Differential and pseudodifferential operators on manifolds

Course on analytic geometry (Jean-Pierre Demailly)

1. Differential operators on vector bundles

We first describe some basic concepts concerning differential operators (symbol, composition, ellipticity, adjoint), in the general context of vector bundles. Assume given a differentiable manifold M of class C^∞, $\dim_{\mathbb{R}} M = m$, and let E, F be vector bundles on M, over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, such that $\operatorname{rank} E = r$, $\operatorname{rank} F = r'$.

1.1. Definition. *A (linear) differential operator of degree δ from E to F is a $\mathbb{K}$-linear operator $P : C^\infty(M, E) \to C^\infty(M, F)$, $u \mapsto Pu$ of the form*

$$Pu(x) = \sum_{|\alpha| \leqslant \delta} a_\alpha(x) D^\alpha u(x),$$

where $E_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^r$, $F_{\restriction\Omega} \simeq \Omega \times \mathbb{K}^{r'}$ are local trivializations on an open chart $\Omega \subset M$ with local coordinates $(x_1, \ldots, x_m)$, and the coefficients $a_\alpha(x)$ are $r' \times r$ matrices $\big(a_{\alpha\lambda\mu}(x)\big)_{1 \leqslant \lambda \leqslant r',\, 1 \leqslant \mu \leqslant r}$ with C^∞ coefficients on Ω. One write here $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}$ as usual, and the matrices $u = (u_\mu)_{1 \leqslant \mu \leqslant r}$, $D^\alpha u = (D^\alpha u_\mu)_{1 \leqslant \mu \leqslant r}$ are viewed as column vectors.

If $t \in \mathbb{K}$ is a parameter and $f \in C^\infty(M, \mathbb{K})$, $u \in C^\infty(M, E)$, an easy calculation shows that $\mathrm{e}^{-tf(x)} P(\mathrm{e}^{tf(x)} u(x))$ is a homogeneous polynomial of degree δ in t, of the form

$$\mathrm{e}^{-tf(x)} P(\mathrm{e}^{tf(x)} u(x)) = t^\delta \sigma_P(x, df(x)) \cdot u(x) + \text{terms } c_j(x) t^j \text{ of degree } j < \delta,$$

where σ_P is a homogeneous polynomial map $T^\star_M \to \mathrm{Hom}(E, F)$ defined by

$$T^\star_{M,x} \ni \xi \mapsto \sigma_P(x, \xi) \in \mathrm{Hom}(E_x, F_x), \qquad \sigma_P(x, \xi) = \sum_{|\alpha| = \delta} a_\alpha(x) \xi^\alpha. \tag{1.2}$$

Then $\sigma_P(x, \xi)$ is a C^∞ function of the variables $(x, \xi) \in T^\star_M$, and this function is independent of the choice of the coordinates or trivializations used for E, F. The function σ_P is called the *principal symbol* of P. The principal symbol of a composition $Q \circ P$ of differential operators $P : C^\infty(M, E) \to C^\infty(M, F)$, $Q : C^\infty(M, F) \to C^\infty(M, G)$ is simply the product

$$\sigma_{Q \circ P}(x, \xi) = \sigma_Q(x, \xi) \sigma_P(x, \xi), \tag{1.3}$$

calculated as a product of matrices (i.e. as a composition of endomorphisms). Differential operators for which the symbol is injective play a very important role :

1.4. Definition. *A differential operator P is said to be elliptic if $\sigma_P(x, \xi) \in \mathrm{Hom}(E_x, F_x)$ is injective for all $x \in M$ and $\xi \in T^\star_{M,x} \smallsetminus \{0\}$.*

Let us now assume that M is oriented and assume given a C^∞ volume form $dV(x) = \gamma(x)\, dx_1 \wedge \cdots \wedge dx_m$ on M, where $\gamma(x) > 0$ is a C^∞ density ; if (M, g) is a Riemannian manifold, one can for instance take The Riemannian volume form $dV = dV_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_m$.

If E is a Euclidian or Hermitian vector bundle, we can define a Hilbert space $L^2(M,E)$ of global sections with values in E, namely the space of forms u with measurable coefficients that are square summable for the scalar product

$$\|u\|^2=\int_M |u(x)|^2\,dV(x), \tag{1.5}$$

$$\langle\!\langle u,v\rangle\!\rangle=\int_M \langle u(x),v(x)\rangle\,dV(x),\qquad u,v\in L^2(M,E). \tag{1.5'}$$

1.6. Definition. *If $P:C^\infty(M,E)\to C^\infty(M,F)$ is a differential operator and if the bundles E, F are Euclidian or Hermitian, there exists a unique differential operator*

$$P^\star:C^\infty(M,F)\to C^\infty(M,E),$$

called the formal adjoint of P, such that for all sections $u\in C^\infty(M,E)$ and $v\in C^\infty(M,F)$ one has an identity

$$\langle\!\langle Pu,v\rangle\!\rangle=\langle\!\langle u,P^\star v\rangle\!\rangle\quad\textit{whenever}\ \ \operatorname{Supp}u\cap\operatorname{Supp}v\Subset M.$$

Proof. The uniqueness is easy to verify, being a consequence of the density of C^∞ form with compact support in $L^2(M,E)$. By a partition of unity argument, we reduce the verification of the existence of $P^\star$ to the proof of its existence locally on M. Now, let $Pu(x)=\sum_{|\alpha|\leqslant\delta}a_\alpha(x)D^\alpha u(x)$ be the description of P relative to the trivializations of E, F associated to an orthonormal frame and to the system of locate coordinates on an open set $\Omega\subset M$. By assuming $\operatorname{Supp}u\cap\operatorname{Supp}v\Subset\Omega$, an integration by parts gives

$$\begin{aligned}
\langle\!\langle Pu,v\rangle\!\rangle&=\int_\Omega\sum_{|\alpha|\leqslant\delta,\lambda,\mu}a_{\alpha\lambda\mu}D^\alpha u_\mu(x)\overline{v}_\lambda(x)\,\gamma(x)\,dx_1\ldots dx_m\\
&=\int_\Omega\sum_{|\alpha|\leqslant\delta,\lambda,\mu}(-1)^{|\alpha|}u_\mu(x)\overline{D^\alpha(\gamma(x)\,\overline{a}_{\alpha\lambda\mu}v_\lambda(x)}\,dx_1\ldots dx_m\\
&=\int_\Omega\langle u,\sum_{|\alpha|\leqslant\delta}(-1)^{|\alpha|}\gamma(x)^{-1}D^\alpha\big(\gamma(x)\,a_\alpha^*v(x)\big)\rangle\,dV(x)
\end{aligned}$$

where a_α^* denotes the adjoint (= conjugate of transpose) of matrix a_α. We thus see that $P^\star$ exists, and is defined in a unique way by

$$P^\star v(x)=\sum_{|\alpha|\leqslant\delta}(-1)^{|\alpha|}\gamma(x)^{-1}D^\alpha\big(\gamma(x)\,a_\alpha^*v(x)\big). \tag{1.7}$$ $\square$

1.8. Remark. The condition $\operatorname{Supp}u\cap\operatorname{Supp}v\Subset M$ avoids boundary terms in the integration by parts. As a consequence, if $(M,\partial M)$ is a manifold with boundary and $\operatorname{Supp}u\cap\operatorname{Supp}v$ touches the boundary, the equality is no longer valid. On the other hand, under the same condition for the supports, the formula still holds whenever products $D^\alpha u_\mu\overline{v}_\lambda$ make sense in the sense of distributions, e.g. if $u\in\mathcal{D}'(M,E)$ and $v\in C^\infty(M,F)$, or vice versa.

Formula (1.7) shows immediately that the principal symbol of $P^\star$ is

$$\sigma_{P^\star}(x,\xi)=(-1)^\delta\sum_{|\alpha|=\delta}a_\alpha^*\,\xi^\alpha=(-1)^\delta\sigma_P(x,\xi)^\star. \tag{1.9}$$

If $\operatorname{rank}E=\operatorname{rank}F$, the operator P is elliptic if and only if $\sigma_P(x,\xi)$ is invertible for $\xi\neq0$, therefore the ellipticity of P is equivalent to that of $P^\star$.

2. Sobolev spaces

The space of tempered distributions $\mathcal{S}'(\mathbb{R}^m)$ is by definition the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ of *rapidly decrasing* C^∞ functions, namely the subspace of functions $f \in C^\infty(\mathbb{R}^m)$ for which all semi-norms $f \mapsto p_{k,\ell}(f) = \sup_{|\alpha|\leqslant \ell}(1+|x|)^k |D^\alpha f(x)|$ are finite – $\mathcal{S}(\mathbb{R}^m)$ is a Fréchet space. The Fourier transform $\widehat{u}$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^m)$ is defined by the usual adjunction property $\widehat{u}(f) = u(\widehat{f})$, where

$$\widehat{f}(\xi) := \int_{\mathbb{R}^m} f(x)\,\mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi}\,d\lambda(x), \quad f \in \mathcal{S}(\mathbb{R}^m).$$

Let us recall the basic Fourier transform formulas, which can be obtained by differentiating under the integral sign, resp. by an integration by parts

$$D^\alpha_\xi \widehat{f}(\xi) = \big((-2\pi\mathrm{i}x)^\alpha f(x)\big)^\wedge, \qquad \widehat{D^\alpha_x f}(\xi) = (2\pi\mathrm{i}\xi)^\alpha \widehat{f}(\xi)$$

for any $f \in \mathcal{S}(\mathbb{R}^m)$. Also, the *Fourier inversion formula* stipulates that

$$\widehat{\widehat{f}}(x) = f(-x) \quad\Longleftrightarrow\quad f(x) = \int_{\mathbb{R}^m} \widehat{f}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi).$$

By adjunction, the same formulas are still valid for all $u \in \mathcal{S}'(\mathbb{R}^m)$. For any real number s, we define the Sobolev space $W^s(\mathbb{R}^m)$ to be the Hilbert space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^m)$ such that the Fourier transform $\widehat{u}$ is a L^2_{loc} function satisfying the estimate

$$\|u\|_s^2 = \int_{\mathbb{R}^m} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\lambda(\xi) < +\infty. \tag{2.1}$$

For $s \in \mathbb{N}$, we have $\widehat{D^\alpha u}(\xi) = (2\pi\mathrm{i}\xi)^\alpha \widehat{u}(\xi)$, therefore, up to equivalence of norms,

$$\|u\|_s^2 \sim \int_{\mathbb{R}^m} \sum_{|\alpha|\leqslant s} |D^\alpha u(x)|^2 d\lambda(x),$$

and $W^s(\mathbb{R}^m)$ is the Hilbert space of functions u such that all derivatives $D^\alpha u$ of order $|\alpha| \leqslant s$ are in $L^2(\mathbb{R}^m)$. It is also easy to see that $W^s(\mathbb{R}^m) \subset L^2(\mathbb{R}^m)$ for $s \geqslant 0$.

We now assume that M is an *oriented* C^∞ manifold of dimension m, equipped with a volume form dV. Let $E \to M$ be a C^∞ Hermitian vector bundle of rank r on M. We denote by

$$W^s_{\mathrm{loc}}(M, E)$$

the (local) *Sobolev space* of sections $u : M \to E$ whose components are locally in $W^s(\mathbb{R}^m)$ on all open charts. More precisely, choose a locally finite covering (Ω_j) of M by relatively compact open coordinate charts $\Omega_j \simeq B(0, r_j) \simeq \mathbb{R}^m$ on which E is trivial. Consider an orthonormal frame $(e_{j,\lambda})_{1\leqslant\lambda\leqslant r}$ of $E_{\restriction\Omega_j}$ and write u in terms of its components, i.e. $u = \sum_{1\leqslant\lambda\leqslant r} u_{j,\lambda}\, e_{j,\lambda}$ in Ω_j. We then define a Fréchet topology on $W^s_{\mathrm{loc}}(M, E)$ by considering the semi-norms

$$\|u\|_{j,s}^2 = \sum_{1\leqslant\lambda\leqslant r} \|\psi_j u_{j,\lambda}\|_s^2 \tag{2.2}$$

where (ψ_j) is a "quadratic partition of unity" subordinate to (Ω_j), i.e. such that $\sum \psi_j^2 = 1$. We show below that the topology defined in this way is independent of the choices made (covering, partition of unity, trivializations of E). If M is compact, the covering (Ω_j) can be taken finite, and one gets a Hilbert space topology associated to the global norm

$$\|u\|_s^2 = \sum_j \|u\|_{j,s}^2 = \sum_{j,\lambda} \|\psi_j u_{j,\lambda}\|_s^2. \tag{2.2$'$}$$

When M is compact, we simply write $W^s_{\mathrm{loc}}(M,E) = W^s(M,E)$. To get the independence of the topology on the choices made, we write $u = \sum_{j,\lambda} \psi_j^2 u_{j,\lambda} e_{j,\lambda}$ and decompose again u with respect to another covering (Ω'_k), $\Omega'_k \simeq \mathbb{R}^m$, another quadratic partition of unity $\sum \psi'^2_k = 1$ and other choices of orthonormal frames $(e'_{k,\lambda})$. The new components $\psi'_k u'_{k,\lambda}$ are finite linear combinations of the $\psi_j u_{j,\lambda}$, multiplied by C^∞ functions with compact support in $\Omega_j \cap \Omega'_k$ (as all such terms involve a factor $\psi_j \psi'_k$). It is then sufficient to apply the following lemma.

2.3. Lemma. *For any $f \in \mathcal{D}(\mathbb{R}^m)$, the multiplication map $u \mapsto fu$ is continuous on $W^s(\mathbb{R}^m)$, i.e. $\|fu\|_s \leqslant C\|u\|_s$ for some constant $C = C_f > 0$.*

Proof. As is well known, we have $\widehat{fu} = \widehat{f} * \widehat{u}$, that is,

$$\widehat{fu}(\xi) = \int_{\mathbb{R}^m} \widehat{f}(\xi - \eta)\, \widehat{u}(\eta)\, d\lambda(\eta).$$

One uses the *Peetre inequality* asserting that

$$(1+|\xi|^2)^s \leqslant 2^{|s|}(1+|\xi-\eta|^2)^{|s|}(1+|\eta|^2)^s \tag{2.4}$$

for all $\xi, \eta \in \mathbb{R}^m$ and all $s \in \mathbb{R}$. Replacing ξ by $\xi + \eta$, the case $s > 0$ is equivalent to

$$1 + |\xi+\eta|^2 \leqslant 2(1+|\xi|^2)(1+|\eta|^2)$$

which is itself an easy consequence of $|\xi+\eta|^2 + |\xi-\eta|^2 = 2(|\xi|^2+|\eta|^2)$. Now, for $s < 0$, the inequality is equivalent to $(1+|\eta|^2)^{|s|} \leqslant 2^{|s|}(1+|\eta-\xi|^2)^{|s|}(1+|\xi|^2)^{|s|}$, resulting from the case $|s| = -s > 0$ by switching ξ and η. The Peetre inequality applied to $s/2$ gives

$$(1+|\xi|)^{s/2}|\widehat{fu}(\xi)| \leqslant 2^{|s|/2} \int_{\mathbb{R}^m} (1+|\xi-\eta|)^{|s|/2}\, |\widehat{f}(\xi-\eta)|\, (1+|\eta|)^{s/2}\, |\widehat{u}(\eta)|\, d\lambda(\eta).$$

The latter integral can be seen as a convolution of $g(\eta) = (1+|\eta|)^{|s|/2}\, |\widehat{f}(\eta)|$ with $h(\eta) = (1+|\eta|)^{s/2}\, |\widehat{u}(\eta)|$. The Young inequality $\|g*h\|_{L^2} \leqslant \|g\|_{L^1}\|h\|_{L^2}$ finally implies

$$\|fu\|_s \leqslant C_f\|u\|_s \quad \text{where} \quad C_f = 2^{|s|/2} \int_{\mathbb{R}^m} (1+|\eta|)^{|s|/2}\, |\widehat{f}(\eta)|\, d\lambda(\eta)$$

and $C_f < +\infty$ since $\widehat{f} \in \mathcal{S}(\mathbb{R}^m)$. □

We now recall two further fundamental facts, namely the Sobolev lemma and the Rellich lemma.

2.5. Sobolev lemma. *For any integer $k \in \mathbb{N}$ and any real number $s > k + \frac{m}{2}$, we have $W^s_{\mathrm{loc}}(M,E) \subset C^k(M,E)$ and the inclusion is continuous.*

Proof. It is enough to consider the case $M = \mathbb{R}^m$ and to show that $W^s(\mathbb{R}^m) \subset C^k(\mathbb{R}^n)$ for $s > k + m/2$. However, by the Fourier inversion formula, $D^\alpha u$ is the Fourier transform of

$(-2\pi\mathrm{i}\xi)^\alpha\widehat{u}(-\xi)$. If we check that $\xi^\alpha\widehat{u}(\xi)\in L^1(\mathbb{R}^m)$, it will follow that $D^\alpha f$ is continuous and $\|D^\alpha f\|_\infty\leqslant(2\pi)^{|\alpha|}\|\xi^\alpha\widehat{u}(\xi)\|_{L^1}$. Now, for $|\alpha|\leqslant k$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}\big\|\xi^\alpha\widehat{u}(\xi)\big\|_{L^1}&\leqslant\int_{\mathbb{R}^m}|\xi|^k\,|\widehat{u}(\xi)|\,d\lambda(\xi)=\int_{\mathbb{R}^m}(1+|\xi|^2)^{-s/2}|\xi|^k\,(1+|\xi|^2)^{s/2}|\widehat{u}(\xi)|\,d\lambda(\xi)\\&\leqslant\Big(\int_{\mathbb{R}^m}\frac{|\xi|^{2k}}{(1+|\xi^2|)^s}\,d\lambda(\xi)\Big)^{1/2}\Big(\int_{\mathbb{R}^m}(1+|\xi^2|)^s\,|\widehat{u}(\xi)|^2\,d\lambda(\xi)\Big)^{1/2},\end{aligned}$$

and we infer $\|D^\alpha u\|_\infty\leqslant C\|u\|_s$ as soon as the first integral of the right hand side is convergent, which is the case if $s>k+m/2$. □

It follows immediately from the Sobolev lemma that

$$\bigcap_{s\geqslant 0}W^s_{\mathrm{loc}}(M,E)=C^\infty(M,E).\tag{2.6}$$

Since $W^s(\mathbb{R}^m)$ and $W^{-s}(\mathbb{R}^m)$ are dual, one can also infer by duality that

$$\bigcup_{s\leqslant 0}W^s_{\mathrm{loc}}(M,E)=\mathcal{D}'(M,E).\tag{2.6$'$}$$

A continuous linear operator $\varphi:F\to G$ of Fréchet spaces is called compact if there exists a neighborhood U of 0 in E such that $\overline{\varphi(U)}$ is compact in V. If F is a Banach space, this just means that for any bounded sequence (x_ν) in E, one can extract a subsequence $(x_{\nu(k)})$ such that $\varphi(x_{\nu(k)})$ converges in G. It is easy to see that a composition $\psi\circ\varphi$ of continuous operators φ,ψ is compact as soon as one of them is compact.

2.7. Rellich lemma. *Let $\Omega\Subset M$ be a relatively compact open subset. Then for all $t>s$, the restriction morphism*

$$W^t_{\mathrm{loc}}(M,E)\to W^s_{\mathrm{loc}}(\Omega,E)$$

is compact. In particular, if M is compact, the inclusion $W^t(M,E)\hookrightarrow W^s(M,E)$ is compact.

Proof. By definition of the topology of $W^s_{\mathrm{loc}}(\Omega,E)$, it is sufficient to show the inclusion morphism $W^t_K(\mathbb{R}^m)\hookrightarrow W^s(\mathbb{R}^m)$ is compact, where $W^t_K(\mathbb{R}^m)$ consists of elements $u\in W^t(\mathbb{R}^m)$ with support in a given compact subset $K\subset\mathbb{R}^m$. Now, for any bounded sequence $u_\nu\in W^t_K(\mathbb{R}^m)$, the Fourier transforms

$$\widehat{u}_\nu(\xi)=\int_{\mathbb{R}^m}u_\nu(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}\,d\lambda(x)$$

are C^∞ functions (they even extend as *entire holomorphic functions* $\widehat{u}_\nu\in\mathcal{O}(\mathbb{C}^m)$ by taking $\xi\in\mathbb{C}^m$), since the convergence of any derivative $D^\alpha\widehat{u}_\nu(\xi)$ is guaranteed by the compactness of the support of u_ν. If $\theta\in\mathcal{D}(\mathbb{R}^m)$ is a real cut-off function equal to 1 on a neighborhood of K and $f\in\mathcal{D}(\mathbb{R}^m)\subset\mathcal{S}(\mathbb{R}^m)$, the formula $u_\nu(f)=\widehat{u}_\nu(f^{\wedge-1})=\widehat{u}_\nu(\widehat{f}(-\eta))$ implies

$$\begin{aligned}D^\alpha\widehat{u}_\nu(\xi)&=(-2\pi\mathrm{i})^{|\alpha|}\int_{\mathbb{R}^m}u_\nu(x)\;\theta(x)\,x^\alpha\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi)\\&=(-2\pi\mathrm{i})^{|\alpha|}\int_{\mathbb{R}^m}\widehat{u}_\nu(\eta)\,\big(\theta(x)\,x^\alpha\big)^\wedge(\xi-\eta)\,d\lambda(\eta).\end{aligned}$$

The Cauchy-Schwarz and Peetre inequalities finally give

$$\begin{aligned}|D^\alpha\widehat{u}_\nu(\xi)|&\leqslant(2\pi)^{|\alpha|}\|u_\nu\|_t\Big(\int_{\mathbb{R}^m}(1+|\eta|^2)^{-t}\,\Big|\big(\theta(x)\,x^\alpha\big)^\wedge(\xi-\eta)\Big|^2\,d\lambda(\eta)\Big)^{1/2}\\&\leqslant(2\pi)^{|\alpha|}2^{|t|/2}(1+|\xi|^2)^{|t|/2}\,\|\theta(x)\,x^\alpha\|_{-t}\,\|u_\nu\|_t.\end{aligned}$$

As $\|u_\nu\|_t$ is bounded and $\theta(x)\,x^\alpha \in \mathcal{D}(\mathbb{R}^m) \subset W^{-t}(\mathbb{R}^m)$, this shows that $\widehat{u}_\nu$ is equicontinuous on every ball $\overline{B}(0,R) \subset \mathbb{R}^m$. By the Ascoli theorem, we can extract a subsequence $\widehat{u}_{\nu(k)}$ that converges uniformly on every ball $\overline{B}(0,R)$. Now

$$\begin{aligned}\|u_{\nu(\ell)} - u_{\nu(k)}\|_s^2 &= \int_{\{|\xi|\leqslant R\}\cup\{|\xi|>R\}} (1+|\xi|^2)^s |\widehat{u}_{\nu(\ell)}(\xi) - \widehat{u}_{\nu(k)}(\xi)|^2\,d\lambda(\xi)\\ &\leqslant (1+R^2)^{\max(s,0)} \int_{|\xi|\leqslant R} |\widehat{u}_{\nu(\ell)}(\xi) - \widehat{u}_{\nu(k)}(\xi)|^2\,d\lambda(\xi) + (1+R^2)^{s-t}\|u_{\nu(\ell)} - u_{\nu(k)}\|_t^2,\end{aligned}$$

and by taking $R > 0$ large, we see that $(u_{\nu(k)})$ is a Cauchy sequence in the Hilbert space $W^s(\mathbb{R}^m)$. Therefore $(u_{\nu(k)})$ is convergent in $W^s(\mathbb{R}^m)$, and we infer that $W^t_K(\mathbb{R}^m) \hookrightarrow W^s(\mathbb{R}^m)$ is compact. □

3. Pseudodifferential operators

If $u \in \mathcal{D}(\mathbb{R}^m)$, the Fourier transform of u is

$$\widehat{u}(\xi) = \int_{\mathbb{R}^m} u(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}\,d\lambda(x) \tag{3.1}$$

and the Fourier inversion formula gives $u(x) = \int_{\mathbb{R}^m} \widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi)$, thus for any differential operator $P = \sum_{|\alpha|\leqslant\delta} a_\alpha(x) D^\alpha$ on $\mathbb{R}^m$, we have

$$Pu(x) = \sum_{|\alpha|\leqslant\delta} a_\alpha(x) D^\alpha u(x) = \int_{\mathbb{R}^m} \sum_{|\alpha|\leqslant\delta} a_\alpha(x)(2\pi\mathrm{i}\xi)^\alpha \widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi). \tag{3.2}$$

We call

$$\sigma(x,\xi) = \sum_{|\alpha|\leqslant\delta} a_\alpha(x)(2\pi\mathrm{i}\xi)^\alpha = \sum_{|\alpha|\leqslant\delta} (2\pi\mathrm{i})^{|\alpha|}\,a_\alpha(x)\,\xi^\alpha$$

the (total) symbol of P. By analogy with (3.2), and in more generality, a *pseudodifferential operator* is defined to be an operator Op_σ of the form

$$\mathrm{Op}_\sigma(u)(x) = \int_{\mathbb{R}^m} \sigma(x,\xi)\,\widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi), \qquad u \in \mathcal{D}(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m), \tag{3.3}$$

where σ belongs to a suitable class of functions on $T^\star_{\mathbb{R}^m}$. The so-called *standard class* of symbols $S^\delta(\mathbb{R}^m)$ is defined as follows : for $\delta \in \mathbb{R}$, $S^\delta(\mathbb{R}^m)$ is the space of C^∞ functions $\sigma(x,\xi)$ on $T^\star_{\mathbb{R}^m}$ such that for any $\alpha, \beta \in \mathbb{N}^m$ and any compact subset $K \subset \mathbb{R}^m$ one has an estimate

$$|D^\alpha_x D^\beta_\xi \sigma(x,\xi)| \leqslant C_{K,\alpha,\beta}(1+|\xi|)^{\delta-|\beta|}, \qquad \forall (x,\xi) \in K \times \mathbb{R}^m, \tag{3.4}$$

where $\delta \in \mathbb{R}$ is regarded as the "degree" of σ. Since $\widehat{u} \in \mathcal{S}(\mathbb{R}^m)$, the integral (3.3) is always convergent under condition (3.4). The case when $\sigma(x,\xi)$ is a polynomial of degree δ in ξ ($\delta \in \mathbb{N}$) precisely corresponds to the case of a differential operator of degree δ. In general, since $\widehat{u}$ belongs to the class $\mathcal{S}(\mathbb{R}^m)$ of rapidly decreasing functions, the integral (3.3) is convergent as well as all its derivative D^α_x, thus $\mathrm{Op}_\sigma(u)$ is a well defined C^∞ function on $\mathbb{R}^m$. In the more general situation of operators acting on a bundle E and having values in a bundle F over a manifold M, we introduce an analogous space of symbols $S^\delta(M;E,F)$. The elements of $S^\delta(M;E,F)$ are functions

$$T^\star_M \ni (x,\xi) \mapsto \sigma(x,\xi) \in \mathrm{Hom}(E_x, F_x)$$

satisfying condition (3.4) with respect to all coordinate systems and all local trivializations of E, F. An associated global operator Op_σ on M can be defined by taking a locally finite covering (Ω_j) of M providing coordinates and trivializations of E, F, a quadratic partition of unity (ψ_j) subordinate to (Ω_j) (i.e. $\sum \psi_j^2 = 1$), and by putting

$$\mathrm{Op}_\sigma(u) = \sum_j \psi_j \,\mathrm{Op}_{\sigma_j}(\psi_j u), \qquad u \in C^\infty(M, E), \tag{3.5}$$

where $\sigma_j \in C^\infty(T^*_{M|\Omega_j}, \mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^{r'})$ is the local expression of σ on Ω_j with respect to the trivializations of E, F. This definition of Op_σ allows to reduce most properties to be checked to the case of $\mathbb{R}^m$ with the trivial bundles $E = F = \mathbb{K}$.

3.6. Remark. The global definition of Op_σ provided by (3.5) does depend on the choice of the partition of unity (ψ_j), and also on the trivializations of E, F. This can be seen already in the case of an ordinary differential operator D^α. In fact, the Leibniz formula shows that

$$\sum_j \psi_j D^\alpha(\psi_j u) = D^\alpha u + \sum_{\beta \leqslant \alpha,\, \beta \neq \alpha} \binom{\alpha}{\beta} \psi_j D^{\alpha-\beta} \psi_j \, D^\beta u$$

differs from the top degree term $D^\alpha u$ by variable lower order terms. The important fact, however, is that the principal symbol is what one expects, and it is always possible to add lower order terms to Op_σ to correct any discrepancies. One can e.g. rely on the following elementary fact about asymptotic expansions of symbols.

3.7. Asymptotic expansions. *Let $\sigma_\nu \in S^{\delta-\nu}(M; E, F)$, $\nu \in \mathbb{N}$, be any sequence of symbols. Then there exists $\tau \in S^\delta(M; E, F)$ such that $\tau - \sum_{\nu=0}^k \sigma_\nu \in S^{\delta-k-1}(M; E, F)$ for all $k \in \mathbb{N}$.*

Proof. By a partition of unity argument, it is sufficient to prove the result in the case $M = \mathbb{R}^m$, $E = F = \mathbb{K}$. Let $\theta(\xi) \geqslant 0$ be a C^∞ cutoff function on $\mathbb{R}^m$, equal to 0 for $|\xi| \leqslant 1$ and to 1 for $|\xi| \geqslant 2$. We set

$$\tau(x,\xi) = \sigma_0(x,\xi) + \sum_{\nu=1}^{+\infty} \theta(\varepsilon_\nu \xi)\sigma_\nu(x,\xi)$$

where $\varepsilon_\nu \in \,]0,1]$ is a sequence that decays sufficiently fast to 0. In fact, by (3.4) there is an increasing sequence of constants $C_N > 0$ such that

$$|D_x^\alpha D_\xi^\beta \sigma_\nu(x,\xi)| \leqslant C_N(1+|\xi|)^{\delta-\nu-|\beta|}$$

for all $|\alpha|, |\beta|, \nu \leqslant N$ and $x \in \mathbb{R}^m$ such that $|x| \leqslant N$. Also, since θ vanishes on $B(0,1)$ and its derivatives vanish on $\mathbb{R}^m \smallsetminus B(0,2)$, there is a constant C'_N such that $|D^\gamma\theta(\xi)| \leqslant C'_N|\xi|(1+|\xi|)^{-|\gamma|}$ for all $\xi \in \mathbb{R}^m$ and $|\gamma| \leqslant N$, thus

$$|D^\gamma(\xi \mapsto \theta(\varepsilon_\nu\xi))| = \varepsilon_\nu^{|\gamma|}|D^\gamma\theta(\varepsilon_\nu\xi)| \leqslant \frac{C'_N\varepsilon_\nu^{|\gamma|}\varepsilon_\nu|\xi|}{(1+\varepsilon_\nu|\xi|)^{|\gamma|}} \leqslant C'_N\frac{\varepsilon_\nu^{1+|\gamma|}(1+|\xi|)}{(\varepsilon_\nu+\varepsilon_\nu|\xi|)^{|\gamma|}} \leqslant C'_N\varepsilon_\nu(1+|\xi|)^{1-|\gamma|}.$$

By the Leibniz formula, for any $|\alpha|, |\beta| \leqslant N$ and $x \in \mathbb{R}^N$ with $|x| \leqslant N$, we get

$$\begin{aligned}\left|D_x^\alpha D_\xi^\beta\Big(\sum_{\nu=1}^{+\infty}\theta(\varepsilon_\nu\xi)\sigma_\nu(x,\xi)\Big)\right| &\leqslant \sum_{\nu=1}^{+\infty}\sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}C'_N\varepsilon_\nu(1+|\xi|)^{1-|\gamma|}C_{\max(N,\nu)}(1+|\xi|)^{\delta-\nu-|\beta-\gamma|}\\ &\leqslant 2^N C'_N\sum_{\nu=1}^{+\infty}\varepsilon_\nu C_{\max(N,\nu)}\,(1+|\xi|)^{\delta-|\beta|},\end{aligned}$$

and the convergence of the series is achieved by taking e.g. $\varepsilon_\nu = 2^{-\nu}C_\nu^{-1}$. This already implies $\tau \in S^\delta(\mathbb{R}^m)$. Next,

$$\tau(x,\xi)-\sum_{\nu=0}^{k}\sigma_\nu(x,\xi) = \theta_{k+1}(\varepsilon_{k+1}\xi)\sigma_{k+1}(x,\xi)+\sum_{\nu=1}^{k}(1-\theta_\nu(\varepsilon_\nu\xi))\sigma_\nu(x,\xi)+\sum_{\nu=k+2}^{+\infty}\theta_\nu(\varepsilon_\nu\xi))\sigma_\nu(x,\xi).$$

Here, the first term coincides with $\sigma_{k+1}(x,\xi)$ for $|\xi| > 2/\varepsilon_{k+1}$, thus it belongs to $S^{\delta-k-1}(\mathbb{R}^m)$. The same is true for the summation $\sum_{\nu=1}^{k}$ which has compact support in ξ. Finally, the above estimates imply

$$\left|D_x^\alpha D_\xi^\beta\bigg(\sum_{\nu=k+2}^{+\infty}\theta(\varepsilon_\nu\xi)\sigma_\nu(x,\xi)\bigg)\right| \leqslant 2^N C'_N \sum_{\nu=k+2}^{+\infty}\varepsilon_\nu C_{\max(N,\nu)}\,(1+|\xi|)^{\delta-k-1-|\beta|},$$

hence the summation $\sum_{\nu=k+2}^{+\infty}$ is also in $S^{\delta-k-1}(\mathbb{R}^m)$. □

The basic results pertaining to the theory of pseudodifferential operators are summarized below.

3.8. Schwartz kernel of a pseudodifferential operator. Condition (3.4) implies that $\xi \mapsto \sigma(x,\xi)$ as well as all its derivatives $\xi \mapsto D_x^\alpha\sigma(x,\xi)$ are of polynomial growth at infinity. This implies that they are in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^m)$, and can be assigned a partial Fourier transform $y \mapsto \widehat{\sigma}_\xi(x,y)$ with respect to the second argument, that is in $\mathcal{S}'(\mathbb{R}^m)$ and depends smoothly on x. The Fourier transform of $\sigma(x,\xi)\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}$ is $K_\sigma(x,y) = \widehat{\sigma}_\xi(x,y-x)$, and for $M = \mathbb{R}^m$, the usually adjunction formula $\widehat{g}(u) = g(\widehat{u})$ implies

$$\mathrm{Op}_\sigma(u) = \int_{y\in M} K_\sigma(x,y)\,u(y)\,dV(y), \quad \forall u \in \mathcal{D}(M), \tag{3.9}$$

or equivalently, viewing $\mathrm{Op}_\sigma(u)$ as a distribution,

$$\int_{x\in M}\mathrm{Op}_\sigma(u)(x)\,f(x)\,dV(x) = \int_{M\times M} K_\sigma(x,y)\,f(x)\,u(y)\,dV(x)\,dV(y), \quad \forall f,u \in \mathcal{D}(M).$$

The distribution $K_\sigma(x,y) \in \mathcal{D}'(\mathbb{R}^m\times\mathbb{R}^m)$ is called the (Schwartz) kernel of Op_σ. More generally, in the case of a manifold, we get by (3.5) a formula similar to (3.9), with a global kernel $K_\sigma(x,y) = \sum_j \psi_j(x)K_{\sigma_j}(x,y)\psi_j(y)$ in $\mathcal{D}'(M\times M)$. Laurent Schwartz has shown (this is the so-called Schwartz kernel theorem) that every continuous linear operator $T : \mathcal{D}(M) \to \mathcal{D}'(M)$ is actually given by such a kernel $K \in \mathcal{D}'(M\times M)$, i.e.

$$Tu(x) = \int_{y\in M} K(x,y)\,u(y)\,dV(y) \Leftrightarrow$$
$$\int_M Tu(x)\,f(x)\,dV(x) = \int_{M\times M} K(x,y)\,f(x)\,u(y)\,dV(x)\,dV(y)$$

for all $f,u \in \mathcal{D}(M)$. A proof can be obtained as a direct application of Grothendieck's theory of topological tensor products (see Chapter 9 of online book).

3.10. Action of pseudodifferential operators on Sobolev spaces. *For $\sigma \in S^\delta(M;E,F)$, the operator Op_σ can be extended uniquely as a continuous linear operator*

$$\mathrm{Op}_\sigma : W^s_{\mathrm{loc}}(M,E) \to W^{s-\delta}_{\mathrm{loc}}(M,F).$$

Proof. Thanks to the presence of a term ψ_j on both sides in (3.5), we are reduced to the case (3.3) where $M = \mathbb{R}^m$ and $x \mapsto u(x)$, $x \mapsto \sigma(x,\xi)$ have support in a compact subset $K \subset \mathbb{R}^m$. The result is easily seen to be true when $\mathrm{Op}_\sigma = \sum_{|\alpha|\leqslant\delta} a_\alpha(x)D^\alpha$ is a differential operator. In fact, since multiplication by a_α is continuous on $W^t(\mathbb{R}^m)$ by Lemma 2.3, we only have to show that $D^\alpha : W^s(\mathbb{R}^m) \to W^{s-\delta}(\mathbb{R}^m)$ is continuous for $|\alpha| \leqslant \delta$, but this is obvious from the identity $\widehat{D^\alpha u}(\xi) = (2\pi\mathrm{i}\xi)^\alpha \widehat{u}(\xi)$. In general, we just compute the Fourier transform of $\mathrm{Op}_\sigma(u)$. By Fubini, for $u \in \mathcal{D}(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m)$, we find

$$\mathrm{Op}_\sigma(u)^\wedge(\eta) = \int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^m}\sigma(x,\xi)\,\widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi)\Big)\mathrm{e}^{-2\pi\mathrm{i}x\cdot\eta}\,d\lambda(x) = \int_{\mathbb{R}^m}\widehat{\sigma}_x(\eta-\xi,\xi)\,\widehat{u}(\xi)\,d\lambda(\xi)$$

where $\widehat{\sigma}_x$ is the partial Fourier transform of $\sigma(x,\xi)$ with respect to x. Inequality (3.4) implies

$$|(2\pi\mathrm{i}\eta)^\alpha D_\xi^\beta\widehat{\sigma}_x(\eta,\xi)| = \Big|\int_{\mathbb{R}^m} D_x^\alpha D_\xi^\beta\sigma(x,\xi)\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\eta}\,d\lambda(x)\Big| \leqslant \lambda(K)\,C_{K,\alpha,\beta}(1+|\xi|)^{\delta-|\beta|},$$

thus we have an estimate

$$|D_\xi^\beta\widehat{\sigma}_x(\eta,\xi)| \leqslant C_{\beta,N}(1+|\eta|^2)^{-N/2}\,(1+|\xi|^2)^{(\delta-|\beta|)/2} \quad \text{for every } N\in\mathbb{N}. \tag{3.11}$$

At this point, we need (3.11) only for $\beta = 0$. The Peetre inequality gives

$$\begin{aligned}(1+|\eta|^2)^{(s-\delta)/2}&\,\big|\,\mathrm{Op}_\sigma(u)^\wedge(\eta)\big|\\ &\leqslant 2^{|s-\delta|/2}\int_{\mathbb{R}^m}(1+|\eta-\xi|^2)^{|s-\delta|/2}(1+|\xi|^2)^{(s-\delta)/2}\,\big|\widehat{\sigma}_x(\eta-\xi,\xi)\big|\,\big|\widehat{u}(\xi)\big|\,d\lambda(\xi)\\ &\leqslant C'_{N,s,\delta}\int_{\mathbb{R}^m}(1+|\eta-\xi|^2)^{(|s-\delta|-N)/2}(1+|\xi|^2)^{s/2}\,\big|\widehat{u}(\xi)\big|\,d\lambda(\xi).\end{aligned}$$

We choose $N > |s-\delta| + m$, e.g. $N = \lfloor\,|s-\delta|\,\rfloor + m + 1$, so that $t := N - |s-\delta| > m$. Then

$$\int_{\mathbb{R}^m}(1+|\eta-\xi|^2)^{-t/2}\,d\lambda(\xi) = \int_{\mathbb{R}^m}(1+|\xi|^2)^{-t/2}\,d\lambda(\xi) < +\infty,$$

and we get a probability measure $d\mu_\eta(\xi) = c\,(1+|\eta-\xi|^2)^{-t/2}\,d\lambda(\xi)$ for a certain constant $c > 0$. By Cauchy-Schwarz applied to the measure $d\mu_\eta(\xi)$ and to the functions $f(\xi) = 1$, $g(\xi) = (1+|\xi|^2)^{s/2}\,|\widehat{u}(\xi)|$, we find

$$(1+|\eta|^2)^{s-\delta}\,\big|\,\mathrm{Op}_\sigma(u)^\wedge(\eta)\big|^2 \leqslant C''_{s,\delta}\int_{\mathbb{R}^m} c\,(1+|\eta-\xi|^2)^{-t/2}\,(1+|\xi|^2)^s\big|\widehat{u}(\xi)\big|^2\,d\lambda(\xi).$$

By the Fubini theorem, a final integration with respect to η implies $\|\,\mathrm{Op}_\sigma(u)\|_{s-\delta}^2 \leqslant C''_{s,\delta}\|u\|_s^2$. Since $\mathcal{D}(\mathbb{R}^m)$ is dense in $W^s(\mathbb{R}^m)$ (exercise!), the existence of a unique continuous extension $\mathrm{Op}_\sigma : W^s(\mathbb{R}^m) \to W^{s-\delta}(\mathbb{R}^m)$ follows. □

3.12. Regularizing operators. In particular, if $\sigma \in S^{-\infty}(M;E,F) := \bigcap_{\delta<0} S^\delta(M;E,F)$, then Op_σ is a continuous operator sending an arbitrary distribution $\mathcal{D}'(M,E)$ into $C^\infty(M,F)$. More precisely, since $\xi \mapsto \sigma(x,\xi)$ is rapidly decreasing locally uniformly in x with all derivatives D_x^α the Schwartz kernel $K_\sigma(x,y) = \widehat{\sigma}_\xi(x,y-x)$ is then easily seen to be smooth, and conversely, if we are given a kernel $K(x,y)$ in $C^\infty(M\times M)$ with "proper support" in $M\times M$ in the sense that for every point $x_0 \in X$, there is a neighborhood W of x_0 and a compact set $L \subset M$ such that $K(x,y) = 0$ on $W \times (M \smallsetminus L)$, the formula

$$Ru(x) = \int_M K(x,y)\,u(y)\,dV(y), \quad u \in \mathcal{D}'(M) \tag{3.13}$$

defines a continuous operator $R : \mathcal{D}'(M) \to C^\infty(M)$. In the case $M = \mathbb{R}^m$, we have

$$Ru(x) = \int_M \widehat{K}_y(x,-\xi)\,\widehat{u}(\xi)\,dV(\xi), \quad \forall u \in \mathcal{D}'(M),$$

thus R is the pseudodifferential operator associated with the symbol $\sigma(x,\xi) = \widehat{K}_y(x,-\xi)\,\mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}$. The properness condition for K implies that $\widehat{K}_y(x,\xi)$ is rapidly decreasing in ξ, thus σ is in the class $\mathcal{R} := S^{-\infty}(M;E,F)$. Such an operator is called a *regularizing operator*.

Regularizing operators play very nicely in regularity theory, in the sense that adding such operators does not interfere with the regularity of distributions at stake. They can somehow be considered as "negligible" in pseudodifferential calculus; it is thus very frequent to make calculations only modulo $\mathcal{R}$.

3.14. Composition of pseudodifferential operators associated to standard symbols. *For any $\sigma \in S^\delta(M;E,F)$ and $\sigma' \in S^{\delta'}(M;F,G)$, $\delta,\delta' \in \mathbb{R}$, there exists a "composed symbol" $\sigma' \diamond \sigma \in S^{\delta+\delta'}(M;E,G)$ such that $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_\sigma = \mathrm{Op}_{\sigma'\diamond\sigma}$. Moreover*

$$\sigma' \diamond \sigma - \sigma' \cdot \sigma \in S^{\delta+\delta'-1}(M;E,G).$$

More precisely, we have an asymptotic expansion

$$\sigma' \diamond \sigma(x,\xi) \sim \sum_\ell \sum_{|\alpha|=\ell} \frac{1}{(2\pi\mathrm{i})^{|\alpha|}\alpha!}\,D_\xi^\alpha \sigma'(x,\xi)\,D_x^\alpha \sigma(x,\xi) \mod \mathcal{R}$$

where the ℓ-th term is in $S^{\delta+\delta'-\ell}(M;E,G)$.

Proof. When $\mathrm{Op}_{\sigma'}$ and Op_σ are differential operators (i.e. $\sigma'(x,\xi)$ and $\sigma(x,\xi)$ are polynomials in ξ), the asymptotic expansion is an exact formula, and is just a finite sum; this follows easily from the Leibniz formula. The general manifold case requires to compute locally finite sums of the form

$$\sum_{j,k} \psi_k\,\mathrm{Op}_{\sigma'_k}\big(\psi_k\psi_j\,\mathrm{Op}_{\sigma_j}(\psi_j u)\big)$$

where $u \in W^s_{\mathrm{loc}}(M)$. The (j,k)-term is non zero only in case $\mathrm{Supp}(\psi_j) \cap \mathrm{Supp}(\psi_k) \neq \emptyset$. One can arrange the covering (Ω_j) and the partition of unity in such a way that whenever this happens $\mathrm{Supp}(\psi_j) \cup \mathrm{Supp}(\psi_k)$ is contained in Ω_j or Ω_k (e.g. by taking geodesic balls $\Omega_j \simeq B(0,r_j)$, so that $B(0,r_j/4)$ still covers M, and $\mathrm{Supp}(\psi_j) \subset B(0,r_j/4)$). Then, by using a diffeomorphism $\Omega_j \simeq \mathbb{R}^m$, it is sufficient to consider the case where $M = \mathbb{R}^m$, and to consider the composition of pseudodifferential operators associated with the symbols $\psi_k(x)\sigma'_k(x,\xi)$ and $\psi_k\psi_j(x)\sigma_j(x,\xi)$. Hence, we can also assume that $\sigma'(x,\xi)$ and $\sigma(x,\xi)$ have compact support in x, being non zero only when $x \in K \subset \mathbb{R}^m$. In that case, for $u \in \mathcal{D}(\mathbb{R}^n)$ we have absolutely convergent integrals of rapidly decreasing functions

$$\mathrm{Op}_{\sigma'}(u)(x) = \int_{\mathbb{R}^m} \sigma'(x,\eta)\,\widehat{u}(\eta)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta),$$

$$\mathrm{Op}_\sigma(u)^\wedge(\eta) = \int_{\mathbb{R}^m} \widehat{\sigma}_x(\eta-\xi,\xi)\,\widehat{u}(\xi)\,d\lambda(\xi),$$

the first one being equivalent to definition (3.3) with ξ replaced by η, and the second one already stated in 3.10. By substituting the second formula in the first, we find

$$\begin{aligned}\mathrm{Op}_{\sigma'}\circ\mathrm{Op}_\sigma(u)(x) &= \int_{\mathbb{R}^m}\sigma'(x,\eta)\bigg(\int_{\mathbb{R}^m}\widehat{\sigma}_x(\eta-\xi,\xi)\,\widehat{u}(\xi)\,d\lambda(\xi)\bigg)\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta)\\ &= \int_{\mathbb{R}^m}\bigg(\int_{\mathbb{R}^m}\sigma'(x,\eta)\,\widehat{\sigma}_x(\eta-\xi,\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta)\bigg)\widehat{u}(\xi)\,d\lambda(\xi)\\ &= \int_{\mathbb{R}^m}\bigg(\int_{\mathbb{R}^m}\sigma'(x,\eta+\xi)\,\widehat{\sigma}_x(\eta,\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta)\bigg)\widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,d\lambda(\xi),\end{aligned}$$

where, in the last line, we have performed a change of variable $\eta\mapsto\eta+\xi$. This shows that $\mathrm{Op}_\sigma\circ\mathrm{Op}_\sigma$ is associated to the symbol

$$\sigma'\diamond\sigma(x,\xi)=\int_{\mathbb{R}^m}\sigma'(x,\eta+\xi)\,\widehat{\sigma}_x(\eta,\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta), \tag{3.15}$$

and we have to prove that $\sigma'\diamond\sigma\in S^{\delta+\delta'}(M;E,G)$. For this, we compute

$$D_x^\alpha D_\xi^\beta\sigma'\diamond\sigma(x,\xi)=\sum_{\mu,\nu}\binom{\alpha}{\mu}\binom{\beta}{\nu}\int_{\mathbb{R}^m}D_x^\mu D_\xi^\nu\sigma'(x,\eta+\xi)\,D_\xi^{\beta-\nu}\widehat{\sigma}_x(\eta,\xi)\,(2\pi\mathrm{i}\eta)^{\alpha-\mu}\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta).$$

As we have already shown (cf. (3.11)), we have estimates

$$|D_\xi^{\beta-\nu}\widehat{\sigma}_x(\eta,\xi)|\leqslant C_{\beta-\nu,N}(1+|\eta|^2)^{-N/2}\,(1+|\xi|^2)^{(\delta-|\beta|+|\nu|)/2},$$

and also

$$\begin{aligned}|D_x^\mu D_\xi^\nu\sigma'(x,\eta+\xi)| &\leqslant C'_{\mu,\nu}(1+|\eta+\xi|^2)^{(\delta'-|\nu|)/2}\\ &\leqslant C'_{\mu,\nu}\,2^{|\delta'-|\nu||/2}(1+|\xi|^2)^{(\delta'-|\nu|)/2}(1+|\eta|^2)^{|\delta'-|\nu||/2}\end{aligned}$$

by the Peetre inequality. Therefore

$$\big|D_x^\alpha D_\xi^\beta\sigma'\diamond\sigma(x,\xi)\big|\leqslant C''_{\alpha,\beta,N}\int_{\mathbb{R}^m}(1+|\xi|^2)^{(\delta+\delta'-|\beta|)/2}(1+|\eta|^2)^{(|\delta'-|\nu||+|\alpha|-|\mu|-N)/2}\,d\lambda(\eta).$$

For N large enough to ensure the convergence of the integral, we get the expected upper bound

$$\big|D_x^\alpha D_\xi^\beta\sigma'\diamond\sigma(x,\xi)\big|\leqslant C'''_{\alpha,\beta}(1+|\xi|)^{\delta+\delta'-|\beta|}.$$

Furthermore, we have a Taylor expansion

$$\sigma'(x,\eta+\xi)=\sum_{|\alpha|\leqslant p}\frac{1}{\alpha!}D_\xi^\alpha\sigma'(x,\xi)\,\eta^\alpha+\int_0^1\sum_{|\alpha|=p+1}\frac{1}{\alpha!}D_\xi^\alpha\sigma'(x,\xi+t\eta)\,\eta^\alpha\,(p+1)(1-t)^p dt$$

The integral term is bounded by

$$C_p^{(4)}(1+|\xi+t\eta|^2)^{(\delta'-p-1)/2}\leqslant C_p^{(5)}(1+|\xi|^2)^{(\delta'-p-1)/2}(1+|\eta|^2)^{|\delta'-p-1|/2}.$$

If we plug the Taylor expansion in (3.15), this remainder term contributes for

$$C_{p,N}^{(6)}\int_{\mathbb{R}^m}(1+|\xi|^2)^{(\delta+\delta'-p-1)/2}(1+|\eta|^2)^{(|\delta'-p-1|-N)/2}\,d\lambda(\eta)\leqslant C_p^{(7)}(1+|\xi|)^{\delta+\delta'-p-1},$$

while the main terms yield

$$\int_{\mathbb{R}^m} \sum_{|\alpha|\leqslant p} \frac{1}{\alpha!} D_\xi^\alpha \sigma'(x,\xi)\,\eta^\alpha\,\widehat{\sigma}_x(\eta,\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\eta}\,d\lambda(\eta) = \sum_{|\alpha|\leqslant p} \frac{1}{(2\pi\mathrm{i})^{|\alpha|}\alpha!} D_\xi^\alpha \sigma'(x,\xi)\,D_x^\alpha \sigma(x,\xi)$$

by the Fourier inversion formula. It is easy to check that the terms with $|\alpha| = \ell$ are in $S^{\delta+\delta'-\ell}(M;E,F)$. Statement 3.14 is proved. □

3.16. Adjoint of a pseudodifferential operator. *Assume that E, F are equipped with smooth Hermitian metrics h_E, h_F. Given $\sigma \in S^\delta(M;E,F)$, there exists an adjoint symbol $\sigma^\dagger \in S^\delta(M;F,E)$ such that Op_σ et $\mathrm{Op}_{\sigma^\dagger}$ are (formally) adjoint, i.e. for every $u \in \mathcal{D}(M,E)$, $v \in \mathcal{D}(M,F)$*

$$\int_M \langle \mathrm{Op}_\sigma(u), v\rangle_F\,dV = \int_M \langle u, \mathrm{Op}_{\sigma^\dagger}(v)\rangle_E\,dV.$$

Proof. We first compute $\sigma^\dagger$ in the trivial case $M = \mathbb{R}^m$, $E = F = \mathbb{K}$ with the trivial metric, when $x \mapsto \sigma(x,\xi)$ has compact support. Then

$$\int_M \langle \mathrm{Op}_\sigma(u), v\rangle\,dV = \int_{\mathbb{R}^m} \mathrm{Op}_\sigma(u)(x)\overline{v(x)}\,d\lambda(x) = \int_{\mathbb{R}^m\times\mathbb{R}^m} \sigma(x,\xi)\,\widehat{u}(\xi)\,\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\,\overline{v(x)}\,d\lambda(x)\,d\lambda(\xi).$$

A partial Fourier transform with respect to x gives

$$\int_M \langle \mathrm{Op}_\sigma(u), v\rangle\,dV = \int_{\mathbb{R}^m\times\mathbb{R}^m} \widehat{\sigma}_x(\eta-\xi,\xi)\,\widehat{u}(\xi)\,\overline{\widehat{v}(\eta)}\,d\lambda(\xi)\,d\lambda(\eta).$$

Similarly, $\langle\!\langle u, \mathrm{Op}_{\sigma^\dagger}(v)\rangle\!\rangle = \overline{\langle\!\langle \mathrm{Op}_{\sigma^\dagger}(v), u\rangle\!\rangle}$, and by switching u and v, ξ and η, σ and $\sigma^\dagger$, we get

$$\int_M \langle \mathrm{Op}_{\sigma^\dagger}(v), u\rangle\,dV = \int_{\mathbb{R}^m\times\mathbb{R}^m} \widehat{\sigma_x^\dagger}(\xi-\eta,\eta)\,\widehat{v}(\eta)\,\overline{\widehat{u}(\xi)}\,d\lambda(\xi)\,d\lambda(\eta).$$

The required relation between σ and $\sigma^\dagger$ is $\widehat{\sigma_x^\dagger}(\xi-\eta,\eta) = \overline{\widehat{\sigma}_x(\eta-\xi,\xi)}$, or equivalently, after a one-to-one substitution $(\eta,\xi) \mapsto (\xi,\xi+\eta)$,

$$\widehat{\sigma_x^\dagger}(\eta,\xi) = \overline{\widehat{\sigma}_x(-\eta,\xi+\eta)}.$$

In the higher rank case $E = \mathbb{K}^r$, $F = \mathbb{K}^{r'}$ we find

$$\widehat{\sigma_x^\dagger}(\eta,\xi) = \widehat{\sigma}_x(-\eta,\xi+\eta)^* \tag{3.17}$$

where A^* is the complex adjoint of a matrix A. We leave the reader check that this defines $\sigma^\dagger \in S^\delta(M;F,E)$, again by means of the Peetre inequality and by the fast decay of $\widehat{\sigma}_x(\eta,\xi)$ in η. The general case follows from the fact that $(\psi_j\,\mathrm{Op}_{\sigma_j}\,\psi_j)^* = \psi_j(\mathrm{Op}_{\sigma_j})^*\psi_j = \psi_j\,\mathrm{Op}_{\sigma_j^\dagger}\,\psi_j$, when orthormal trivializations of E, F are taken in each chart. □

4. Fundamental results on elliptic operators

The concept of ellipticity is easily extended to pseudodifferential operators by expressing that the symbol $\sigma(x,\xi)$ is "uniformly" injective when $\xi \to \infty$.

4.1. Definition. *A pseudodifferential operator* Op_σ *of degree* δ *is called elliptic if it can be defined by a symbol* $\sigma \in S^\delta(M,E,F)$ *such that*

$$|\sigma(x,\xi)\cdot u| \geqslant c|\xi|^\delta |u|, \quad \forall (x,\xi) \in T^\star_M, \quad \forall u \in E_x$$

for $|\xi|$ *large enough, the estimation being uniform for* $x \in M$.

If E and F have the same rank, the ellipticity condition implies that $\sigma(x,\xi)$ is invertible for ξ large. By taking a suitable C^∞ truncating function $\theta(\xi) \geqslant 0$ equal to 0 for $|\xi| \leqslant R$ large and to 1 for $|\xi| \geqslant 2R$, one sees that the function $\sigma'(x,\xi) = \theta(\xi)\sigma(x,\xi)^{-1}$ defines a symbol in the space $S^{-\delta}(M;F,E)$. Also, since

$$1 - \sigma'(x,\xi)\,\sigma(x,\xi) = 1 - \theta(\xi) \in \mathcal{D}(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m),$$

this difference is a regularizing operator. According to 3.14, we have $\mathrm{Id} - \mathrm{Op}_{\sigma'} \circ \mathrm{Op}_\sigma = \mathrm{Op}_\rho$, where $\rho \in S^{-1}(M;E,E)$, and thus $\rho^{\diamond j} \in S^{-j}(M;E,E)$. Choose a symbol τ equivalent to the asymptotic expansion $\mathrm{Id} + \rho + \rho^{\diamond 2} + \cdots + \rho^{\diamond j} + \cdots$ (this is possible by 3.7). Then Op_τ is an inverse of $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_\sigma = \mathrm{Id} - \mathrm{Op}_\rho$ modulo $\mathcal{R}$. It is then clear that one obtains an inverse $\mathrm{Op}_{\tau\diamond\sigma'}$ of Op_σ modulo $\mathcal{R}$. An easy consequence of this observation is the following :

4.2. Gårding inequality. *Assume given* $P : C^\infty(M,E) \to C^\infty(M,F)$ *an elliptic differential operator of degree* δ*, where* $\operatorname{rank} E = \operatorname{rank} F = r$*, and let* $\widetilde{P}$ *be an extension of* P *with distributional coefficient sections. For all* $u \in W^0(M,E)$ *such that* $\widetilde{P}u \in W^s(M,F)$*, one then has* $u \in W^{s+\delta}(M,E)$ *and*

$$\|u\|_{s+\delta} \leqslant C_s(\|\widetilde{P}u\|_s + \|u\|_0),$$

where C_s *is a positive constant depending only on* s.

Proof. Since P is elliptic of degree δ, there exists a symbol $\sigma \in S^{-\delta}(M;F,E)$ such that $\mathrm{Op}_\sigma \circ \widetilde{P} = \mathrm{Id} + R$, $R \in \mathcal{R}$. Then $\|\mathrm{Op}_\sigma(v)\|_{s+\delta} \leqslant C\|v\|_s$ by applying 3.10. Consequently, in setting $v = \widetilde{P}u$, we see that $u = \mathrm{Op}_\sigma(\widetilde{P}u) - Ru$ satisfies the desired estimate. □

5. Finiteness theorem

We conclude these notes with the proof of the following fundamental finiteness theorem, which is the starting point of L^2 Hodge theory. We assume throughout this section that M is compact, in general finiteness cannot hold on non compact manifolds.

5.1. Finiteness theorem. *Assume given* E*,* F *Hermitian vector bundles on a compact manifold* M*, such that* $\operatorname{rank} E = \operatorname{rank} F = r$*, and let* $P : C^\infty(M,E) \to C^\infty(M,F)$ *be an elliptic differential operator of degree* δ*. Then :*

i) $\operatorname{Ker} P$ *is finite dimensional.*

ii) $P\big(C^\infty(M,E)\big)$ *is closed and of finite codimension in* $C^\infty(M,F)$ *; moreover, if* $P^\star$ *is the formal adjoint of* P*, there exists a decomposition*

$$C^\infty(M,F) = P\big(C^\infty(M,E)\big) \oplus \operatorname{Ker} P^\star$$

as an orthogonal direct sum in $W^0(M,F) = L^2(M,F)$.

Proof. (i) The Gårding inequality shows that $\|u\|_{s+\delta} \leqslant C_s \|u\|_0$ for all $u \in \operatorname{Ker} P$. By the Sobolev lemma, this implies that $\operatorname{Ker} P$ is closed in $W^0(M,E)$. Moreover, the $\| \ \|_0$-closed unit ball of $\operatorname{Ker} P$ is contained in the $\| \ \|_\delta$-boule of radius C_0, therefore it is compact according to the Rellich lemma. The Riesz Theorem implies $\dim \operatorname{Ker} P < +\infty$.

(ii) We first show that the extension

$$\widetilde{P} : W^{s+\delta}(M,E) \to W^s(M,F)$$

has a closed image all s. For any $\varepsilon > 0$, there exists a finite number of elements $v_1, \ldots, v_N \in W^{s+\delta}(M,F)$, $N = N(\varepsilon)$, such that

$$\|u\|_0 \leqslant \varepsilon \|u\|_{s+\delta} + \sum_{j=1}^{N} |\langle\langle u, v_j \rangle\rangle_0|. \tag{5.2}$$

Indeed, the set

$$K_{(v_j)} = \Big\{ u \in W^{s+\delta}(M,F) \ ; \ \varepsilon \|u\|_{s+\delta} + \sum_{j=1}^{N} |\langle\langle u, v_j \rangle\rangle_0| \leqslant 1 \Big\}$$

is relatively compact in $W^0(M,F)$ and $\bigcap_{(v_j)} \overline{K}_{(v_j)} = \{0\}$. It follows that there are elements (v_j) such that the $\overline{K}_{(v_j)}$ are contained in the unit ball of $W^0(M,E)$, as required. Substituting the $\|u\|_0$ by its upper bound (5.2) in the Gårding inquality, we obtain

$$(1 - C_s \varepsilon) \|u\|_{s+\delta} \leqslant C_s \Big(\|\widetilde{P} u\|_s + \sum_{j=1}^{N} |\langle\langle u, v_j \rangle\rangle_0| \Big).$$

Define $T = \big\{ u \in W^{s+\delta}(M,E) \ ; \ u \perp v_j, \ 1 \leqslant j \leqslant n \big\}$ and put $\varepsilon = 1/2C_s$. It follows that

$$\|u\|_{s+\delta} \leqslant 2 C_s \|\widetilde{P} u\|_s, \qquad \forall u \in T.$$

This implies that $\widetilde{P}(T)$ is closed. As a consequence

$$\widetilde{P}\big(W^{s+\delta}(M,E)\big) = \widetilde{P}(T) + \operatorname{Span}\big(\widetilde{P}(v_1), \ldots, \widetilde{P}(v_N)\big)$$

is closed in $W^s(M,E)$. Consider now the case $s = 0$. Since $C^\infty(M,E)$ is dense in $W^\delta(M,E)$, we see that in $W^0(M,E) = L^2(M,E)$, one has

$$\Big(\widetilde{P}\big(W^\delta(M,E)\big) \Big)^\perp = \Big(P\big(C^\infty(M,E)\big) \Big)^\perp = \operatorname{Ker} \widetilde{P^\star}.$$

We have thus proven that

$$W^0(M,E) = \widetilde{P}\big(W^\delta(M,E)\big) \oplus \operatorname{Ker} \widetilde{P^\star}. \tag{5.3}$$

Since $P^\star$ is also elliptic, it follows that $\operatorname{Ker} \widetilde{P^\star}$ is finite dimensional and that $\operatorname{Ker} \widetilde{P^\star} = \operatorname{Ker} P^\star$ is contained in $C^\infty(M,F)$. By applying the Gårding inequality, the decomposition formula (5.3) gives

$$W^s(M,E) = \widetilde{P}\big(W^{s+\delta}(M,E)\big) \oplus \operatorname{Ker} P^\star, \tag{5.4}$$

$$C^\infty(M,E) = P\big(C^\infty(M,E)\big) \oplus \operatorname{Ker} P^\star. \tag{5.5}$$

We finish this section by the construction of the Green operator associated with a self-adjoint elliptic operator.

5.6. Theorem. *Assume given E a Hermitian vector bundle of rank r over a compact manifold M, and $P : C^\infty(M,E) \to C^\infty(M,E)$ a self-adjoint elliptic differential operator of degree δ. Then if H denotes the orthogonal projection $H : C^\infty(M,E) \to \operatorname{Ker} P$, there exists a unique operator G on $C^\infty(M,E)$ such that*

$$PG + H = GP + H = \mathrm{Id}\,.$$

Moreover G is a pseudodifferential operator of degree $-\delta$, called Green operator associated to P.

Proof. According to Theorem 5.1, $\operatorname{Ker} P = \operatorname{Ker} P^\star$ is finite dimensional, and $\operatorname{Im} P = (\operatorname{Ker} P)^\perp$. It then follows that the restriction of P to $(\operatorname{Ker} P)^\perp$ is a bijective operator. One defines G toe be $0 \oplus P^{-1}$ relative to the orthogonal decomposition $C^\infty(M,E) = \operatorname{Ker} P \oplus (\operatorname{Ker} P)^\perp$. The relations $PG + H = GP + H = \mathrm{Id}$ are then obvious, as well as the uniqueness of G. Moreover, G is continuous in the Fréchet space topology of $C^\infty(M,E)$, according to the Banach theorem. One also uses the fact that there exists a pseudodifferential operator Q of order $-\delta$ which is an inverse of P modulo $\mathcal{R}$, i.e., $PQ = \mathrm{Id} + R$, $R \in \mathcal{R}$. It then follows that

$$Q = (GP + H)Q = G(\mathrm{Id} + R) + HQ = G + GR + HQ,$$

where GR and HQ are regularizing operators (H is a regularizing operator of finite rank defined by the kernel $\sum \varphi_s(x) \otimes \varphi_s^\star(y)$, where (φ_s) is a base of eigenfunctions of $\operatorname{Ker} P \subset C^\infty(M,E)$). Consequently $G = Q \bmod \mathcal{R}$ and G is a pseudodifferential operator of degree $-\delta$. □

5.7. Corollary. *Under the hypotheses of Theorem* 5.6*, the eigenvalues of P form a real sequence (λ_k) such that $\lim_{k\to+\infty} |\lambda_k| = +\infty$, the eigenspaces V_{λ_k} of P are finite dimensional, and one has a Hilbert space* (*completed*) *orthogonal direct sum*

$$L^2(M,E) = \widehat{\bigoplus_k} V_{\lambda_k}.$$

For any integer $m \in \mathbb{N}$, an element $u = \sum_k u_k \in L^2(M,E)$ is in $W^{m\delta}(X,E)$ if and only if $\sum |\lambda_k|^{2m} \|u_k\|^2 < +\infty$.

Proof. The Green operator extends to a self-adjoint operator

$$\widetilde{G} : L^2(M,E) \to L^2(M,E)$$

which factors through $W^\delta(M,E)$, and is therefore compact. This operator defines an inverse to $\widetilde{P} : W^\delta(M,E) \to L^2(M,E)$ on $(\operatorname{Ker} P)^\perp$. The spectral theory of compact self-adjoint operators shows that the eigenvalues μ_k of $\widetilde{G}$ form a real sequence μ_k tending to 0 and that $L^2(M,E)$ is a Hilbert orthogonal direct sum of eigenspaces. The corresponding eigenvalues of $\widetilde{P}$ are $\lambda_k = \mu_k^{-1}$ if $\mu_k \neq 0$, and according to the ellipticity of $P - \lambda_k \mathrm{Id}$, the eigenspaces $V_{\lambda_k} = \operatorname{Ker}(P - \lambda_k \mathrm{Id})$ are finite dimensional and contained in $C^\infty(M,E)$. Finally, if $u = \sum_k u_k \in L^2(M,E)$, the Gårding inequality shows that $u \in W^{m\delta}(M,E)$ if and only if $\widetilde{P}^m u \in L^2(M,E) = W^0(M,E)$, which easily gives the condition $\sum |\lambda_k|^{2m} \|u_k\|^2 < +\infty$. □