Introduction to analytic geometry (course by Jean-Pierre Demailly)
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1. (*The $\partial\overline{\partial}$-lemma*).

Let (X,ω) be a compact Kähler manifold. Let u be a smooth d-closed (p,q)-form (i.e. $\partial u = \overline{\partial} u = 0$. Show that the following properties are equivalent:
(a) u is d-exact (i.e. there is a smooth form v of degree $k-1$, where $k = p+q$, such that $u = dv$).
(b) u is ∂-exact (i.e. there is a smooth form v of bidegree $(p-1,q)$ such that $u = \partial v$).
(c) u is $\overline{\partial}$-exact (i.e. there is a smooth form v of bidegree $(p,q-1)$ such that $u = \overline{\partial} v$).
(d) u is Aeppli exact (i.e. there are smooth forms v, v' of bidegrees $(p-1,q)$ and $(p,q-1)$ such that $u = \partial v + \overline{\partial} v'$).
(e) u is $\partial\overline{\partial}$-exact (i.e. there is a smooth form s of bidegree $(p-1,q-1)$ such that $u = \partial\overline{\partial} s$).
(f) $u \perp \mathcal{H}^{p,q}(X,\mathbb{C})$ (orthogonality to harmonic (p,q)-forms).
Hint. It is easy to show that (e) implies (a), (b), (c) and (d). Also, each of the four properties (a), (b), (c), (d) implies (f), so it is enough to show that (f) implies (e). For this, assuming (f), show that u can be decomposed as $\partial v + \partial^* w$ and observe that $\partial^* w$ must be zero. Then use a three term decomposition of v with respect to $\overline{\partial}$ and $\overline{\partial}^*$ and conclude.

2. (*Bott-Chern cohomology*).

If X is any complex manifold, one defines $H^{p,q}_{\mathrm{BC}}(X,\mathbb{C})$ to be the quotient of the space of (p,q)-forms u such that $\partial u = \overline{\partial} u = 0$ by $\partial\overline{\partial}$-exact (p,q)-forms $\partial\overline{\partial} v$.

(a) Show that there are always natural morphisms

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^{p,q}_{\overline{\partial}}(X,\mathbb{C}), \qquad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \to H^k_{\mathrm{DR}}(X,\mathbb{C}).$$

(b) When (X,ω) is compact Kähler, show that the above natural morphisms are isomorphisms and correspond to the Hodge decomposition in terms of ω-harmonic forms. Derive from there that the Hodge decomposition is independent of the choice of the Kähler metric ω.

3. (*First Chern class and Lelong-Poincaré equation*).

(a) Let $L \to X$ be a holomorphic line bundle on any complex manifold X, and h a smooth Hermitian metric on L. One defines the first Chern class of L, denoted $c_1(L)$, to be the cohomology class $\{\frac{i}{2\pi}\Theta_{L,h}\}$ associated with the curvature tensor $\Theta_{L,h}$ of the Chern connection. Show that this defines a real cohomology class in $H^{1,1}_{\mathrm{BC}}(X,\mathbb{C})$, and also a De Rham cohomology class in $H^2_{\mathrm{DR}}(X,\mathbb{R})$, independent of the metric h.
Hint. If h' is another metric, there exists a global weight $\psi \in C^\infty(X,\mathbb{R})$ such that $h' = he^{-\psi}$. Then relate $\Theta_{L,h'}$ and $\Theta_{L,h}$.

(b) If X is compact Kähler of dimension n, one defines the degree of L with respect to ω to be

$$\deg_\omega(L) := \int_X c_1(L) \wedge \{\omega\}^{n-1} = \int_X \frac{i}{2\pi}\Theta_{L,h} \wedge \omega^{n-1}.$$

This degree is independent of h. Show that this is true more generally if ω satisfies $\partial\overline{\partial}(\omega^{n-1}) = 0$ (such a metric is called a Gauduchon metric; Paul Gauduchon has shown in 1977 that such a metric exists on any compact complex manifold).

(c) Let s be a meromorphic section of L that does not vanish identically on any connected component of X (this means that locally $s = fe$ where e is a non vanishing holomorphic section and $f = g/h$ a meromorphic section). Let $D = \sum m_j D_j$, $m_j \in \mathbb{Z}$, be the divisor of zeroes and poles of s. Prove the Lelong-Poincaré formula, asserting that

$$\frac{i}{2\pi}\partial\overline{\partial}\log|s|^2_h = [D] - \frac{1}{2\pi}\Theta_{L,h}$$

in the sense of currents. Infer that $c_1(L)$ coincides with the cohomology class of $[D]$ in $H^{1,1}_{\mathrm{BC}}(X,\mathbb{C})$. In particular, in dimension $n=1$,

$$\deg(L) := \int_X c_1(X) = \deg D := \sum m_j \in \mathbb{Z}.$$

4. (*Projective space and the $\mathcal{O}(m)$ line bundles*)

On complex projective space $\mathbb{P}^n = \mathbb{C}^{n+1} \smallsetminus \{0\}$, $n \geq 1$, one defines the "tautological line bundle" $\mathcal{O}(-1)$ to be the subbundle of the trivial bundle $\mathbb{C}^{n+1}$ such the fiber at a point $x = [z] = [z_0 : z_1 : \ldots : z_n]$ consists of the line $\mathbb{C}z$, i.e. $\mathcal{O}(1)_{[z]} = \mathbb{C}z$. Its "tautological metric" is the one induced by the standard Hermitian metric of $\mathbb{C}^{n+1}$, $|z|^2 = \sum |z_j|^2$. Finally, one defines $\mathcal{O}(1)$ to be the dual $\mathcal{O}(-1)^* = \mathcal{O}(-1)^{-1}$ and $\mathcal{O}(m)$, $m \in \mathbb{Z}$, to be the m-th tensor product of $\mathcal{O}(1)$ (or the $(-m)$-th tensor product of $\mathcal{O}(-1)$).

(a) If $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ is a homogeneous polynomial of degree m on $\mathbb{C}^{n+1}$, i.e. $P(z) = a \cdot z^m$ where $z^m \in S^m(\mathbb{C}^{n+1})$ and $a = \sum a_\alpha (e^*)^\alpha \in S^m(\mathbb{C}^{n+1})^*$ with respect to the canonical basis (e_j) of $\mathbb{C}^{n+1}$, then P can be seen as a section of $\mathcal{O}(m)$. In particular, every linear form in $(\mathbb{C}^{n+1})^*$ defines a section of $\mathcal{O}(1) = \mathcal{O}(-1)^*$.

(b) Conversely, if σ is a holomorphic section of $\mathcal{O}(m)$ over $\mathbb{P}^n$, then $f(z) = \sigma([z]) \cdot z^m$ (which should be interpreted as a division $\sigma([z])/z^m$ for $m < 0$) defines a holomorphic function on $\mathbb{C}^{n+1} \smallsetminus \{0\}$ such that $f(\lambda z) = \lambda^m f(z)$. Then must be a homogeneous polynomial of degree m for $m \geq 0$, and must vanish identically if $m < 0$.

(c) Show that the curvature of $\mathcal{O}(1)$ with respect to its standard Hermitian metric yields $\frac{i}{2\pi}\Theta_{\mathcal{O}(1),\mathrm{std}} = \frac{i}{2\pi}\partial\overline{\partial}\log|z|^2 = \omega_{\mathrm{FS}}$, i.e. the Fubini-Study metric. In particular $\mathcal{O}(1)$ has positive curvature.

(d) If σ is a holomorphic section of $\mathcal{O}(m)$ and Y its zero divisor (in particular, if Y is a smooth degree m hypersurface), then

$$\int_Y \omega_{\mathrm{FS}}^{n-1} = \int_{\mathbb{P}^n} [Y] \wedge \omega_{\mathrm{FS}}^{n-1} = m \int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n.$$

Hint. Use the Lelong-Poincaré formula.

(e) Use (d) and induction on n to infer that $\int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = 1$ by taking Y to be the hyperplane $z_n = 0$.

5. (*Riemann-Roch formula for compact Riemann surfaces, i.e. compact complex curves*)

On a curve, a divisor $D = \sum m_j [p_j]$ is a just a sum of points $p_j \in X$ with multiplicities $m_j \in \mathbb{Z}$. The line bundle (or rather, invertible sheaf) $\mathcal{O}_X(D)$ consists of the sheaf of germs of meromorphic functions f such that $\mathrm{div}(f) + D \geq 0$ (or $f = 0$).

(a) One defines the "tautological meromorphic section" s_D of $\mathcal{O}_X(D)$ to be given by the meromorphic function 1. Show that it is holomorphic as a section of $\mathcal{O}_X(D)$ if and only if $D \geq 0$ and, in general, that its zero and pole divisor, as a section of $\mathcal{O}_X(D)$ (and not as a meromorphic function!), is equal to D.

For a line bundle $L \to X$ on a compact complex manifold, one defines its Euler characteristic to be

$$\chi(X,L) = \sum_{q=0}^{n} (-1)^q h^q(X,L) - h^1(X,L), \qquad h^q(X,L) = \dim H^q(X,L) = \dim H^{0,q}_{\overline{\partial}}(X,L),$$

thus, if $\dim X = 1$ (which we assume from now on), $\chi(X,L) = h^0(X,L) - h^1(X,L)$, and by Serre duality, $h^1(X,L) = h^0(X, \Omega^1_X \otimes L^{-1})$.

(b) Using the long exact sequence of cohomology associated with the short exact sequence

$$0 \to L \otimes \mathcal{O}_X(-[p]) \to L \to Q_p \to 0$$

where Q_p is the skyscraper sheaf of fiber $\simeq \mathbb{C}$ at p, show that $\chi(X, L \otimes \mathcal{O}(-[p])) = \chi(X,L) - 1$. Similarly, show that $\chi(X, L \otimes \mathcal{O}([p])) = \chi(X,L) + 1$ by using the exact sequence $0 \to L \to L \otimes \mathcal{O}_X([p]) \to Q_p \to 0$.

(c) Infer from (b) the fundamental *Riemann-Roch formula*

$$\chi(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g$$

where $g = h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X)$ is the genus of X.

(d) Write down the long cohomology exact sequence associated with $O \to L \to L \otimes \mathcal{O}_X(m[p]) \to Q_{m,p} \to 0$ (where the stalk of $Q_{m,p}$ at p is $\simeq \mathbb{C}^m$), and prove that $H^0(X, L \otimes \mathcal{O}_X(m[p])) \neq 0$ for $m > h^1(X,L)$. If $D' \geq 0$ is the zero divisor of $\sigma \in H^0(X, L \otimes \mathcal{O}_X(m[p]))$, show that $L \simeq \mathcal{O}_X(D)$ with $D = D' - m[p]$. Infer that one has $\chi(X,L) = \deg(L) + 1 - g$, where $\deg(L) = \deg(D) = \deg(D') - m$.