



Kähler manifolds and transcendental techniques in algebraic geometry

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Complex manifolds / (p, q) -forms

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- A complex n -dimensional manifold is given by coordinate charts equipped with **local holomorphic coordinates** (z_1, z_2, \dots, z_n) .
- A differential form u of type (p, q) can be written as a sum

$$u(z) = \sum_{|J|=p, |K|=q} u_{JK}(z) dz_J \wedge d\bar{z}_K$$

where $J = (j_1, \dots, j_p)$, $K = (k_1, \dots, k_q)$,

$$dz_J = dz_{j_1} \wedge \dots \wedge dz_{j_p}, \quad d\bar{z}_K = d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q}.$$

Complex manifolds / Currents

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- The current T is said to be **positive** if the distribution $\sum \lambda_j \bar{\lambda}_k T_{JK}$ is a positive real measure for all $(\lambda_J) \in \mathbb{C}^N$ (so that $T_{KJ} = \overline{T}_{JK}$, hence $\overline{T} = T$).
- The coefficients T_{JK} are then **complex measures** – and the diagonal ones T_{JJ} are **positive real measures**.

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- The coefficients T_{JK} are then **complex measures** – and the diagonal ones T_{JJ} are **positive real measures**.
- T is said to be **closed** if $dT = 0$ in the sense of distributions.

Complex manifolds / Basic examples of Currents

- The current of integration over a codimension p analytic cycle $A = \sum c_j A_j$ is defined by duality as $[A] = \sum c_j [A_j]$ with

$$\langle [A_j], u \rangle = \int_{A_j} u|_{A_j}$$

for every $(n-p, n-p)$ test form u on X .

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- Hessian forms of plurisubharmonic functions :

$$\varphi \text{ plurisubharmonic} \Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) \geq 0$$

then

$$T = i\partial\bar{\partial}\varphi \quad \text{is a closed positive } (1,1)\text{-current.}$$

Complex manifolds / Kähler metrics

- A **Kähler metric** is a smooth **positive definite** $(1, 1)$ -form

$$\omega(z) = i \sum_{1 \leq j, k \leq n} \omega_{jk}(z) dz_j \wedge d\bar{z}_k \quad \text{such that } d\omega = 0.$$

- The manifold X is said to be **Kähler** (or of **Kähler type**) if it possesses at least one Kähler metric ω [Kähler 1933]

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- The manifold X is said to be **Kähler** (or of **Kähler type**) if it possesses at least one Kähler metric ω [Kähler 1933]
- Every complex analytic and locally closed submanifold $X \subset \mathbb{P}_{\mathbb{C}}^N$ in projective space is Kähler when equipped with the restriction of the **Fubini-Study metric**

$$\omega_{FS} = \frac{i}{2\pi} \log(|z_0|^2 + |z_1|^2 + \dots + |z_N|^2).$$

- Especially projective algebraic varieties are Kähler.

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- Cohomology classes [forms / currents yield same groups]
 α d -closed k -form/current to $\mathbb{C} \mapsto \{\alpha\} \in H^k(X, \mathbb{C})$
 α $\bar{\partial}$ -closed (p, q) -form/current to $F \mapsto \{\alpha\} \in H^{p,q}(X, F)$

Dolbeault isomorphism (Dolbeault - Grothendieck 1953)

$$\begin{aligned} H^{0,q}(X, F) &\simeq H^q(X, \mathcal{O}(F)), \\ H^{p,q}(X, F) &\simeq H^q(X, \Omega_X^p \otimes \mathcal{O}(F)) \end{aligned}$$

Hodge decomposition theorem

- **Theorem.** *If (X, ω) is compact Kähler, then*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}).$$

- *Each group $H^{p,q}(X, \mathbb{C})$ is isomorphic to the space of (p, q) **harmonic forms** α with respect to ω , i.e. $\Delta_\omega \alpha = 0$.*

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- **Hodge Conjecture** [a millenium problem!].
*If X is a projective algebraic manifold,
Hodge (p, p) -classes $= H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$
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are generated by **classes of algebraic cycles of codimension p with \mathbb{Q} -coefficients**.*
- **(Claire Voisin, 2001)** \exists 4-dimensional complex torus X possessing a non trivial Hodge class of type $(2, 2)$, such that every coherent analytic sheaf \mathcal{F} on X satisfies $c_2(\mathcal{F}) = 0$.

Idea of proof of Claire Voisin's counterexample

The idea is to show the existence of a 4-dimensional complex torus $X = \mathbb{C}^4/\Lambda$ which does not contain any analytic subset of positive dimension, and such that the Hodge classes of degree 4 are perpendicular to ω^{n-2} for a suitable choice of the Kähler metric ω .

The lattice Λ is explicitly found via a number theoretic **construction of Weil** based on the number field $\mathbb{Q}[i]$, also considered by S. Zucker.

The theorem of existence of **Hermitian Yang-Mills connections** for stable bundles combined with Lübke's inequality then implies $c_2(\mathcal{F}) = 0$ for every coherent sheaf \mathcal{F} on the torus.

Kodaira embedding theorem

Theorem. *X a compact complex n -dimensional manifold.
Then the following properties are equivalent.*

- *X can be embedded in some projective space $\mathbb{P}_{\mathbb{C}}^N$ as a closed analytic submanifold (and such a submanifold is automatically algebraic by Chow's theorem).*
- *X carries a hermitian holomorphic line bundle (L, h) with positive definite smooth curvature form $i\Theta_{L,h} > 0$.
For $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$,*

$$i\Theta_{L,h} = i\partial\bar{\partial}\varphi = -i\partial\bar{\partial}\log h,$$

$$c_1(L) = \left\{ \frac{i}{2\pi} \Theta_{L,h} \right\}.$$

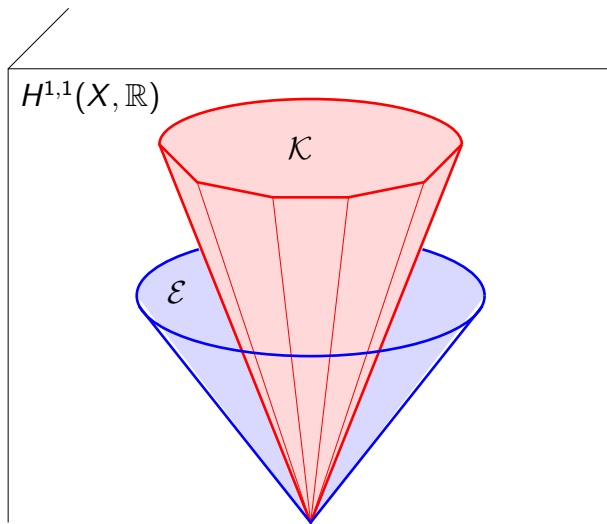
- *X possesses a Hodge metric, i.e., a Kähler metric ω such that $\{\omega\} \in H^2(X, \mathbb{Z})$.*

Positive cones

Definition. Let X be a compact Kähler manifold.

- The **Kähler cone** is the set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of Kähler forms. This is an open convex cone.
- The **pseudo-effective cone** is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1, 1)$ currents. This is a closed convex cone.
(by weak compactness of bounded sets of currents).
- Always true: $\overline{\mathcal{K}} \subset \mathcal{E}$.
- One can have: $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
if X is the surface obtained by blowing-up \mathbb{P}^2 in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Kähler (red) cone and pseudoeffective (blue) cone



Neron Severi parts of the cones

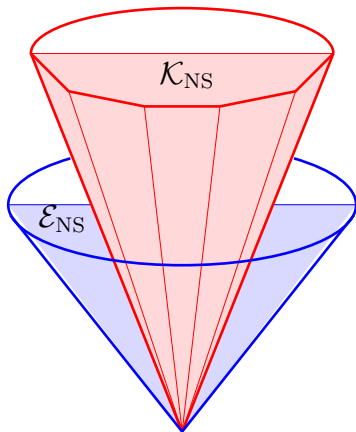
In case X is projective, it is interesting to consider the “algebraic part” of our “transcendental cones” \mathcal{K} and \mathcal{E} , which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X, \mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$\begin{aligned}\mathrm{NS}(X) &:= H^{1,1}(X, \mathbb{R}) \cap (H^2(X, \mathbb{Z})/\{\text{torsion}\}), \\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \\ \mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X).\end{aligned}$$

Neron Severi parts of the cones

$$H^{1,1}(X, \mathbb{R})$$

$$\text{NS}_{\mathbb{R}}(X)$$

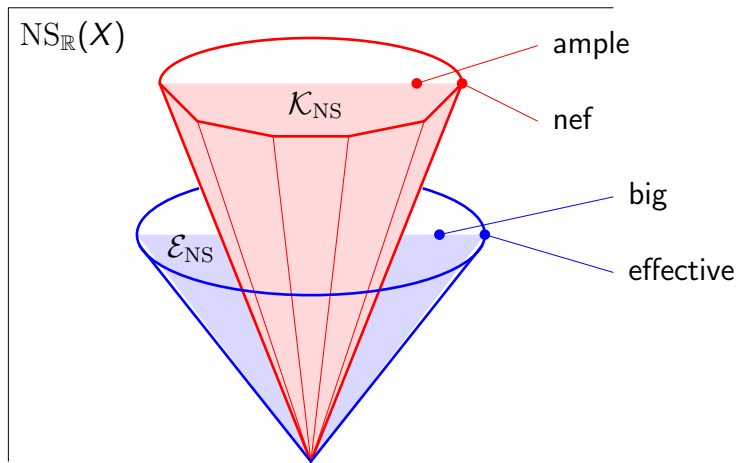


Theorem (Kodaira+successors, D90). Assume X projective.

- \mathcal{K}_{NS} is the open cone generated by **ample** (or **very ample**) divisors A (Recall that a divisor A is said to be very ample if the linear system $H^0(X, \mathcal{O}(A))$ provides an embedding of X in projective space).
- The closed cone $\overline{\mathcal{K}}_{\text{NS}}$ consists of the closure of the cone of **nef divisors** D (or nef line bundles L), namely effective integral divisors D such that $D \cdot C \geq 0$ for every curve C .
- \mathcal{E}_{NS} is the closure of the cone of **effective divisors**, i.e. divisors $D = \sum c_j D_j$, $c_j \in \mathbb{R}_+$.
- The interior $\mathcal{E}_{\text{NS}}^\circ$ is the cone of **big divisors**, namely divisors D such that $h^0(X, \mathcal{O}(kD)) \geq c k^{\dim X}$ for k large.

Proof: L^2 estimates for $\bar{\partial}$ / Bochner-Kodaira technique

ample / nef / effective / big divisors



Approximation of currents, Zariski decomposition

- **Definition.** *On X compact Kähler, a **Kähler current** T is a closed positive $(1, 1)$ -current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.*

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 - **Theorem.** $\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha = \{T\}$, $T =$ a Kähler current.
- We say that \mathcal{E}° is the cone of **big** $(1, 1)$ -classes.

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We say that \mathcal{E}° is the cone of **big $(1, 1)$ -classes**.

- **Theorem (D92).** Any Kähler current T can be written

$$T = \lim T_m$$

where $T_m \in \alpha = \{T\}$ has **logarithmic poles**, i.e.

\exists a **modification** $\mu_m : \tilde{X}_m \rightarrow X$ such that

$$\mu_m^* T_m = [E_m] + \beta_m$$

where E_m is an effective \mathbb{Q} -divisor on \tilde{X}_m with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on \tilde{X}_m .

Idea of proof of analytic Zariski decomposition (1)

Locally one can write $T = i\partial\bar{\partial}\varphi$ for some strictly plurisubharmonic potential φ on X . The approximating potentials φ_m of φ are defined as

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \left\{ f \in \mathcal{O}(\Omega); \int_{\Omega} |f|^2 e^{-2m\varphi} dV < +\infty \right\}.$$

The Ohsawa-Takegoshi L^2 extension theorem (applied to extension from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$. On the other hand $\varphi = \lim_{m \rightarrow +\infty} \varphi_m$ by a Bergman kernel trick and by the mean value inequality.

Idea of proof of analytic Zariski decomposition (2)

The Hilbert basis $(g_{\ell,m})$ is a family of local generators of the multiplier ideal sheaf $\mathcal{I}(mT) = \mathcal{I}(m\varphi)$. The modification $\mu_m : \tilde{X}_m \rightarrow X$ is obtained by blowing-up this ideal sheaf, with

$$\mu_m^* \mathcal{I}(mT) = \mathcal{O}(-mE_m).$$

for some effective \mathbb{Q} -divisor E_m with normal crossings on \tilde{X}_m . Now, we set $T_m = i\partial\bar{\partial}\varphi_m$ and $\beta_m = \mu_m^* T_m - [E_m]$. Then $\beta_m = i\partial\bar{\partial}\psi_m$ where

$$\psi_m = \frac{1}{2m} \log \sum_{\ell} |g_{\ell,m} \circ \mu_m / h|^2 \quad \text{locally on } \tilde{X}_m$$

and h is a generator of $\mathcal{O}(-mE_m)$, and we see that β_m is a smooth semi-positive form on \tilde{X}_m . The construction can be made global by using a gluing technique, e.g. via partitions of unity, and β_m can be made Kähler by a perturbation argument.

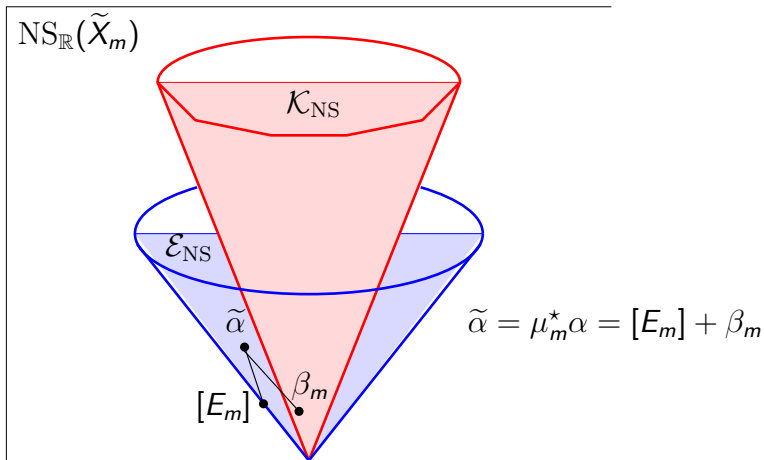
Algebraic analogue

The more familiar algebraic analogue would be to take $\alpha = c_1(L)$ with a big line bundle L and to blow-up the base locus of $|mL|$, $m \gg 1$, to get a \mathbb{Q} -divisor decomposition

$$\mu_m^* L \sim E_m + D_m, \quad E_m \text{ effective, } D_m \text{ free.}$$

Such a blow-up is usually referred to as a “log resolution” of the linear system $|mL|$, and we say that $E_m + D_m$ is an approximate Zariski decomposition of L . We will also use this terminology for Kähler currents with logarithmic poles.

Analytic Zariski decomposition



Characterization of the Fujiki class \mathcal{C}

Theorem (Demailly-Păun 2004). *A compact complex manifold X is bimeromorphic to a Kähler manifold \tilde{X} (or equivalently, dominated by a Kähler manifold \tilde{X}) if and only if it carries a Kähler current T .*

Proof. If $\mu: \tilde{X} \rightarrow X$ is a modification and $\tilde{\omega}$ is a Kähler metric on \tilde{X} , then $T = \mu_* \tilde{\omega}$ is a Kähler current on X .

Conversely, if T is a Kähler current, we take $\tilde{X} = \tilde{X}_m$ and $\tilde{\omega} = \beta_m$ for m large enough.

Definition. *The class of compact complex manifolds X bimeromorphic to some Kähler manifold \tilde{X} is called the Fujiki class \mathcal{C} .*

Hodge decomposition still holds true in \mathcal{C} .

Numerical characterization of the Kähler cone

Theorem (Demailly-Păun 2004).

Let X be a compact Kähler manifold. Let

$$\mathcal{P} = \left\{ \alpha \in H^{1,1}(X, \mathbb{R}) ; \int_Y \alpha^p > 0, \forall Y \subset X, \dim Y = p \right\}.$$

“cone of numerically positive classes”.

*Then the Kähler cone \mathcal{K} is
one of the connected components of \mathcal{P} .*

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Corollary (Projective case).

If X is projective algebraic, then $\mathcal{K} = \mathcal{P}$.

*Note: this is a “transcendental version” of the
Nakai-Moishezon criterion.*

Example (non projective) for which $\mathcal{K} \subsetneq \mathcal{P}$.

Take $X =$ generic complex torus $X = \mathbb{C}^n/\Lambda$.

Then X **does not possess any analytic subset** except finite subsets and X itself.

Hence $\mathcal{P} = \{\alpha \in H^{1,1}(X, \mathbb{R}); \int_X \alpha^n > 0\}$.

Since $H^{1,1}(X, \mathbb{R})$ is in one-to-one correspondence with constant hermitian forms, \mathcal{P} is the set of hermitian forms on \mathbb{C}^n such that $\det(\alpha) > 0$, i.e.

possessing an even number of negative eigenvalues.

\mathcal{K} is the component with all eigenvalues > 0 .

Proof of the theorem : use Monge-Ampère

Fix $\alpha \in \overline{\mathcal{K}}$ so that $\int_X \alpha^n > 0$.

If ω is Kähler, then $\{\alpha + \varepsilon\omega\}$ is a Kähler class $\forall \varepsilon > 0$.

Use the **Calabi-Yau theorem** (Yau 1978) to solve the Monge-Ampère equation

$$(\alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon)^n = f_\varepsilon$$

where $f_\varepsilon > 0$ is a suitably chosen volume form.

Necessary and sufficient condition :

$$\int_X f_\varepsilon = (\alpha + \varepsilon\omega)^n \quad \text{in } H^{n,n}(X, \mathbb{R}).$$

Otherwise, the volume form of the Kähler metric $\alpha_\varepsilon = \alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon$ can be spread **randomly**.

Proof of the theorem : concentration of mass

In particular, the mass of the right hand side f_ε can be spread in an ε -neighborhood U_ε of any given subvariety $Y \subset X$.

If $\text{codim } Y = p$, one can show that

$$(\alpha + \varepsilon\omega + i\partial\bar{\partial}\varphi_\varepsilon)^p \rightarrow \Theta \quad \text{weakly}$$

where Θ positive (p, p) -current and $\Theta \geq \delta[Y]$ for some $\delta > 0$.

Now, “diagonal trick”: apply the above result to

$$\tilde{X} = X \times X, \quad \tilde{Y} = \text{diagonal} \subset \tilde{X}, \quad \tilde{\alpha} = \text{pr}_1^* \alpha + \text{pr}_2^* \alpha.$$

As $\tilde{\alpha}$ is nef on \tilde{X} and $\int_{\tilde{X}} (\tilde{\alpha})^{2n} > 0$, it follows by the above that the class $\{\tilde{\alpha}\}^n$ contains a Kähler current Θ such that $\Theta \geq \delta[\tilde{Y}]$ for some $\delta > 0$. Therefore

$$T := (\text{pr}_1)_*(\Theta \wedge \text{pr}_2^* \omega)$$

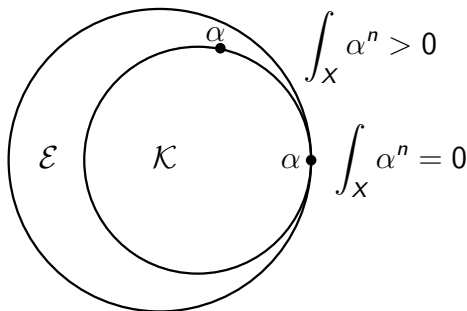
is numerically equivalent to a multiple of α and dominates $\delta\omega$, and we see that T is a Kähler current.

Generalized Grauert-Riemenschneider result

Main conclusion (Demailly-Păun 2004).

Let X be a compact Kähler manifold and let $\{\alpha\} \in \overline{\mathcal{K}}$ such that $\int_X \alpha^n > 0$.

Then $\{\alpha\}$ contains a Kähler current T , i.e. $\{\alpha\} \in \mathcal{E}^\circ$.



Final step of proof

Clearly the open cone \mathcal{K} is contained in \mathcal{P} , hence in order to show that \mathcal{K} is one of the connected components of \mathcal{P} , we need only show that \mathcal{K} is closed in \mathcal{P} , i.e. that $\overline{\mathcal{K}} \cap \mathcal{P} \subset \mathcal{K}$. Pick a class $\{\alpha\} \in \overline{\mathcal{K}} \cap \mathcal{P}$. In particular $\{\alpha\}$ is nef and satisfies $\int_X \alpha^n > 0$. Hence $\{\alpha\}$ contains a Kähler current T .

Now, an induction on dimension using the assumption $\int_Y \alpha^p > 0$ for all analytic subsets Y (we also use resolution of singularities for Y at this step) shows that the restriction $\{\alpha\}|_Y$ is the class of a Kähler current on Y .

We conclude that $\{\alpha\}$ is a Kähler class by results of Paun (PhD 1997), therefore $\{\alpha\} \in \mathcal{K}$.

Variants of the main theorem

Corollary 1 (DP2004). *Let X be a compact Kähler manifold.*

$\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ is Kähler $\Leftrightarrow \exists \omega$ Kähler s.t. $\int_Y \alpha^k \wedge \omega^{p-k} > 0$

for all $Y \subset X$ irreducible and all $k = 1, 2, \dots, p = \dim Y$.

Proof. Argue with $(1-t)\alpha + t\omega$, $t \in [0, 1]$.

Corollary 2 (DP2004). *Let X be a compact Kähler manifold.*

$\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ is nef ($\alpha \in \overline{\mathcal{K}}$) $\Leftrightarrow \forall \omega$ Kähler $\int_Y \alpha \wedge \omega^{p-1} \geq 0$

for all $Y \subset X$ irreducible and all $k = 1, 2, \dots, p = \dim Y$.

Consequence. *the dual of the nef cone $\overline{\mathcal{K}}$ is the closed convex cone in $H_{\mathbb{R}}^{n-1, n-1}(X)$ generated by cohomology classes of currents of the form $[Y] \wedge \omega^{p-1}$ in $H^{n-1, n-1}(X, \mathbb{R})$.*

Deformations of compact Kähler manifolds

A **deformation of compact complex manifolds** is a proper holomorphic map

$$\pi : \mathcal{X} \rightarrow S \quad \text{with smooth fibers } X_t = \pi^{-1}(t).$$

Basic question (Kodaira \sim 1960). Is every compact Kähler manifold X a limit of projective manifolds :

$$X \simeq X_0 = \lim X_{t_\nu}, \quad t_\nu \rightarrow 0, \quad X_{t_\nu} \text{ projective ?}$$

Recent results by Claire Voisin (2004)

- *In any dimension ≥ 4 , $\exists X$ compact Kähler manifold which does not have the homotopy type (or even the homology ring) of a complex projective manifold.*
- *In any dimension ≥ 8 , $\exists X$ compact Kähler manifold such that no compact bimeromorphic model X' of X has the homotopy type of a projective manifold.*

Conjecture on deformation stability of the Kähler property

Theorem (Kodaira and Spencer 1960).

The Kähler property is open with respect to deformation : if X_{t_0} is Kähler for some $t_0 \in S$, then the nearby fibers X_t are also Kähler (where “nearby” is in metric topology).

We expect much more.

Conjecture. Let $\mathcal{X} \rightarrow S$ be a deformation with irreducible base space S such that *some fiber X_{t_0} is Kähler*. Then there should exist a countable union of analytic strata $S_\nu \subset S$, $S_\nu \neq S$, such that

- X_t is Kähler for $t \in S \setminus \bigcup S_\nu$.
- X_t is bimeromorphic to a Kähler manifold (i.e. has a Kähler current) for $t \in \bigcup S_\nu$.

Theorem on deformation stability of Kähler cones

Theorem (Demailly-Păun 2004). *Let $\pi : \mathcal{X} \rightarrow S$ be a deformation of compact Kähler manifolds over an irreducible base S . Then there exists a countable union $S' = \bigcup S_\nu$ of analytic subsets $S_\nu \subsetneq S$, such that the Kähler cones $\mathcal{K}_t \subset H^{1,1}(X_t, \mathbb{C})$ of the fibers $X_t = \pi^{-1}(t)$ are $\nabla^{1,1}$ -invariant over $S \setminus S'$ under parallel transport with respect to the $(1,1)$ -projection $\nabla^{1,1}$ of the Gauss-Manin connection ∇ in the decomposition of*

$$\nabla = \begin{pmatrix} \nabla^{2,0} & * & 0 \\ * & \nabla^{1,1} & * \\ 0 & * & \nabla^{0,2} \end{pmatrix}$$

on the Hodge bundle $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$.

Positive cones in $H^{n-1,n-1}(X)$ and Serre duality

Definition. Let X be a compact Kähler manifold.

- Cone of $(n-1, n-1)$ positive currents

$$\mathcal{N} = \overline{\text{cone}}\{ \{ T \} \in H^{n-1,n-1}(X, \mathbb{R}) ; T \text{ closed } \geq 0 \}.$$

- Cone of effective curves

$$\begin{aligned} \mathcal{N}_{\text{NS}} &= \mathcal{N} \cap \text{NS}_{\mathbb{R}}^{n-1,n-1}(X), \\ &= \overline{\text{cone}}\{ \{ C \} \in H^{n-1,n-1}(X, \mathbb{R}) ; C \text{ effective curve} \}. \end{aligned}$$

- Cone of movable curves : with $\mu : \tilde{X} \rightarrow X$, let

$$\mathcal{M}_{\text{NS}} = \overline{\text{cone}}\{ \{ C \} \in H^{n-1,n-1}(X, \mathbb{R}) ; [C] = \mu_*(H_1 \cdots H_{n-1}) \}$$

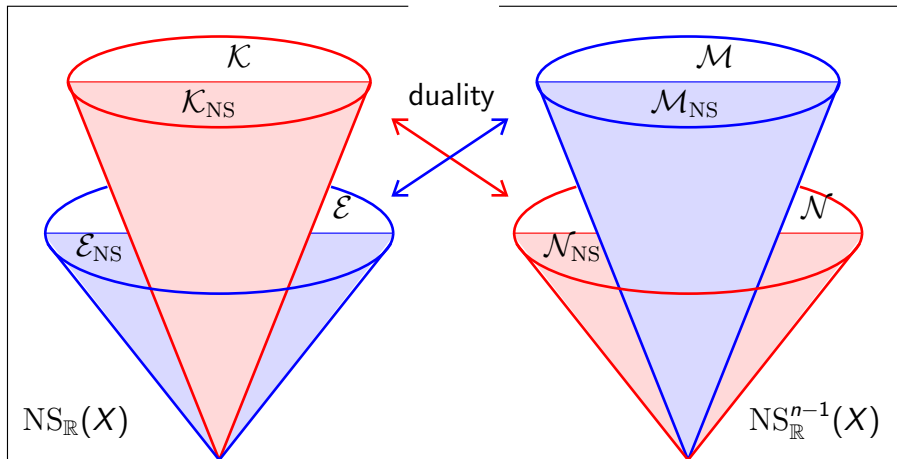
where $H_j =$ ample hyperplane section of \tilde{X} .

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$$\mathcal{M} = \overline{\text{cone}}\{ \{ T \} \in H^{n-1,n-1}(X, \mathbb{R}) ; T = \mu_*(\tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_{n-1}) \}$$

where $\tilde{\omega}_j =$ Kähler metric on \tilde{X} .

Main duality theorem



$$H^{1,1}(X, \mathbb{R}) \leftarrow \text{Serre duality} \rightarrow H^{n-1,n-1}(X, \mathbb{R})$$

Precise duality statement

Recall that the Serre duality pairing is

$$(\alpha^{(p,q)}, \beta^{(n-p,n-q)}) \longmapsto \int_X \alpha \wedge \beta.$$

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If X is compact Kähler, then

\mathcal{K} and \mathcal{N} are dual cones.

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Conjecture (Boucksom-Demailly-Paun-Peternell 2004)

If X is Kähler, then

\mathcal{E} and \mathcal{M} should be dual cones.

Concept of volume (very important !)

Definition (Boucksom 2002).

The *volume* (*movable self-intersection*) of a big class $\alpha \in \mathcal{E}^\circ$ is

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \int_{\tilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^* T = [E] + \beta$ with respect to some modification $\mu : \tilde{X} \rightarrow X$.

If $\alpha \in \mathcal{K}$, then $\mathrm{Vol}(\alpha) = \alpha^n = \int_X \alpha^n$.

Theorem. (Boucksom 2002). If L is a big line bundle and $\mu_m^*(mL) = [E_m] + [D_m]$ (where $E_m = \text{fixed part}$, $D_m = \text{moving part}$), then

$$\mathrm{Vol}(c_1(L)) = \lim_{m \rightarrow +\infty} \frac{n!}{m^n} h^0(X, mL) = \lim_{m \rightarrow +\infty} D_m^n.$$

Approximate Zariski decomposition

In other words, the volume measures the amount of sections and the growth of the degree of the images of the rational maps

$$\Phi_{|mL|} : X \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$$

By Fujita 1994 and Demailly-Ein-Lazarsfeld 2000, one has

Theorem. *Let L be a big line bundle on the projective manifold X . Let $\epsilon > 0$. Then there exists a modification $\mu : X_{\epsilon} \rightarrow X$ and a decomposition $\mu^*(L) = E + \beta$ with E an effective \mathbb{Q} -divisor and β a big and nef \mathbb{Q} -divisor such that*

$$\mathrm{Vol}(L) - \epsilon \leq \mathrm{Vol}(\beta) \leq \mathrm{Vol}(L).$$

Movable intersection theory

Theorem (Boucksom 2002) *Let X be a compact Kähler manifold and*

$$H_{\geq 0}^{k,k}(X) = \{ \{T\} \in H^{k,k}(X, \mathbb{R}) ; T \text{ closed } \geq 0 \}.$$

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- $\forall k = 1, 2, \dots, n, \exists$ *canonical “movable intersection product”*

$$\mathcal{E} \times \dots \times \mathcal{E} \rightarrow H_{\geq 0}^{k,k}(X), \quad (\alpha_1, \dots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_{k-1} \cdot \alpha_k \rangle$$

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such that $\text{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever α is a big class.

- *The product is increasing, homogeneous of degree 1 and superadditive in each argument, i.e.*

$$\langle \alpha_1 \cdots (\alpha'_j + \alpha''_j) \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha'_j \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha''_j \cdots \alpha_k \rangle.$$

It coincides with the ordinary intersection product when the $\alpha_j \in \overline{\mathcal{K}}$ are nef classes.

Movable intersection theory (continued)

- For $k = 1$, one gets a “divisorial Zariski decomposition”

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Movable intersection theory (continued)

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where :

- *$N(\alpha)$ is a uniquely defined effective divisor which is called the “negative divisorial part” of α . The map $\alpha \mapsto N(\alpha)$ is homogeneous and subadditive ;*
- *$\langle\alpha\rangle$ is “nef outside codimension 2”.*

Construction of the movable intersection product

First assume that all classes α_j are big, i.e. $\alpha_j \in \mathcal{E}^\circ$. Fix a smooth closed $(n - k, n - k)$ *semi-positive* form u on X . We select Kähler currents $T_j \in \alpha_j$ with logarithmic poles, and simultaneous **more and more accurate** log-resolutions $\mu_m : \tilde{X}_m \rightarrow X$ such that

$$\mu_m^* T_j = [E_{j,m}] + \beta_{j,m}.$$

We define

$$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle = \lim_{m \rightarrow +\infty} \uparrow \{ (\mu_m)_* (\beta_{1,m} \wedge \beta_{2,m} \wedge \cdots \wedge \beta_{k,m}) \}$$

as a weakly convergent subsequence. The main point is to show that there is actually convergence and that the **limit is unique in cohomology** ; this is based on “monotonicity properties” of the Zariski decomposition.

Generalized abundance conjecture

Definition. For a class $\alpha \in H^{1,1}(X, \mathbb{R})$, the numerical dimension $\nu(\alpha)$ is

- $\nu(\alpha) = -\infty$ if α is not pseudo-effective,
- $\nu(\alpha) = \max\{p \in \mathbb{N} ; \langle \alpha^p \rangle \neq 0\} \in \{0, 1, \dots, n\}$ if α is pseudo-effective.

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Conjecture (“generalized abundance conjecture”). For an arbitrary compact Kähler manifold X , the Kodaira dimension should be equal to the numerical dimension :

$$\kappa(X) = \nu(c_1(K_X)).$$

Remark. The generalized abundance conjecture holds true when $\nu(c_1(K_X)) = -\infty, 0, n$ (cases $-\infty$ and n being easy).

Orthogonality estimate

Theorem. *Let X be a projective manifold.*

Let $\alpha = \{T\} \in \mathcal{E}_{\text{NS}}^\circ$ be a big class represented by a Kähler current T , and consider an approximate Zariski decomposition

$$\mu_m^* T_m = [E_m] + [D_m]$$

Then

$$(D_m^{n-1} \cdot E_m)^2 \leq 20 (C\omega)^n (\text{Vol}(\alpha) - D_m^n)$$

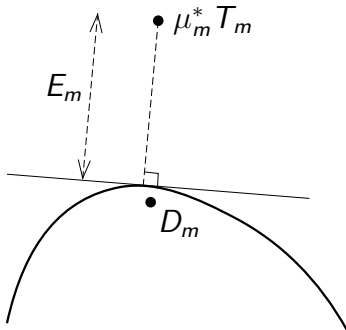
where $\omega = c_1(H)$ is a Kähler form and $C \geq 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ is nef).

By going to the limit, one gets

Corollary. $\alpha \cdot \langle \alpha^{n-1} \rangle - \langle \alpha^n \rangle = 0.$

Schematic picture of orthogonality estimate

The proof is similar to the case of projecting a point onto a convex set, where the segment to closest point is orthogonal to tangent plane.



Proof of duality between \mathcal{E}_{NS} and \mathcal{M}_{NS}

Theorem (Boucksom-Demailly-Păun-Peternell 2004).

For X projective, a class α is in \mathcal{E}_{NS} (pseudo-effective) if and only if it is dual to the cone \mathcal{M}_{NS} of moving curves.

Proof of the theorem. We want to show that $\mathcal{E}_{\text{NS}} = \mathcal{M}_{\text{NS}}^{\vee}$. By obvious positivity of the integral pairing, one has in any case

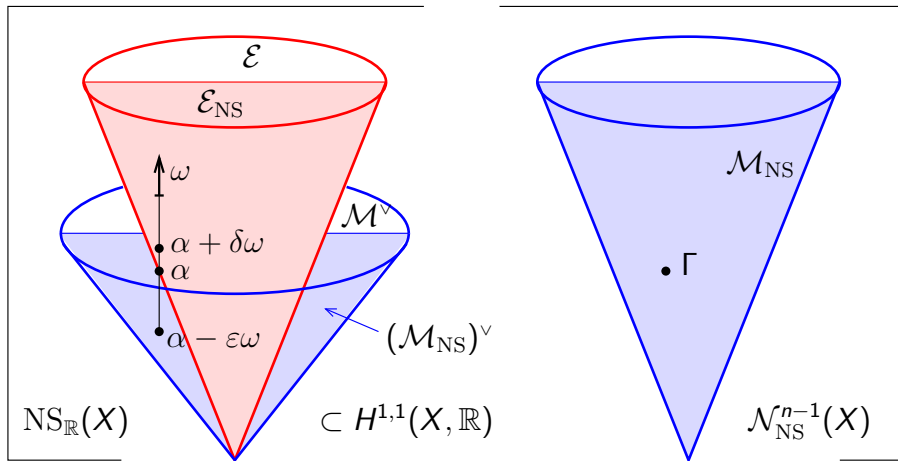
$$\mathcal{E}_{\text{NS}} \subset (\mathcal{M}_{\text{NS}})^{\vee}.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\text{NS}}$ on the boundary of \mathcal{E}_{NS} which is in the interior of $\mathcal{M}_{\text{NS}}^{\vee}$. Hence

$$(*) \quad \alpha \cdot \Gamma \geq \varepsilon \omega \cdot \Gamma$$

for every moving curve Γ , while $\langle \alpha^n \rangle = \text{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called **uniruled** if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}_{\mathbb{C}}^1$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)

*A projective manifold X is **not uniruled** if and only if K_X is pseudo-effective, i.e. $K_X \in \mathcal{E}_{\text{NS}}$.*

Proof (of the non trivial implication). If $K_X \notin \mathcal{E}_{\text{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard “**bend-and-break**” lemma of Mori then implies that there is family Γ_t of **rational curves** with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Plurigenera and the Minimal Model Program

Fundamental question. Prove that every birational class of non uniruled algebraic varieties contains a “minimal” member X with mild singularities, where “minimal” is taken in the sense of avoiding unnecessary blow-ups; minimality actually means that K_X is nef and not just pseudo-effective (pseudo-effectivity is known by the above results).

This requires performing certain birational transforms known as **flips**, and one would like to know whether

- a) flips are indeed possible (“**existence of flips**”),
- b) the process terminates (“**termination of flips**”).

Thanks to Kawamata 1992 and Shokurov (1987, 1996), this has been proved in dimension 3 at the end of the 80’s and more recently in dimension 4 (C. Hacon and J. McKernan also introduced in 2005 a new induction procedure).

Finiteness of the canonical ring

Basic questions.

- *Finiteness of the canonical ring:*
Is the canonical ring $R = \bigoplus H^0(X, mK_X)$ of a variety of general type always finitely generated ?
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Is there a bound r_n depending only on dimension $\dim X = n$, such that the pluricanonical map Φ_{mK_X} of a variety of general type yields a birational embedding in projective space for $m \geq r_n$?

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- *Invariance of plurigenera:*
Are plurigenera $p_m = h^0(X, mK_X)$ always invariant under deformation ?

Recent results on extension of sections

The following is a very slight extension of results by M. Păun (2005) and B. Claudon (2006), which are themselves based on the ideas of Y.T. Siu 2000 and S. Takayama 2005.

Theorem. *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a family of projective manifolds over the unit disk, and let $(L_j, h_j)_{0 \leq j \leq m-1}$ be (singular) hermitian line bundles with semipositive curvature currents $i\Theta_{L_j, h_j} \geq 0$ on \mathcal{X} . Assume that*

- the restriction of h_j to the central fiber X_0 is well defined (i.e. not identically $+\infty$).*
- additionally the multiplier ideal sheaf $\mathcal{I}(h_j|_{X_0})$ is trivial for $1 \leq j \leq m-1$.*

Then any section σ of $\mathcal{O}(mK_{\mathcal{X}} + \sum L_j)|_{X_0} \otimes \mathcal{I}(h_0|_{X_0})$ over the central fiber X_0 extends to \mathcal{X} .

Proof / invariance of plurigenera

The proof relies on a clever iteration procedure based on the Ohsawa-Takegoshi L^2 extension theorem, and a convergence process of an analytic nature (no algebraic proof at present !)

The special case of the theorem obtained by taking all bundles L_j trivial tells us in particular that any pluricanonical section σ of $mK_{\mathcal{X}}$ over X_0 extends to \mathcal{X} . By the upper semi-continuity of $t \mapsto h^0(X_t, mK_{X_t})$, this implies

Corollary (Siu 2000). *For any projective family $t \mapsto X_t$ of algebraic varieties, the plurigenera $p_m(X_t) = h^0(X_t, mK_{X_t})$ do not depend on t .*