On the cohomology of pseudoeffective line bundles

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in honor of Professor Yum-Tong Siu
on the occasion of his 70th birthday

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- Study sections and cohomology of holomorphic line bundles $L \to X$ on compact Kähler manifolds, without assuming any strict positivity of the curvature

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  – solution of MMP (BCHM 2006), D-Hacon-Păun (2010)

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Any subspace $V_m \subset H^0(X, L^{\otimes m})$ define a meromorphic map

$$\begin{array}{rcll} \Phi_{mL} : X \smallsetminus Z_m & \longrightarrow & \mathbb{P}(V_m) & \text{(hyperplanes of } V_m) \\ x & \longmapsto & H_x = \{\sigma \in V_m\,;\ \sigma(x) = 0\} & \end{array}$$

where Z_m = base locus $B(mL) = \bigcap \sigma^{-1}(0)$.

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One has

$$(\Theta_{L,h})_{|X \smallsetminus B} = \frac{1}{m} \Phi_{mL}^* \omega_{\mathrm{FS}} \text{ where } \Phi_{mL} : X \smallsetminus B \to \mathbb{P}(V_m) \simeq \mathbb{P}^{N_m}.$$

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Definition

- L is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, (possibly singular) such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X, in the sense of currents.

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The well-known Kodaira embedding theorem states that L is positive if and only if L is ample, namely:
$Z_m = B(mL) = \emptyset$ and

$$\Phi_{|mL|} : X \to \mathbb{P}(H^0(X, L^{\otimes m}))$$

is an embedding for $m \geq m_0$ large enough.

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Definitions

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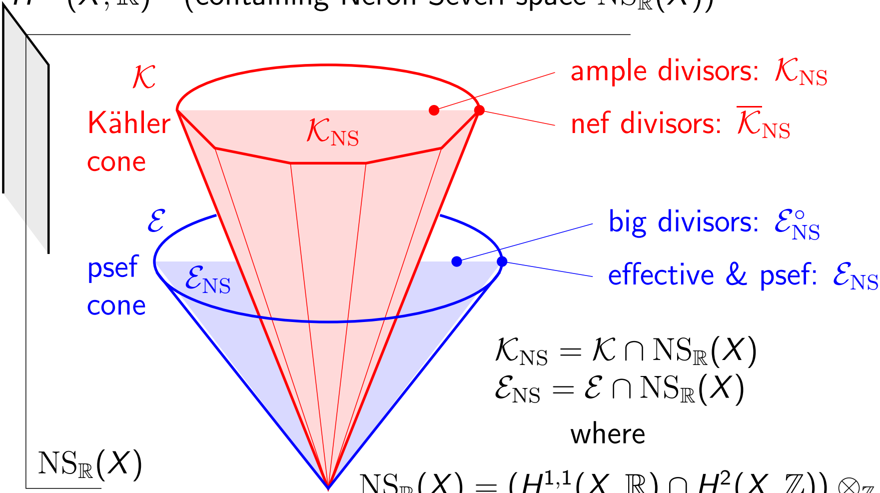
  (by weak compactness of bounded sets of currents).
- $\overline{\mathcal{K}}$ is the cone of "nef classes". One has $\overline{\mathcal{K}} \subset \mathcal{E}$.
- It may happen that $\overline{\mathcal{K}} \subsetneq \mathcal{E}$:
  if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Ample / nef / effective / big divisors

Positive cones can be visualized as follows :

$H^{1,1}(X, \mathbb{R})$ (containing Neron-Severi space $\mathrm{NS}_{\mathbb{R}}(X)$)



$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X)$$
$$\mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X)$$

where

$$\mathrm{NS}_{\mathbb{R}}(X) = (H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{R}$$

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

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Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.

We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

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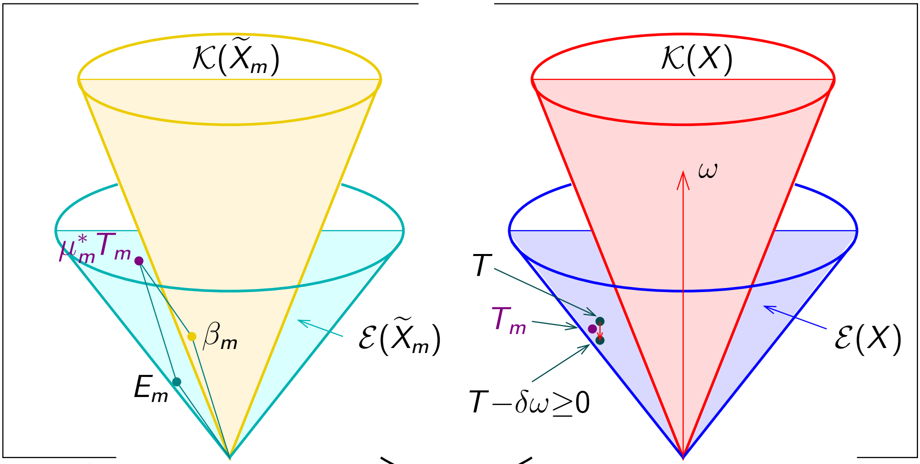
Theorem on approximate Zariski decomposition (D, '92)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles, i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$ where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

Schematic picture of Zariski decomposition



$\mathrm{NS}_{\mathbb{R}}(\widetilde{X}_m) \subset H^{1,1}(\widetilde{X}_m, \mathbb{R})$

$\mu_m : \widetilde{X}_m \longrightarrow X$ (blow-up)

$\mathrm{NS}_{\mathbb{R}}(X) \subset H^{1,1}(X, \mathbb{R})$

Idea of proof of analytic Zariski decomposition

- Write locally

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- Approximate T (again locally) as

$$T_m = i\partial\overline{\partial}\varphi_m, \qquad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \left\{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2 e^{-2m\varphi}dV < +\infty\right\}.$$

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- Further, $\varphi = \lim_{m\to+\infty} \varphi_m$ by the mean value inequality.

"Movable" intersection of currents

Let $\quad \mathcal{P}(X) =$ closed positive $(1,1)$-currents on X

$$H^{k,k}_{\geq 0}(X) = \{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\}.$$

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Theorem (Boucksom PhD 2002, Junyan Cao PhD 2012)

$\forall k = 1, 2, \ldots, n$, $\exists$ canonical "movable intersection product"

$$\mathcal{P} \times \cdots \times \mathcal{P} \to H^{k,k}_{\geq 0}(X), \quad (T_1, \ldots, T_k) \mapsto \langle T_1 \cdot T_2 \cdots T_k \rangle$$

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Method. $T_j = \lim_{\varepsilon \to 0} T_j + \varepsilon\omega$, can assume T_j Kähler. Approximate each T_j by Kähler currents $T_{j,m}$ with logarithmic poles,take a simultaneous log-resolution $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \beta_{j,m}.$$

and define

$$\langle T_1 \cdot T_2 \cdots T_k \rangle = \lim_{m \to +\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}.$$

Volume and numerical dimension of currents

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Special case. The volume of a class $\alpha \in H^{1,1}(X, \mathbb{R})$ is

$$\mathrm{Vol}(\alpha) = \sup_{T \in \alpha} \langle T^n \rangle \quad \text{if } \alpha \in \mathcal{E}^\circ \text{ (big class)},$$
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$$\mathrm{nd}(T) = \max\left\{p \in \mathbb{N}\,;\ \langle T^p \rangle \neq 0 \quad \text{in } H^{p,p}_{\geq 0}(X)\right\}.$$

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Numerical dimension of a hermitian line bundle (L, h)

$$\mathrm{nd}(L, h) = \mathrm{nd}(\Theta_{L,h}).$$

Generalized abundance conjecture

Numerical dimension of a class $\alpha \in H^{1,1}(X,\mathbb{R})$

If α is not pseudoeffective, set $\mathrm{nd}(\alpha) = -\infty$, otherwise

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Generalized abundance conjecture

For X compact Kähler, K_X is abundant, i.e. $\kappa(X) = \mathrm{nd}(K_X)$.

Hard Lefschetz theorem with pseudoeffective coefficients

Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, and for $h = e^{-\varphi}$, let $\mathcal{I}(h) = \mathcal{I}(\varphi)$ be the multiplier ideal sheaf:

$$\mathcal{I}(\varphi)_x := \left\{ f \in \mathcal{O}_{X,x} \,;\, \exists V \ni x,\ \int_V |f|^2 e^{-\varphi} dV_\omega < +\infty \right\}.$$

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$$\Theta_{L,h} \geq \varepsilon\omega \implies H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0 \ \text{ for } q \geq 1.$$

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Hard Lefschetz theorem (D-Peternell-Schneider 2001)

Assume merely $\Theta_{L,h} \geq 0$. Then, the Lefschetz map : $u \mapsto \omega^q \wedge u$ induces a surjective morphism :

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathcal{I}(h)).$$

Idea of proof of Hard Lefschetz theorem

Main tool. "Equisingular approximation theorem":

$$\varphi = \lim \downarrow \varphi_\nu \quad \Rightarrow \quad h = \lim h_\nu$$

with:

- $\varphi_\nu \in C^\infty(X \smallsetminus Z_\nu)$, where Z_ν is an increasing sequence of analytic sets,
- $\mathcal{I}(h_\nu) = \mathcal{I}(h), \quad \forall \nu$,
- $\Theta_{L,h_\nu} \geq -\varepsilon_\nu \omega$.

(Again, the proof uses in several ways the Ohsawa-Takegoshi theorem).

Idea of proof of Hard Lefschetz theorem

Main tool. "Equisingular approximation theorem":

$$\varphi = \lim \downarrow \varphi_\nu \quad \Rightarrow \quad h = \lim h_\nu$$

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Then, use the fact that $X \smallsetminus Z_\nu$ is Kähler complete, so one can apply (non compact) harmonic form theory on $X \smallsetminus Z_\nu$, and pass to the limit to get rid of the errors ε_ν.

Generalized Nadel vanishing theorem

Theorem (Junyan Cao, PhD 2012)

Let X be compact Kähler, and let (L, h) be pseudoeffective on X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \quad \text{for } q \geq n - \mathrm{nd}(L, h) + 1,$$

where

$$\mathcal{I}_+(h) = \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon}) = \lim_{\varepsilon \to 0} \mathcal{I}((1+\varepsilon)\varphi)$$

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Remark 2. In the projective case, one can use a hyperplane section argument, provided one first shows that $\mathrm{nd}(L, h)$ coincides with H. Tsuji's algebraic definition ($\dim Y = p$) :

$$\mathrm{nd}(L, h) = \max\left\{p \in \mathbb{N}\,;\ \exists Y^p \subset X,\ h^0(Y, (L^{\otimes m} \otimes \mathcal{I}(h^m))_{|Y}) \geq cm^p\right\}.$$

Proof of generalized Nadel vanishing (projective case)

Hyperplane section argument (projective case). Take $A =$ very ample divisor, $\omega = \Theta_{A,h_A} > 0$, and $Y = A_1 \cap \ldots \cap A_{n-p}$, $A_j \in |A|$. Then

$$\langle \Theta_{L,h}^p \rangle \cdot Y = \int_X \langle \Theta_{L,h}^p \rangle \cdot Y = \int_X \langle \Theta_{L,h}^p \rangle \wedge \omega^{n-p} > 0.$$

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Lemma (J. Cao)

When (L,h) is big, i.e. $\langle \Theta^n_{L,h} \rangle > 0$, there exists a metric $\widetilde{h}$ such that $\mathcal{I}(\widetilde{h}) = \mathcal{I}_+(h)$ with $\Theta_{L,\widetilde{h}} \geq \varepsilon\omega$ [Riemann-Roch].

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Conclude by induction on $\dim X$ and the exact cohomology sequence for the restriction to a hyperplane section.

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Kähler case. Assume $c_1(L)$ nef for simplicity. Then $c_1(L) + \varepsilon\omega$ Kähler. By Yau's theorem, solve Monge-Ampère equation:

$$\exists h_\varepsilon \text{ on } L, \quad (\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n = C_\varepsilon \omega^n.$$

Here $C_\varepsilon \geq \binom{n}{p}\langle\Theta^p_{L,h}\rangle \cdot (\varepsilon\omega)^{n-p} \sim C\varepsilon^{n-p}$, $p = \mathrm{nd}(L,h)$.

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so $\lambda_{q+1} \ldots \lambda_n \leq C$ on a large open set $U \subset X$ and

$$\lambda_q^q \geq \lambda_1 \ldots \lambda_q \geq c\varepsilon^{n-p} \quad \Rightarrow \quad \lambda_q \geq c\varepsilon^{(n-p)/q} \text{ on } U,$$

$$\textstyle\sum_{j=1}^q (\lambda_j - \varepsilon) \geq \lambda_q - q\varepsilon \geq c\varepsilon^{(n-p)/q} - q\varepsilon > 0 \text{ for } q > n-p.$$

Final step: use Bochner-Kodaira formula

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Then one has to show that one can take the limit by assuming integrability with $e^{-(1+\delta)\varphi}$, thus introducing $\mathcal{I}_+(h)$.

Application to Kähler geometry

Definition (Campana)

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It is expected that simple compact Kähler manifolds are either generic complex tori, generic hyperkähler manifolds and their finite quotients, up to modification.

On simple Kähler 3-folds

Theorem (Campana - D - Verbitsky, 2013)

Let X be a compact Kähler 3-fold without any positive dimensional analytic subset $A \subsetneq X$. Then
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- Hilbert polynomial $P(m) = \chi(X, K_X^{\otimes m})$ is bounded, hence $\chi(X, \mathcal{O}_X) = 0$.
- Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism.

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