Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 8, 05/12/2019

1. (*Properties of compact Kähler manifolds*).

If X is a manifold (resp. a complex manifold), one defines the Betti numbers, resp. the Hodge numbers to be

$$b_k(X) = \dim_{\mathbb{R}} H^k_{\mathrm{DR}}(X,\mathbb{R}), \qquad \text{resp. } h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X,\mathbb{C}).$$

(a) Let Z be any complex manifold. Show that the conjugation of forms induces a conjugate linear isomorphism from $H^{p,q}_{\overline{\partial}}(Z,\mathbb{C})$ onto $H^{q,p}_{\overline{\partial}}(Z,\mathbb{C})$. Derive from there that for (X,ω) compact Kähler, one has

$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X).$$

(b) If (X,ω) is compact Kähler, then $b_1(X)$ is even. More generally the odd Betti numbers $b_{2k+1}(X)$ are even. As a consequence, the Hopf manifolds $X = \mathbb{C}^n \smallsetminus \{0\}/\Gamma$ where Γ is an infinite cyclic group of homotheties $z \mapsto \alpha^k z$, $0 < \alpha < 1$, cannot be Kähler.

(c) If u is a holomorphic p-form on (X,ω) compact Hermitian, check that

$$\int_X |u|^2_\omega dV_\omega = C_{n,p}\, i^{p^2} \int_X u \wedge \overline{u} \wedge \omega^{n-p}.$$

with $C_{n,p} > 0$. Infer from there that one must have $du = 0$ if ω is Kähler or if $p = n-1$.

2. (*Compact Riemann surfaces of low genus*).

A compact Riemann surface X has $b_1(X) = 2g$, and one can define g to be the genus of X.

(a) If X is diffeomorphic to the sphere S^2, then $g = 0$. Conversely, if $g = 0$, show that $H^{0,1}_{\mathbb{C}}(X,\mathbb{C}) = 0$. Derive from there that X possesses a meromorphic function f with a simple pole, and infer that f has degree 1 and must be an isomorphism from $\mathbb{P}^1$ onto X.
Hint. Solve an equation $\overline{\partial} u = \overline{\partial}(\theta(z))/z$ where z is a local holomorphic coordinate centered at a point p, and θ is a cutoff function equal to 1 near p, with compact support in the neighborhood V where z is defined, and set $f(z) = \theta(z)/z - u(z)$.

(b) Assume $g \geq 1$. If $K_X = T^*_X = \Omega^1_X$, show that $H^0(X,\Omega^1_X) = H^{1,0}(X,\mathbb{C})$ has dimension g. Take a non zero element $u \in H^0(X,\Omega^1_X)$ and consider the injection of sheaves $\mathcal{O}_X \hookrightarrow \Omega^1_X$, $f \mapsto fu$.
– Show that the quotient sheaf $Q = \Omega^1_X/\mathcal{O}_X$ is a skyscraper sheaf with support in the finite set of zeroes of u, and that the dimension of the stalk at a zero of u is equal to the multiplicity of that zero.
– Look at the long exact sequence of cohomology groups associated with the short exact sequence of sheaves $0 \to \mathcal{O}_X \to \Omega^1_X \to Q \to 0$. Using Serre duality, infer that $H^0(X,Q) = 2g-2$, and that u has $2g-2$ zeroes counted with multiplicities.
Hint: recall that a skyscraper sheaf only has non zero cohomology in degree 0.

(c) If $g = 1$, show that X is isomorphic to an elliptic curve.
Hint. Using (b), infer that Ω^1_X and T_X are trivial. If ξ is a non vanishing vector field, consider the flow $\Phi : \mathbb{C} \times X \to X$ of ξ, which is an additive holomorphic action of $(\mathbb{C},+)$ on X. The goal is to show that for any point $x_0 \in X$, the map $\mathbb{C} \to X$, $t \mapsto \Phi(t,x_0)$ yields an isomorphism $\mathbb{C}/\Lambda \to X$, where Λ is a lattice.
– The orbit of any point $x_0 \in X$ is open.
– The orbits of two points x_0, x_1 are either equal or disjoint.
– Use the connectedness of X to infer that there is only one orbit, and that the map $t \mapsto \Phi_t(x_0)$ cannot be injective.
– Consider $\Lambda = \{t \in \mathbb{C} \,/\, \varphi_t(x_0) = x_0\}$. Show that Λ is a discrete co-compact subgroup of $\mathbb{C}$.