Complements:
– Closed Hilbertian operators
– Complete Riemannian manifolds

§ 1. Closed Hilbertian operators

We expose here some basic results of Von Neumann's theory of unbounded operators on Hilbert spaces. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be complex Hilbert spaces. We consider a linear operator T defined on a subspace $\operatorname{Dom} T \subset \mathscr{H}_1$ (called the domain of T) into $\mathscr{H}_2$. The operator T is said to be *densely defined* if $\operatorname{Dom} T$ is dense in $\mathscr{H}_1$, and *closed* if its graph

$$\operatorname{Gr} T = \big\{(x, Tx)\ ;\ x \in \operatorname{Dom} T\big\}$$

is closed in $\mathscr{H}_1 \times \mathscr{H}_2$.

Assume now that T is closed and densely defined. The adjoint $T^\star$ of T (in Von Neumann's sense) is constructed as follows: $\operatorname{Dom} T^\star$ is the set of $y \in \mathscr{H}_2$ such that the linear form

$$\operatorname{Dom} T \ni x \longmapsto \langle Tx, y\rangle_2$$

is bounded in $\mathscr{H}_1$-norm. Since $\operatorname{Dom} T$ is dense, there exists for every y in $\operatorname{Dom} T^\star$ a unique element $T^\star y \in \mathscr{H}_1$ such that $\langle Tx, y\rangle_2 = \langle x, T^\star y\rangle_1$ for all $x \in \operatorname{Dom} T^\star$. It is immediate to verify that $\operatorname{Gr} T^\star = \big(\operatorname{Gr}(-T)\big)^\perp$ in $\mathscr{H}_1 \times \mathscr{H}_2$. It follows that $T^\star$ is closed and that every pair $(u, v) \in \mathscr{H}_1 \times \mathscr{H}_2$ can be written

$$(u, v) = (x, -Tx) + (T^\star y, y), \qquad x \in \operatorname{Dom} T, \quad y \in \operatorname{Dom} T^\star.$$

Take in particular $u = 0$. Then

$$x + T^\star y = 0, \qquad v = y - Tx = y + TT^\star y, \qquad \langle v, y\rangle_2 = \|y\|_2^2 + \|T^\star y\|_1^2.$$

If $v \in (\operatorname{Dom} T^\star)^\perp$ we get $\langle v, y\rangle_2 = 0$, thus $y = 0$ and $v = 0$. Therefore $T^\star$ is densely defined and our discussion implies:

(1.1) Theorem [Von Neumann 1929]). *If $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ is a closed and densely defined operator, then its adjoint $T^\star$ is also closed and densely defined and $(T^\star)^\star = T$. Furthermore, we have the relation* $\operatorname{Ker} T^\star = (\operatorname{Im} T)^\perp$ *and its dual* $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Im} T^\star}$. □

Consider now two closed and densely defined operators T, S :

$$\mathscr{H}_1 \xrightarrow{\ T\ } \mathscr{H}_2 \xrightarrow{\ S\ } \mathscr{H}_3$$

such that $S \circ T = 0$. By this, we mean that the range $T(\operatorname{Dom} T)$ is contained in $\operatorname{Ker} S \subset \operatorname{Dom} S$, in such a way that there is no problem for defining the composition $S \circ T$. The starting point of all L^2 estimates is the following abstract existence theorem.

(1.2) Theorem. *There are orthogonal decompositions*

$$\mathcal{H}_2 = (\operatorname{Ker} S \cap \operatorname{Ker} T^\star) \oplus \overline{\operatorname{Im} T} \oplus \overline{\operatorname{Im} S^\star},$$
$$\operatorname{Ker} S = (\operatorname{Ker} S \cap \operatorname{Ker} T^\star) \oplus \overline{\operatorname{Im} T}.$$

In order that $\operatorname{Im} T = \operatorname{Ker} S$*, it suffices that*

$$\|T^\star x\|_1^2 + \|Sx\|_3^2 \geqslant C\|x\|_2^2, \qquad \forall x \in \operatorname{Dom} S \cap \operatorname{Dom} T^\star \tag{1.3}$$

for some constant $C > 0$*. In that case, for every* $v \in \mathcal{H}_2$ *such that* $Sv = 0$*, there exists* $u \in \mathcal{H}_1$ *such that* $Tu = v$ *and*

$$\|u\|_1^2 \leqslant \frac{1}{C}\|v\|_2^2.$$

In particular

$$\overline{\operatorname{Im} T} = \operatorname{Im} T = \operatorname{Ker} S, \qquad \overline{\operatorname{Im} S^\star} = \operatorname{Im} S^\star = \operatorname{Ker} T^\star.$$

Proof. Since S is closed, the kernel $\operatorname{Ker} S$ is closed in $\mathcal{H}_2$. The relation $(\operatorname{Ker} S)^\perp = \overline{\operatorname{Im} S^\star}$ implies

$$\mathcal{H}_2 = \operatorname{Ker} S \oplus \overline{\operatorname{Im} S^\star} \tag{1.4}$$

and similarly $\mathcal{H}_2 = \operatorname{Ker} T^\star \oplus \overline{\operatorname{Im} T}$. However, the assumption $S \circ T = 0$ shows that $\overline{\operatorname{Im} T} \subset \operatorname{Ker} S$, therefore

$$\operatorname{Ker} S = (\operatorname{Ker} S \cap \operatorname{Ker} T^\star) \oplus \overline{\operatorname{Im} T}. \tag{1.5}$$

The first two equalities in Th. 1.2 are then equivalent to the conjunction of (1.4) and (1.5).

Now, under assumption (1.3), we are going to show that the equation $Tu = v$ is always solvable if $Sv = 0$. Let $x \in \operatorname{Dom} T^\star$. One can write

$$x = x' + x'' \quad \text{where} \quad x' \in \operatorname{Ker} S \quad \text{and} \quad x'' \in (\operatorname{Ker} S)^\perp \subset (\operatorname{Im} T)^\perp = \operatorname{Ker} T^\star.$$

Since $x, x'' \in \operatorname{Dom} T^\star$, we have also $x' \in \operatorname{Dom} T^\star$. We get

$$\langle v, x\rangle_2 = \langle v, x'\rangle_2 + \langle v, x''\rangle_2 = \langle v, x'\rangle_2$$

because $v \in \operatorname{Ker} S$ and $x'' \in (\operatorname{Ker} S)^\perp$. As $Sx' = 0$ and $T^\star x'' = 0$, the Cauchy-Schwarz inequality combined with (1.3) implies

$$|\langle v, x\rangle_2|^2 \leqslant \|v\|_2^2\ \|x'\|_2^2 \leqslant \frac{1}{C}\|v\|_2^2\ \|T^\star x'\|_1^2 = \frac{1}{C}\|v\|_2^2\ \|T^\star x\|_1^2.$$

This shows that the linear form $T^\star_X \ni x \longmapsto \langle x, v\rangle_2$ is continuous on $\operatorname{Im} T^\star \subset \mathcal{H}_1$ with norm $\leqslant C^{-1/2}\|v\|_2$. By the Hahn-Banach theorem, this form can be extended to a continuous linear form on $\mathcal{H}_1$ of norm $\leqslant C^{-1/2}\|v\|_2$, i.e. we can find $u \in \mathcal{H}_1$ such that $\|u\|_1 \leqslant C^{-1/2}\|v\|_2$ and

$$\langle x, v\rangle_2 = \langle T^\star x, u\rangle_1, \qquad \forall x \in \operatorname{Dom} T^\star.$$

This means that $u \in \operatorname{Dom}(T^\star)^\star = \operatorname{Dom} T$ and $v = Tu$. We have thus shown that $\operatorname{Im} T = \operatorname{Ker} S$, in particular $\operatorname{Im} T$ is closed. The dual equality $\operatorname{Im} S^\star = \operatorname{Ker} T^\star$ follows by considering the dual pair $(S^\star, T^\star)$. □

§ 2. Complete Riemannian manifolds

Let (M, g) be a riemannian manifold of dimension m, with metric

$$g(x) = \sum g_{jk}(x)\, dx_j \otimes dx_k, \qquad 1 \leqslant j, k \leqslant m.$$

The length of a path $\gamma\ :\ [a, b] \longrightarrow M$ is by definition

$$\ell(\gamma) = \int_a^b |\gamma'(t)|_g dt = \int_a^b \Big(\sum_{j,k} g_{jk}\big(\gamma(t)\big)\, \gamma_j'(t)\gamma_k'(t)\Big)^{1/2} dt.$$

The geodesic distance of two points $x, y \in M$ is

$$\delta(x, y) = \inf_\gamma \ell(\gamma) \qquad \text{over paths } \gamma \text{ with } \ \gamma(a) = x, \quad \gamma(b) = y,$$

if x, y are in the same connected component of M, $\delta(x, y) = +\infty$ otherwise. It is easy to check that δ satisfies the usual axioms of distances: for the separation axiom, use the fact that if y is outside some closed coordinate ball $\overline{B}$ of radius r centered at x and if $g \geqslant c|dx|^2$ on $\overline{B}$, then $\delta(x, y) \geqslant c^{1/2} r$. In addition, δ satisfies the axiom:

$$\text{(2.1)} \qquad \text{for every } x, y \in M, \qquad \inf_{z \in M} \max\{\delta(x, z), \delta(y, z)\} = \frac{1}{2}\delta(x, y).$$

In fact for every $\varepsilon > 0$ there is a path γ such that $\gamma(a) = x$, $\gamma(b) = y$, $\ell(\gamma) < \delta(x, y) + \varepsilon$ and we can take z to be at mid-distance between x and y along γ. A metric space E with a distance δ satisfying the additional axiom (2.1) will be called a *geodesic* metric space (Gromov calls them “length spaces”). It is then easy to see by dichotomy that any two points $x, y \in E$ can be joined by a chain of points $x = x_0, x_1, \ldots, x_N = y$ such that $\delta(x_j, x_{j+1}) < \varepsilon$ and $\sum \delta(x_j, x_{j+1}) < \delta(x, y) + \varepsilon$.

(2.2) Lemma (Hopf-Rinow)**.** *Let (E, δ) be a geodesic metric space. Then the following properties are equivalent:*

a) *E is locally compact and complete*;

b) *all closed geodesic balls $\overline{B}(x_0, r)$ are compact.*

Proof. Since any Cauchy sequence is bounded, it is immediate that b) implies a). We now check that a) $\Longrightarrow$ b). Fix x_0 and define R to be the supremum of all $r > 0$ such that $\overline{B}(x_0, r)$ is compact. Since E is locally compact, we have $R > 0$. Suppose that $R < +\infty$. Then $\overline{B}(x_0, r)$ is compact for every $r < R$. Let y_ν be a sequence of points in $\overline{B}(x_0, R)$. Fix an integer p. As $\delta(x_0, y_\nu) \leqslant R$, axiom (2.1) shows that we can find points $z_\nu \in M$ such that $\delta(x_0, z_\nu) \leqslant (1 - 2^{-p})R$ and $\delta(z_\nu, y_\nu) \leqslant 2^{1-p}R$. Since $\overline{B}(x_0, (1 - 2^{-p})R)$ is compact, there is a subsequence $(z_{\nu(p,q)})_{q \in \mathbb{N}}$ converging to a limit point w_p with $\delta(z_{\nu(p,q)}, w_p) \leqslant 2^{-q}$. We proceed by induction on p and take $\nu(p+1, q)$ to be a subsequence of $\nu(p, q)$. Then

$$\delta(y_{\nu(p,q)}, w_p) \leqslant \delta(y_{\nu(p,q)}, z_{\nu(p,q)}) + \delta(z_{\nu(p,q)}, w_p) \leqslant 2^{1-p}R + 2^{-q}.$$

Since $(y_{\nu(p+1,q)})$ is a subsequence of $(y_{\nu(p,q)})$, we infer that $\delta(w_p, w_{p+1}) \leqslant 3\, 2^{-p}R$ by letting q tend to $+\infty$. By the completeness hypothesis, the Cauchy sequence (w_p) converges to a limit point $w \in M$, and the above inequalities show that $(y_{\nu(p,p)})$ converges to $w \in \overline{B}(x_0,R)$. Therefore $\overline{B}(x_0,R)$ is compact. Now, each point $y \in \overline{B}(x_0,R)$ can be covered by a compact ball $\overline{B}(y,\varepsilon_y)$, and the compact set $\overline{B}(x_0,R)$ admits a finite covering by concentric balls $B(y_j,\varepsilon_{y_j}/2)$. Set $\varepsilon = \min \varepsilon_{y_j}$. Every point $z \in \overline{B}(x_0,R+\varepsilon/2)$ is at distance $\leqslant \varepsilon/2$ of some point $y \in \overline{B}(x_0,R)$, hence at distance $\leqslant \varepsilon/2 + \varepsilon_{y_j}/2$ of some point y_j, in particular $\overline{B}(x_0,R+\varepsilon/2) \subset \bigcup \overline{B}(y_j,\varepsilon_{y_j})$ is compact. This is a contradiction, so $R = +\infty$. □

The following standard definitions and properties will be useful in order to deal with the completeness of the metric.

(2.3) Definitions.

a) *A riemannian manifold (M,g) is said to be complete if (M,δ) is complete as a metric space.*

b) *A continuous function $\psi : M \to \mathbb{R}$ is said to be exhaustive if for every $c \in \mathbb{R}$ the sublevel set $M_c = \{x \in M \ ; \ \psi(x) < c\}$ is relatively compact in M.*

c) *A sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of M is said to be exhaustive if $M = \bigcup K_\nu$ and if K_ν is contained in the interior of $K_{\nu+1}$ for all ν (so that every compact subset of M is contained in some K_ν).*

(2.4) Lemma. *The following properties are equivalent:*

a) *(M,g) is complete;*

b) *there exists an exhaustive function $\psi \in \mathscr{C}^\infty(M,\mathbb{R})$ such that $|d\psi|_g \leqslant 1$;*

c) *there exists an exhaustive sequence $(K_\nu)_{\nu\in\mathbb{N}}$ of compact subsets of M and functions $\psi_\nu \in \mathscr{C}^\infty(M,\mathbb{R})$ such that*

$$\begin{aligned} &\psi_\nu = 1 \quad \text{in a neighborhood of } K_\nu, \qquad \operatorname{Supp} \psi_\nu \subset K^\circ_{\nu+1}, \\ &0 \leqslant \psi_\nu \leqslant 1 \quad \text{and} \quad |d\psi_\nu|_g \leqslant 2^{-\nu}. \end{aligned}$$

Proof. a) $\Longrightarrow$ b). Without loss of generality, we may assume that M is connected. Select a point $x_0 \in M$ and set $\psi_0(x) = \frac{1}{2}\delta(x_0,x)$. Then ψ_0 is a Lipschitz function with constant $\frac{1}{2}$, thus ψ_0 is differentiable almost everywhere on M and $|d\psi_0|_g \leqslant \frac{1}{2}$. We can find a smoothing ψ of ψ_0 such that $|d\psi|_g \leqslant 1$ and $|\psi - \psi_0| \leqslant 1$. Then ψ is an exhaustion function of M.

b) $\Longrightarrow$ c). Choose ψ as in a) and a function $\rho \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ such that $\rho = 1$ on $]-\infty, 1.1]$, $\rho = 0$ on $[1.9,+\infty[$ and $0 \leqslant \rho' \leqslant 2$ on $[1,2]$. Then

$$K_\nu = \{x \in M \ ; \ \psi(x) \leqslant 2^{\nu+1}\}, \qquad \psi_\nu(x) = \rho\big(2^{-\nu-1}\psi(x)\big)$$

satisfy our requirements.

c) $\Longrightarrow$ b). Set $\psi = \sum 2^\nu(1-\psi_\nu)$.

b) $\Longrightarrow$ a). The inequality $|d\psi|_g \leqslant 1$ implies $|\psi(x) - \psi(y)| \leqslant \delta(x,y)$ for all $x, y \in M$, so all δ-balls must be relatively compact in M. □