Algebraic embeddings of complex and almost complex structures

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(based on joint work with Hervé Gaussier)

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A question raised by Fedor Bogomolov

Rough question

Can one produce an arbitrary compact complex manifold $X$, resp. an arbitrary compact Kähler manifold $X$ by means of a "purely algebraic construction"?
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Assume that $X^{2n}$ is a compact $C^\infty$ real even dimensional manifold that is embedded in $Z$, as follows:
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Can one produce an arbitrary compact complex manifold $X$, resp. an arbitrary compact Kähler manifold $X$ by means of a “purely algebraic construction”? 

Let $Z$ be a projective algebraic manifold, $\dim_{\mathbb{C}} Z = N$, equipped with a subbundle (or rather subsheaf) $\mathcal{D} \subset \mathcal{O}_Z(T_Z)$. 

Assume that $X^{2n}$ is a compact $C^\infty$ real even dimensional manifold that is embedded in $Z$, as follows:

(i) $f : X \hookrightarrow Z$ is a smooth (say $C^\infty$) embedding

(ii) $\forall x \in X$, $f_* T_{X,x} \oplus \mathcal{D}_{f(x)} = T_{Z,f(x)}$.

(iii) $f(X) \cap \mathcal{D}_{\text{sing}} = \emptyset$.

We say that $X \hookrightarrow (Z, \mathcal{D})$ is a transverse embedding.
Construction of an almost complex structure

\[ f_* T_{X,x} = T_{M,f(x)} \cong T_{Z,f(x)}/D_{f(x)} \]

is in a natural way a complex vector space

⇒ almost complex structure \( J_f \)
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$\Rightarrow$ almost complex structure $J_f$

Observation 1 (André Haefliger)

If $D \subset T_Z$ is an algebraic foliation, i.e. $[D, D] \subset D$, then the almost complex structure $J_f$ on $X$ induced by $(Z, D)$ is integrable.
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If \( D \subset T_Z \) is an algebraic foliation, i.e. \([D, D] \subset D\), then the almost complex structure \( J_f \) on \( X \) induced by \((Z, D)\) is integrable.

**Proof:** Any 2 charts yield a holomorphic transition map \( U \rightarrow V \)
⇒ holomorphic atlas
Observation 2

If $D \subset T_Z$ is an algebraic foliation and $f_t : X \hookrightarrow (Z, D)$ is an isotopy of transverse embeddings, $t \in [0, 1]$, then all complex structures $(X, J_{f_t})$ are biholomorphic.
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Proof:
A conjecture of Bogomolov

To each triple \((Z, D, \alpha)\) where

- \(Z\) is a complex projective manifold
- \(D \subset T_Z\) is an algebraic foliation
- \(\alpha\) is an isotopy class of transverse embeddings \(f : X \hookrightarrow (Z, D)\)

one can thus associate a biholomorphism class \((X, J_f)\).

Conjecture (from RIMS preprint of Bogomolov, 1995)

One can construct in this way every compact complex manifold \(X\).

Additional question 1

What if \((X, \omega)\) is Kähler? Can one embed in such a way that \(\omega\) is the pull-back of a transversal Kähler structure on \((Z, D)\)?

Additional question 2

Can one describe the non injectivity of the “Bogomolov functor” \((Z, D, \alpha) \mapsto (X, J_f)\), i.e. moduli spaces of such embeddings?
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There exist large classes of examples!

**Example 1: tori**

If $Z$ is an Abelian variety and $N \geq 2n$, every $n$-dimensional compact complex torus $X = \mathbb{C}^n / \Lambda$ can be embedded transversally to a linear codimension $n$ foliation $\mathcal{D}$ on $Z$.

**Example 2: LVMB manifolds**

One obtains a rich class, named after Lopez de Medrano, Verjovsky, Meersseman, Bosio, by considering foliations on $\mathbb{P}^N$ given by a commutative Lie subalgebra of the Lie algebra of $\text{PGL}(N+1, \mathbb{C})$.

The corresponding transverse varieties produced include e.g. Hopf surfaces and the Calabi-Eckmann manifolds $S^2_p \times S^2_q$. 

\[
\begin{array}{c}
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\hline \\
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![Diagram of a foliation on a torus](image)

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What about the almost complex case?

Easier question: drop the integrability assumption

Can one realize every compact almost complex manifold \((X, J)\) by a transverse embedding into a projective algebraic pair \((Z, \mathcal{D})\), \(\mathcal{D} \subset T_Z\), so that \(J = J_f\)?

Not surprisingly, there are constraints, and \(Z\) cannot be "too small". But how large exactly?

Let \(\Gamma^\infty(X, Z, \mathcal{D})\) the Fréchet manifold of transverse embeddings \(f: X \hookrightarrow (Z, \mathcal{D})\) and \(J^\infty(X)\) the space of smooth almost complex structures on \(X\).

Further question: When is \(f \mapsto \text{J}_f\), \(\Gamma^\infty(X, Z, \mathcal{D}) \to J^\infty(X)\) a submersion?

Note: technically one has to consider rather Banach spaces of \(C^{r+\alpha}\) Hölder regularity.
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Variation formula for $J_f$

First, the tangent space to the Fréchet manifold $\Gamma^\infty(X, Z, D)$ at a point $f$ consists of

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Let $[\cdot, \cdot]$ be the Lie bracket of vector fields in $T_Z$,

$$\theta : D \times D \to T_Z/D, \quad (\xi, \eta) \mapsto [\xi, \eta] \mod D$$

be the torsion tensor of the holomorphic distribution $D$, and $\nu \mapsto \bar{\partial} J_f \nu$ the $\bar{\partial}$ operator of the almost complex structure $(X, J_f)$. 
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be the torsion tensor of the holomorphic distribution $D$, and $v \mapsto \bar{\partial}_{J_f} v$ the $\bar{\partial}$ operator of the almost complex structure $(X, J_f)$. Then the differential of the natural map $f \mapsto J_f$ along any infinitesimal variation $w = u + f^* v : X \rightarrow f^* T_Z = f^* D \oplus f^* T_X$ of $f$ is given by

$$dJ_f(w) = 2J_f \left( f_*^{-1} \theta(\bar{\partial}_{J_f} f, u) + \bar{\partial}_{J_f} v \right)$$
Sufficient condition for submersivity

**Theorem (D - Gaussier, 2014)**

Let \( f : X \hookrightarrow (Z, D) \) be a smooth transverse embedding. Assume that \( f \) and the torsion tensor \( \theta \) of \( D \) satisfy the following additional conditions:

1. \( f \) is a totally real embedding, i.e. \( \overline{\partial} f(x) \in \text{End}_{\mathbb{C}}(T_{X,x}, T_{Z,f(x)}) \) is injective at every point \( x \in X \);

2. for every \( x \in X \) and every \( \eta \in \text{End}_{\mathbb{C}}(T_{X}) \), there exists a vector \( \lambda \in D_{f(x)} \) such that \( \theta(\overline{\partial} f(x) \cdot \xi, \lambda) = \eta(\xi) \) for all \( \xi \in T_{X} \).

Then there is a neighborhood \( U \) of \( f \) in \( \Gamma^{\infty}(X, Z, D) \) and a neighborhood \( V \) of \( Jf \) in \( J^{\infty}(X) \) such that \( U \rightarrow V, f \mapsto Jf \) is a submersion.

**Remark.** A necessary condition for (ii) to be possible is that \( \text{rank} \, D = N - n \geq n^2 = \text{dim} \, \text{End}(T_{X}) \), i.e. \( N \geq n + n^2 \).
Existence of universal embedding spaces

Theorem (D - Gaussier, 2014)

For all integers $n \geq 1$ and $k \geq 4n$, there exists a complex affine algebraic manifold $Z_{n,k}$ of dimension $N = 2k + 2(k^2 + n(k - n))$ possessing a real structure (i.e. an anti-holomorphic algebraic involution) and an algebraic distribution $D_{n,k} \subset T_{Z_{n,k}}$ of codimension $n$, with the following property:

for every compact $n$-dimensional almost complex manifold $(X, J)$ admits an embedding $f : X \hookrightarrow Z_{n,k}$ transverse to $D_{n,k}$ and contained in the real part of $Z_{n,k}$, such that $J = Jf$.

The choice $k = 4n$ yields the explicit embedding dimension $N = 38n^2 + 8n$ (and a quadratic bound $N = O(n^2)$ is optimal by what we have seen previously).

Hint. $Z_{n,k}$ is produced by a fiber space construction mixing Grassmannians and twistor spaces.
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for every compact \( n \)-dimensional almost complex manifold \( (X, J) \) admits an embedding \( f : X \hookrightarrow Z_{n,k}^\mathbb{R} \) transverse to \( \mathcal{D}_{n,k} \) and contained in the real part of \( Z_{n,k} \), such that \( J = J_f \).
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**Hint.** $Z_{n,k}$ is produced by a fiber space construction mixing Grassmannians and twistor spaces.
**First observation.** There exists a $C^\infty$ embedding $\varphi : X \hookrightarrow \mathbb{R}^{2k}$, $k \geq 4n$, by the Whitney embedding theorem, and one can assume $N_{\varphi(X)} = (T_{\varphi(X)})^\perp$ to carry a complex structure for $k \geq 8n$; otherwise take $\Phi = \varphi \times \varphi : X \hookrightarrow \mathbb{R}^{2k} \times \mathbb{R}^{2k}$ and observe that

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Second step. Assuming $(N_x, J')$ almost complex, let $Z^\mathbb{R}_{n,k}$ be the set of triples $(x, S, J)$ such that $S \in \text{Gr}^{\mathbb{R}}(2k, 2n)$, $\text{codim } S = 2n$, $J \in \text{End}(\mathbb{R}^{2k})$, $J^2 = -\text{Id}$, $J(S) \subset S$. Define

$$f : X \rightarrow Z^\mathbb{R}_{n,k}, \quad x \mapsto (\varphi(x), N_\varphi(x), \varphi(x), \tilde{J}(x))$$

where $\tilde{J}$ is induced by $J(x) \oplus J'(x)$ on $\varphi_\ast T_x \oplus N_x$. 

Third step. Complexify $Z^\mathbb{R}_{n,k}$ as a variety $Z^n_{n,k} = Z^\mathbb{C}_{n,k}$ and define an algebraic distribution $D^n_{n,k} \subset T Z^n_{n,k}$.
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Definition of $Z_{n,k}$ and $D_{n,k}$

We let $Z_{n,k} = Z_{n,k}^\mathbb{C}$ be the set of triples

$$(z, S, J) \in \mathbb{C}^{2k} \times \text{Gr}^\mathbb{C}(2k, 2n) \times \text{End}(\mathbb{C}^{2k})$$

with $J^2 = -\text{Id}$, $J(S) = S$. Moreover we assume that we have “balanced” decompositions

$$S = S' \oplus S'', \quad \text{dim } S' = \text{dim } S'' = n,$$
$$\mathbb{C}^{2k} = \Sigma' \oplus \Sigma'', \quad \text{dim } \Sigma' = \text{dim } \Sigma'' = k$$

for the $i$ and $-i$ eigenspaces of $J|_S$ and $J$, $S' \subset \Sigma'$, $S'' \subset \Sigma''$. 

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Finally, if $\pi = \text{pr}_1 : Z_{n,k} \to \mathbb{C}^{2k}$ is the first projection, we take $D_{n,k}$ at point $w = (z, S, J)$ to be

$$D_{n,k,w} := (d\pi)^{-1}(S' \oplus \Sigma'').$$
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Since $\mathbb{C}^{2k} = \Sigma' \oplus \Sigma''$, we have

$$(T_{Z_{n,k}}/D_{n,k})_w \cong \Sigma'/S',$$

which on real points, is isomorphic to $(S^\mathbb{R})_\perp$. 

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Consider the case of a compact almost complex symplectic manifold \((X, J, \omega)\) where the symplectic form \(\omega\) is assumed to be \(J\)-compatible, i.e. \(J^*\omega = \omega\) and \(\omega(\xi, J\xi) > 0\).
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**Definition**

We say that a closed semipositive \((1,1)\)-form \(\beta\) on \(Z\) is a transverse Kähler structure to \(D \subset T_Z\) if \(\text{Ker} \beta \subset D\), i.e., if \(\beta\) induces a Kähler form on germs of complex submanifolds transverse to \(D\).
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**Theorem (D - Gaussier, 2014)**

There also exist universal embedding spaces for compact almost complex symplectic manifolds, i.e. a certain triple \((Z, D, \beta)\) as above, such that every \((X, J, \omega), \dim_{\mathbb{C}} X = n, \{\omega\} \in H^2(X, \mathbb{Z})\), embeds transversally by \(f : X \hookrightarrow (Z, D, \beta)\), in such a way that

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J = J_f \quad \text{and} \quad \omega = f^*\beta.
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\[ J = J_{f} \text{ and } \omega = f^{*}\beta. \]

**Proof.** Use the Tischler symplectic embedding \(X \hookrightarrow (\mathbb{P}^{2n+1}, \omega_{FS})\).
Integrability condition for an almost complex structure

Recall that the Nijenhuis tensor of an almost complex structure $J$ is

$$N_J(\zeta, \eta) = 4 \text{Re} [\zeta^{0,1}, \eta^{0,1}]^{1,0} = [\zeta, \eta] - [J\zeta, J\eta] + J[\zeta, J\eta] + J[J\zeta, \eta].$$
Integrability condition for an almost complex structure

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$$N_J(\zeta, \eta) = 4 \Re [\zeta^{0,1}, \eta^{0,\bar{1}}]^{1,0} = [\zeta, \eta] - [J\zeta, J\eta] + J[\zeta, J\eta] + J[J\zeta, \eta].$$

The Newlander-Nirenberg theorem states that

$(X, J)$ is complex analytic if and only if $N_J \equiv 0$. 

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The Newlander-Nirenberg theorem states that $(X, J)$ is complex analytic if and only if $N_J \equiv 0$.

In fact, we have the following relation between the torsion form $\theta$ of a distribution and the Nijenhuis tensor of the related transverse structure:

**Nijenhuis tensor formula**

If $\theta$ denotes the torsion of $(Z, D)$, the Nijenhuis tensor of the almost complex structure $J_f$ induced by a transverse embedding $f : X \hookrightarrow (Z, D)$ is given by $\forall z \in X, \forall \zeta, \eta \in T_zX$

$$N_{J_f}(\zeta, \eta) = 4 \theta(\overline{\partial}_{J_f} f(z) \cdot \zeta, \overline{\partial}_{J_f} f(z) \cdot \eta).$$
Solution of a weak Bogomolov conjecture

Theorem (D - Gaussier, 2014)

There exist universal embedding spaces \((W, \mathcal{E}, \mathcal{S}) = (W_{n,k}, \mathcal{E}_{n,k}, \mathcal{S}_{n,k})\)
where \(\dim W_{n,k} < \dim Z_{n,k} + n(\dim Z_{n,k} - 2n) = O(nk^2) = O(n^3)\),
and \(\mathcal{S} \subset \mathcal{E} \subset T_W\) are algebraic subsheaves satisfying \([\mathcal{S}, \mathcal{S}] \subset \mathcal{E}\)
(partial integrability), such that every compact \(\mathbb{C}\)-manifold \((X, J)\)
of given dimension \(n\) embeds transversally by \(f : X \hookrightarrow (W_{n,k}, \mathcal{E}_{n,k})\),
i.e. \(J = J_f\), with the additional constraint \(\text{Im}(\bar{\partial}f) \subset \mathcal{S}_{n,k}\).
**Theorem (D - Gaussier, 2014)**

There exist universal embedding spaces \((W, E, S) = (W_{n,k}, E_{n,k}, S_{n,k})\) where \(\dim W_{n,k} < \dim Z_{n,k} + n(\dim Z_{n,k} - 2n) = O(nk^2) = O(n^3)\), and \(S \subset E \subset T_W\) are algebraic subsheaves satisfying \([S, S] \subset E\) (partial integrability), such that every compact \(\mathbb{C}\)-manifold \((X, J)\) of given dimension \(n\) embeds transversally by \(f : X \hookrightarrow (W_{n,k}, E_{n,k})\), i.e. \(J = J_f\), with the additional constraint \(\text{Im}(\overline{\partial} f) \subset S_{n,k}\).

**Proof.** By the Nijenhuis tensor formula, since \(\overline{\partial} J_f f\) is injective with values in \(D_{n,k}\), we see that \(S = \overline{\partial} J_f f(T_{X,x}) \subset D_{n,k,f(x)}\) must be an \(n\)-dimensional complex subspace of \(D_{n,k,x} \subset T_{Z,f(x)}\) that is totally isotropic for \(\theta\), i.e. \(\theta|_{S \times S} \equiv 0\).
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There exist universal embedding spaces \((W, \mathcal{E}, S) = (W_{n,k}, \mathcal{E}_{n,k}, S_{n,k})\) where \(\dim W_{n,k} < \dim Z_{n,k} + n(\dim Z_{n,k} - 2n) = O(nk^2) = O(n^3)\), and \(S \subset \mathcal{E} \subset T_W\) are algebraic subsheaves satisfying \([S, S] \subset \mathcal{E}\) (partial integrability), such that every compact \(\mathbb{C}\)-manifold \((X, J)\) of given dimension \(n\) embeds transversally by \(f : X \hookrightarrow (W_{n,k}, \mathcal{E}_{n,k})\), i.e. \(J = J_f\), with the additional constraint \(\text{Im}(\overline{\partial} f) \subset S_{n,k}\).

Proof. By the Nijenhuis tensor formula, since \(\overline{\partial}_{J_f} f\) is injective with values in \(\mathcal{D}_{n,k}\), we see that \(S = \overline{\partial}_{J_f} f(T_X, x) \subset \mathcal{D}_{n,k,f(x)}\) must be an \(n\)-dimensional complex subspace of \(\mathcal{D}_{n,k,x} \subset T_{Z,f(x)}\) that is totally isotropic for \(\theta\), i.e. \(\theta|_{S \times S} \equiv 0\).

We let \(W_{n,k} \subset \text{Gr}(\mathcal{D}_{n,k}, n)\) be the subvariety of the Grassmannian bundle consisting of the \(\theta\)-isotropic \(n\)-subspaces, and lift \(\mathcal{D}_{n,k} \subset T_{Z,n,k}\) to \(\mathcal{E}_{n,k} \subset T_{W_{n,k}}, S_{n,k}\) being the tautological isotropic subbundle.
In complex dimension 2, it is known that there exist compact almost complex manifolds that cannot be given a complex structure: by Van de Ven (1966), for $X$ a complex surface, 

$$p = c_1^2(X), \quad q = c_2(X)$$

is in the region \{ $p \leq 8q$, $p + q \equiv 0(12)$ \}, but the only restriction for $X$ almost complex is $p + q \equiv 0(12)$.

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**Yau’s challenge and $S^6$**

For $n \geq 3$, find a compact almost complex $n$-fold that cannot be given a complex structure.

The sphere $S^6$ can be realized as the set of octonions $x \in O$ such that $x^2 = -1$ ($\iff \text{Re} \ x = 0$ and $|x| = 1$).

A natural non integrable almost complex structure is then given by $Jxh = xh$, $h \in T_{S^6}$, $x \iff \text{Re} \ h = 0$ and $xh + hx = 0$.

$S^6$ is strongly suspected of not carrying a complex structure!
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$$J_x h = xh, \; h \in T_{S^6,x} \iff \Re h = 0 \text{ and } xh + hx = 0.$$

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Application to complex structures on $S^6$

The octonion embedding $f : S^6 \hookrightarrow \mathbb{O} = \mathbb{R}^{2k}$, $k = 4$ (which has trivial rank 2 normal bundle), yields a universal embedding $\varphi : S^6 \rightarrow Z_{3,4}$ where $\dim Z_{3,4} = 46$, $\text{rank} \ D_{3,4} = 43$ (corank 3).
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By passing to the Grassmannian bundle we get a map $\psi : S^6 \to W_{3,4}$ where $\dim W_{3,4} < 46 + 3 \times 40 = 166$, $W_{3,4}$ being equipped with bundles $\mathcal{E}_{3,4} \supset S_{3,4}$ of respective coranks 3 and 43, and at the homotopy level the question is whether $\overline{\partial} \psi \subset \mathcal{E}_{3,4}$ can be retracted to a section with values in $S_{3,4}$ over the whole $S^6$. If the answer is negative, this would prove that there are no complex structures on $S^6$ (it is well known that $S^6$ admits only two almost complex structures up to homotopy, $J_0$ given by the octonions and its conjugate $-J_0$).

In general, this approach could yield topological obstructions for an almost complex structure to be homotopic to a complex structure.
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In general, this approach could yield topological obstructions for an almost complex structure to be homotopic to a complex structure.
What about Bogomolov’s original conjecture?

**Proposition (reduction of the conjecture to another one !)**

Assume that holomorphic foliations can be approximated by *Nash algebraic foliations* uniformly on compact subsets of any polynomially convex open subset of $\mathbb{C}^N$.

Proof:

$\Phi(U)$ Runge \exists $\Phi : U \to Z$ holomorphic embedding into $Z$ affine algebraic (Stout).

J.-P. Demailly (Grenoble), CIME 2018 on non Kähler geometry Embeddings of complex and almost complex structures 18/19
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The proof uses the Grauert technique of embedding $X$ as a totally real submanifold of $X \times \overline{X}$, and taking a Stein neighborhood $U \supset \Delta$. 

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Proof:

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\exists \Phi : U \rightarrow Z \text{ holomorphic embedding into } Z \text{ affine algebraic (Stout)}. 
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Thank you for your attention