Geometric constructions in relation with algebraic and transcendental numbers

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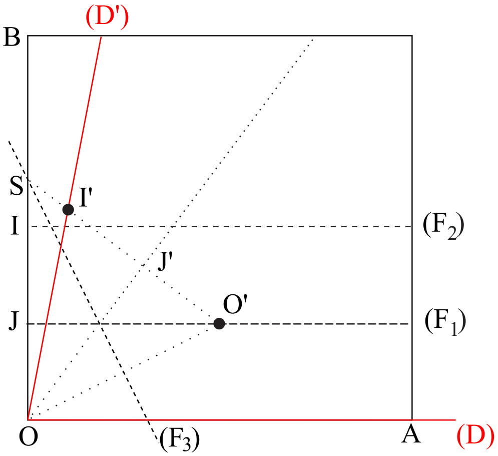
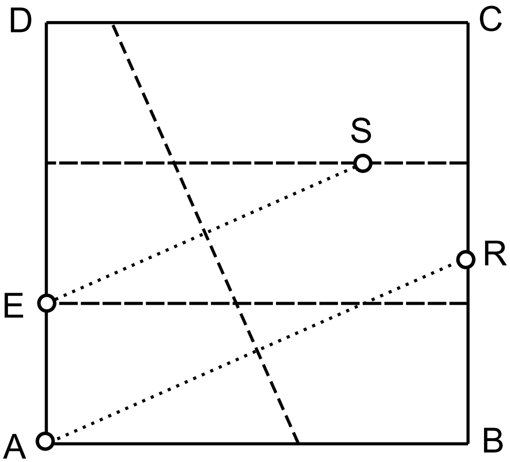
In Japan, on the other hand, there is a rich tradition of making origamis : it is the art of folding paper and maker nice geometric constructions out of such foldings.

Trisection of an angle with origamis



Folding along $(F3)$ so that O is brought to $O' \in (F_1)$ and I is brought to $I' \in D'$ constructs the trisection of angle(D, D') !

Cube root of 2 with origamis



Exercise. Show that this construction can be used to produce $\sqrt[3]{2}$ (side of the square is 3).

“Axioms” for ruler and compasses

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  Then enlarge S into $S' \supset S$ by constructing lines and circles according to the following rules:

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- Axiom (RC2). Given 2 lines, 1 line and a circle, or 2 circles constructed from RC1, S' contains all points of intersection of these.
- Question : Describe the set of points $\mathrm{Constr}_{\mathrm{RC}}(\mathrm{S})$ which can be constructed from S in finitely many steps.

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- Addition in $\mathbb{C}$

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- Multiplication in $\mathbb{C}$

$$\begin{aligned}(x + iy) \times (x' + iy') &= xx' + ixy' + iyx' + i \times i \times yy' \\ &= xx' + ixy' + iyx' + (-1) \times yy' \\ &= (xx' - yy') + i(xy' + yx')\end{aligned}$$

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- Interpretation of addition in $\mathbb{C}$ Addition corresponds to adding vectors in the plane: use a parallelogram
- Interpretation of multiplication in $\mathbb{C}$ Introduce $|\mathrm{z}| = \sqrt{\mathrm{x}^2 + \mathrm{y}^2}$ and $\arg(\mathrm{z}) = \mathrm{angle}(0\mathrm{x}, \mathrm{Oz})$. Then

$$|\mathrm{zz}'| = |\mathrm{z}|\ |\mathrm{z}'|, \qquad \arg(\mathrm{zz}') = \arg(\mathrm{z}) + \arg(\mathrm{z}') \mod 2\pi$$

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$$\sqrt{z} = \pm\Big(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} + \varepsilon i\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\Big)$$

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- Theorem (d'Alembert-Gauss) Every polynomial of degree d $a_d z^d + a_{d-1} z^{d-1} + \ldots + a_1 z + a_0$ with coefficients in $\mathbb{C}$ has exactly d roots when counted with multiciplicities.

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- Definition One says that $z \in \mathbb{C}$ is an algebraic number if it is a solution of a polynomial with $a_j \in \mathbb{Q}$ (or $a_j \in \mathbb{Z}$), a transcendental number otherwise

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  $e^\pi = (e^{i\pi})^{-i} = (-1)^{-i}$.
- Unknown whether e/π is transcendental, not even known that $e/\pi \notin \mathbb{Q}$!

Subfields of the field of complex numbers

- A subset $\mathbb{F} \subset \mathbb{C}$ is called a field (but there is a more general concept than just for numbers...) if $\mathbb{F}$ contains $0, 1$, and is stable by addition, subtraction, multiplication and division, (i.e. for $z, w \in \mathbb{F}$, we have $z + w \in \mathbb{F}$, $z - w \in \mathbb{F}$, $zw \in \mathbb{F}$, $z/w \in \mathbb{F}$ if $w \neq 0$)

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- The set denoted $\mathbb{Q}[\sqrt{2}]$ of numbers of the form $x + y\sqrt{2}$, $x, y \in \mathbb{Q}$ is a field :
$$(x + y\sqrt{2})^{-1} = \frac{x - y\sqrt{2}}{x^2 - 2y^2} = \frac{x}{x^2 - 2y^2} - \frac{y}{x^2 - 2y^2}\sqrt{2}$$

Algebraic number fields

- Similarly the set denoted $\mathbb{Q}[\sqrt{-2}]$ of numbers of the form $x + y\sqrt{-2} = x + iy\sqrt{2}$, $x, y \in \mathbb{Q}$ is a field (exercise !)
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- The set $\mathbb{Q}[\sqrt[3]{2}]$ of numbers of the form $x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2$, $x, y, z \in \mathbb{Q}$ is a field (cubic field)
This is a bit harder to prove.
Hint: calculate ω^3 where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and then show that the product

$$(x + y\omega\sqrt[3]{2} + z(\omega\sqrt[3]{2})^2)(x + y\omega^2\sqrt[3]{2} + z(\omega^2\sqrt[3]{2})^2)$$

no longer involves ω and yields a rational number when multiplied by $x + y\sqrt[3]{2} + z(\sqrt[3]{2})^2$.

Degree of a number field

- One can show (but this is yet harder) that if α, β, $\gamma, \ldots$ are algebraic numbers, then the sets $\mathbb{Q}[\alpha]$, $\mathbb{Q}[\alpha, \beta]$, $\mathbb{Q}[\alpha, \beta, \gamma]$ of polynomials $P(\alpha)$, $P(\alpha, \beta)$, $P(\alpha, \beta, \gamma)$ (...) with rational coefficients are fields.

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- If $\mathbb{F} \subset \mathbb{G}$ are fields and every element $y \in \mathbb{G}$ can be written in a unique way $y = x_1\alpha_1 + \ldots + x_p\alpha_p$ for $x_i \in \mathbb{F}$ and certain (well chosen) elements $\alpha_i \in \mathbb{G}$, one says that $\mathbb{G}$ has (finite) degree p over $\mathbb{F}$, with basis (α_j) over $\mathbb{F}$, and one writes $[\mathbb{G} : \mathbb{F}] = p$

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- Example: $[\mathbb{Q}[\sqrt[2]{:}\mathbb{Q}] = 2$ and $[\mathbb{Q}[\sqrt[3]{2} : \mathbb{Q}] = 3$.
- Exercise. If $\mathbb{G} = \mathbb{F}[\alpha]$ where $\alpha \in \mathbb{G}$, $\alpha \notin \mathbb{F}$ and α satisfies an equation of degree 2 with coefficients in $\mathbb{F}$, then $[\mathbb{G} : \mathbb{F}] = 2$. Idem for degree d if α does not satisfy any equation of lower order (take $\alpha_j = \alpha^j$, $0 \leq j \leq d-1$).

Successive extensions of fields

- **Theorem.** If $\mathbb{F} \subset \mathbb{G} \subset \mathbb{K}$ are fields then

$$[\mathbb{K} : \mathbb{F}] = [\mathbb{K} : \mathbb{G}] \times [\mathbb{G} : \mathbb{F}]$$

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- Proof. Write $p = [\mathbb{G} : \mathbb{F}]$ and $q = [\mathbb{K} : \mathbb{G}]$.
  Every $z \in \mathbb{K}$ can be written in a unique way
  $z = \sum_k y_k \beta_k, \quad y_k \in \mathbb{G}$ for a basis $\beta_1, \ldots, \beta_q \in \mathbb{K}$,
  and each $y_k \in \mathbb{G}$ can then be written in a unique way
  $y_k = \sum_j x_{jk} \alpha_j, \quad x_{jk} \in \mathbb{F}$ for a basis $\alpha_1, \ldots, \alpha_p \in \mathbb{G}$,
  so, uniquely in terms of the $\alpha_j \beta_k$ (check it!)
  $z = \sum_{j,k} x_{jk} \alpha_j \beta_k$.
  Thus $(\alpha_j \beta_k)$ is a basis of $\mathbb{K}$ over $\mathbb{F}$ and $[\mathbb{K} : \mathbb{F}] = pq$.

Re-interpretation of constructions with ruler and compasses

- We start from a set of points S in the plane (of at least two points) and interpret them as complex numbers in coordinates. By a rotation, change of origin and change of unit, we mant assume that two of these numbers are $s_1 = 0$, $s_2 = 1$, the other ones are complex numbers $s_3 \ldots, s_n$, $n = \sharp S$.

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- Basic observation. The set of points constructible from S by ruler and compasses is stable by addition, multiplication, inverse, and also by conjugation and square root.
- The set $\mathbb{Q}(S)$ of all rational fractions $P(s_3, \ldots, s_n)/Q(s_3, \ldots, s_n)$ is a field (equal to $\mathbb{Q}$ if we start from only two points).

Necessary and sufficient condition for constructibility

- When we construct a bigger set $S' \subset S$ with ruler and compasses, we only solve linear and quadratic equations (intersections of lines and/or circles) with coefficients in $\mathbb{Q}(S)$ for the first step.

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- In general, our construction consists of producing a "tower of quadratic extensions"

$$\mathbb{Q}(S) = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \ldots \subset \mathbb{F}_k = \mathbb{Q}(S')$$

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  where each field $\mathbb{F}_{j+1} = \mathbb{F}_j[\alpha_j]$ is obtained by adjoining a point α_j satisfying at most a quadratic equation.
- Remark. The "quadratic tower" condition is necessary and sufficient: any such tower starting with $\mathbb{Q}(S)$ consists of points which are constructible step by step from S.

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- Proof. – We are using n-th rooths of 1, i.e. the field $\mathbb{Q}[\omega]$, $\omega^{n-1} + \ldots + \omega + 1 = 0$, of degree $d \leq n - 1$.
  – Degree can be $d < n - 1$ (example $d = 2$ for $n = 6$).
  – Reduction to the case $n = p^r$ is a prime power
  – When $n = p^r$, ω is of degree $d = (p - 1)p^r$ exactly (this has to be proved!). Thus either $p = 2$ or $r = 1$ and $p - 1$ has to be a pover of 2, i.e. $p = 2^s + 1$, and then s itself has to be a power of 2.

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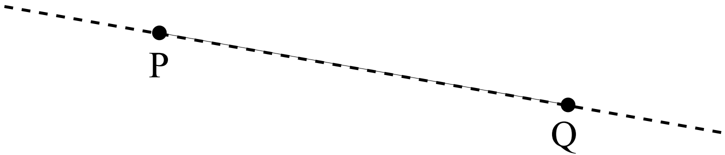
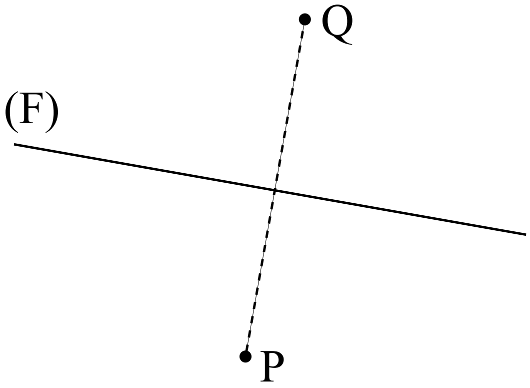
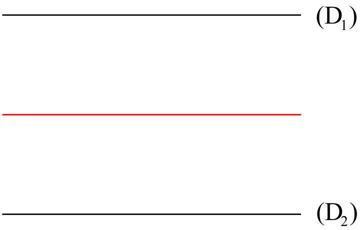


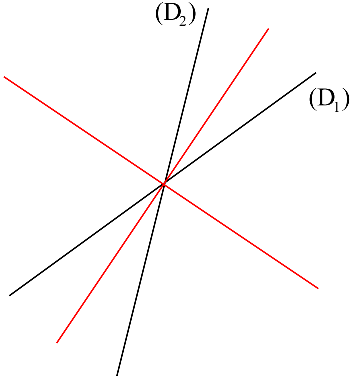
Axiom O2. Given two points P, Q, one can fold paper to bring P to Q (through the median line of segment $[P, Q]$.



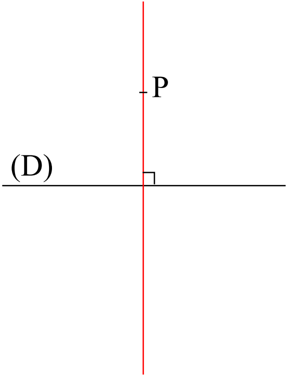
Axioms of construction by origamis (3)

Axiom O3. Given two lines (D_1), (D_2) one can fold paper to bring (D_1) onto (D_2) (through one of the bissecting lines)

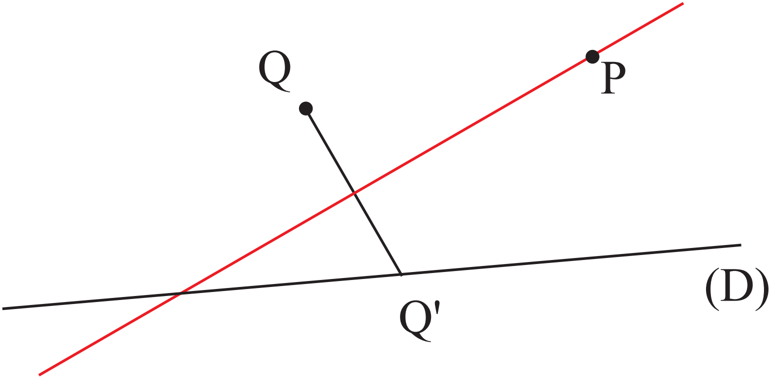




Axiom O4. Given one point P and a line (D), one can fold through point P in such a way that (D) is brought to itself (thus perpendicular to (D) through P)

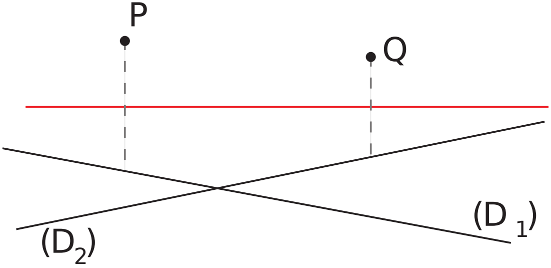


Axiom O5. Given a line (D) and two points P, Q, one can (whenever possible) fold paper through P in such a way that Q is brought to a point of (D).



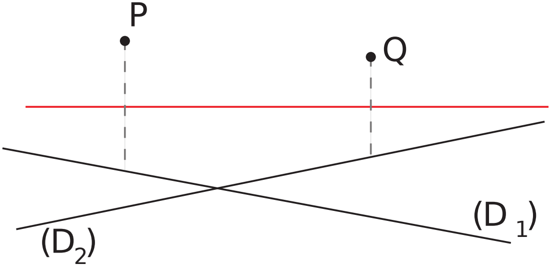
Axioms of construction by origamis (6)

Axiom O6. Given two lines (D_1) and (D_2) and two points P, Q, one can (whenever possible) fold paper to bring P to a point of (D_1) and Q to a point of (D_2)



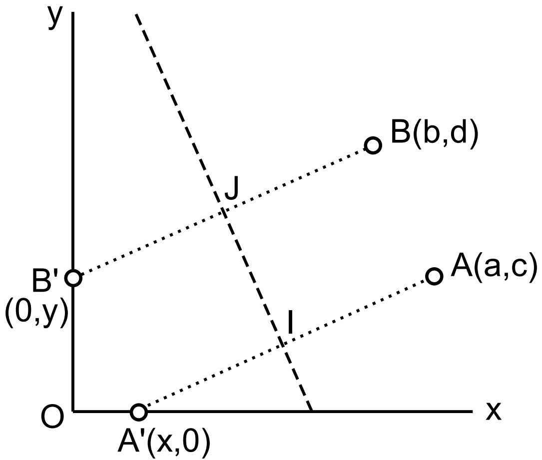
Axioms of construction by origamis (6)

Axiom O6. Given two lines (D_1) and (D_2) and two points P, Q, one can (whenever possible) fold paper to bring P to a point of (D_1) and Q to a point of (D_2)



In fact, axiom O6 can be seen to imply all others. As in the case of compass and ruler, one can see that the axioms allow to take arbitrary integer multiples or quotients, as well as addition, multiplication or division of complex numbers.

Origamis and cubic equations



Problem. Bring $A(a,c)$ given onto $A' \in Ox$ and $B(b,d)$ given onto $B' \in Oy$ by folding.

Origamis and cubic equations (calculation)

One gets $t = \mathrm{slope}(AA') = \mathrm{slope}(BB') \Rightarrow t = \dfrac{d-y}{b} = \dfrac{c}{a-x}$

$I\Big(\dfrac{a+x}{2},\dfrac{c}{2}\Big)$ $J\Big(\dfrac{b}{2},\dfrac{d+y}{2}\Big)$, slope $(IJ) = \dfrac{-1}{t} = \dfrac{d+y-c}{b-(a+x)}$
Therefore

$$x = a - \frac{c}{t}, \quad y = d - bt, \quad \frac{-1}{t} = \frac{2d-c-bt}{b-2a+\frac{c}{t}}$$

whence the equation

$$bt^3 + (c-2d)t^2 + (2a-b)t - c = 0$$

which is equivalent to the most general cubic equation $t^3 + pt^2 + qt + r = 0$ by putting

$$a = \frac{q+1}{2}, \quad b = 1, \quad c = -r, \quad d = -\frac{p+r}{2}.$$

Necessary and sufficient condition for constructibility by origamis

- Theorem: a set S' can be constructed by origamis from $S = \{0, 1, s_3, \ldots, s_n\}$ if and only if there is a tower of field extensions

$$\mathbb{Q}(S) = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \ldots \subset \mathbb{F}_k = \mathbb{Q}(S')$$

where each extension $\mathbb{F}_{j+1} = \mathbb{F}_j[\alpha_j]$ is a quadratic or cubic extension.

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  where each extension $\mathbb{F}_{j+1} = \mathbb{F}_j[\alpha_j]$ is a quadratic or cubic extension.

- Corollary. A polygon with n sides can be constructed with origamis if and only if $n = 2^k 3^\ell p_1 \ldots p_m$ where each p_j is a prime number with the property that each $p_j - 1 = 2^{a_j} 3^{b_j}$.

References

James King, Origami constructible numbers, 2004 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.123.6667&rep=rep1&type=pdf

Michael Filaseta Lecture Notes, Section 6, the transcendence of e and π, http://www.math.sc.edu/~filaseta/gradcourses/Math785/Math785Notes6.pdf

Serge Lang, Algebra, Graduate Texts in Mathematics 211, Springer, 3rd edition, 2002.