Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 6, 13/11/2019

1. (*Basic homotopy formula for De Rham cohomology*).

Let X a real n-dimensional real manifold. Smooth differential forms of degree p on the product $\widetilde X = [0,1]\times X$ are denoted by

$$u(t,x)=\sum_{|I|=p}u'_I(t,x)\,dx_I\sum_{|I|=p-1}u''_I(t,x)\,dt\wedge dx_I,\quad (t,x)\in[0,1]\times X,$$

(or in abridged form $u=u'+dt\wedge u''$). One defines an operator

$$K:C^\infty(\widetilde X,\Lambda^pT^*_{\widetilde X})\to C^\infty(X,\Lambda^{p-1}T^*_X),\quad u\mapsto v=Ku$$

by

$$v(x)=\int_0^1 dt\wedge u''(t,x)=\sum_{|I|=p-1}\Big(\int_0^1 u''_I(t,x)\,dt\Big)dx_I.$$

(a) Let $i_t:X\to\widetilde X=[0,1]\times X$ be the "injection at time t", i.e. $x\mapsto(t,x)$. Check the fundamental formula $dKu+Kdu=i_1^*u-i_0^*u$ for every $u\in C^\infty(\widetilde X,\Lambda^pT^*_{\widetilde X})$. Infer that the induced morphisms $i_0^*,i_1^*:H^p_{\rm DR}(\widetilde X,\mathbb R)\to H^p_{\rm DR}(X,\mathbb R)$ are equal (and actually, that i_t^* is independent of t).

(b) Let X,Y be real C^∞ differential manifolds and $f,g:X\to Y$ be smooth maps. One says that f,g are (differentiably) homotopic if there exists a C^∞ map $h:[0,1]\times X\to Y$ such that $f(x)=h(0,x)$ and $g(x)=h(1,x)$ [h is then called a differentiable homotopy between f and g; check that this is an equivalence relation]. Show that the pull-back morphisms $f^*,g^*:H^p_{\rm DR}(Y,\mathbb R)\to H^p_{\rm DR}(X,\mathbb R)$ are equal.
Hint. Consider the composition $H=Kh^*:C^\infty(Y,\Lambda^pT^*_Y)\to C^\infty(X,\Lambda^{p-1}T^*_X)$.

(c) (*Poincaré lemma*) A manifold X is said to have the homotopy type of a point if Id_X and any constant map $X\to\{x_0\}\subset X$ are smoothly homotopic (examples: X a convex or star-shaped open set in $\mathbb R^n$). Using (b), show that the De Rham cohomology of such a manifold satisfies $H^p_{\rm DR}(X,\mathbb R)=0$ for $p\ge1$ and $H^0_{\rm DR}(X,\mathbb R)=\mathbb R$.

2. (*A proof of the Dolbeault-Grothendieck lemma on polydisks*).

Let $\Omega_R=\prod_{1\le j\le n}D(0,R_j)$ be a polydisk in $\mathbb C^n$, $r_j\in\,]0,+\infty]$, and let $\Omega_r=\prod_{1\le j\le n}D(0,r_j)$, $0<r_j<R_j$, be a smaller polydisk. The first goal is to prove that for every C^∞ (p,q)-form v in Ω_R with $q\ge1$ and $\overline\partial v=0$, there exists a C^∞ $(p,q-1)$-form u with $\overline\partial u=v$ in Ω_r.

(a) Show that it is enough to consider the case $p=0$.
Hint. $\overline\partial$ does not interact with the $(p,0)$ terms dz_I, so that one just gets a direct sum of $\binom{n}{p}$ identical $(0,\bullet)$ $\overline\partial$-complexes [but this is true only on a manifold X where T_X is trivial, such as $\mathbb C^n$].

(b) One defines $\mathcal C_{q,k}(\Omega_r)$ to be the space of smooth forms v of bidegree $(0,q)$ on Ω_r that involve only $d\overline z_1,\ldots,d\overline z_k$ in their expansion $\sum_{|J|=q}v_J(z)\,d\overline z_J$, and satisfy $\overline\partial v=0$.

- For a C^∞ function g on a 1-dimensional disk $D(0,R)$ and θ a suitable cutoff function, prove that $f=\frac{1}{\pi z}*(\theta g)$ solves the equation $\frac{\partial f}{\partial\overline z}=g$ on $D(0,r)$, $r<R$.
- If $v\in\mathcal C_{q,k}(\Omega_R)$, show that the coefficients v_J are holomorphic in $z_{k+1},\ldots,z_n$ and by writing $v=v'+d\overline z_k\wedge v''$ where v', v'' involve only $d\overline z_1,\ldots,d\overline z_{k-1}$, prove that $v-\overline\partial(\frac{1}{\pi z_k}*_k\theta(z_k)v''(z))\in\mathcal C_{q,k-1}(\Omega_r)$. [Here $*_k$ means a convolution computed term by term for the coefficients $\theta(z_k)v''_J(z)$, with respect to the single variable $z_k\in\mathbb C$].
- Prove the first goal by descending induction on k.

(c) Show that one can actually obtain a solution on Ω_R itself.
Hint. Solve with u_ν defined on Ω_{r_ν}, where r_ν converges to R. For $q=1$, show that one can achieve convergence by adding suitable polynomials. For $q\ge2$, use induction on q, and make corrections on u_ν with $\overline\partial$ of compactly supported forms to achieve convergence.

3. (*Elementary ellipticity results for the Laplace operator* $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$ *on* $\mathbb{R}^n$).

(a) Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $\Omega_\varepsilon = \{z \in \Omega\,;\ d(z, \complement\Omega) > \varepsilon\}$. If $u \in \mathcal{D}'(\Omega)$ and ρ_ε is a family of regularizing kernels associated with a radial function ρ, show that $u * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and that $u * \rho_\varepsilon$ converges weakly to u on every relatively compact open subset $\Omega' \Subset \Omega$.
Hint. First show that $u * \rho_\varepsilon$ is a continuous function on every compact subset $K \subset \Omega_\varepsilon$.

(b) If $f \in C^2(\Omega)$ is a harmonic function (i.e. $\Delta f = 0$), show that $f * \rho_\varepsilon = f$ on Ω_ε.
Hint. Use Stokes's theorem to see that for $B(x_0, R) \subset \Omega$, the average $\mu(r)$ of f on the sphere $S(x_0, r)$ satisfies $\mu'(r) = 0$ on $]0, R]$, and, as a consequence, that $\mu(r) = f(x_0)$ for all such r. Then apply the definition of a convolution.

(c) Let $u \in \mathcal{D}'(\Omega)$ be such that $\Delta u = 0$. Show that in fact $u \in C^\infty(\Omega)$, i.e. every harmonic function or distribution is automatically C^∞.
Hint. Consider $u * \rho_\varepsilon$ and $u * \rho_\varepsilon * \rho_\eta$ and infer that these functions do not depend on ε or η.

(d) If $u \in \mathcal{D}'(\Omega)$ satisfies $du = 0$, then u is a constant. Similarly, if $u \in \mathcal{D}'(\Omega)$ satisfies $\overline{\partial} u = 0$ on $\Omega \subset \mathbb{C}^n$, then u is a holomorphic function in the usual sense.

(e) (*The Newton kernel*) For $n \neq 2$, show that there exists a constant c_n such that $\Delta N = \delta_0$ on $\mathbb{R}^n$, where $N(x) = c_n |x|^{2-n}$ and $|x|^2 = x_1^2 + \cdots + x_n^2$ (What happens for $n = 2$?).
Hint. Compute explicity ΔN_ε with $N_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1-n/2}$ and let $\varepsilon \to 0$.

(f) (*"Hypoellipticity" of the Laplace operator*) Assume that $f \in \mathcal{D}'(\Omega)$ satisfies $\Delta f = g$ with $g \in C^p(\Omega)$. Show that $f \in C^{p+1}(\Omega)$.
Hint. In a neighborhood of a point $x_0 \in \Omega$, consider $h = f - (\theta g) * N$ where θ is a suitable cutoff function, and compute Δh.

4. (*Proof of the Poincaré and Dolbeault-Grothendieck lemmas for currents*).

(a) Show that the proof given in exercise 2 also works for De Rham cohomology on parallelepipeds $\prod\,]a_i, b_i[\ \subset \mathbb{R}^n$, replacing the fundamental solution $\frac{1}{\pi z}$ of $\frac{\partial}{\partial \overline{z}}$ by the Heaviside function H, which is a fundamental solution of $\frac{d}{dx}$.

(b) Check that these proofs also work with currents, for the De Rham complex as well as for the Dolbeault complex.

5. ($\overline{\partial}$ *cohomology with compact supports*).

For any sheaf of abelian groups $\mathcal{F}$ on a topological space X, sheaf cohomology theory leads to the definition of "cohomology groups with compact support" $H_c^q(X, \mathcal{F})$. In case $(X, \mathcal{O}_X)$ is a complex manifold, by general facts of that theory, $H_c^q(X, \mathcal{O}_X)$ coincides with the cohomology of the $\overline{\partial}$-complex of smooth $(0, q)$-forms; alternatively, one can adopt this fact as a definition of $H_c^q(X, \mathcal{O}_X)$.

(a) Prove that $H_c^1(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}}^n) = 0$ for $n \geq 2$.
Hint. Let v be a smooth $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{C}^n$ with compact support contained in a large polydisk $\overline{D}(0, R)$ and consider a smooth function u on $\mathbb{C}^n$ such that $\overline{\partial} u = v$ (such a u exists by exercise 2). Using the Cauchy formula, show that there exists a function $f \in \mathcal{O}(\mathbb{C}^n)$ such that $f = u$ on $\mathbb{C}^n \smallsetminus \overline{D}(0, R)$ and infer the result.

(b) In the case of dimension $n = 1$ and $\Omega = \mathbb{C}$, show that $H_c^1(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ is infinite dimensional.
Hint. For any $(0,1)$-form $v(z) = g(z) d\overline{z}$ with compact support, one can find a compact supported function u such that $\overline{\partial} u(z) = v(z) = g(z) d\overline{z}$ only in case $\int_{\mathbb{C}} g(z)\, z^p\, d\lambda(z) = 0$ for all $p \in \mathbb{N}$. Conversely, if this is the case, show (by a power series expansion) that $u = \frac{1}{\pi z} * g$ has compact support.

(c) More generally, let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an open subset such that $\mathbb{C}^n \smallsetminus \Omega$ has no compact connected component. Show that $H_c^1(\Omega, \mathcal{O}_\Omega) = 0$.
Hint. Derive the result from the case of $\mathbb{C}^n$, extending forms with 0 in the complement of Ω.

(d) (*Hartogs extension theorem*) Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be an open set and $K \subset \Omega$ a compact set such that $\Omega \smallsetminus K$ is connected. Show that every holomorphic function $f \in \mathcal{O}(\Omega \smallsetminus K)$ admits a unique holomorphic extension $\widetilde{f}$ to Ω. Is the result true for $n = 1$?
Hint. Let $\theta \in \mathcal{D}(\Omega)$, $0 \leq \theta \leq 1$, such that $\theta = 1$ in a neighborhood of K. Observe that for the $(0,1)$-form $v = \overline{\partial}((1 - \theta) f)$, extended by 0 on $\mathbb{C}^n \smallsetminus (\Omega \smallsetminus K)$, there exists a smooth compactly supported function u such that $\overline{\partial} u = v$ on $\mathbb{C}^n$.