Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 5, 08/11/2019

1. *Cauchy-Pompeiu formula* (this is a generalization, published by Pompeiu in 1905, of the classical Cauchy formula from 1825). Let $\Omega \subset \mathbb{C}$ be a bounded open subset such that the boundary $\partial\Omega$ is piecewise C^1, and let $f : \overline{\Omega} \to \mathbb{C}$ be a C^1 function on the closure $\overline{\Omega}$. Then

(a) $$\int_{\partial\Omega} f(z)\,dz = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar z}\,d\lambda \quad \text{and}$$

(b) $$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-z_0}dz - \frac{1}{\pi}\iint_{\Omega} \frac{1}{z-z_0}\frac{\partial f}{\partial \bar z}\,d\lambda$$

for every $z_0 \in \Omega$, where $d\lambda = dx \wedge dy$ is the Lebesgue measure on $\mathbb{C}$ (and $z = x + iy$).

Hint. For (a), apply the Stokes formule with $\alpha = f(z)\,dz$, and compute $d\alpha$. For (b), replace f with $g(z) = \frac{f(z)}{z-z_0}$ and Ω with $\Omega_\varepsilon = \Omega \smallsetminus \overline{D}(z_0, \varepsilon)$. Then take the limit as $\varepsilon \to 0$, by means of a change of variable $z = z_0 + \varepsilon e^{i\theta}$ on the (oriented) circle bounding the disc $D(z_0, \varepsilon)$. One has to check that the function $z \mapsto \frac{1}{z-z_0}$ is actually integrable for the Lebesgue measure in a neighborhood of z_0.

(c) Derive from the Pompeiu formula that, in the sense of distributions, one has $\frac{\partial}{\partial \bar z}(\frac{1}{\pi z}) = \delta_0$ on $\mathbb{C}$.

(d) Prove formula (c) in another way, by checking first that $(**)$ $\frac{\partial}{\partial z}(\log|z|^2) = \frac{1}{z}$ on $\mathbb{C}$, and then by applying the formula for $\partial\overline{\partial}\log|z|^2$ given in the course.
Hint. Formula $(**)$ is obviously valid on $\mathbb{C}^*$. To show its validity on $\mathbb{C}$, compute the derivative of $(1-\rho(z/\varepsilon))\log|z|^2$ where $\rho \geq 0$ is a C^∞ function with support in the disc $D(0, 1)$, equal to 1 on $D(0, 1/2)$, and show that one can pass to the weak limit as $\varepsilon \to 0$.

(e) Show that the equation $\frac{\partial}{\partial \bar z}u = v$ has a solution $u \in C^p(\mathbb{C}, \mathbb{C})$ for every given function $v \in C^p_c(\mathbb{C})$ with compact support. What are the other solutions ?
Hint. In general, for any constant coefficient differential operator $P(D)$ possessing a “fundamental solution” E on $\mathbb{R}^n$ (i.e. a distribution E such that $P(D)E = \delta_0$), any equation $P(D)u = v$ with a compactly supported function or distribution v in the right hand side, admits the convolution $u = E*v$ as a solution.

(f) If $\Omega \subset \mathbb{C}$ is a disc $D(0, R)$, show that the equation $\frac{\partial}{\partial \bar z}u = v$ has a solution $u \in C^p(\Omega, \mathbb{C})$ for every given function $v \in C^p(\Omega, \mathbb{C})$.
Hint. Multiply v by a cutoff function to get v_k with compact support in $\Omega_k = D(0, R - 2^{-k})$, equal to v on Ω_{k-1}. For $\widetilde{u}_k = E * v_k$, show that the differences $\widetilde{u}_{k+1} - \widetilde{u}_k$ are holomorphic on Ω_{k-1}, and observe (by truncating their power series at 0) that one can find inductively polynomials $P_k \in \mathbb{C}[z]$ such that the functions $u_k = \widetilde{u}_k + P_k$ converge uniformly on every compact subset of Ω towards a solution $u \in C^p(\Omega)$.

2. (a) Let $\varphi \in C^2(D(0, R), \mathbb{R})$, where $D(0, R) \subset \mathbb{C}$. Show that for all $0 < r_0 < r < R$, the following *generalized Jensen formula* holds:

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{i\theta})d\theta - \frac{1}{2\pi}\int_0^{2\pi} \varphi(r_0e^{i\theta})d\theta = \frac{1}{\pi}\int_{r_0}^{r} \frac{dt}{t}\int_{D(0,t)} \Delta\varphi\,d\lambda,$$

where Δ is the usual Laplace operator on $\mathbb{C} \simeq \mathbb{R}^2$.
Hint. Compute the derivative $\frac{d}{dr}\int_0^{2\pi} \varphi(re^{i\theta})d\theta$ by differentiation under the integral sign, and use Stokes' theorem to relate that integral to $\int_{D(0,t)} \Delta\varphi\,d\lambda$.

(b) Derive from (a) that when $\varphi \in C^2(\Omega, \mathbb{R})$ is subharmonic and $D(z_0, R) \subset \Omega$, the mean value function $\mu(z_0, r) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + re^{i\theta})d\theta$ is non decreasing on $[0, R[$, and in fact, that $t \mapsto \mu(z_0, e^t)$ is convex (one says that $r \mapsto \mu(z_0, r)$ is a convex non decreasing function of $\log r$).

(c) Conversely, if for every $z_0 \in \Omega$ there exists a sequence $r_\nu \to 0$ (possibly depending on z_0) such that $\mu(z_0, r_\nu) \geq \varphi(z_0)$, then $\Delta\varphi \geq 0$ on Ω, i.e. φ is subharmonic.