On the computational complexity of mathematical functions

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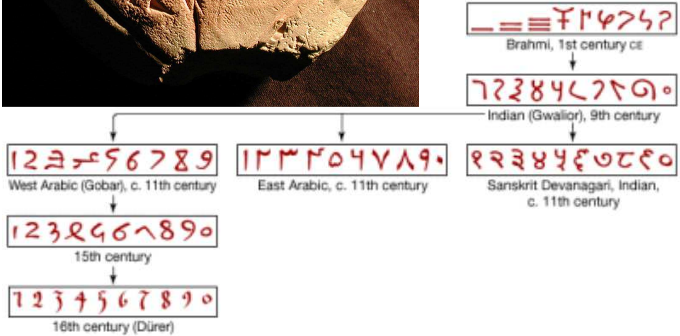
Computing, a very old concern

Babylonian mathematical tablet allowing the computation of $\sqrt{2}$ (1800 – 1600 BC)

Decimal numeral system invented in India ($\sim$ 500BC ?) :



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The next progress was the discovery of the first infinite series formula by Madhava (circa 1350 – 1450), a prominent mathematician-astronomer from Kerala (formula rediscovered in the XVIIe century by Leibniz and Gregory) :

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

Convergence is unfortunately very slow, but Madhava was able to improve convergence and reached in this way 11 decimal places.

Ramanujan's formula for π

Srinivasa Ramanujan (1887 – 1920), a self-taught mathematical prodigee. His work dealt mainly with arithmetics and function theory

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{+\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \qquad (1910).$$

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Each term is approximately 10^8 times smaller than the preceding one, so the convergence is very fast.

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  – tries to classify problems according to their difficulty
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– quadratic complexity when $\#\,\text{steps} \leq C\,N^2$

– quasi-linear complexity when $\#\,\text{steps} \leq C_\varepsilon\,N^{1+\varepsilon}$, $\forall \varepsilon > 0$.

First observations about complexity

- Addition has linear complexity:
consider decimal numbers of the form $0.a_1a_2a_3\ldots a_N$, $0.b_1b_2b_3\ldots b_N$, we have

$$\sum_{1\le n\le N} a_n 10^{-n} + \sum_{1\le n\le N} b_n 10^{-n} = \sum_{1\le n\le N} (a_n+b_n) 10^{-n},$$

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- What about multiplication ?

$$\sum_{1\le k\le N} a_k 10^{-k} \times \sum_{1\le \ell\le N} b_\ell 10^{-\ell} = \sum_{1\le n\le N} c_n 10^{-n}, \quad c_n = \sum_{k+\ell=n} a_k b_\ell.$$

Calculation of each c_n requires at most N elementary multiplications and $N-1$ additions and corresponding carries, thus the algorithm requires less than $N\times 3N$ steps.

Thus multiplication has at most quadratic complexity.

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Karatsuba's idea: for $N = 2q$ even, split $x = 0.a_1a_2\ldots a_N$ as

$$x = x' + 10^{-q}x'', \quad x' = 0.a_1a_2\ldots a_q, \quad x'' = 0.a_{q+1}a_{q+2}\ldots a_{2q}$$

and similarly $y = 0.b_1b_2\ldots b_N = y' + 10^{-q}y''$. To calculate xy, one would normally need $x'y'$, $x''y''$ and $x'y'' + x''y'$ which take 4 multiplications and 1 addition of q-digit numbers. However, one can use only 3 multiplications by calculating

$$x'y', \quad x''y'', \quad x'y'' + x''y' = x'y' + x''y'' - (x' - x'')(y' - y'')$$

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$$x'y', \quad x''y'', \quad x'y'' + x''y' = x'y' + x''y'' - (x' - x'')(y' - y'')$$

(at the expense of 4 additions). One then proceeds inductively to conclude that the time $T(N)$ needed for $N = 2^s$ satisfies

$$T(2^s) \leq 3\,T(2^{s-1}) + 4\,2^{s-1}.$$

Optimal complexity of multiplication

It is an easy exercise to conclude by induction that $T(2^s) \leq 6\,3^s - 4\,2^s$ if one assumes $T(1) = 1$, and so

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The Schönage-Strassen algorithm is based on the use of discrete Fourier transforms. The theory comes from Joseph Fourier, the founder of my university in 1810 ...

Joseph Fourier

Joseph Fourier (1768 – 1830)
in his suit of member of
Académie des Sciences,
of which he became
"Secrétaire Perpétuel"
(Head) in 1822.

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In 1824, he predicts the green house effect !

Heat equation and Fourier series

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In fact all periodic phenomena can be described in this way. This is the basis of the modern theory of signal processing and electromagnetism.

Discrete Fourier transform

Let $(a_n)_{0\le n<N}$ be a finite sequence of numbers and let u be a primitive N-th root of unity, i.e.

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Main formulas of Fourier theory

Fourier transform of a convolution:

For $a = (a_n)$ and $b = (b_n)$ define $c = a * b$ to be the sequence

$$c_n = \sum_{p+q=n \mod N} a_p b_q \quad \text{“convolution of } a \text{ and } b.\text{”}$$

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Proof. $\displaystyle \sum_s c_s u^{sn} = \sum_s \Big(\sum_{k+\ell=s} a_k b_\ell \Big) u^{sn} = \sum_{k,\ell} a_k u^{kn} b_\ell u^{\ell n} = \widehat{a}_n \widehat{b}_n.$

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Proof. $\widehat{\widehat{a}}_n = \sum_k \Big(\sum_\ell a_\ell u^{k\ell} \Big) u^{kn} = \sum_\ell a_\ell \Big(\sum_k u^{k(n+\ell)} \Big)$ and $\sum_k u^{k(n+\ell)} = 0$ if $\ell \neq -n$ and $\sum_k u^{k(n+\ell)} = N$ if $\ell = -n$.

Fast Fourier Transform (FFT)

Consequence: To calculate the convolution $c = a * b$ (which is what we need to calculate $\sum a_k 10^{-k} \sum b_\ell 10^{-\ell}$), one calculates the Fourier transforms $(\widehat{a}_n)$, $(\widehat{b}_n)$, then $\widehat{c}_n = \widehat{a}_n \widehat{b}_n$, which gives back $(-c_{-n})$ and thus (c_n) by Fourier inversion.

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This looks complicated, but the Fourier transform can be computed extremely fast !!

FFT algorithm: assume that $N = 2^s$ (in our example $N = 65536 = 2^{16}$) and define inductively $\alpha_{n,0} = a_n$ and

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By considering the binary decomposition $n = \sum n_k 2^k$, $0 \le k < s$, of any integer $n = 0 \ldots N-1$, one sees that $\alpha_{n,s} = \widehat{a}_n$. The calculation requires only s steps, each of which requires N additions and $2N$ mutiplications (using $u^{2^{k+1} n} = (u^{2^k n})^2$), so in total we consume only $3sN = 3N \log_2 N$ operations !

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Approximate division can be obtained solely from multiplication! If x_0 is a rough approximation of $1/a$, then the sequence

$$x_{n+1} = 2x_n - ax_n^2$$

satisfies $1 - ax_{n+1} = (1 - ax_n)^2$, and so inductively $1 - ax_n = (1 - ax_0)^{2^n}$ will converge extremely fast to 0. In fact if $|1 - ax_0| < 1/10$ and $n \sim \log_2 N$, we get already N correct digits. Hence we need iterating only $\log_2 N$ times the sequence, and so division is also quasi-linear in time.

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Similarly, square roots can be approximated by using only multiplications and divisions, thanks to the "Babylonian algorithm":

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad x_0 > 0$$

What about π ?

In fact Carl-Friedrich Gauss (another mathematical prodigee...) discovered around 1797 the following formula for the arithmetic-geometric mean:
start from real numbers $a, b > 0$ and define inductively $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

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$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}.$$

Then (a_n) and (b_n) converge (extremely fast, only $\sim \log_2 N$ steps to get N correct digits) towards

$$M(a,b) = \frac{2\pi}{I(a,b)} \quad \text{where } I(a,b) = \int_0^{2\pi} \frac{dx}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}}$$

(an "elliptic integral").

The Brent-Salamin formula

Using this and another formula due to Legendre (1752 – 1833), Brent and Salamin found in 1976 a remarkable formula for π. Define

$$c_n = \sqrt{a_n^2 - b_n^2}$$

in the arithmetic-geometric sequence. Then

$$\pi = \frac{4\,M(1, 1/\sqrt{2})^2}{1 - \sum_{n=1}^{+\infty} 2^{n+1} c_n^2}.$$

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This formula has been used several times to break the world record, which seems to be 5 trillions digits since 2010 (however, there exist so efficient quadratic complexity formulas that they are still competitive at that level...)

Complexity of matrix multiplication

Question. How many steps are necessary to compute the product $C = AB$ of two $n \times n$ matrices, assuming that each elementary multiplication or addition takes 1 step?

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leads to calculate n^2 coefficients, each of which requires n multiplications and $(n-1)$ additions, so in total $n^2(2n-1) \sim 2n^3$ operations.

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The fastest known algorithm, due to Coppersmith and Winograd in 1987 has $\#\text{steps} \le C\, n^{2.38}$ (quite complicated!)