**Introduction to analytic geometry (course by Jean-Pierre Demailly)**
**Sheet number 7, 21/11/2019**

*The goal of this sheet is to detail the proof of the Lelong-Poincaré formula.*

**1.** (*Multiplicities in the divisor of a holomorphic function*).

(a) Let $f \in \mathcal{O}_{X,x}$ be a non zero germ of holomorphic function at a point $x \in X$ in a complex $n$-dimensional manifold $X$. One defines the vanishing order $\mathrm{ord}_x(f)$ of $f$ at point $x$ to be the largest integer $k$ such that $D^\alpha f(x) = 0$ for all mult-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| < k$, i.e. $f(z) = O(|z-x|^k)$ as $z \to x$. Show that $\mathrm{ord}_x(fg) = \mathrm{ord}_x(f)\,\mathrm{ord}_x(g)$ for $f, g \in \mathcal{O}_{X,x}$, $f, g \not\equiv 0$.

(b) Assume that $x \in Z_{\mathrm{reg}}$ where $Z = f^{-1}(0)$. Since $\mathrm{codim}\, Z = 1$, by the definition of what is a smooth codim 1 submanifold, one can find local holomorphic coordinates $(z_1, \ldots, z_n)$ centered at $x$ on an open neighborhood $V$ of $x$, such that $Z \cap V = \{z \in V \,/\, z_1 = 0\}$. Then show that there exists and integer $m \geq 1$ such that $f(z) = z_1^m g(z)$ with a non vanishing function $g \in \mathcal{O}_X(V)$ (after possibly shrinking $V$).
*Hint.* The division process is obtained by looking at the Taylor expansion of $f$, and leads to a function $g$ that is no longer divisible by $z_1$. If $g(x) = 0$, show that the smoothness (and local irreducibility) of the germ of $Z$ at $x$ is contradicted.

(c) Prove that the integer $m$ occuring in (b) does not depend on the choice of coordinates (subject to the condition $Z \cap V = \{z \in V \,/\, z_1 = 0\}$), and infer that one can attach a "multiplicity" $m_j$ of $f$ along each connected component $Z'_j$ of $Z_{\mathrm{reg}}$, such that $m_j = \mathrm{ord}_x(f)$ for every point $x \in Z'_j$.

(d) For a monomial $f(z) = z_1^{a_1} \cdots z_n^{a_p}$ on $\mathbb{C}^n$, evaluate $\mathrm{ord}_x(f)$ for every point $x \in \mathbb{C}^n$.

**2.** Let $M$ be a $n$-dimensional real differentiable manifold, and $T = \sum_{|I|=n-p} T_I(x)\, dx_I \in \mathcal{D}'_p(M)$ be a dimension $p$ current. One says that $T$ is a *normal current* if both $T$ and $dT$ are of order 0, i.e. there coefficients are Radon measures.

(a) Assume that the support $\mathrm{Supp}(T)$ is contained in a closed submanifold $S \subset M$ and near a point $a \in S$, take coordinates $(x_1, \ldots, x_n)$ on a neighborhood $V \ni a$, centered at $a$, such that

$$S \cap V = \{x \in V \,/\, x_1 = \cdots = x_q = 0\}, \qquad q = \mathrm{codim}\, S.$$

If $T$ is of order 0, show that $x_j T = 0$, $1 \leq j \leq q$ (and give a counterexample if $T$ is not of order 0).
*Hint.* For a counterexample, take $S = \{0\} \subset M = \mathbb{R}$ and $T = \delta'_0$ (derivative of the Dirac measure).

(b) With the notation of (a), if $T$ is normal, show that $dx_j \wedge T = 0$, $1 \leq j \leq q$, and infer that one must have $T = 0$ if $\dim S < p$, i.e. $q > n - p$.

(c) If $X$ is a complex manifold and $T$ is a normal bidimension $(p,p)$ current with support in an analytic subset $W$ with $\dim_{\mathbb{C}} W < p$, show that $T = 0$.
*Hint.* Use a stratification $W_j \subset W$ by analytic subsets, $\dim W_j \leq j$, such that $W_j \smallsetminus W_{j-1}$ is a closed submanifold of dimension $j$ in $X \smallsetminus W_{j-1}$, $0 \leq j \leq s = \dim W$, and show by descending induction on $j$ that $T_{|X \smallsetminus W_j} = 0$.

**3.** (a) Show that on $\mathbb{C}^n$, the $(1,1)$-current $\frac{i}{\pi}\partial\overline{\partial} \log |z_1|$ is equal to the current of integration on the hyperplane $H_1 = \{z_1 = 0\}$.
*Hint.* Evaluate on a test form of bidegree $f = \sum_{|I|=|J|=n-1} f_{IJ}(z)\, dz_I \wedge d\overline{z}_J$, and show that the only term that is to be taken into account is $\frac{\partial^2}{\partial z_1 \partial \overline{z}_1} f_{IJ}(z)$ with $I = J = \{2, \ldots, n\}$.

(b) Let $X$ be a complex $n$-dimensional manifold and $f \in \mathcal{O}(X)$ that does not vanish identically on any connected component of $X$. Define $Z = f^{-1}(0)$ et $W = Z_{\mathrm{sing}}$, so that $\dim W \leq n - 2$. Show that

$$\frac{i}{\pi}\partial\overline{\partial} \log |f| = [Z_f] = \sum m_j [Z_j] \quad \text{on } X \smallsetminus W, \quad \text{where } Z_j = \overline{Z'_j}.$$

*Hint.* Combine 1 (b,c) and 3 (a).

(c) Assuming known that $\frac{i}{\pi}\partial\overline{\partial} \log |f|$ and $[Z_f]$ are closed $\geq 0$ bidegree $(1,1)$ currents, prove that one actually has $\frac{i}{\pi}\partial\overline{\partial} \log |f| - [Z_f] = 0$ using 2 (c).