodaira-Nakano vanishing theorem ([AN54]). *If F is a positive line bundle on a compact complex manifold X, then*

$$H^{p,q}(X, F) = H^q(X, \Omega_X^p \otimes F) = 0 \qquad \textit{for} \ \ p + q \geq n + 1. \qquad \square$$

More generally, if F is a Griffiths positive (or ample) vector bundle of rank $r \geq 1$, Le Potier [LP75] proved that $H^{p,q}(X, F) = 0$ for $p + q \geq n + r$. The proof is not a direct consequence of the Bochner technique. A rather easy proof has been found by M. Schneider [Sch74], using the Leray spectral sequence associated to the projectivized bundle $\mathbb{P}(F) \to X$.

3. L^2 Estimates and Existence Theorems

The starting point is the following L^2 existence theorem, which is essentially due to Hörmander [Hör65, 66], and Andreotti-Vesentini [AV65]. We will only outline the main ideas, referring e.g. to [Dem82] for a detailed exposition of the technical situation considered here.

(3.1) Theorem. *Let (X, ω) be a Kähler manifold. Here X is not necessarily compact, but we assume that the geodesic distance δ_ω is complete on X. Let F be a hermitian vector bundle of rank r over X, and assume that the curvature operator $A = A^{p,q}_{F,\omega} = [\mathrm{i}\Theta(F), \Lambda_\omega]$ is positive definite everywhere on $\Lambda^{p,q}T^\star_X \otimes F$, $q \geq 1$. Then for any form $g \in L^2(X, \Lambda^{p,q}T^\star_X \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle A^{-1}g, g\rangle\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ such that $D''f = g$ and*

$$\int_X |f|^2\, dV_\omega \leq \int_X \langle A^{-1}g, g\rangle\, dV_\omega.$$

Proof. The assumption that δ_ω is complete implies the existence of cut-off functions ψ_ν with arbitrarily large compact support such that $|d\psi_\nu| \leq 1$ (take ψ_ν to be a function of the distance $x \mapsto \delta_\omega(x_0, x)$, which is an almost everywhere differentiable 1-Lipschitz function, and regularize if necessary). From this, it follows that very form $u \in L^2(X, \Lambda^{p,q}T^\star_X \otimes F)$ such that $D''u \in L^2$ and $D''^\star u \in L^2$ in the sense of distribution theory is a limit of a sequence of smooth forms u_ν with compact support, in such a way that $u_\nu \to u$, $D''u_\nu \to D''u$ and $D''^\star u_\nu \to D''^\star u$ in L^2 (just take u_ν to be a regularization of $\psi_\nu u$). As a consequence, the basic a priori inequality (2.8) extends to arbitrary forms u such that $u,\ D''u, D''^\star u \in L^2$. Now, consider the Hilbert space orthogonal decomposition

$$L^2(X, \Lambda^{p,q}T^\star_X \otimes F) = \operatorname{Ker} D'' \oplus (\operatorname{Ker} D'')^\perp,$$

observing that $\operatorname{Ker} D''$ is weakly (hence strongly) closed. Let $v = v_1 + v_2$ be the decomposition of a smooth form $v \in \mathcal{D}^{p,q}(X,F)$ with compact support according to this decomposition (v_1, v_2 do not have compact support in general!). Since $(\operatorname{Ker} D'')^\perp \subset \operatorname{Ker} D''^\star$ by duality and $g, v_1 \in \operatorname{Ker} D''$ by hypothesis, we get $D''^\star v_2 = 0$ and

$$|\langle g, v\rangle|^2 = |\langle g, v_1\rangle|^2 \le \int_X \langle A^{-1}g, g\rangle\, dV_\omega \int_X \langle Av_1, v_1\rangle\, dV_\omega$$

thanks to the Cauchy-Schwarz inequality. The a priori inequality (2.8) applied to $u = v_1$ yields

$$\int_X \langle Av_1, v_1\rangle\, dV_\omega \le \|D''v_1\|^2 + \|D''^\star v_1\|^2 = \|D''^\star v_1\|^2 = \|D''^\star v\|^2.$$

Combining both inequalities, we find

$$|\langle g, v\rangle|^2 \le \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)\|D''^\star v\|^2$$

for every smooth (p,q)-form v with compact support. This shows that we have a well defined linear form

$$w = D''^\star v \longmapsto \langle v, g\rangle, \qquad L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F) \supset D''^\star(\mathcal{D}^{p,q}(F)) \longrightarrow \mathbb{C}$$

on the range of $D''^\star$. This linear form is continuous in L^2 norm and has norm $\le C$ with

$$C = \Big(\int_X \langle A^{-1}g, g\rangle\, dV_\omega\Big)^{1/2}.$$

By the Hahn-Banach theorem, there is an element $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ with $||f|| \le C$, such that $\langle v, g\rangle = \langle D''^\star v, f\rangle$ for every v, hence $D''f = g$ in the sense of distributions. The inequality $||f|| \le C$ is equivalent to the last estimate in the theorem. □

The above L^2 existence theorem can be applied in the fairly general context of *weakly pseudoconvex* manifolds. By this, we mean a complex manifold X such that there exists a smooth psh exhaustion function ψ on X (ψ is said to be an exhaustion if for every $c > 0$ the sublevel set $X_c = \psi^{-1}(c)$ is relatively compact, i.e. $\psi(z)$ tends to $+\infty$ when z is taken outside larger and larger compact subsets of X). In particular, every compact complex manifold X is weakly pseudoconvex (take $\psi = 0$), as well as every Stein manifold, e.g. affine algebraic submanifolds of $\mathbb{C}^N$ (take $\psi(z) = |z|^2$), open balls $X = B(z_0, r)$ $\big($take $\psi(z) = 1/(r - |z - z_0|^2)\big)$, convex open subsets, etc. Now, a basic observation is that every weakly pseudoconvex Kähler manifold (X, ω) carries a *complete* Kähler metric: let $\psi \ge 0$ be a psh exhaustion function and set

$$\omega_\varepsilon = \omega + \varepsilon\, \mathrm{i}d'd''\psi^2 = \omega + 2\varepsilon(2\mathrm{i}\psi d'd''\psi + \mathrm{i}d'\psi \wedge d''\psi).$$

Then $|d\psi|_{\omega_\varepsilon} \le 1/\varepsilon$ and $|\psi(x) - \psi(y)| \le \varepsilon^{-1}\delta_{\omega_\varepsilon}(x,y)$. It follows easily from this estimate that the geodesic balls are relatively compact, hence $\delta_{\omega_\varepsilon}$ is complete for every $\varepsilon > 0$. Therefore, the L^2 existence theorem can be applied to each Kähler metric ω_ε, and by passing to the limit it can even be applied to the non necessarily complete metric ω. An important special case is the following

(3.2) Theorem. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a hermitian line bundle and let*

$$\gamma_1(x) \le \ldots \le \gamma_n(x)$$

be the curvature eigenvalues (i.e. the eigenvalues of $\mathrm{i}\Theta(F)$ with respect to the metric ω) at every point. Assume that the curvature is positive, i.e. $\gamma_1 > 0$ everywhere. Then for any form $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes F)$ satisfying $D''g = 0$ and $\int_X \langle(\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega < +\infty$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2\, dV_\omega \le \int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|g|^2\, dV_\omega.$$

Proof. Indeed, for $p = n$, Formula 2.11 shows that

$$\langle Au, u\rangle \ge (\gamma_1 + \ldots + \gamma_q)|u|^2,$$

hence $\langle A^{-1}u, u\rangle \ge (\gamma_1 + \ldots + \gamma_q)^{-1}|u|^2$. □

An important observation is that the above theorem still applies when the hermitian metric on F is a singular metric with positive curvature in the sense of currents. In fact, by standard regularization techniques (convolution of psh functions by smoothing kernels), the metric can be made smooth and the solutions obtained by (3.1) or (3.2) for the smooth metrics have limits satisfying the desired estimates. Especially, we get the following

(3.3) Corollary. *Let (X,ω) be a Kähler manifold, $\dim X = n$. Assume that X is weakly pseudoconvex. Let F be a holomorphic line bundle equipped with a singular metric whose local weights are denoted $\varphi \in L^1_{\mathrm{loc}}$. Suppose that*

$$\mathrm{i}\Theta(F) = 2\mathrm{i}d'd''\varphi \ge \varepsilon\omega$$

for some $\varepsilon > 0$. Then for any form $g \in L^2(X, \Lambda^{n,q}T^\star_X \otimes F)$ satisfying $D''g = 0$, there exists $f \in L^2(X, \Lambda^{p,q-1}T^\star_X \otimes F)$ such that $D''f = g$ and

$$\int_X |f|^2 e^{-\varphi}\, dV_\omega \le \frac{1}{q\varepsilon}\int_X |g|^2 e^{-\varphi}\, dV_\omega. \qquad \square$$

Here we denoted somewhat incorrectly the metric by $|f|^2e^{-\varphi}$, as if the weight φ were globally defined on X (of course, this is so only if F is globally trivial). We will use this notation anyway, because it clearly describes the dependence of the L^2 norm on the psh weights.

In order to apply Corollary 3.3 in a fruitful way, it is usually necessary to select φ with suitable logarithmic poles along an analytic set. The basic construction of such a function is provided by the following Lemma.

(3.4) Lemma. *Let X be a compact complex manifold X equipped with a Kähler metric $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)dz_j\wedge d\overline{z}_k$ and let $Y\subset X$ be an analytic subset of X. Then there exist globally defined quasi-plurisubharmonic potentials ψ and $(\psi_\varepsilon)_{\varepsilon\in]0,1]}$ on X, satisfying the following properties.*

(i) *The function ψ is smooth on $X\smallsetminus Y$, satisfies $i\partial\overline{\partial}\psi\ge -A\omega$ for some $A>0$, and ψ has logarithmic poles along Y, i.e., locally near Y*

$$\psi(z)\sim \log\sum_k |g_k(z)| + O(1)$$

where (g_k) is a local system of generators of the ideal sheaf $\mathcal{I}_Y$ of Y in X.

(ii) *We have $\psi=\lim_{\varepsilon\to 0}\downarrow\psi_\varepsilon$ where the ψ_ε are C^∞ and possess a uniform Hessian estimate*

$$i\partial\overline{\partial}\psi_\varepsilon\ge -A\omega \qquad \text{on } X.$$

(iii) *Consider the family of hermitian metrics*

$$\omega_\varepsilon := \omega + \frac{1}{2A}i\partial\overline{\partial}\psi_\varepsilon \ge \frac{1}{2}\omega.$$

For any point $x_0\in Y$ and any neighborhood U of x_0, the volume element of ω_ε has a uniform lower bound

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^n \ge \delta(U)>0,$$

where $V_\varepsilon=\{z\in X\,;\ \psi(z)<\log\varepsilon\}$ is the "tubular neighborhood" of radius ε around Y.

(iv) *For every integer $p\ge 0$, the family of positive currents ω_ε^p is bounded in mass. Moreover, if Y contains an irreducible component Y' of codimension p, there is a uniform lower bound*

$$\int_{U\cap V_\varepsilon}\omega_\varepsilon^p\wedge\omega^{n-p}\ge \delta_p(U)>0$$

in any neighborhood U of a regular point $x_0\in Y'$. In particular, any weak limit Θ of ω_ε^p as ε tends to 0 satisfies $\Theta\ge\delta'[Y']$ for some $\delta'>0$.

Proof. By compactness of X, there is a covering of X by open coordinate balls B_j, $1\le j\le N$, such that $\mathcal{I}_Y$ is generated by finitely many holomorphic functions $(g_{j,k})_{1\le k\le m_j}$ on a neighborhood of $\overline{B}_j$. We take a partition of unity (θ_j) subordinate to (B_j) such that $\sum\theta_j^2=1$ on X, and define

$$\psi(z)=\frac{1}{2}\log\sum_j\theta_j(z)^2\sum_k|g_{j,k}(z)|^2,$$

$$\psi_\varepsilon(z)=\frac{1}{2}\log(e^{2\psi(z)}+\varepsilon^2)=\frac{1}{2}\log\Big(\sum_{j,k}\theta_j(z)^2|g_{j,k}(z)|^2+\varepsilon^2\Big).$$

Moreover, we consider the family of $(1,0)$-forms with support in B_j such that

$$\gamma_{j,k} = \theta_j \partial g_{j,k} + 2 g_{j,k} \partial\theta_j.$$

Straightforward calculations yield

$$\begin{aligned}
&\overline\partial\psi_\varepsilon = \frac{1}{2}\frac{\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2},\\
(3.5)\qquad &i\partial\overline\partial\psi_\varepsilon = \frac{i}{2}\bigg(\frac{\sum_{j,k}\gamma_{j,k}\wedge\overline{\gamma_{j,k}}}{e^{2\psi}+\varepsilon^2} - \frac{\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}\wedge\sum_{j,k}\theta_j g_{j,k}\overline{\gamma_{j,k}}}{(e^{2\psi}+\varepsilon^2)^2}\bigg),\\
&\qquad + i\,\frac{\sum_{j,k}|g_{j,k}|^2(\theta_j\partial\overline\partial\theta_j - \partial\theta_j\wedge\overline\partial\theta_j)}{e^{2\psi}+\varepsilon^2}.
\end{aligned}$$

As $e^{2\psi} = \sum_{j,k}\theta_j^2|g_{j,k}|^2$, the first big sum in $i\partial\overline\partial\psi_\varepsilon$ is nonnegative by the Cauchy-Schwarz inequality; when viewed as a hermitian form, the value of this sum on a tangent vector $\xi\in T_X$ is simply

$$(3.6)\quad \frac{1}{2}\Bigg(\frac{\sum_{j,k}|\gamma_{j,k}(\xi)|^2}{e^{2\psi}+\varepsilon^2} - \frac{\big|\sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)\big|^2}{(e^{2\psi}+\varepsilon^2)^2}\Bigg) \ge \frac{1}{2}\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^2}\sum_{j,k}|\gamma_{j,k}(\xi)|^2.$$

Now, the second sum involving $\theta_j\partial\overline\partial\theta_j - \partial\theta_j\wedge\overline\partial\theta_j$ in (3.5) is uniformly bounded below by a fixed negative hermitian form $-A\omega$, $A\gg 0$, and therefore $i\partial\overline\partial\psi_\varepsilon \ge -A\omega$. Actually, for every pair of indices (j,j') we have a bound

$$C^{-1} \le \sum_k |g_{j,k}(z)|^2 / \sum_k |g_{j',k}(z)|^2 \le C \qquad \text{on } \overline B_j\cap\overline B_{j'},$$

since the generators $(g_{j,k})$ can be expressed as holomorphic linear combinations of the $(g_{j',k})$ by Cartan's theorem A (and vice versa). It follows easily that all terms $|g_{j,k}|^2$ are uniformly bounded by $e^{2\psi}+\varepsilon^2$. In particular, ψ and ψ_ε are quasi-plurisubharmonic, and we see that (i) and (ii) hold true. By construction, the real $(1,1)$-form $\omega_\varepsilon := \omega + \frac{1}{2A}i\partial\overline\partial\psi_\varepsilon$ satisfies $\omega_\varepsilon \ge \frac{1}{2}\omega$, hence it is Kähler and its eigenvalues with respect to ω are at least equal to $1/2$.

Assume now that we are in a neighborhood U of a regular point $x_0\in Y$ where Y has codimension p. Then $\gamma_{j,k} = \theta_j\partial g_{j,k}$ at x_0, hence the rank of the system of $(1,0)$-forms $(\gamma_{j,k})_{k\ge1}$ is at least equal to p in a neighborhood of x_0. Fix a holomorphic local coordinate system $(z_1,\ldots,z_n)$ such that $Y=\{z_1=\ldots=z_p=0\}$ near x_0, and let $S\subset T_X$ be the holomorphic subbundle generated by $\partial/\partial z_1,\ldots,\partial/\partial z_p$. This choice ensures that the rank of the system of $(1,0)$-forms $(\gamma_{j,k|S})$ is everywhere equal to p. By (1,3) and the minimax principle applied to the p-dimensional subspace $S_z\subset T_{X,z}$, we see that the p-largest eigenvalues of ω_ε are bounded below by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$.

However, we can even restrict the form defined in (3.6) to the $(p-1)$-dimensional subspace $S\cap\operatorname{Ker}\tau$ where $\tau(\xi) := \sum_{j,k}\theta_j\overline{g_{j,k}}\gamma_{j,k}(\xi)$, to see that the $(p-1)$-largest eigenvalues of ω_ε are bounded below by $c/(e^{2\psi}+\varepsilon^2)$, $c>0$. The p-th eigenvalue is then bounded by $c\varepsilon^2/(e^{2\psi}+\varepsilon^2)^2$ and the remaining $(n-p)$-ones by $1/2$. From this we infer

$$\begin{aligned}
\omega_\varepsilon^n &\ge c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\omega^n \qquad \text{near } x_0,\\
\omega_\varepsilon^p &\ge c\,\frac{\varepsilon^2}{(e^{2\psi}+\varepsilon^2)^{p+1}}\Big(i\sum_{1\le\ell\le p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}}\Big)^p
\end{aligned}$$

where $(\gamma_{j,k_\ell})_{1\le\ell\le p}$ is a suitable p-tuple extracted from the $(\gamma_{j,k})$, such that $\bigcap_\ell \operatorname{Ker}\gamma_{j,k_\ell}$ is a smooth complex (but not necessarily holomorphic) subbundle of codimension p of T_X; by the definition of the forms $\gamma_{j,k}$, this subbundle must coincide with T_Y along Y. From this, properties (iii) and (iv) follow easily; actually, up to constants, we have $e^{2\psi}+\varepsilon^2\sim|z_1|^2+\ldots+|z_p|^2+\varepsilon^2$ and

$$i\sum_{1\le\ell\le p}\gamma_{j,k_\ell}\wedge\overline{\gamma_{j,k_\ell}}\ge c\,i\partial\overline{\partial}(|z_1|^2+\ldots+|z_p|^2)-O(\varepsilon)i\partial\overline{\partial}|z|^2\qquad\text{on } U\cap V_\varepsilon,$$

hence, by a straightforward calculation,

$$\omega_\varepsilon^p\wedge\omega^{n-p}\ge c\big(i\partial\overline{\partial}\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\big)^p\wedge\big(i\partial\overline{\partial}(|z_{p+1}|^2+\ldots+|z_n|^2)\big)^{n-p}$$

on $U\cap V_\varepsilon$; notice also that $\omega_\varepsilon^n\ge 2^{-(n-p)}\omega_\varepsilon^p\wedge\omega^{n-p}$, so any lower bound for the volume of $\omega_\varepsilon^p\wedge\omega^{n-p}$ will also produce a bound for the volume of ω_ε^n. As it is well known, the (p,p)-form

$$\Big(\frac{i}{2\pi}\partial\overline{\partial}\log(|z_1|^2+\ldots+|z_p|^2+\varepsilon^2)\Big)^p\qquad\text{on } \mathbb{C}^n$$

can be viewed as the pull-back to $\mathbb{C}^n=\mathbb{C}^p\times\mathbb{C}^{n-p}$ of the Fubini-Study volume form of the complex p-dimensional projective space of dimension p containing $\mathbb{C}^p$ as an affine Zariski open set, rescaled by the dilation ratio ε. Hence it converges weakly to the current of integration on the p-codimensional subspace $z_1=\ldots=z_p=0$. Moreover the volume contained in any compact tubular cylinder

$$\{|z'|\le C\varepsilon\}\times K''\subset\mathbb{C}^p\times\mathbb{C}^{n-p}$$

depends only on C and K (as one sees after rescaling by ε). The fact that ω_ε^p is uniformly bounded in mass can be seen easily from the fact that

$$\int_X\omega_\varepsilon^p\wedge\omega^{n-p}=\int_X\omega^n,$$

as ω and ω_ε are in the same Kähler class. Let Θ be any weak limit of ω_ε^p. By what we have just seen, Θ carries non zero mass on every p-codimensional component Y' of Y, for instance near every regular point. However, standard results of the theory of currents (support theorem and Skoda's extension result) imply that $\mathbf{1}_{Y'}\Theta$ is a closed positive current and that $\mathbf{1}_{Y'}\Theta=\lambda[Y']$ is a nonnegative multiple of the current of integration on Y'. The fact that the mass of Θ on Y' is positive yields $\lambda>0$. Lemma 3.4 is proved. □

4. Multiplier Ideal Sheaves

We now introduce the concept of *multiplier ideal sheaf*, following A. Nadel [Nad89]. The main idea actually goes back to the fundamental works of Bombieri [Bom70] and H. Skoda [Sko72a].

(4.1) Definition. *Let φ be a psh function on an open subset $\Omega\subset X$; to φ is associated the ideal subsheaf $\mathcal{I}(\varphi)\subset\mathcal{O}_\Omega$ of germs of holomorphic functions $f\in\mathcal{O}_{\Omega,x}$ such that*

$|f|^2e^{-\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x.

The zero variety $V(\mathcal{I}(\varphi))$ is thus the set of points in a neighborhood of which $e^{-\varphi}$ is non integrable. Of course, such points occur only if φ has logarithmic poles. This is made precise as follows.

(4.2) Definition. *A psh function φ is said to have a logarithmic pole of coefficient γ at a point $x \in X$ if the Lelong number*

$$\nu(\varphi, x) := \liminf_{z\to x} \frac{\varphi(z)}{\log|z-x|}$$

is non zero and if $\nu(\varphi, x) = \gamma$.

(4.3) Lemma (Skoda [Sko72a]). *Let φ be a psh function on an open set Ω and let $x \in \Omega$.*

a) *If $\nu(\varphi, x) < 2$, then $e^{-\varphi}$ is integrable in a neighborhood of x, in particular $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega,x}$.*

b) *If $\nu(\varphi, x) \geq 2(n+s)$ for some integer $s \geq 0$, then $e^{-\varphi} \geq C|z-x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\varphi)_x \subset \mathfrak{m}^{s+1}_{\Omega,x}$, where $\mathfrak{m}_{\Omega,x}$ is the maximal ideal of $\mathcal{O}_{\Omega,x}$.*

c) *The zero variety $V(\mathcal{I}(\varphi))$ of $\mathcal{I}(\varphi)$ satisfies*

$$E_{2n}(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_2(\varphi)$$

where $E_c(\varphi) = \{x \in X\,;\, \nu(\varphi, x) \geq c\}$ is the c-sublevel set of Lelong numbers of φ.

Proof. a) Set $\Theta = dd^c\varphi$ and $\gamma = \nu(\Theta, x) = \nu(\varphi, x)$. Let χ be a cut-off function with support in a small ball $B(x, r)$, equal to 1 in $B(x, r/2)$. As $(dd^c \log|z|)^n = \delta_0$, we get

$$\begin{aligned}\varphi(z) &= \int_{B(x,r)} \chi(\zeta)\varphi(\zeta)(dd^c \log|\zeta - z|)^n \\ &= \int_{B(x,r)} dd^c(\chi(\zeta)\varphi(\zeta)) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1}\end{aligned}$$

for $z \in B(x, r/2)$. Expanding $dd^c(\chi\varphi)$ and observing that $d\chi = dd^c\chi = 0$ on $B(x, r/2)$, we find

$$\varphi(z) = \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge \log|\zeta - z|(dd^c \log|\zeta - z|)^{n-1} + \text{smooth terms}$$

on $B(x, r/2)$. Fix r so small that

$$\int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - x|)^{n-1} \leq \nu(\Theta, x, r) < 2.$$

By continuity, there exists $\delta, \varepsilon > 0$ such that

$$I(z) := \int_{B(x,r)} \chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1} \le 2 - \delta$$

for all $z \in B(x, \varepsilon)$. Applying Jensen's convexity inequality to the probability measure

$$d\mu_z(\zeta) = I(z)^{-1}\chi(\zeta)\Theta(\zeta) \wedge (dd^c \log|\zeta - z|)^{n-1},$$

we find

$$-\varphi(z) = \int_{B(x,r)} I(z) \log|\zeta - z|^{-1}\, d\mu_z(\zeta) + O(1) \quad \Longrightarrow$$
$$e^{-\ varphi(z)} \le C \int_{B(x,r)} |\zeta - z|^{-I(z)}\, d\mu_z(\zeta).$$

As

$$d\mu_z(\zeta) \le C_1|\zeta - z|^{-(2n-2)}\Theta(\zeta) \wedge (dd^c|\zeta|^2)^{n-1} = C_2|\zeta - z|^{-(2n-2)} d\sigma_\Theta(\zeta),$$

we get

$$e^{-\varphi(z)} \le C_3 \int_{B(x,r)} |\zeta - z|^{-2+\delta-(2n-2)} d\sigma_\Theta(\zeta),$$

and the Fubini theorem implies that $e^{-2\varphi(z)}$ is integrable on a neighborhood of x.

b) If $\nu(\varphi, x) = \gamma$, the convexity properties of psh functions, namely, the convexity of $\log r \mapsto \sup_{|z-x|=r} \varphi(z)$ implies that

$$\varphi(z) \le \gamma \log|z - x|/r_0 + M,$$

where M is the supremum on $B(x, r_0)$. Hence there exists a constant $C > 0$ such that $e^{-2\varphi(z)} \ge C|z - x|^{-2\gamma}$ in a neighborhood of x. The desired result follows from the identity

$$\int_{B(0,r_0)} \frac{\left|\sum a_\alpha z^\alpha\right|^2}{|z|^{2\gamma}} dV(z) = \text{Const} \int_0^{r_0} \Big(\sum |a_\alpha|^2 r^{2|\alpha|}\Big) r^{2n-1-2\gamma}\, dr,$$

which is an easy consequence of Parseval's formula. In fact, if γ has integral part $[\gamma] = n + s$, the integral converges if and only if $a_\alpha = 0$ for $|\alpha| \le s$.

c) is just a simple formal consequence of a) and b). □

(4.3) Proposition ([Nad89])**.** *For any psh function φ on $\Omega \subset X$, the sheaf $\mathcal{I}(\varphi)$ is a coherent sheaf of ideals over Ω. Moreover, if Ω is a bounded Stein open set, the sheaf $\mathcal{I}(\varphi)$ is generated by any Hilbert basis of the L^2 space $\mathcal{H}^2(\Omega, \varphi)$ of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-\varphi}\, d\lambda < +\infty$.*

Proof. Since the result is local, we may assume that Ω is a bounded pseudoconvex open set in $\mathbb{C}^n$. By the strong noetherian property of coherent sheaves, the family of sheaves generated by finite subsets of $\mathcal{H}^2(\Omega, \varphi)$ has a maximal element on each compact subset of Ω, hence $\mathcal{H}^2(\Omega, \varphi)$ generates a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_\Omega$. It is clear that $\mathcal{J} \subset \mathcal{I}(\varphi)$; in order to prove the equality, we need only check that $\mathcal{J}_x + \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1} = \mathcal{I}(\varphi)_x$ for every integer s, in view of the Krull lemma. Let

$f \in \mathcal{I}(\varphi)_x$ be defined in a neighborhood V of x and let θ be a cut-off function with support in V such that $\theta = 1$ in a neighborhood of x. We solve the equation $d''u = g := d''(\theta f)$ by means of Hörmander's L^2 estimates 3.3, where F is the trivial line bundle $\Omega \times \mathbb{C}$ equipped with the strictly psh weight

$$\widetilde{\varphi}(z) = \varphi(z) + 2(n+s)\log|z-x| + |z|^2.$$

We get a solution u such that $\int_\Omega |u|^2 e^{-\varphi}|z-x|^{-2(n+s)}d\lambda < \infty$, thus $F = \theta f - u$ is holomorphic, $F \in \mathcal{H}^2(\Omega, \varphi)$ and $f_x - F_x = u_x \in \mathcal{I}(\varphi)_x \cap \mathfrak{m}_{\Omega,x}^{s+1}$. This proves the coherence. Now, $\mathcal{J}$ is generated by any Hilbert basis of $\mathcal{H}^2(\Omega, \varphi)$, because it is well-known that the space of sections of any coherent sheaf is a Fréchet space, therefore closed under local L^2 convergence. □

The multiplier ideal sheaves satisfy the following basic functoriality property with respect to direct images of sheaves by modifications.

(4.5) Proposition. *Let $\mu : X' \to X$ be a modification of non singular complex manifolds (i.e. a proper generically 1:1 holomorphic map), and let φ be a psh function on X. Then*

$$\mu_\star\big(\mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu)\big) = \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi).$$

Proof. Let $n = \dim X = \dim X'$ and let $S \subset X$ be an analytic set such that $\mu : X' \smallsetminus S' \to X \smallsetminus S$ is a biholomorphism. By definition of multiplier ideal sheaves, $\mathcal{O}(K_X) \otimes \mathcal{I}(\varphi)$ is just the sheaf of holomorphic n-forms f on open sets $U \subset X$ such that $\mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-\varphi} \in L^1_{\mathrm{loc}}(U)$. Since φ is locally bounded from above, we may even consider forms f which are a priori defined only on $U \smallsetminus S$, because f will be in $L^2_{\mathrm{loc}}(U)$ and therefore will automatically extend through S. The change of variable formula yields

$$\int_U \mathrm{i}^{n^2} f \wedge \overline{f}\, e^{-\varphi} = \int_{\mu^{-1}(U)} \mathrm{i}^{n^2} \mu^\star f \wedge \overline{\mu^\star f}\, e^{-\varphi\circ\mu},$$

hence $f \in \Gamma(U, \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi))$ iff $\mu^\star f \in \Gamma(\mu^{-1}(U), \mathcal{O}(K_{X'}) \otimes \mathcal{I}(\varphi \circ \mu))$. Proposition 4.5 is proved. □

(4.6) Remark. If φ has analytic singularities, i.e. if there are holomorphic functions $(f_j)_{1 \le j \le N}$ and a constant $\alpha > 0$ such that

$$\varphi(z) = \frac{\alpha}{2}\log(|f_1(z)|^2 + \ldots + |f_N(z)|^2) + O(1)$$

in a neighborhood of every point, the computation of $\mathcal{I}(\varphi)$ can be reduced to a purely algebraic problem.

The first observation is that $\mathcal{I}(\varphi)$ can be computed easily if φ has the form $\varphi = \sum \alpha_j \log|g_j|$ where $D_j = g_j^{-1}(0)$ are nonsingular irreducible divisors with normal crossings. Then $\mathcal{I}(\varphi)$ is the sheaf of functions h on open sets $U \subset X$ such that

$$\int_U |h|^2 \prod |g_j|^{-2\alpha_j} dV < +\infty.$$

Since locally the g_j can be taken to be coordinate functions from a local coordinate system $(z_1,\ldots,z_n)$, the condition is that h is divisible by $\prod g_j^{m_j}$ where $m_j-\alpha_j>-1$ for each j, i.e. $m_j\ge\lfloor\alpha_j\rfloor$ (integer part). Hence

$$\mathcal{I}(\varphi)=\mathcal{O}(-\lfloor D\rfloor)=\mathcal{O}(-\sum\lfloor\alpha_j\rfloor D_j)$$

where $\lfloor D\rfloor$ denotes the integral part of the $\mathbb{Q}$-divisor $D=\sum\alpha_jD_j$.

Now, consider the general case of analytic singularities and suppose that $\varphi\sim\frac{\alpha}{2}\log\big(|f_1|^2+\cdots+|f_N|^2\big)$ near the poles. By the remarks after Definition 1.10, we may assume that the (f_j) are generators of the integrally closed ideal sheaf $\mathcal{J}=\mathcal{J}(\varphi/\alpha)$, defined as the sheaf of holomorphic functions h such that $|h|\le C\exp(\varphi/\alpha)$. In this case, the computation is made as follows (see also L. Bonavero's work [Bon93], where similar ideas are used in connection with "singular" holomorphic Morse inequalities).

First, one computes a smooth modification $\mu:\widetilde{X}\to X$ of X such that $\mu^\star\mathcal{J}$ is an invertible sheaf $\mathcal{O}(-D)$ associated with a normal crossing divisor $D=\sum\lambda_jD_j$, where (D_j) are the components of the exceptional divisor of $\widetilde{X}$ (take the blow-up X' of X with respect to the ideal $\mathcal{J}$ so that the pull-back of $\mathcal{J}$ to X' becomes an invertible sheaf $\mathcal{O}(-D')$, then blow up again by Hironaka [Hir64] to make X' smooth and D' have normal crossings). Now, we have $K_{\widetilde{X}}=\mu^\star K_X+R$ where $R=\sum\rho_jD_j$ is the zero divisor of the Jacobian function J_μ of the blow-up map. By the direct image formula 4.5, we get

$$\mathcal{I}(\varphi)=\mu_\star\big(\mathcal{O}(K_{\widetilde{X}}-\mu^\star K_X)\otimes\mathcal{I}(\varphi\circ\mu)\big)=\mu_\star\big(\mathcal{O}(R)\otimes\mathcal{I}(\varphi\circ\mu)\big).$$

Now, $(f_j\circ\mu)$ are generators of the ideal $\mathcal{O}(-D)$, hence

$$\varphi\circ\mu\sim\alpha\sum\lambda_j\log|g_j|$$

where g_j are local generators of $\mathcal{O}(-D_j)$. We are thus reduced to computing multiplier ideal sheaves in the case where the poles are given by a $\mathbb{Q}$-divisor with normal crossings $\sum\alpha\lambda_jD_j$. We obtain $\mathcal{I}(\varphi\circ\mu)=\mathcal{O}(-\sum\lfloor\alpha\lambda_j\rfloor D_j)$, hence

$$\mathcal{I}(\varphi)=\mu_\star\mathcal{O}_{\widetilde{X}}\big(\sum(\rho_j-\lfloor\alpha\lambda_j\rfloor)D_j\big).\qquad\square$$

(4.7) Exercise. Compute the multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with $\varphi=\log(|z_1|^{\alpha_1}+\ldots+|z_p|^{\alpha_p})$ for arbitrary real numbers $\alpha_j>0$.
Hint: using Parseval's formula and polar coordinates $z_j=r_je^{\mathrm{i}\theta_j}$, show that the problem is equivalent to determining for which p-tuples $(\beta_1,\ldots,\beta_p)\in\mathbb{N}^p$ the integral

$$\int_{[0,1]^p}\frac{r_1^{2\beta_1}\ldots r_p^{2\beta_p}\,r_1dr_1\ldots r_pdr_p}{r_1^{2\alpha_1}+\ldots+r_p^{2\alpha_p}}=\int_{[0,1]^p}\frac{t_1^{(\beta_1+1)/\alpha_1}\ldots t_p^{(\beta_p+1)/\alpha_p}}{t_1+\ldots+t_p}\frac{dt_1}{t_1}\ldots\frac{dt_p}{t_p}$$

is convergent. Conclude from this that $\mathcal{I}(\varphi)$ is generated by the monomials $z_1^{\beta_1}\ldots z_p^{\beta_p}$ such that $\sum(\beta_p+1)/\alpha_p>1$. (This exercise shows that the analytic definition of $\mathcal{I}(\varphi)$ is sometimes also quite convenient for computations). $\square$

Let F be a line bundle over X with a singular metric h of curvature current $\Theta_h(F)$. If φ is the weight representing the metric in an open set $\Omega \subset X$, the ideal sheaf $\mathcal{I}(\varphi)$ is independent of the choice of the trivialization and so it is the restriction to Ω of a global coherent sheaf $\mathcal{I}(h)$ on X. We will sometimes still write $\mathcal{I}(h) = \mathcal{I}(\varphi)$ by abuse of notation. In this context, we have the following fundamental vanishing theorem, which is probably one of the most central results of analytic and algebraic geometry (as we will see later, it contains the Kawamata-Viehweg vanishing theorem as a special case).

(4.8) Nadel vanishing theorem ([Nad89], [Dem93b]). *Let (X, ω) be a Kähler weakly pseudoconvex manifold, and let F be a holomorphic line bundle over X equipped with a singular hermitian metric h of weight φ. Assume that $\mathrm{i}\Theta_h(F) \geq \varepsilon\omega$ for some continuous positive function ε on X. Then*

$$H^q\big(X, \mathcal{O}(K_X + F) \otimes \mathcal{I}(h)\big) = 0 \qquad \textit{for all } q \geq 1.$$

Proof. Let $\mathcal{L}^q$ be the sheaf of germs of (n, q)-forms u with values in F and with measurable coefficients, such that both $|u|^2 e^{-\varphi}$ and $|d''u|^2 e^{-\varphi}$ are locally integrable. The d'' operator defines a complex of sheaves $(\mathcal{L}^\bullet, d'')$ which is a resolution of the sheaf $\mathcal{O}(K_X + F) \otimes \mathcal{I}(\varphi)$: indeed, the kernel of d'' in degree 0 consists of all germs of holomorphic n-forms with values in F which satisfy the integrability condition; hence the coefficient function lies in $\mathcal{I}(\varphi)$; the exactness in degree $q \geq 1$ follows from Corollary 3.3 applied on arbitrary small balls. Each sheaf $\mathcal{L}^q$ is a $\mathcal{C}^\infty$-module, so $\mathcal{L}^\bullet$ is a resolution by acyclic sheaves. Let ψ be a smooth psh exhaustion function on X. Let us apply Corollary 3.3 globally on X, with the original metric of F multiplied by the factor $e^{-\chi\circ\psi}$, where χ is a convex increasing function of arbitrary fast growth at infinity. This factor can be used to ensure the convergence of integrals at infinity. By Corollary 3.3, we conclude that $H^q\big(\Gamma(X, \mathcal{L}^\bullet)\big) = 0$ for $q \geq 1$. The theorem follows. □

(4.9) Corollary. *Let (X, ω), F and φ be as in Theorem 4.8 and let $x_1, \ldots, x_N$ be isolated points in the zero variety $V(\mathcal{I}(\varphi))$. Then there is a surjective map*

$$H^0(X, K_X + F) \twoheadrightarrow \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X + L)_{x_j} \otimes \big(\mathcal{O}_X/\mathcal{I}(\varphi)\big)_{x_j}.$$

Proof. Consider the long exact sequence of cohomology associated to the short exact sequence $0 \to \mathcal{I}(\varphi) \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}(\varphi) \to 0$ twisted by $\mathcal{O}(K_X + F)$, and apply Theorem 4.8 to obtain the vanishing of the first H^1 group. The asserted surjectivity property follows. □

(4.10) Corollary. *Let (X, ω), F and φ be as in Theorem 4.8 and suppose that the weight function φ is such that $\nu(\varphi, x) \geq 2(n + s)$ at some point $x \in X$ which is an isolated point of $E_2(\varphi)$. Then $H^0(X, K_X + F)$ generates all s-jets at x.*

Proof. The assumption is that $\nu(\varphi, y) < 2$ for y near x, $y \neq x$. By Skoda's lemma 4.3 b), we conclude that $e^{-\varphi}$ is integrable at all such points y, hence $\mathcal{I}(\varphi)_y = \mathcal{O}_{X,y}$, whilst $\mathcal{I}(\varphi)_x \subset \mathfrak{m}_{X,x}^{s+1}$ by 4.3 a). Corollary 4.10 is thus a special case of 4.9. □

The philosophy of these results (which can be seen as generalizations of the Hörmander-Bombieri-Skoda theorem [Bom70], [Sko72a, 75]) is that the problem of constructing holomorphic sections of $K_X + F$ can be solved by constructing suitable hermitian metrics on F such that the weight φ has isolated poles at given points x_j. The following result gives a somewhat general result in this direction.

(4.11) Theorem. *Let X be a compact complex manifold, E a holomorphic vector bundle and (F, h_F) a hermitian line bundle with a smooth metric h such that*

$$\omega = \mathrm{i}\Theta_{h_F}(F) > 0.$$

Let φ be a quasi-psh function on X, i.e. a function φ such that $\mathrm{i}d'd''\varphi \geq -C\omega$ for some constant $C > 0$. Then

a) *There exists an integer m_0 such that*

$$H^q(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)) = 0$$

for $q \geq 1$ and $m \geq m_0$.

b) *The restriction map*

$$H^0(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)) \longrightarrow H^0(X, E \otimes F^{\otimes m} \otimes \mathcal{O}_X/\mathcal{I}(\varphi))$$

is surjective for $m \geq m_0$.

b) *The vector bundle $E \otimes F^{\otimes m}$ generates its sections (or jets of any order s) for $m \geq m_0(s)$ large enough.*

Proof. a) Put an arbitrary smooth hermitian metric h_E en E and consider the singular hermitian metric $h_E \cdot h_F^m \cdot e^{-\varphi}$ on $E \otimes F^{\otimes m}$. The L^2 holomorphic sections of $E \otimes F^{\otimes m}$ are exactly the sections of the sheaf $E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi)$. On the other hand, the curvature of the metric is

$$\mathrm{i}\Theta_{h_E}(E) + (m\,\mathrm{i}\Theta_{h_F}(F) + \mathrm{i}d'd''\varphi) \otimes \mathrm{Id}_E$$

and therefore the curvature is Nakano > 0 for m large. This implies the vanishing of $H^q(X, E \otimes F^{\otimes m} \otimes \mathcal{I}(\varphi))$.

b) The vanishing of H^1 itself implies the surjevtivity statement on the H^0 groups by the same argument as in Corollary (4.9).

c) Clearly, one can construct a quasi-psh function φ with a single logarithmic pole at a point $x \in X$ by taking

$$\varphi(z) = \theta(z)(n + s - 1) \log \sum |z_j - x_j|^2$$

in some local coordinates near x, where θ is a cut-off function with support in the coordinate open set. Then $\mathcal{I}(\varphi) = \mathfrak{m}_x^s$ and we conclude by b). □

(4.12) Remark. Assume that X is compact and that F is a positive line bundle on X. Let $\{x_1, \ldots, x_N\}$ be a finite set. Show that there are constants $C_1, C_2 \geq 0$ depending only on X and E, F such that $H^0(X, E \otimes F^{\otimes m})$ interpolates given jets of order s_j at x_j for $m \geq C_1 \sum s_j + C_2$. To see this, we take a quasi-psh weight

$$\varphi(z) = \sum \theta_j(z)(n + s_j - 1) \log |w^{(j)}(z)|$$

with respect to coordinate systems $(w_k^{(j)}(z))_{1 \leq k \leq n}$ centered at x_j. The cut-off functions can be taken of a fixed radius (bounded away from 0) with respect to a finite collection of coordinate patches covering X. It is then easy to see that $\mathrm{i}d'd''\varphi \geq -C(\sum s_j + 1)\omega$. □

(4.13) Theorem (Kodaira [Kod54]). *Let X be a compact complex manifold. A line bundle L on X is ample if and only if L is positive. In particular, a manifold X posessing a positive line bundle is projective, and can be embedded in projective space via the canonical map $\Phi_{|mL|} : X \to \mathbb{P}^N$ for N large.*

Proof. If the line bundle L is ample, then by definition the canonical map $\Phi_{|mL|} : X \to \mathbb{P}^N$ is an embedding for m large, and $mL = \Phi_{|mL|}^{-1}\mathcal{O}(1)$. This implies that L can be equipped with a metric of positive curvature, as we saw in (1.15). Conversely, if L possesses a metric with positive curvature, then by Theorem 4.11 c), there exists an integer m_0 such that for $m \geq m_0$ sections in $H^0(X, mL)$ separate any pair of points $\{x, y\} \subset X$ and generate 1-jets of sections at every point $x \in X$. However, as is easily seen, separation of points is equivalent to the injectivity of the map $\Phi_{|mL|}$, and the generation of 1-jets is equivalent to the fact that $\Phi_{|mL}$ is an immersion. □

(4.14) Theorem (Chow [Chw49]). *Let $X \subset \mathbb{P}^N$ be a* (*closed*) *complex analytic subset of $\mathbb{P}^N$. Then X is algebraic and can be defined as the common zero set of a finite collection of homogeneous polynomials $P_j(z_0, z_1, \ldots, z_N) = 0$, $1 \leq j \leq k$.*

Proof. By Lemma 3.4 (i), there exists a quasi psh function ψ with logarithmic poles along X. Then $\mathcal{I}(2N\psi) \subset \mathcal{I}_X$, since $e^{-2N\psi}$ is certainly not integrable along Z. For $x \in \mathbb{P}^N \smallsetminus X$, we consider a quasi-psh weight

$$\varphi_x(z) = \psi(z) + \theta_x(z)(2N) \log |z - x|$$

with a suitable cut-off function with support on a neighborhood of x possessing holomorphic coordinates, in such a way that $\mathcal{I}(\varphi_x)_x = \mathfrak{m}_{\mathbb{P}^N,x}$. We can arrange that $\mathrm{i}\partial\overline{\partial}\varphi_x \geq -C\omega$ uniformly for all x. Then we have vanishing of $H^1(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}(\varphi_x))$ for $m \geq m_0$ and therefore we get a surjective map

$$H^0(\mathbb{P}^N, \mathcal{O}(m)) \to H^0(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{O}_{\mathbb{P}^N}/\mathcal{I}(\varphi_x)).$$

As a consequence, we can find a homogeneous polynomial of degree m which takes value 1 at x and vanishes on X (as prescribed by the ideal sheaf $\mathcal{I}(\psi)$). □

In the case of submanifolds, one can of course prove a slightly more precise result, by demanding that the polynomials P_j have non vanishing differentials along X.

(4.15) Theorem (Chow [Chw49]). *Let $X \subset \mathbb{P}^N$ be a complex analytic submanifold of dimension n. Then X is projective algebraic and can be defined by a collection of homogeneous polynomials $P_j(z_0, z_1, \ldots, z_N) = 0$, $1 \le j \le k$, such that the system of differentials (dP_j) has rank equal to* $\operatorname{codim} X$ *at every point of X.*

Proof. Put $r = \operatorname{codim} X = N - n$. There exists a quasi-psh function φ which has logarithmic poles along X. To see this, just take an open overing of $\mathbb{P}^N$ by open sets U_j where X is defined by $w_1^{(j)} = \ldots = w_r^{(j)} = 0$ and $(w_k^{(j)})_{1 \le k \le N}$ is a suitable coordinate system on U_j. As in the proof of Lemma 3.4, the function

$$\varphi(z) = \log\Big(\sum \theta_j^2(z)\big(|w_1^{(j)}|^2 + \ldots + |w_r^{(j)}|^2\big)\Big)$$

is quasi-psh if the functions θ_j are cut-off functions with support in U_j such that $\sum \theta_j^2 = 1$. An easy calculation also shows that $\mathcal{I}((r+1)\varphi) = \mathcal{I}_X^2$ where $\mathcal{I}_X$ is the reduced ideal sheaf of X in $\mathbb{P}^N$. Hence for $m \ge m_0$ large enough, we have

$$H^q(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X^2) = 0$$

for $q \ge 1$. The long-exact sequence associated with

$$0 \to \mathcal{I}_X^2 \to \mathcal{I}_X \to \mathcal{I}_X/\mathcal{I}_X^2 \to 0$$

twisted by $\mathcal{O}(m)$ implies the surjectivity of

$$H^0(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X) \to H^0(X, \mathcal{O}(m) \otimes \mathcal{I}_X/\mathcal{I}_X^2).$$

However, $\mathcal{I}_X/\mathcal{I}_X^2$ can be identified with the conormal bundle N_X^* (where $N_X = T_{\mathbb{P}^N|X}/T_X$), and we infer from this that $H^0(X, \mathcal{O}(m) \otimes \mathcal{I}_X/\mathcal{I}_X^2)$ is generated by its global sections for $m \ge m_1$ large enough by Theorem 4.11 c). This means that we can generate any 1-differential of N_X^* at any point $x \in X$ as the differential of a section

$$P \in H^0(\mathbb{P}^N, \mathcal{O}(m) \otimes \mathcal{I}_X) \subset H^0(\mathbb{P}^N, \mathcal{O}(m)),$$

i.e. a homogeneous polynomial of degree m vanishing on X, for $m \ge \max(m_0, m_1)$. □

(4.16) Corollary (Kodaira [Kod54]). *Let X be a compact complex manifold and* $\dim_{\mathbb{C}} X = n$. *The following conditions are equivalent.*

a) *X is projective algebraic, i.e. X can be embedded as an algebraic submanifold of the complex projective space $\mathbb{P}^N$ for N large.*

b) *X carries a positive line bundle L.*

c) *X carries a Hodge metric, i.e. a Kähler metric ω with rational cohomology class $\{\omega\} \in H^2(X, \mathbb{Q})$.*

Proof. a) $\Leftrightarrow$ b). This follows from Theorems 4.13 and 4.15 combined.

b) $\Rightarrow$ c). Take $\omega = \frac{\mathrm{i}}{2\pi}\Theta(L)$; then $\{\omega\}$ is the image of $c_1(L) \in H^2(X, \mathbb{Z})$.

c) $\Rightarrow$ b). We can multiply $\{\omega\}$ by a common denominator of its coefficients and suppose that $\{\omega\}$ is in the image of $H^2(X,\mathbb{Z})$. Then a classical result due to A. Weil shows that there exists a hermitian line bundle (L,h) such that $\frac{\mathrm{i}}{2\pi}\Theta_h(L)=\omega$. This bundle L is then positive. □

(4.17) Exercise (solution of the Levi problem)**.** Show that the following two properties are equivalent.

a) X is strongly pseudoconvex, i.e. X admits a strongly psh exhaustion function.

b) X is Stein, i.e. the global holomorphic functions $H^0(X,\mathcal{O}_X)$ separate points and yield local coordinates at any point, and X is holomorphically convex (this means that for any discrete sequence z_ν there is a function $f\in H^0(X,\mathcal{O}_X)$ such that $|f(z_\nu)|\to\infty$). □

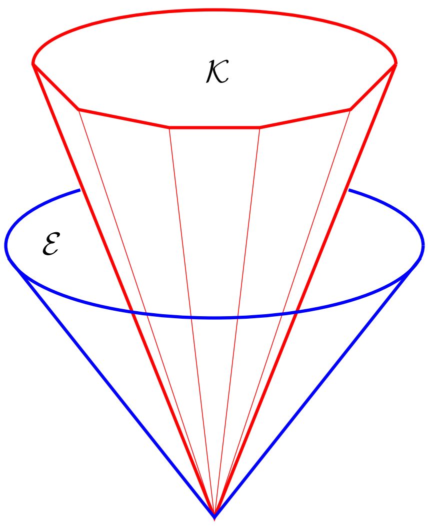
5. Nef and Pseudo-Effective Cones

We now introduce important concepts of positivity for cohomology classes of type $(1,1)$.

(5.1) Definition. *Let X be a compact Kähler manifold.*

(i) *The Kähler cone is the set $\mathcal{K}\subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an* <u>*open convex cone*</u>*.*

(ii) *The pseudo-effective cone is the set $\mathcal{E}\subset H^{1,1}(X,\mathbb{R})$ of cohomology classes $\{T\}$ of closed positive currents of type $(1,1)$. This is a* <u>*closed convex cone*</u>*.*



$\overline{\mathcal{K}}$ = nef cone in $H^{1,1}(X,\mathbb{R})$

$\mathcal{E}$ = pseudo-effective cone in $H^{1,1}(X,\mathbb{R})$

The openness of $\mathcal{K}$ is clear by definition, and the closedness of $\mathcal{E}$ follows from the fact that bounded sets of currents are weakly compact (as follows from the similar weak compacteness property for bounded sets of positive measures). It is then clear that $\overline{\mathcal{K}}\subset\mathcal{E}$.

In spite of the fact that cohomology groups can be defined either in terms of forms or currents, it turns out that the cones $\overline{\mathcal{K}}$ and $\mathcal{E}$ are in general different. To see this, it is enough to observe that a Kähler class $\{\alpha\}$ satisfies $\int_Y \alpha^p > 0$ for every p-dimensional analytic set. On the other hand, if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisopr $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

In case X is projective, it is interesting to consider also the algebraic analogues of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

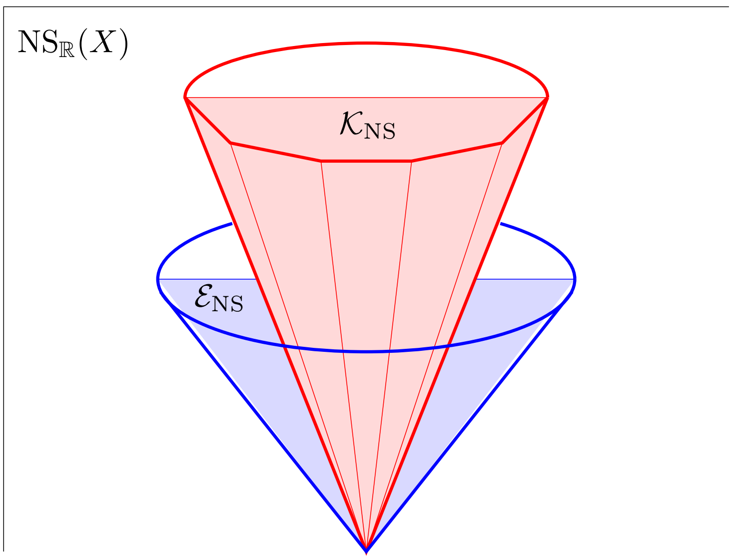
$$\begin{aligned}\mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big),\\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.\end{aligned}$$

All classes of real divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}$, lie by definition in $\mathrm{NS}_{\mathbb{R}}(X)$. Notice that the integral lattice $H^2(X,\mathbb{Z})/\{\text{torsion}\}$ need not hit at all the subspace $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$ in the Hodge decomposition, hence in general the Picard number

$$\rho(X) = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{NS}(X) = \dim_{\mathbb{R}} \mathrm{NS}_{\mathbb{R}}(X)$$

satisfies $\rho(X) \le h^{1,1} = \dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R})$, but the equality can be strict (actually, it is well known that a generic complex torus $X = \mathbb{C}^n/\Lambda$ satisfies $\rho(X) = 0$ and $h^{1,1} = n^2$). In order to deal with the case of algebraic varieties we introduce

$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \qquad \mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).$$



A very important fact is that the "Neron-Severi part" of any of the open or closed transcendental cones $\mathcal{K}$, $\mathcal{E}$, $\overline{\mathcal{K}}$, $\mathcal{E}^\circ$ is algebraic, i.e. can be characterized in simple algebraic terms.

(5.2) Theorem. *Let X be a projective manifold. Then*

(i) $\mathcal{E}_{\mathrm{NS}}$ *is the closure of the cone generated by classes of* effective *divisors, i.e. divisors* $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.

(ii) $\mathcal{K}_{\mathrm{NS}}$ *is the open cone generated by classes of* ample (*or very ample*) *divisors* A (*Recall that a divisor* A *is said to be very ample if the linear system* $H^0(X, \mathcal{O}(A))$ *provides an embedding of* X *in projective space*).

(iii) *The interior* $\mathcal{E}^{\circ}_{\mathrm{NS}}$ *is the cone generated by classes of* big *divisors, namely divisors* D *such that* $h^0(X, \mathcal{O}(kD)) \geq c\, k^{\dim X}$ *for* k *large.*

(iv) *The closed cone* $\overline{\mathcal{K}}_{\mathrm{NS}}$ *consists of the closure of the cone generated by* nef *divisors* D *(or nef line bundles* L*), namely effective integral divisors* D *such that* $D\cdot C \geq 0$ *for every curve* C.

Sketch of proof. These results were already observed (maybe in a slightly different terminology) in [Dem90]. If we denote by $\mathcal{K}_{\mathrm{alg}}$ the open cone generated by ample divisors, resp. by $\mathcal{E}_{\mathrm{alg}}$ the closure of the cone generated by effective divisors, it is obvious that

$$\mathcal{K}_{\mathrm{alg}} \subset \mathcal{K}_{\mathrm{NS}}, \qquad \mathcal{E}_{\mathrm{alg}} \subset \mathcal{E}_{\mathrm{NS}}.$$

As was to be expected, the interesting part lies in the converse inclusions. The inclusion $K_{\mathrm{NS}} \subset \mathcal{K}_{\mathrm{alg}}$ is more or less equivalent to the Kodaira embedding theorem : if a rational class $\{\alpha\}$ is in $\mathcal{K}$, then some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle L whose curvature form is Kähler. Therefore L is ample and $\{\alpha\} \in \mathcal{K}_{\mathrm{alg}}$; property (ii) follows.

Similarly, if we take a rational class $\{\alpha\} \in \mathcal{E}^{\circ}_{\mathrm{NS}}$, then it is still in $\mathcal{E}$ by subtracting a small multiple $\varepsilon\omega$ of a Kähler class, hence some multiple of $\{\alpha\}$ is the first Chern class of a hermitian line bundle (L, h) with curvature form

$$T = \Theta_h(L) := -\frac{i}{2\pi} i\partial\overline{\partial} \log h \geq \varepsilon\omega.$$

Let us apply Theorem 4.11 to a metric of the form $h^k e^{-m\psi}$ on $L^{\otimes k}$, where ψ has logarithmic poles at given points x_j in X. It is then easily shown that $L^{\otimes k}$ admits sections which have given m-jets at the point x_j, provided that $k \geq Cm$, $C \gg 1$, and the x_j are chosen outside the Lelong sublevel sets of $\log h$. From this we get $h^0(X, L^{\otimes k}) \geq m^n/n! \geq ck^n$, hence the linear system kL can be represented by a big divisor. This implies (iii) and also that $\mathcal{E}^{\circ}_{\mathrm{NS}} \subset \mathcal{E}_{\mathrm{alg}}$. Therefore $\mathcal{E}_{\mathrm{NS}} \subset \mathcal{E}_{\mathrm{alg}}$ by passing to the closure ; (i) follows. The statement (iv) about nef divisors follows e.g. from [Kle66], [Har70], since every nef divisor is a limit of a sequence of ample rational divisors. □

As a natural extrapolation of the algebraic situation, we say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though these classes are not necessarily integral). Property 5.2 (i) also explains the terminology used for the pseudo-effective cone.

6. Numerical Characterization of the Kähler Cone

We describe here the main results obtained in [DP03]. The upshot is that the Kähler cone depends only on the intersection product of the cohomology ring, the Hodge structure and the homology classes of analytic cycles. More precisely, we have:

(6.1) Theorem. *Let X be a compact Kähler manifold. Let $\mathcal{P}$ be the set of real $(1,1)$ cohomology classes $\{\alpha\}$ which are numerically positive on analytic cycles, i.e. such that $\int_Y \alpha^p > 0$ for every irreducible analytic set Y in X, $p = \dim Y$. Then the Kähler cone $\mathcal{K}$ of X is one of the connected components of $\mathcal{P}$.*

(6.2) Special case. *If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.*

These results (which are new even in the projective case) can be seen as a generalization of the well-known Nakai-Moishezon criterion. Recall that the Nakai-Moishezon criterion provides a necessary and sufficient criterion for a line bundle to be ample: *a line bundle $L \to X$ on a projective algebraic manifold X is ample if and only if*

$$L^p \cdot Y = \int_Y c_1(L)^p > 0,$$

for every algebraic subset $Y \subset X$, $p = \dim Y$.

It turns out that the numerical conditions $\int_Y \alpha^p > 0$ also characterize arbitrary transcendental Kähler classes when X is projective: this is precisely the meaning of the special case 6.2.

(6.3) Example. The following example shows that the cone $\mathcal{P}$ need not be connected (and also that the components of $\mathcal{P}$ need not be convex, either). Let us consider for instance a complex torus $X = \mathbb{C}^n/\Lambda$. It is well-known that a generic torus X does not possess any analytic subset except finite subsets and X itself. In that case, the numerical positivity is expressed by the single condition $\int_X \alpha^n > 0$. However, on a torus, $(1,1)$-classes are in one-to-one correspondence with constant hermitian forms α on $\mathbb{C}^n$. Thus, for X generic, $\mathcal{P}$ is the set of hermitian forms on $\mathbb{C}^n$ such that $\det(\alpha) > 0$, and Theorem 6.1 just expresses the elementary result of linear algebra saying that the set $\mathcal{K}$ of positive definite forms is one of the connected components of the open set $\mathcal{P} = \{\det(\alpha) > 0\}$ of hermitian forms of positive determinant (the other components, of course, are the sets of forms of signature (p,q), $p+q=n$, q even. They are not convex when $p > 0$ and $q > 0$).

Sketch of proof of Theorems 6.1 and 6.2. By definition a *Kähler current* is a closed positive current T of type $(1,1)$ such that $T \geq \varepsilon\omega$ for some smooth Kähler metric ω and $\varepsilon > 0$ small enough. The crucial steps of the proof of Theorem 6.1 are contained in the following statements.

(6.4) Proposition (Paun [Pau98a, 98b]). *Let X be a compact complex manifold (or more generally a compact complex space). Then*

(i) *The cohomology class of a closed positive* $(1,1)$*-current* $\{T\}$ *is nef if and only if the restriction* $\{T\}_{|Z}$ *is nef for every irreducible component* Z *in any of the Lelong sublevel sets* $E_c(T)$.

(ii) *The cohomology class of a Kähler current* $\{T\}$ *is a Kähler class* (*i.e. the class of a* smooth *Kähler form*) *if and only if the restriction* $\{T\}_{|Z}$ *is a Kähler class for every irreducible component* Z *in any of the Lelong sublevel sets* $E_c(T)$.

The proof of Proposition 6.4 is not extremely hard if we take for granted the fact that Kähler currents can be approximated by Kähler currents with logarithmic poles, a fact which was first proved in [Dem92] (see also section 9 below). Thus in (ii), we may assume that $T = \alpha + \mathrm{i}\partial\overline{\partial}\varphi$ is a current with analytic singularities, where φ is a quasi-psh function with logarithmic poles on some analytic set Z, and φ smooth on $X \smallsetminus Z$. Now, we proceed by an induction on dimension (to do this, we have to consider analytic spaces rather than with complex manifolds, but it turns out that this makes no difference for the proof). Hence, by the induction hypothesis, there exists a smooth potential ψ on Z such that $\alpha_{|Z} + \mathrm{i}\partial\overline{\partial}\psi > 0$ along Z. It is well known that one can then find a potential $\widetilde{\psi}$ on X such that $\alpha + \mathrm{i}\partial\overline{\partial}\widetilde{\psi} > 0$ in a neighborhood V of Z (but possibly non positive elsewhere). Essentially, it is enough to take an arbitrary extension of ψ to X and to add a large multiple of the square of the distance to Z, at least near smooth points; otherwise, we stratify Z by its successive singularity loci, and proceed again by induction on the dimension of these loci. Finally, we use a a standard gluing procedure : the current $T = \alpha + \mathrm{i}\max_\varepsilon(\varphi, \widetilde{\psi} - C)$, $C \gg 1$, will be equal to $\alpha + \mathrm{i}\partial\overline{\partial}\varphi > 0$ on $X \smallsetminus V$, and to a smooth Kähler form on V. □

The next (and more substantial step) consists of the following result which is reminiscent of the Grauert-Riemenschneider conjecture ([Siu84], [Dem85]).

(6.5) Theorem ([DP03]). *Let* X *be a compact Kähler manifold and let* $\{\alpha\}$ *be a nef class* (*i.e.* $\{\alpha\} \in \overline{\mathcal{K}}$). *Assume that* $\int_X \alpha^n > 0$. *Then* $\{\alpha\}$ *contains a Kähler current* T, *in other words* $\{\alpha\} \in \mathcal{E}^\circ$.

Step 1. The basic argument is to prove that for every irreducible analytic set $Y \subset X$ of codimension p, the class $\{\alpha\}^p$ contains a closed positive (p,p)-current Θ such that $\Theta \geq \delta[Y]$ for some $\delta > 0$. For this, we use in an essentail way the Calabi-Yau theorem [Yau78] on solutions of Monge-Ampère equations, which yields the following result as a special case:

(6.6) Lemma ([Yau78]). *Let* (X,ω) *be a compact Kähler manifold and* $n = \dim X$. *Then for any smooth volume form* $f > 0$ *such that* $\int_X f = \int_X \omega^n$, *there exist a Kähler metric* $\widetilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ *in the same Kähler class as* ω, *such that* $\widetilde{\omega}^n = f$. □

We exploit this by observing that $\alpha + \varepsilon\omega$ is a Kähler class, and by solving the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\overline{\partial}\varphi_\varepsilon)^n = C_\varepsilon \omega_\varepsilon^n \tag{6.6a}$$

where (ω_ε) is the family of Kähler metrics on X produced by Lemma 3.4 (iii), such that their volume is concentrated in an ε-tubular neighborhood of Y.

$$C_\varepsilon = \frac{\int_X \alpha_\varepsilon^n}{\int_X \omega_\varepsilon^n} = \frac{\int_X (\alpha + \varepsilon\omega)^n}{\int_X \omega^n} \geq C_0 = \frac{\int_X \alpha^n}{\int_X \omega^n} > 0.$$

Let us denote by

$$\lambda_1(z) \leq \ldots \leq \lambda_n(z)$$

the eigenvalues of $\alpha_\varepsilon(z)$ with respect to $\omega_\varepsilon(z)$, at every point $z \in X$ (these functions are continuous with respect to z, and of course depend also on ε). The equation (6.6a) is equivalent to the fact that

$$\lambda_1(z) \ldots \lambda_n(z) = C_\varepsilon \tag{6.6b}$$

is constant, and the most important observation for us is that the constant C_ε is bounded away from 0, thanks to our assumption $\int_X \alpha^n > 0$.

Fix a regular point $x_0 \in Y$ and a small neighborhood U (meeting only the irreducible component of x_0 in Y). By Lemma 3.4, we have a uniform lower bound

$$\int_{U \cap V_\varepsilon} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) > 0. \tag{6.6c}$$

Now, by looking at the p smallest (resp. $(n-p)$ largest) eigenvalues λ_j of α_ε with respect to ω_ε, we find

$$\alpha_\varepsilon^p \geq \lambda_1 \ldots \lambda_p \, \omega_\varepsilon^p, \tag{6.6d}$$

$$\alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p \geq \frac{1}{n!} \lambda_{p+1} \ldots \lambda_n \, \omega_\varepsilon^n, \tag{6.6e}$$

The last inequality (6.6e) implies

$$\int_X \lambda_{p+1} \ldots \lambda_n \, \omega_\varepsilon^n \leq n! \int_X \alpha_\varepsilon^{n-p} \wedge \omega_\varepsilon^p = n! \int_X (\alpha + \varepsilon\omega)^{n-p} \wedge \omega^p \leq M$$

for some constant $M > 0$ (we assume $\varepsilon \leq 1$, say). In particular, for every $\delta > 0$, the subset $E_\delta \subset X$ of points z such that $\lambda_{p+1}(z) \ldots \lambda_n(z) > M/\delta$ satisfies $\int_{E_\delta} \omega_\varepsilon^n \leq \delta$, hence

$$\int_{E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \leq 2^{n-p} \int_{E_\delta} \omega_\varepsilon^n \leq 2^{n-p}\delta. \tag{6.6f}$$

The combination of (6.6c) and (6.6f) yields

$$\int_{(U \cap V_\varepsilon) \smallsetminus E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \delta_p(U) - 2^{n-p}\delta.$$

On the other hand (6.6b) and (6.6d) imply

$$\alpha_\varepsilon^p \geq \frac{C_\varepsilon}{\lambda_{p+1} \ldots \lambda_n} \omega_\varepsilon^p \geq \frac{C_\varepsilon}{M/\delta} \omega_\varepsilon^p \qquad \text{on } (U \cap V_\varepsilon) \smallsetminus E_\delta.$$

From this we infer

$$\text{(6.6g)} \quad \int_{U\cap V_\varepsilon} \alpha_\varepsilon^p \wedge \omega^{n-p} \geq \frac{C_\varepsilon}{M/\delta} \int_{(U\cap V_\varepsilon)\smallsetminus E_\delta} \omega_\varepsilon^p \wedge \omega^{n-p} \geq \frac{C_\varepsilon}{M/\delta}(\delta_p(U) - 2^{n-p}\delta) > 0$$

provided that δ is taken small enough, e.g. $\delta = 2^{-(n-p+1)}\delta_p(U)$. The family of (p,p)-forms α_ε^p is uniformly bounded in mass since

$$\int_X \alpha_\varepsilon^p \wedge \omega^{n-p} = \int_X (\alpha + \varepsilon\omega)^p \wedge \omega^{n-p} \leq \text{Const.}$$

Inequality (6.6g) implies that any weak limit Θ of (α_ε^p) carries a positive mass on $U \cap Y$. By Skoda's extension theorem [Sko82], $\mathbf{1}_Y\Theta$ is a closed positive current with support in Y, hence $\mathbf{1}_Y\Theta = \sum c_j[Y_j]$ is a combination of the various components Y_j of Y with coefficients $c_j > 0$. Our construction shows that Θ belongs to the cohomology class $\{\alpha\}^p$. Step 1 of Theorem 6.5 is proved.

Step 2. The second and final step consists in using a "diagonal trick": for this, we apply Step 1 to

$$\widetilde{X} = X \times X, \qquad \widetilde{Y} = \text{diagonal}\, \Delta \subset \widetilde{X}, \qquad \widetilde{\alpha} = \mathrm{pr}_1^*\,\alpha + \mathrm{pr}_2^*\,\alpha.$$

It is then clear that $\widetilde{\alpha}$ is nef on $\widetilde{X}$ and that

$$\int_{\widetilde{X}} (\widetilde{\alpha})^{2n} = \binom{2n}{n}\Big(\int_X \alpha^n\Big)^2 > 0.$$

It follows by Step 1 that the class $\{\widetilde{\alpha}\}^n$ contains a Kähler current Θ of bidegree (n,n) such that $\Theta \geq \delta[\Delta]$ for some $\delta > 0$. Therefore the push-forward

$$T := (\mathrm{pr}_1)_*(\Theta \wedge \mathrm{pr}_2^*\,\omega)$$

is a positive $(1,1)$-current such that

$$T \geq \delta(\mathrm{pr}_1)_*([\Delta] \wedge \mathrm{pr}_2^*\,\omega) = \delta\omega.$$

It follows that T is a Kähler current. On the other hand, T is numerically equivalent to $(\mathrm{pr}_1)_*(\widetilde{\alpha}^n \wedge \mathrm{pr}_2^*\,\omega)$, which is the form given in coordinates by

$$x \mapsto \int_{y\in X} \big(\alpha(x) + \alpha(y)\big)^n \wedge \omega(y) = C\alpha(x)$$

where $C = n\int_X \alpha(y)^{n-1} \wedge \omega(y)$. Hence $T \equiv C\alpha$, which implies that $\{\alpha\}$ contains a Kähler current. Theorem 6.5 is proved. □

End of Proof of Theorems 6.1 and 6.2. Clearly the open cone $\mathcal{K}$ is contained in $\mathcal{P}$, hence in order to show that $\mathcal{K}$ is one of the connected components of $\mathcal{P}$, we need only show that $\mathcal{K}$ is <u>closed</u> in $\mathcal{P}$, i.e. that $\overline{\mathcal{K}} \cap \mathcal{P} \subset \mathcal{K}$. Pick a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X \alpha^n > 0$. By Theorem 6.5 we conclude that $\{\alpha\}$ contains a Kähler current T. However, an induction on dimension using the assumption $\int_Y \alpha^p$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}_{|Y}$ is the class of a Kähler current on Y. We conclude that $\{\alpha\}$ is a Kähler class by 6.4 (ii), therefore $\{\alpha\} \in \mathcal{K}$, as desired. □

The projective case 6.2 is a consequence of the following variant of Theorem 6.1.

(6.7) Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is Kähler if and only if there exists a Kähler metric ω on X such that $\int_Y \alpha^k \wedge \omega^{p-k} > 0$ for all irreducible analytic sets Y and all $k = 1, 2, \ldots, p = \dim Y$.*

Proof. The assumption clearly implies that

$$\int_Y (\alpha + t\omega)^p > 0$$

for all $t \in \mathbb{R}_+$, hence the half-line $\alpha + (\mathbb{R}_+)\omega$ is entirely contained in the cone $\mathcal{P}$ of numerically positive classes. Since $\alpha + t_0\omega$ is Kähler for t_0 large, we conclude that the half-line in entirely contained in the connected component $\mathcal{K}$, and therefore $\alpha \in \mathcal{K}$. □

In the projective case, we can take $\omega = c_1(H)$ for a given very ample divisor H, and the condition $\int_Y \alpha^k \wedge \omega^{p-k} > 0$ is equivalent to

$$\int_{Y \cap H_1 \cap \ldots \cap H_{p-k}} \alpha^k > 0$$

for a suitable complete intersection $Y \cap H_1 \cap \ldots \cap H_{p-k}$, $H_j \in |H|$. This shows that algebraic cycles are sufficient to test the Kähler property, and the special case 6.2 follows. On the other hand, we can pass to the limit in 6.7 by replacing α by $\alpha + \varepsilon\omega$, and in this way we get also a characterization of nef classes.

(6.8) Corollary. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only if there exists a Kähler metric ω on X such that $\int_Y \alpha^k \wedge \omega^{p-k} \geq 0$ for all irreducible analytic sets Y and all $k = 1, 2, \ldots, p = \dim Y$.*

By a formal convexity argument, one can derive from 6.7 or 6.8 the following interesting consequence about the dual of the cone $\mathcal{K}$. We will not give the proof here, because it is just a simple tricky argument which does not require any new analysis.

(6.9) Theorem. *Let X be a compact Kähler manifold. A $(1,1)$ cohomology class $\{\alpha\}$ on X is nef if and only for every irreducible analytic set Y in X, $p = \dim X$ and every Kähler metric ω on X we have $\int_Y \alpha \wedge \omega^{p-1} \geq 0$. In other words, the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1,n-1}(X, \mathbb{R})$, where Y runs over the collection of irreducible analytic subsets of X and $\{\omega\}$ over the set of Kähler classes of X.*

Our main Theorem 6.1 has an important application to the deformation theory of compact Kähler manifolds.

(6.10) Theorem. *Let $\pi : \mathcal{X} \to S$ be a deformation of compact Kähler manifolds over an irreducible base S. Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t =$*

$\pi^{-1}(t)$ *are invariant over* $S \smallsetminus S'$ *under parallel transport with respect to the* $(1,1)$*-projection* $\nabla^{1,1}$ *of the Gauss-Manin connection* ∇ *in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

We moreover conjecture that for an arbitrary deformation $\mathcal{X} \to S$ of compact complex manifolds, the Kähler property is open with respect to the countable Zariski topology on the base S of the deformation.

Let us recall the general fact that all fibers X_t of a deformation over a connected base S are diffeomorphic, since $\mathcal{X} \to S$ is a locally trivial differentiable bundle. This implies that the cohomology bundle

$$S \ni t \mapsto H^k(X_t, \mathbb{C})$$

is locally constant over the base S. The corresponding (flat) connection of this bundle is called the Gauss-Manin connection, and will be denoted here by ∇. As is well known, the Hodge filtration

$$F^p(H^k(X_t, \mathbb{C})) = \bigoplus_{r+s=k, r\geq p} H^{r,s}(X_t, \mathbb{C})$$

defines a *holomorphic* subbundle of $H^k(X_t, \mathbb{C})$ (with respect to its locally constant structure). On the other hand, the Dolbeault groups are given by

$$H^{p,q}(X_t, \mathbb{C}) = F^p(H^k(X_t, \mathbb{C})) \cap \overline{F^{k-p}(H^k(X_t, \mathbb{C}))}, \qquad k = p+q,$$

and they form *real analytic* subbundles of $H^k(X_t, \mathbb{C})$. We are interested especially in the decomposition

$$H^2(X_t, \mathbb{C}) = H^{2,0}(X_t, \mathbb{C}) \oplus H^{1,1}(X_t, \mathbb{C}) \oplus H^{0,2}(X_t, \mathbb{C})$$

and the induced decomposition of the Gauss-Manin connection acting on H^2

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & * \\ * & \nabla^{1,1} & * \\ * & * & \nabla^{0,2} \end{pmatrix}.$$

Here the stars indicate suitable bundle morphisms – actually with the lower left and upper right stars being zero by Griffiths' transversality property, but we do not really care here. The notation $\nabla^{p,q}$ stands for the induced (real analytic, not necessarily flat) connection on the subbundle $t \mapsto H^{p,q}(X_t, \mathbb{C})$.

Sketch of Proof of Theorem 6.10. The result is local on the base, hence we may assume that S is contractible. Then the family is differentiably trivial, the Hodge bundle $t \mapsto H^2(X_t, \mathbb{C})$ is the trivial bundle and $t \mapsto H^2(X_t, \mathbb{Z})$ is a trivial lattice. We use the existence of a relative cycle space $C^p(\mathcal{X}/S) \subset C^p(\mathcal{X})$ which consists of all cycles contained in the fibres of $\pi : X \to S$. It is equipped with a canonical holomorphic projection

$$\pi_p : C^p(\mathcal{X}/S) \to S.$$

We then define the S_ν's to be the images in S of those connected components of $C^p(\mathcal{X}/S)$ which do not project onto S. By the fact that the projection is proper on each component, we infer that S_ν is an analytic subset of S. The definition of the S_ν's imply that the cohomology classes induced by the analytic cycles $\{[Z]\}$, $Z \subset X_t$, remain exactly the same for all $t \in S \smallsetminus S'$. This result implies in its turn that the conditions defining the numerically positive cones $\mathcal{P}_t$ remain the same, except for the fact that the spaces $H^{1,1}_{\mathbb{R}}(X_t) \subset H^2(X_t, \mathbb{R})$ vary along with the Hodge decomposition. At this point, a standard calculation implies that the $\mathcal{P}_t$ are invariant by parallel transport under $\nabla^{1,1}$. This is done as follows.

Since S is irreducible and S' is a countable union of analytic sets, it follows that $S \smallsetminus S'$ is arcwise connected by piecewise smooth analytic arcs. Let

$$\gamma : [0,1] \to S \smallsetminus S', \qquad u \mapsto t = \gamma(u)$$

be such a smooth arc, and let $\alpha(u) \in H^{1,1}(X_{\gamma(u)}, \mathbb{R})$ be a family of real $(1,1)$-cohomology classes which are constant by parallel transport under $\nabla^{1,1}$. This is equivalent to assuming that

$$\nabla(\alpha(u)) \in H^{2,0}(X_{\gamma(u)}, \mathbb{C}) \oplus H^{0,2}(X_{\gamma(u)}, \mathbb{C})$$

for all u. Suppose that $\alpha(0)$ is a numerically positive class in $X_{\gamma(0)}$. We then have

$$\alpha(0)^p \cdot \{[Z]\} = \int_Z \alpha(0)^p > 0$$

for all p-dimensional analytic cycles Z in $X_{\gamma(0)}$. Let us denote by

$$\zeta_Z(t) \in H^{2q}(X_t, \mathbb{Z}), \qquad q = \dim X_t - p,$$

the family of cohomology classes equal to $\{[Z]\}$ at $t = \gamma(0)$, such that $\nabla\zeta_Z(t) = 0$ (i.e. constant with respect to the Gauss-Manin connection). By the above discussion, $\zeta_Z(t)$ is of type (q,q) for all $t \in S$, and when $Z \subset X_{\gamma(0)}$ varies, $\zeta_Z(t)$ generates all classes of analytic cycles in X_t if $t \in S \smallsetminus S'$. Since ζ_Z is ∇-parallel and $\nabla\alpha(u)$ has no component of type $(1,1)$, we find

$$\frac{d}{du}(\alpha(u)^p \cdot \zeta_Z(\gamma(u)) = p\alpha(u)^{p-1} \cdot \nabla\alpha(u) \cdot \zeta_Z(\gamma(u)) = 0.$$

We infer from this that $\alpha(u)$ is a numerically positive class for all $u \in [0,1]$. This argument shows that the set $\mathcal{P}_t$ of numerically positive classes in $H^{1,1}(X_t, \mathbb{R})$ is invariant by parallel transport under $\nabla^{1,1}$ over $S \smallsetminus S'$.

By a standard result of Kodaira-Spencer [KS60] relying on elliptic PDE theory, every Kähler class in X_{t_0} can be deformed to a nearby Kähler class in nearby fibres X_t. This implies that the connected component of $\mathcal{P}_t$ which corresponds to the Kähler cone $\mathcal{K}_t$ must remain the same. The theorem is proved. □

As a by-product of our techniques, especially the regularization theorem for currents, we also get the following result for which we refer to [DP03].

(6.11) Theorem. *A compact complex manifold carries a Kähler current if and only if it is bimeromorphic to a Kähler manifold* (*or equivalently, dominated by a Kähler manifold*)*.*

This class of manifolds is called the *Fujiki class* $\mathcal{C}$. If we compare this result with the solution of the Grauert-Riemenschneider conjecture, it is tempting to make the following conjecture which would somehow encompass both results.

(6.12) Conjecture. *Let X be a compact complex manifold of dimension n. Assume that X possesses a nef cohomology class $\{\alpha\}$ of type $(1,1)$ such that $\int_X \alpha^n > 0$. Then X is in the Fujiki class $\mathcal{C}$.* [*Also, $\{\alpha\}$ would contain a Kähler current, as it follows from Theorem 6.5 if Conjecture 6.12 is proved*]*.*

We want to mention here that most of the above results were already known in the cases of complex surfaces (i.e. in dimension 2), thanks to the work of N. Buchdahl [Buc99, 00] and A. Lamari [Lam99a, 99b].

Shortly after the original [DP03] manuscript appeared in April 2001, Daniel Huybrechts [Huy01] informed us Theorem 6.1 can be used to calculate the Kähler cone of a very general hyperkähler manifold: the Kähler cone is then equal to a suitable connected component of the positive cone defined by the Beauville-Bogomolov quadratic form. In the case of an arbitrary hyperkähler manifold, S.Boucksom [Bou02] later showed that a $(1,1)$ class $\{\alpha\}$ is Kähler if and only if it lies in the positive part of the Beauville-Bogomolov quadratic cone and moreover $\int_C \alpha > 0$ for all <u>*rational curves*</u> $C \subset X$ (see also [Huy99]).

7. Cones of Curves

In a dual way, we consider in $H^{n-1,n-1}_{\mathbb{R}}(X)$ the cone $\mathcal{N}$ generated by classes of positive currents T of type $(n-1,n-1)$ (i.e., of bidimension $(1,1)$). In the projective case, we also consider the intersection

By extension, we will say that $\overline{\mathcal{K}}$ is the cone of *nef* $(1,1)$-cohomology classes (even though they are not necessarily integral). We now turn ourselves to cones in cohomology of bidegree $(n-1,n-1)$.

(7.1) Definition. *Let X be a compact Kähler manifold.*

(i) *We define $\mathcal{N}$ to be the* (*closed*) *convex cone in $H^{n-1,n-1}_{\mathbb{R}}(X)$ generated by classes of positive currents T of type $(n-1,n-1)$* (*i.e., of bidimension $(1,1)$*)*.*

(ii) *We define the cone $\mathcal{M} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ of <u>movable classes</u> to be the closure of the convex cone generated by classes of currents of the form*

$$\mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$$

where $\mu : \widetilde{X} \to X$ is an arbitrary modification (*one could just restrict oneself to compositions of blow-ups with smooth centers*)*, and the $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Clearly $\mathcal{M} \subset \mathcal{N}$.*

(iii) *Correspondingly, we introduce the intersections*

$$\mathcal{N}_{\mathrm{NS}} = \mathcal{N} \cap N_1(X), \qquad \mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X),$$

in the space of integral bidimension $(1,1)$*-classes*

$$N_1(X) := (H^{n-1,n-1}_{\mathbb{R}}(X) \cap H^{2n-2}(X,\mathbb{Z})/\mathrm{tors}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(iv) *If X is projective, we define* $\mathrm{NE}(X)$ *to be the convex cone generated by all effective curves. Clearly* $\overline{\mathrm{NE}(X)} \subset \mathcal{N}_{\mathrm{NS}}$.

(v) *If X is projective, we say that C is a* <u>*strongly movable curve*</u> *if*

$$C = \mu_\star(\widetilde{A}_1 \cap \ldots \cap \widetilde{A}_{n-1})$$

for suitable very ample divisors $\widetilde{A}_j$ on $\widetilde{X}$, where $\mu : \widetilde{X} \to X$ is a modification. We let $\mathrm{SME}(X)$ *to be the convex cone generated by all strongly movable* (*effective*) *curves. Clearly* $\overline{\mathrm{SME}(X)} \subset \mathcal{M}_{\mathrm{NS}}$.

(vi) *We say that C is a* <u>*movable curve*</u> *if $C = C_{t_0}$ is a member of an analytic family $(C_t)_{t\in S}$ such that $\bigcup_{t\in S} C_t = X$ and, as such, is a reduced irreducible* 1*-cycle. We let* $\mathrm{ME}(X)$ *to be the convex cone generated by all movable* (*effective*) *curves.*

The upshot of this definition lies in the following easy observation.

(7.2) Proposition. *Let X be a compact Kähler manifold. Consider the Poincaré duality pairing*

$$H^{1,1}(X,\mathbb{R}) \times H^{n-1,n-1}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta) \longmapsto \int_X \alpha \wedge \beta.$$

Then the duality pairing takes nonnegative values

(i) *for all pairs $(\alpha,\beta) \in \overline{\mathcal{K}} \times \mathcal{N}$;*

(ii) *for all pairs $(\alpha,\beta) \in \mathcal{E} \times \mathcal{M}$.*

(iii) *for all pairs (α,β) where $\alpha \in \mathcal{E}$ and $\beta = [C_t] \in \mathrm{ME}(X)$ is the class of a movable curve.*

Proof. (i) is obvious. In order to prove (ii), we may assume that $\beta = \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1})$ for some modification $\mu : \widetilde{X} \to X$, where $\alpha = \{T\}$ is the class of a positive $(1,1)$-current on X and $\widetilde{\omega}_j$ are Kähler forms on $\widetilde{X}$. Then

$$\int_X \alpha \wedge \beta = \int_X T \wedge \mu_\star(\widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1}) = \int_X \mu^* T \wedge \widetilde{\omega}_1 \wedge \ldots \wedge \widetilde{\omega}_{n-1} \geq 0.$$

Here, we have used the fact that a closed positive $(1,1)$-current T always has a pull-back $\mu^\star T$, which follows from the fact that if $T = i\partial\overline{\partial}\varphi$ locally for some plurisubharmonic function in X, we can set $\mu^\star T = i\partial\overline{\partial}(\varphi \circ \mu)$. For (iii), we suppose $\alpha = \{T\}$ and $\beta = \{[C_t]\}$. Then we take an open covering (U_j) on X such that $T = i\partial\overline{\partial}\varphi_j$ with suitable plurisubharmonic functions φ_j on U_j. If we select a smooth partition of unity $\sum \theta_j = 1$ subordinate to (U_j), we then get

$$\int_X \alpha \wedge \beta = \int_{C_t} T_{|C_t} = \sum_j \int_{C_t \cap U_j} \theta_j i\partial\overline{\partial}\varphi_{j|C_t} \geq 0.$$

For this to make sense, it should be noticed that $T_{|C_t}$ is a well defined closed positive $(1,1)$-current (i.e. measure) on C_t for almost every $t \in S$, in the sense of Lebesgue measure. This is true only because (C_t) covers X, thus $\varphi_{j|C_t}$ is not identically $-\infty$ for almost every $t \in S$. The equality in the last formula is then shown by a regularization argument for T, writing $T = \lim T_k$ with $T_k = \alpha + i\partial\overline{\partial}\psi_k$ and a decreasing sequence of smooth almost plurisubharmonic potentials $\psi_k \downarrow \psi$ such that the Levi forms have a uniform lower bound $i\partial\overline{\partial}\psi_k \geq -C\omega$ (such a sequence exists by [Dem92]). Then, writing $\alpha = i\partial\overline{\partial}v_j$ for some smooth potential v_j on U_j, we have $T = i\partial\overline{\partial}\varphi_j$ on U_j with $\varphi_j = v_j + \psi$, and this is the decreasing limit of the smooth approximations $\varphi_{j,k} = v_j + \psi_k$ on U_j. Hence $T_{k|C_t} \to T_{|C_t}$ for the weak topology of measures on C_t. □

If $\mathcal{C}$ is a convex cone in a finite dimensional vector space E, we denote by $\mathcal{C}^\vee$ the dual cone, i.e. the set of linear forms $u \in E^\star$ which take nonnegative values on all elements of $\mathcal{C}$. By the Hahn-Banach theorem, we always have $\mathcal{C}^{\vee\vee} = \overline{\mathcal{C}}$.

Proposition 7.2 leads to the natural question whether the cones $(\mathcal{K}, \mathcal{N})$ and $(\mathcal{E}, \mathcal{M})$ are dual under Poincaré duality. This question is addressed in the next section. Before doing so, we observe that the algebraic and transcendental cones of $(n-1, n-1)$ cohomology classes are related by the following equalities.

(7.3) Theorem. *Let X be a projective manifold. Then*

(i) $\overline{\mathrm{NE}(X)} = \mathcal{N}_{\mathrm{NS}}$.

(ii) $\overline{\mathrm{SME}(X)} = \overline{\mathrm{ME}(X)} = \mathcal{M}_{\mathrm{NS}}$.

Proof. (i) It is a standard result of algebraic geometry (see e.g. [Har70]), that the cone of effective cone $\mathrm{NE}(X)$ is dual to the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors, hence

$$\mathcal{N}_{\mathrm{NS}} \supset \overline{\mathrm{NE}(X)} = \mathcal{K}^\vee.$$

On the other hand, (7.3) (i) implies that $\mathcal{N}_{\mathrm{NS}} \subset \mathcal{K}^\vee$, so we must have equality and (i) follows.

Similarly, (ii) requires a duality statement which will be established only in the next sections, so we postpone the proof. □

8. Main Duality Results

It is very well-known that the cone $\overline{\mathcal{K}_{\mathrm{NS}}}$ of nef divisors is dual to the cone $\mathcal{N}_{\mathrm{NS}}$ of effective curves if X is projective. The transcendental case can be stated as follows.

(8.1) Theorem (Demailly-Paun, 2001)**.** *If X is Kähler, the cones $\overline{\mathcal{K}} \subset H^{1,1}(X, \mathbb{R})$ and $\mathcal{N} \subset H^{n-1,n-1}_{\mathbb{R}}(X)$ are dual by Poincaré duality, and $\mathcal{N}$ is the closed convex cone* *generated by classes* $[Y] \wedge \omega^{p-1}$ *where* $Y \subset X$ *ranges over* p*-dimensional analytic subsets,* $p = 1, 2, \ldots, n$*, and* ω *ranges over Kähler forms.*

Proof. Indeed, Prop. 7.4 shows that the dual cone $\mathcal{K}^\vee$ contains $\mathcal{N}$ which itself contains the cone $\mathcal{N}'$ of all classes of the form $\{[Y] \wedge \omega^{p-1}\}$. The main result of [DP03] conversely shows that the dual of $(\mathcal{N}')^\vee$ is equal to $\overline{\mathcal{K}}$, so we must have

$$\mathcal{K}^\vee = \overline{\mathcal{N}'} = \mathcal{N}. \qquad \square$$

The other important duality result is the following characterization of pseudo-effective classes, proved in [BDPP03] (the "only if" part already follows from 7.4 (iii)).

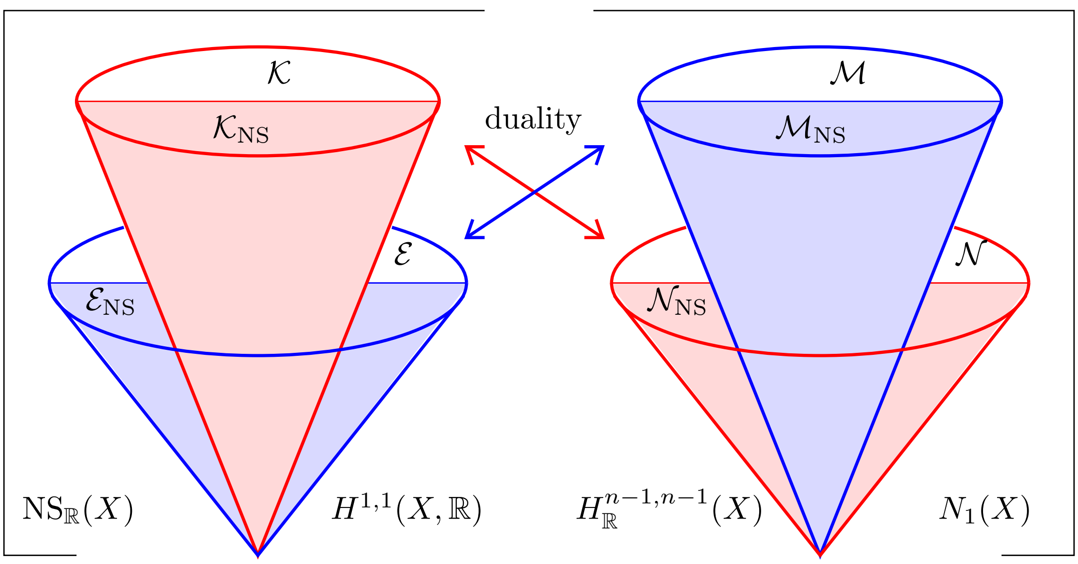
(8.2) Theorem. *If* X *is projective, then a class* $\alpha \in \mathrm{NS}_\mathbb{R}(X)$ *is pseudo-effective if (and only if) it is in the dual cone of the cone* $\mathrm{SME}(X)$ *of strongly movable curves.*

In other words, a line bundle L is pseudo-effective if (and only if) $L \cdot C \geq 0$ for all *movable curves*, i.e., $L \cdot C \geq 0$ for every very generic curve C (not contained in a countable union of algebraic subvarieties). In fact, by definition of $\mathrm{SME}(X)$, it is enough to consider only those curves C which are images of generic complete intersection of very ample divisors on some variety $\widetilde{X}$, under a modification $\mu : \widetilde{X} \to X$.

By a standard blowing-up argument, it also follows that a line bundle L on a normal Moishezon variety is pseudo-effective if and only if $L \cdot C \geq 0$ for every movable curve C.

The Kähler analogue should be :

(8.3) Conjecture. *For an arbitrary compact Kähler manifold* X*, the cones* $\mathcal{E}$ *and* $\mathcal{M}$ *are dual.*



The relation between the various cones of movable curves and currents in (7.5) is now a rather direct consequence of Theorem 8.2. In fact, using ideas hinted in [DPS96], we can say a little bit more. Given an irreducible curve $C \subset X$, we consider its normal "bundle" $N_C = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_C)$, where $\mathcal{I}$ is the ideal sheaf of C. If C is a general member of a covering family (C_t), then N_C is nef. Now [DPS96] says that the dual cone of the pseudo-effective cone of X contains the closed cone spanned by curves with nef normal bundle, which in turn contains the cone of movable curves. In this way we get :

(8.4) Theorem. *Let X be a projective manifold. Then the following cones coincide.*

(i) *the cone $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$;*

(ii) *the closed cone $\overline{\mathrm{SME}(X)}$ of strongly movable curves;*

(iii) *the closed cone $\overline{\mathrm{ME}(X)}$ of movable curves;*

(iv) *the closed cone $\overline{\mathrm{ME}_{\mathrm{nef}}(X)}$ of curves with nef normal bundle.*

Proof. We have already seen that

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathrm{ME}_{\mathrm{nef}}(X) \subset (\mathcal{E}_{\mathrm{NS}})^{\vee}$$

and

$$\mathrm{SME}(X) \subset \mathrm{ME}(X) \subset \mathcal{M}_{\mathrm{NS}} \subset (\mathcal{E}_{\mathrm{NS}})^{\vee}$$

by 7.4 (iii). Now Theorem 8.2 implies $(\mathcal{M}_{\mathrm{NS}})^{\vee} = \overline{\mathrm{SME}(X)}$, and 8.4 follows. □

(8.5) Corollary. *Let X be a projective manifold and L a line bundle on X.*

(i) *L is pseudo-effective if and only if $L \cdot C \geq 0$ for all curves C with nef normal sheaf N_C.*

(ii) *If L is big, then $L \cdot C > 0$ for all curves C with nef normal sheaf N_C.*

8.5 (i) strenghtens results from [PSS99]. It is however not yet clear whether $\mathcal{M}_{\mathrm{NS}} = \mathcal{M} \cap N_1(X)$ is equal to the closed cone of curves with *ample* normal bundle (although we certainly expect this to be true).

The most important special case of Theorem 8.2 is

(8.6) Theorem. *If X is a projective manifold and is not uniruled, then K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$.*

Proof. If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, Theorem 7.2 shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The "bend-and-break" lemma then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled. □

A stronger result is expected to be true, namely :

(8.7) Conjecture (special case of the "abundance conjecture")**.** *If K_X is pseudo-effective, then $\kappa(X) \geq 0$.*

9. Approximation of psh functions by logarithms of holomorphic functions

The fundamental tool is the Ohsawa-Takegoshi extension theorem in the following form ([OT87], see also [Dem00]).

(9.1) Theorem. *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let $Y \subset X$ be a nonsingular complex submanifold defined by a section s of some hermitian vector bundle E with* bounded *curvature tensor on Ω. Assume that s is everywhere transverse to the zero section and that $|s| \le e^{-1}$ on Ω. Then there is a constant $C > 0$ (depending only on E), with the following property: for every psh function φ on Ω, every holomorphic function f on Y with $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} e^{-\varphi} dV_Y < +\infty$, there exists an extension F of f to Ω such that*

$$\int_\Omega \frac{|F|^2}{|s|^{2r}(-\log|s|)^2} e^{-\varphi} dV_\Omega \le C \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} e^{-\varphi} dV_Y.$$

Here we simply take Y to be a point $\{z_0\}$. In this case, the theorem says that we can find $F \in \mathcal{O}(\Omega)$ with a prescribed value $F(z_0)$, such that

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \le C |F(z_0)|^2.$$

We now show that every psh function on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$ can be approximated very accurately by psh functions with analytic singularities. The main idea is taken from [Dem92].

(9.2) Theorem. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |\sigma_\ell|^2$ where (σ_ℓ) is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

a) $\varphi(z) - \dfrac{C_1}{m} \le \varphi_m(z) \le \sup\limits_{|\zeta - z| < r} \varphi(\zeta) + \dfrac{1}{m} \log \dfrac{C_2}{r^n}$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and

b) $\nu(\varphi, z) - \dfrac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof. Note that $\sum |\sigma_\ell(z)|^2$ is the square of the norm of the evaluation linear form $f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$. As φ is locally bounded above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |\sigma_\ell|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover we have

$$\varphi_m(z) = \sup_{f \in B(1)} \frac{1}{m} \log |f(z)|$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$. For $r < d(z, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$\begin{aligned}|f(z)|^2 &\le \frac{1}{\pi^n r^{2n}/n!}\int_{|\zeta - z|<r} |f(\zeta)|^2 d\lambda(\zeta)\\ &\le \frac{1}{\pi^n r^{2n}/n!}\exp\Big(2m \sup_{|\zeta-z|<r}\varphi(\zeta)\Big)\int_\Omega |f|^2 e^{-2m\varphi}d\lambda.\end{aligned}$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z) \le \sup_{|\zeta-z|<r}\varphi(\zeta) + \frac{1}{2m}\log\frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in a) is proved. Conversely, the Ohsawa-Takegoshi extension theorem applied to the 0-dimensional subvariety $\{z\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi}d\lambda \le C_3|a|^2 e^{-2m\varphi(z)},$$

where C_3 only depends on n and $\operatorname{diam}\Omega$. We fix a such that the right hand side is 1. This gives the other inequality

$$\varphi_m(z) \ge \frac{1}{m}\log|a| = \varphi(z) - \frac{\log C_3}{2m}.$$

The above inequality implies $\nu(\varphi_m, z) \le \nu(\varphi, z)$. In the opposite direction, we find

$$\sup_{|x-z|<r}\varphi_m(x) \le \sup_{|\zeta-z|<2r}\varphi(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n}.$$

Divide by $\log r$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a psh function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\varphi_m, x) \ge \nu(\varphi, x) - \frac{n}{m}. \qquad \square$$

Theorem 9.2 implies in a straighforward manner a deep result of [Siu74] on the analyticity of the Lelong number sublevel sets.

(9.3) Corollary. *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number sublevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \ge c\big\}$$

is an analytic subset of X.

Proof. Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in 9.2 b) imply that

$$E_c(\varphi) = \bigcap_{m \geq m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $\sigma_\ell^{(\alpha)}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

(9.4) Regularization theorem for currents. *Let X be a compact complex manifold equipped with a hermitian metric ω. Let $T = \alpha + i\partial\overline{\partial}\psi$ be a closed $(1,1)$-current on X, where α is smooth and ψ is a quasi-plurisubharmonic function. Assume that $T \geq \gamma$ for some real $(1,1)$-form γ on X with real coefficients. Then there exists a sequence $T_k = \alpha + i\partial\overline{\partial}\psi_k$ of closed $(1,1)$-currents such that*

(i) *ψ_k (and thus T_k) is smooth on the complement $X \smallsetminus Z_k$ of an analytic set Z_k, and the Z_k's form an increasing sequence*

$$Z_0 \subset Z_1 \subset \ldots \subset Z_k \subset \ldots \subset X.$$

(ii) *There is a uniform estimate $T_k \geq \gamma - \delta_k \omega$ with $\lim \downarrow \delta_k = 0$ as k tends to $+\infty$.*

(iii) *The sequence (ψ_k) is non increasing, and we have $\lim \downarrow \psi_k = \psi$. As a consequence, T_k converges weakly to T as k tends to $+\infty$.*

(iv) *Near Z_k, the potential ψ_k has logarithmic poles, namely, for every $x_0 \in Z_k$, there is a neighborhood U of x_0 such that $\psi_k(z) = \lambda_k \log \sum_\ell |g_{k,\ell}|^2 + O(1)$ for suitable holomorphic functions $(g_{k,\ell})$ on U and $\lambda_k > 0$. Moreover, there is a (global) proper modification $\mu_k : \widetilde{X}_k \to X$ of X, obtained as a sequence of blow-ups with smooth centers, such that $\psi_k \circ \mu_k$ can be written locally on $\widetilde{X}_k$ as*

$$\psi_k \circ \mu_k(w) = \lambda_k \big(\sum n_\ell \log |\widetilde{g}_\ell|^2 + f(w) \big)$$

where $(\widetilde{g}_\ell = 0)$ are local generators of suitable (global) divisors D_ℓ on $\widetilde{X}_k$ such that $\sum D_\ell$ has normal crossings, n_ℓ are positive integers, and the f's are smooth functions on $\widetilde{X}_k$.

Sketch of proof. We briefly indicate the main ideas, since the proof can only be reconstructed by patching together arguments which appeared in different places (although the core the proof is entirely in [Dem92]). After replacing T with $T - \alpha$, we can assume that $\alpha = 0$ and $T = i\partial\overline{\partial}\psi \geq \gamma$. Given a small $\varepsilon > 0$, we select a covering of X by open balls B_j together with holomorphic coordinates $(z^{(j)})$ and real numbers β_j such that

$$0 \leq \gamma - \beta_j \, i\partial\overline{\partial}|z^{(j)}|^2 \leq \varepsilon \, i\partial\overline{\partial}|z^{(j)}|^2 \qquad \text{on } B_j$$

(this can be achieved just by continuity of γ, after diagonalizing γ at the center of the balls). We now take a partition of unity (θ_j) subordinate to (B_j) such that $\sum \theta_j^2 = 1$, and define

$$\psi_k(z) = \frac{1}{2k} \log \sum_j \theta_j^2 e^{2k\beta_j |z^{(j)}|^2} \sum_{\ell \in \mathbb{N}} |g_{j,k,\ell}|^2$$

where $(g_{j,k,\ell})$ is a Hilbert basis of the Hilbert space of holomorphic functions f on B_j such that

$$\int_{B_j} |f|^2 e^{-2k(\psi - \beta_j |z^{(j)}|^2)} < +\infty.$$

Notice that by the Hessian estimate $i\partial\overline{\partial}\psi \geq \gamma \geq \beta_j\, i\partial\overline{\partial}|z^{(j)}|^2$, the weight involved in the L^2 norm is plurisubharmonic. It then follows from the proof of Proposition 3.7 in [Dem92] that all properties (i)–(iv) hold true, except possibly the fact that the sequence ψ_k can be chosen to be non increasing, and the existence of the modification in (iv). However, the multiplier ideal sheaves of the weights $k(\psi - \beta_j|z^{(j)}|^2)$ are generated by the $(g_{j,k,\ell})_\ell$ on B_j, and these sheaves glue together into a global coherent multiplier ideal sheaf $\mathcal{I}(k\psi)$ on X (see [DEL99]); the modification μ_k is then obtained by blowing-up the ideal sheaf $\mathcal{I}(k\psi)$ so that $\mu_k^*\mathcal{I}(k\psi)$ is an invertible ideal sheaf associated with a normal crossing divisor (Hironaka [Hir64]). The fact that ψ_k can be chosen to be non increasing follows from a quantitative version of the "subadditivity of multiplier ideal sheaves" which is proved in Step 3 of the proof of Theorem 2.2.1 in [DPS01] (see also ([DEL99]). (Anyway, this property will not be used here, so the reader may wish to skip the details). □

10. Zariski Decomposition and Movable Intersections

Let X be compact Kähler and let $\alpha \in \mathcal{E}^\circ$ be in the *interior* of the pseudo–effective cone. In analogy with the algebraic context such a class α is called "big", and it can then be represented by a *Kähler current* T, i.e. a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

(10.1) Theorem (Demailly [Dem92], [Bou02, 3.1.24]. *If T is a Kähler current, then one can write $T = \lim T_m$ for a sequence of Kähler currents T_m which have logarithmic poles with coefficients in $\frac{1}{m}\mathbb{Z}$, i.e. there are modifications $\mu_m : X_m \to X$ such that*

$$\mu_m^\star T_m = [E_m] + \beta_m$$

where E_m is an effective $\mathbb{Q}$-divisor on X_m with coefficients in $\frac{1}{m}\mathbb{Z}$ (the "fixed part") and β_m is a closed semi-positive form (the "movable part").

Proof. This is a direct consequence of the results of section 9. Locally we can write $T = i\partial\overline{\partial}\varphi$ for some strictly plurisubharmonic potential φ. By the Bergman kernel trick and the Ohsawa-Takegoshi L^2 extension theorem, we get local approximations

$$\varphi = \lim \varphi_m, \qquad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the set of holomorphic functions which are L^2 with respect to the weight $e^{-2m\varphi}$. This Hilbert basis is also a family of local generators of the globally defined multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. Then $\mu_m : X_m \to X$ is obtained by blowing-up this ideal sheaf, so that

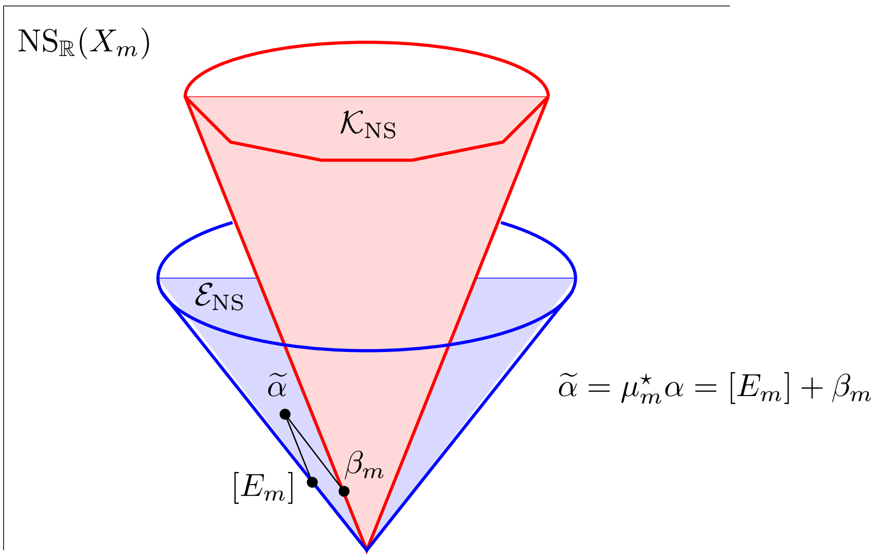
$$\mu_m^\star\mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

We should notice that by approximating $T - \frac{1}{m}\omega$ instead of T, we can replace β_m by $\beta_m + \frac{1}{m}\mu^\star\omega$ which is a big class on X_m ; by playing with the multiplicities of the components of the exceptional divisor, we could even achieve that β_m is a Kähler class on X_m, but this will not be needed here. □

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a $\mathbb{Q}$-divisor decomposition

$$\mu_m^\star L \sim E_m + D_m, \qquad E_m \text{ effective}, \quad D_m \text{ free}.$$

Such a blow-up is usually referred to as a "log resolution" of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L. We will also use this terminology for Kähler currents with logarithmic poles.



(10.2) Definition. *We define the* <u>*volume*</u>*, or* <u>*movable self-intersection*</u> *of a big class $\alpha \in \mathcal{E}^\circ$ to be*

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^\star T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$.

By Fujita [Fuj94] and Demailly-Ein-Lazarsfeld [DEL00], if L is a big line bundle, we have

$$\mathrm{Vol}(c_1(L)) = \lim_{m\to+\infty} D_m^n = \lim_{m\to+\infty} \frac{n!}{m^n} h^0(X, mL),$$

and in these terms, we get the following statement.

(10.3) Proposition. *Let L be a big line bundle on the projective manifold X. Let $\epsilon > 0$. Then there exists a modification $\mu : X_\epsilon \to X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective $\mathbb{Q}$-divisor and β a big and nef $\mathbb{Q}$-divisor such that*

$$\mathrm{Vol}(L) - \varepsilon \le \mathrm{Vol}(\beta) \le \mathrm{Vol}(L).$$

It is very useful to observe that the supremum in Definition 10.2 is actually achieved by a collection of currents whose singularities satisfy a filtering property. Namely, if $T_1 = \alpha + i\partial\overline{\partial}\varphi_1$ and $T_2 = \alpha + i\partial\overline{\partial}\varphi_2$ are two Kähler currents with logarithmic poles in the class of α, then

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \max(\varphi_1, \varphi_2) \tag{10.4}$$

is again a Kähler current with weaker singularities than T_1 and T_2. One could define as well

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad \varphi = \frac{1}{2m}\log(e^{2m\varphi_1} + e^{2m\varphi_2}), \tag{10.4'}$$

where $m = \mathrm{lcm}(m_1, m_2)$ is the lowest common multiple of the denominators occuring in T_1, T_2. Now, take a simultaneous log-resolution $\mu_m : X_m \to X$ for which the singularities of T_1 and T_2 are resolved as $\mathbb{Q}$-divisors E_1 and E_2. Then clearly the associated divisor in the decomposition $\mu_m^\star T = [E] + \beta$ is given by $E = \min(E_1, E_2)$. By doing so, the volume $\int_{X_m} \beta^n$ gets increased, as we shall see in the proof of Theorem 10.5 below.

(10.5) Theorem (Boucksom [Bou02]). *Let X be a compact Kähler manifold. We denote here by $H^{k,k}_{\ge 0}(X)$ the cone of cohomology classes of type (k,k) which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k, n-k)$.*

(i) *For each integer $k = 1, 2, \ldots, n$, there exists a canonical "movable intersection product"*

$$\mathcal{E} \times \cdots \times \mathcal{E} \to H^{k,k}_{\ge 0}(X), \quad (\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

such that $\mathrm{Vol}(\alpha) = \langle \alpha^n \rangle$ *whenever α is a big class.*

(ii) *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

$$\langle \alpha_1 \cdots (\alpha'_j + \alpha''_j) \cdots \alpha_k \rangle \ge \langle \alpha_1 \cdots \alpha'_j \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha''_j \cdots \alpha_k \rangle.$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

(iii) *The movable intersection product satisfies the Teissier-Hovanskii inequalities*

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \ge (\langle \alpha_1^n \rangle)^{1/n} \ldots (\langle \alpha_n^n \rangle)^{1/n} \qquad (\textit{with } \langle \alpha_j^n \rangle = \mathrm{Vol}(\alpha_j)\,).$$

(iv) *For $k = 1$, the above "product" reduces to a (non linear) projection operator*

$$\mathcal{E} \to \mathcal{E}_1, \qquad \alpha \to \langle \alpha \rangle$$

onto a certain convex subcone $\mathcal{E}_1$ of $\mathcal{E}$ such that $\overline{\mathcal{K}} \subset \mathcal{E}_1 \subset \mathcal{E}$. Moreover, there is a "divisorial Zariski decomposition"

$$\alpha = \{N(\alpha)\} + \langle\alpha\rangle$$

where $N(\alpha)$ *is a uniquely defined effective divisor which is called the "negative divisorial part" of* α. *The map* $\alpha \mapsto N(\alpha)$ *is homogeneous and subadditive, and* $N(\alpha) = 0$ *if and only if* $\alpha \in \mathcal{E}_1$.

(v) *The components of* $N(\alpha)$ *always consist of divisors whose cohomology classes are linearly independent, especially* $N(\alpha)$ *has at most* $\rho = \operatorname{rank}_{\mathbb{Z}} \mathrm{NS}(X)$ *components.*

Proof. We essentially repeat the arguments developped in [Bou02], with some simplifications arising from the fact that X is supposed to be Kähler from the start.

(i) First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n-k, n-k)$ *semi-positive* form u on X. We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and a simultaneous log-resolution $\mu : \widetilde{X} \to X$ such that

$$\mu^\star T_j = [E_j] + \beta_j.$$

We consider the direct image current $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ (which is a closed positive current of bidegree (k,k) on X) and the corresponding integrals

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \geq 0.$$

If we change the representative T_j with another current T'_j, we may always take a simultaneous log-resolution such that $\mu^\star T'_j = [E'_j] + \beta'_j$, and by using (10.4′) we can always assume that $E'_j \leq E_j$. Then $D_j = E_j - E'_j$ is an effective divisor and we find $[E_j] + \beta_j \equiv [E'_j] + \beta'_j$, hence $\beta'_j \equiv \beta_j + [D_j]$. A substitution in the integral implies

$$\begin{aligned}
&\int_{\widetilde{X}} \beta'_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad = \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u + \int_{\widetilde{X}} [D_1] \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \\
&\qquad \geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.
\end{aligned}$$

Similarly, we can replace successively all forms β_j by the β'_j, and by doing so, we find

$$\int_{\widetilde{X}} \beta'_1 \wedge \beta'_2 \wedge \ldots \wedge \beta'_k \wedge \mu^\star u \geq \int_{\widetilde{X}} \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k \wedge \mu^\star u.$$

We claim that the closed positive currents $\mu_\star(\beta_1 \wedge \ldots \wedge \beta_k)$ are uniformly bounded in mass. In fact, if ω is a Kähler metric in X, there exists a constant $C_j \geq 0$ such that $C_j\{\omega\} - \alpha_j$ is a Kähler class. Hence $C_j\omega - T_j \equiv \gamma_j$ for some Kähler form γ_j on X. By pulling back with μ, we find $C_j\mu^\star\omega - ([E_j] + \beta_j) \equiv \mu^\star\gamma_j$, hence

$$\beta_j \equiv C_j\mu^\star\omega - ([E_j] + \mu^\star\gamma_j).$$

By performing again a substitution in the integrals, we find

$$\int_{\widetilde{X}} \beta_1 \wedge \ldots \wedge \beta_k \wedge \mu^\star u \leq C_1 \ldots C_k \int_{\widetilde{X}} \mu^\star\omega^k \wedge \mu^\star u = C_1 \ldots C_k \int_X \omega^k \wedge u$$

and this is true especially for $u = \omega^{n-k}$. We can now arrange that for each of the integrals associated with a countable dense family of forms u, the supremum is achieved by a sequence of currents $(\mu_m)_\star(\beta_{1,m} \wedge \ldots \wedge \beta_{k,m})$ obtained as direct images by a suitable sequence of modifications $\mu_m : \widetilde{X}_m \to X$. By extracting a subsequence, we can achieve that this sequence is weakly convergent and we set

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m\to+\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}$$

(the monotonicity is not in terms of the currents themselves, but in terms of the integrals obtained when we evaluate against a smooth closed semi-positive form u). By evaluating against a basis of positive classes $\{u\} \in H^{n-k,n-k}(X)$, we infer by Poincaré duality that the class of $\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle$ is uniquely defined (although, in general, the representing current is not unique).

(ii) It is indeed clear from the definition that the movable intersection product is homogeneous, increasing and superadditive in each argument, at least when the α_j's are in $\mathcal{E}^\circ$. However, we can extend the product to the closed cone $\mathcal{E}$ by monotonicity, by setting

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{\delta\downarrow 0} \downarrow \langle (\alpha_1 + \delta\omega) \cdot (\alpha_2 + \delta\omega) \cdots (\alpha_k + \delta\omega) \rangle$$

for arbitrary classes $\alpha_j \in \mathcal{E}$ (again, monotonicity occurs only where we evaluate against closed semi-positive forms u). By weak compactness, the movable intersection product can always be represented by a closed positive current of bidegree (k,k).

(iii) The Teissier-Hovanskii inequalities are a direct consequence of the fact that they hold true for nef classes, so we just have to apply them to the classes $\beta_{j,m}$ on $\widetilde{X}_m$ and pass to the limit.

(iv) When $k = 1$ and $\alpha \in \mathcal{E}^0$, we have

$$\alpha = \lim_{m\to+\infty} \{(\mu_m)_\star T_m\} = \lim_{m\to+\infty} (\mu_m)_\star[E_m] + \{(\mu_m)_\star \beta_m\}$$

and $\langle \alpha \rangle = \lim_{m\to+\infty} \{(\mu_m)_\star \beta_m\}$ by definition. However, the images $F_m = (\mu_m)_\star F_m$ are effective $\mathbb{Q}$-divisors in X, and the filtering property implies that F_m is a decreasing sequence. It must therefore converge to a (uniquely defined) limit $F = \lim F_m := N(\alpha)$ which is an effective $\mathbb{R}$-divisor, and we get the asserted decomposition in the limit.

Since $N(\alpha) = \alpha - \langle \alpha \rangle$ we easily see that $N(\alpha)$ is subadditive and that $N(\alpha) = 0$ if α is the class of a smooth semi-positive form. When α is no longer a big class, we define

$$\langle \alpha \rangle = \lim_{\delta\downarrow 0} \downarrow \langle \alpha + \delta\omega \rangle, \qquad N(\alpha) = \lim_{\delta\downarrow 0} \uparrow N(\alpha + \delta\omega)$$

(the subadditivity of N implies $N(\alpha + (\delta+\varepsilon)\omega) \le N(\alpha + \delta\omega)$). The divisorial Zariski decomposition follows except maybe for the fact that $N(\alpha)$ might be a convergent countable sum of divisors. However, this will be ruled out when (v) is proved. As $N(\bullet)$ is subadditive and homogeneous, the set $\mathcal{E}_1 = \{\alpha \in \mathcal{E} \,;\, N(\alpha) = 0\}$ is a closed convex conne, and we find that $\alpha \mapsto \langle \alpha \rangle$ is a projection of $\mathcal{E}$ onto $\mathcal{E}_1$ (according to [Bou02], $\mathcal{E}_1$ consists of those pseudo-effective classes which are "nef in codimension 1").

(v) Let $\alpha \in \mathcal{E}^\circ$, and assume that $N(\alpha)$ contains linearly dependent components F_j. Then already all currents $T \in \alpha$ should be such that $\mu^\star T = [E] + \beta$ where $F = \mu_\star E$ contains those linearly dependent components. Write $F = \sum \lambda_j F_j$, $\lambda_j > 0$ and assume that

$$\sum_{j \in J} c_j F_j \equiv 0$$

for a certain non trivial linear combination. Then some of the coefficients c_j must be negative (and some other positive). Then E is numerically equivalent to

$$E' \equiv E + t\mu^\star\Big(\sum \lambda_j F_j\Big),$$

and by choosing $t > 0$ appropriate, we obtain an effective divisor E' which has a zero coefficient on one of the components $\mu^\star F_{j_0}$. By replacing E with $\min(E, E')$ via (10.4′), we eliminate the component $\mu^\star F_{j_0}$. This is a contradiction since $N(\alpha)$ was supposed to contain F_{j_0}. □

(10.6) Definition. *For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, we define the* <u>*numerical dimension*</u> *$\nu(\alpha)$ to be $\nu(\alpha) = -\infty$ if α is not pseudo-effective, and*

$$\nu(\alpha) = \max\{p \in \mathbb{N}\ ;\ \langle \alpha^p \rangle \neq 0\}, \qquad \nu(\alpha) \in \{0, 1, \ldots, n\}$$

if α is pseudo-effective.

By the results of [DP03], a class is big ($\alpha \in \mathcal{E}^\circ$) if and only if $\nu(\alpha) = n$. Classes of numerical dimension 0 can be described much more precisely, again following Boucksom [Bou02].

(10.7) Theorem. *Let X be a compact Kähler manifold. Then the subset $\mathcal{D}_0$ of irreducible divisors D in X such that $\nu(D) = 0$ is countable, and these divisors are rigid as well as their multiples. If $\alpha \in \mathcal{E}$ is a pseudo-effective class of numerical dimension 0, then α is numerically equivalent to an effective $\mathbb{R}$-divisor $D = \sum_{j \in J} \lambda_j D_j$, for some finite subset $(D_j)_{j \in J} \subset \mathcal{D}_0$ such that the cohomology classes $\{D_j\}$ are linearly independent and some $\lambda_j > 0$. If such a linear combination is of numerical dimension 0, then so is any other linear combination of the same divisors.*

Proof. It is immediate from the definition that a pseudo-effective class is of numerical dimension 0 if and only if $\langle \alpha \rangle = 0$, in other words if $\alpha = N(\alpha)$. Thus $\alpha \equiv \sum \lambda_j D_j$ as described in 10.7, and since $\lambda_j \langle D_j \rangle \leq \langle \alpha \rangle$, the divisors D_j must themselves have numerical dimension 0. There is at most one such divisor D in any given cohomology class in $NS(X) \cap \mathcal{E} \subset H^2(X, \mathbb{Z})$, otherwise two such divisors $D \equiv D'$ would yield a blow-up $\mu : \widetilde{X} \to X$ resolving the intersection, and by taking $\min(\mu^\star D, \mu^\star D')$ via (10.4′), we would find $\mu^\star D \equiv E + \beta$, $\beta \neq 0$, so that $\{D\}$ would not be of numerical dimension 0. This implies that there are at most countably many divisors of numerical dimension 0, and that these divisors are rigid as well as their multiples. □

The above general concept of numerical dimension leads to a very natural formulation of the abundance conjecture for non-minimal (Kähler) varieties.

(10.8) Generalized abundance conjecture. *For an arbitrary compact Kähler manifold X, the Kodaira dimension should be equal to the numerical dimension :*

$$\kappa(X) = \nu(X) := \nu(c_1(K_X)).$$

This appears to be a fairly strong statement. In fact, it is not difficult to show that the generalized abundance conjecture would contain the $C_{n,m}$ conjectures.

(10.9) Remark. Using the Iitaka fibration, it is immediate to see that $\kappa(X) \le \nu(X)$.

(10.10) Remark. It is known that abundance holds in the case $\nu(X) = -\infty$ (if K_X is not pseudo-effective, no multiple of K_X can have sections), or in the case $\nu(X) = n$ which implies K_X big ; the latter property follows e.g. from the solution of the Grauert-Riemenschneider conjecture in the form proven in [Dem85] (see also [DP03]).

In the remaining cases, the most tractable situation is the case when $\nu(X) = 0$. In fact Theorem 10.7 then gives $K_X \equiv \sum \lambda_j D_j$ for some effective divisor with numerically independent components, $\nu(D_j) = 0$. It follows that the λ_j are rational and therefore

$$(*) \qquad K_X \sim \sum \lambda_j D_j + F \qquad \text{where } \lambda_j \in \mathbb{Q}^+,\ \nu(D_j) = 0 \text{ and } F \in \operatorname{Pic}^0(X).$$

If we assume additionally that $q(X) = h^{0,1}(X)$ is zero, then mK_X is linearly equivalent to an integral divisor for some multiple m, and it follows immediately that $\kappa(X) = 0$. The case of a general projective manifold with $\nu(X) = 0$ and positive irregularity $q(X) > 0$ has been solved by Campana-Peternell [CP04], Corollary 3.7. It would be interesting to understand the Kähler case as well.

11. The Orthogonality Estimate

The goal of this section is to show that, in an appropriate sense, approximate Zariski decompositions are almost orthogonal.

(11.1) Theorem. *Let X be a projective manifold, and let $\alpha = \{T\} \in \mathcal{E}^\circ_{\mathrm{NS}}$ be a big class represented by a Kähler current T. Consider an approximate Zariski decomposition*

$$\mu_m^\star T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \le 20\,(C\omega)^n \big(\operatorname{Vol}(\alpha) - D_m^n\big)$$

where $\omega = c_1(H)$ is a Kähler form and $C \ge 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

Proof. For every $t \in [0,1]$, we have

$$\operatorname{Vol}(\alpha) = \operatorname{Vol}(E_m + D_m) \ge \operatorname{Vol}(tE_m + D_m).$$

Now, by our choice of C, we can write E_m as a difference of two nef divisors

$$E_m = \mu^\star\alpha - D_m = \mu_m^\star(\alpha + C\omega) - (D_m + C\mu_m^\star\omega).$$

(11.2) Lemma. *For all nef $\mathbb{R}$-divisors A, B we have*

$$\mathrm{Vol}(A - B) \geq A^n - nA^{n-1} \cdot B$$

as soon as the right hand side is positive.

Proof. In case A and B are integral (Cartier) divisors, this is a consequence of the holomorphic Morse inequalities, [Dem01, 8.4]; one can also argue by an elementary estimate of to $H^0(X, mA - B_1 - \ldots - B_m)$ via the Riemann-Roch formula (assuming A and B very ample, $B_1, \ldots, B_m \in |B|$ generic). If A and B are $\mathbb{Q}$-Cartier, we conclude by the homogeneity of the volume. The general case of $\mathbb{R}$-divisors follows by approximation using the upper semi-continuity of the volume [Bou02, 3.1.26]. □

(11.3) Remark. We hope that Lemma 11.2 also holds true on an arbitrary Kähler manifold for arbitrary nef (non necessarily integral) classes. This would follow from a generalization of holomorphic Morse inequalities to non integral classes. However the proof of such a result seems technically much more involved than in the case of integral classes.

(11.4) Lemma. *Let $\beta_1, \ldots, \beta_n$ and $\beta'_1, \ldots, \beta'_n$ be nef classes on a compact Kähler manifold $\widetilde{X}$ such that each difference $\beta'_j - \beta_j$ is pseudo-effective. Then the n-th intersection products satisfy*

$$\beta_1 \cdots \beta_n \leq \beta'_1 \cdots \beta'_n.$$

Proof. We can proceed step by step and replace just one β_j by $\beta' j \equiv \beta_j + T_j$ where T_j is a closed positive $(1,1)$-current and the other classes $\beta'_k = \beta_k$, $k \neq j$ are limits of Kähler forms. The inequality is then obvious. □

End of proof of Theorem 11.1. In order to exploit the lower bound of the volume, we write

$$tE_m + D_m = A - B, \qquad A = D_m + t\mu_m^\star(\alpha + C\omega), \quad B = t(D_m + C\mu_m^\star\omega).$$

By our choice of the constant C, both A and B are nef. Lemma 11.2 and the binomial formula imply

$$\begin{aligned}
\mathrm{Vol}(tE_m + D_m) &\geq A^n - nA^{n-1} \cdot B \\
&= D_m^n + nt\,D_m^{n-1} \cdot \mu_m^\star(\alpha + C\omega) + \sum_{k=2}^{n} t^k \binom{n}{k} D_m^{n-k} \cdot \mu_m^\star(\alpha + C\omega)^k \\
&\qquad - nt\,D_m^{n-1} \cdot (D_m + C\mu_m^\star\omega) \\
&\qquad - nt^2 \sum_{k=1}^{n-1} t^{k-1} \binom{n-1}{k} D_m^{n-1-k} \cdot \mu_m^\star(\alpha + C\omega)^k \cdot (D_m + C\mu_m^\star\omega).
\end{aligned}$$

Now, we use the obvious inequalities

$$D_m \le \mu_m^\star(C\omega), \qquad \mu_m^\star(\alpha + C\omega) \le 2\mu_m^\star(C\omega), \qquad D_m + C\mu_m^\star\omega \le 2\mu_m^\star(C\omega)$$

in which all members are nef (and where the inequality $\le$ means that the difference of classes is pseudo-effective). We use Lemma 11.4 to bound the last summation in the estimate of the volume, and in this way we get

$$\mathrm{Vol}(tE_m + D_m) \ge D_m^n + ntD_m^{n-1}\cdot E_m - nt^2\sum_{k=1}^{n-1} 2^{k+1}t^{k-1}\binom{n-1}{k}(C\omega)^n.$$

We will always take t smaller than $1/10n$ so that the last summation is bounded by $4(n-1)(1+1/5n)^{n-2} < 4ne^{1/5} < 5n$. This implies

$$\mathrm{Vol}(tE_m + D_m) \ge D_m^n + nt\,D_m^{n-1}\cdot E_m - 5n^2t^2(C\omega)^n.$$

Now, the choice $t = \frac{1}{10n}(D_m^{n-1}\cdot E_m)((C\omega)^n)^{-1}$ gives by substituting

$$\frac{1}{20}\frac{(D_m^{n-1}\cdot E_m)^2}{(C\omega)^n} \le \mathrm{Vol}(E_m + D_m) - D_m^n \le \mathrm{Vol}(\alpha) - D_m^n$$

(and we have indeed $t \le \frac{1}{10n}$ by Lemma 11.4), whence Theorem 11.1. Of course, the constant 20 is certainly not optimal. □

(11.5) Corollary. *If $\alpha \in \mathcal{E}_{\mathrm{NS}}$, then the divisorial Zariski decomposition $\alpha = N(\alpha) + \langle\alpha\rangle$ is such that*

$$\langle\alpha^{n-1}\rangle\cdot N(\alpha) = 0.$$

Proof. By replacing α by $\alpha + \delta c_1(H)$, one sees that it is sufficient to consider the case where α is big. Then the orthogonality estimate implies

$$(\mu_m)_\star(D_m^{n-1})\cdot(\mu_m)_\star E_m = D_m^{n-1}\cdot(\mu_m)^\star(\mu_m)_\star E_m \le D_m^{n-1}\cdot E_m \le C(\mathrm{Vol}(\alpha) - D_m^n)^{1/2}.$$

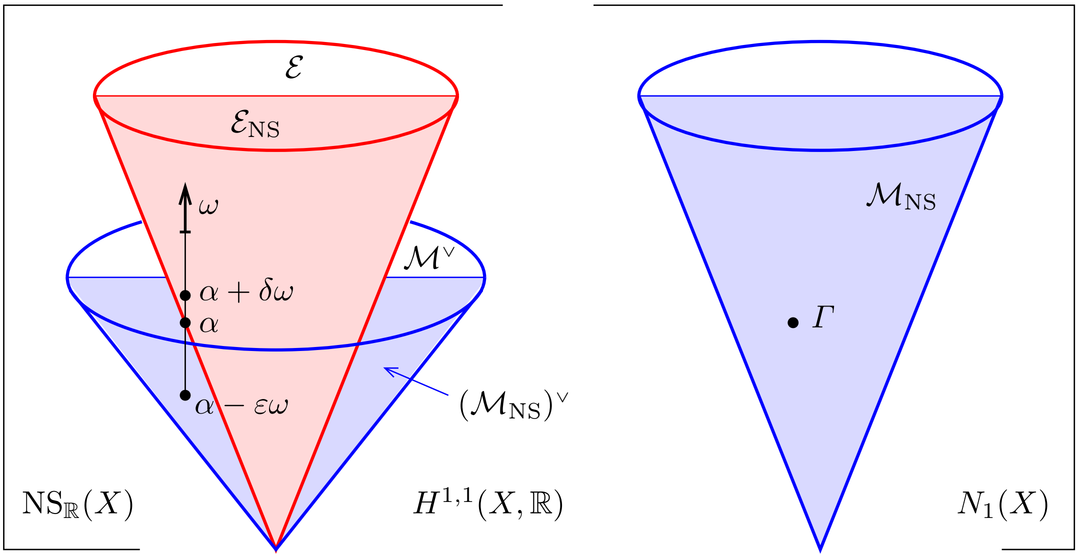
Since $\langle\alpha^{n-1}\rangle = \lim(\mu_m)_\star(D_m^{n-1})$, $N(\alpha) = \lim(\mu_m)_\star E_m$ and $\lim D_m^n = \mathrm{Vol}(\alpha)$, we get the desired conclusion in the limit. □

12. Proof of the Main Duality Theorem

The proof is reproduced from [BDPP03]. We want to show that $\mathcal{E}_{\mathrm{NS}}$ and $\mathrm{SME}(X)$ are dual (Theorem 8.2). By 7.4 (iii) we have in any case

$$\mathcal{E}_{\mathrm{NS}} \subset (\mathrm{SME}(X))^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$ on the boundary of $\mathcal{E}_{\mathrm{NS}}$ which is in the interior of $\mathrm{SME}(X)^\vee$.



Let $\omega = c_1(H)$ be an ample class. Since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class $\alpha + \delta\omega$ is big for every $\delta > 0$, and since $\alpha \in ((\mathrm{SME}(X))^\vee)^\circ$ we still have $\alpha - \varepsilon\omega \in (\mathrm{SME}(X))^\vee$ for $\varepsilon > 0$ small. Therefore

$$\alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma \tag{12.1}$$

for every movable curve Γ. We are going to contradict (12.1). Since $\alpha + \delta\omega$ is big, we have an approximate Zariski decomposition

$$\mu_\delta^\star(\alpha + \delta\omega) = E_\delta + D_\delta.$$

We pick $\Gamma = (\mu_\delta)_\star(D_\delta^{n-1})$. By the Hovanskii-Teissier concavity inequality

$$\omega \cdot \Gamma \geq (\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}.$$

On the other hand

$$\begin{aligned}
\alpha \cdot \Gamma &= \alpha \cdot (\mu_\delta)_\star(D_\delta^{n-1}) \\
&= \mu_\delta^\star\alpha \cdot D_\delta^{n-1} \leq \mu_\delta^\star(\alpha + \delta\omega) \cdot D_\delta^{n-1} \\
&= (E_\delta + D_\delta) \cdot D_\delta^{n-1} = D_\delta^n + D_\delta^{n-1} \cdot E_\delta.
\end{aligned}$$

By the orthogonality estimate, we find

$$\begin{aligned}
\frac{\alpha \cdot \Gamma}{\omega \cdot \Gamma} &\leq \frac{D_\delta^n + \big(20(C\omega)^n(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)\big)^{1/2}}{(\omega^n)^{1/n}(D_\delta^n)^{(n-1)/n}} \\
&\leq C'(D_\delta^n)^{1/n} + C''\frac{(\mathrm{Vol}(\alpha + \delta\omega) - D_\delta^n)^{1/2}}{(D_\delta^n)^{(n-1)/n}}.
\end{aligned}$$

However, since $\alpha \in \partial\mathcal{E}_{\mathrm{NS}}$, the class α cannot be big so

$$\lim_{\delta \to 0} D_\delta^n = \mathrm{Vol}(\alpha) = 0.$$

We can also take D_δ to approximate $\mathrm{Vol}(\alpha+\delta\omega)$ in such a way that $(\mathrm{Vol}(\alpha+\delta\omega)-D_\delta^n)^{1/2}$ tends to 0 much faster than D_δ^n. Notice that $D_\delta^n \ge \delta^n\omega^n$, so in fact it is enough to take

$$\mathrm{Vol}(\alpha+\delta\omega)-D_\delta^n \le \delta^{2n}.$$

This is the desired contradiction by (12.1). □

(12.2) Remark. If holomorphic Morse inequalities were known also in the Kähler case, we would infer by the same proof that "α not pseudo-effective" implies the existence of a blow-up $\mu:\widetilde X\to X$ and a Kähler metric $\widetilde\omega$ on $\widetilde X$ such that $\alpha\cdot\mu_\star(\widetilde\omega)^{n-1}<0$. In the special case when $\alpha=K_X$ is not pseudo-effective, we would expect the Kähler manifold X to be covered by rational curves. The main trouble is that characteristic p techniques are no longer available. On the other hand it is tempting to approach the question via techniques of symplectic geometry :

(12.3) Question. *Let (M,ω) be a compact real symplectic manifold. Fix an almost complex structure J compatible with ω, and for this structure, assume that $c_1(M)\cdot\omega^{n-1}>0$. Does it follow that M is covered by rational J-pseudoholomorphic curves ?*

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(July 30, 2004, revised on November 26, 2007, printed on November 26, 2007)

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