Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 4, 24/10/2019

1. (a) Let X be a differential manifold. Show that the wedge product of forms defines a (non commutative) graded ring structure on De Rham cohomology $H^\bullet(X,\mathbb{R}) = \bigoplus_{p\in\mathbb{N}} H^p(X,\mathbb{R})$.
Hint. The main point is to check that the wedge product is well defined, i.e. that the cohomology class of a product does not depend on the representatives that are chosen for the factors.

(b) If $(X,\mathcal{O}_X)$ is a complex manifold, show in the same way that Dolbeault cohomology $H^{\bullet,\bullet}(X,\mathbb{C}) = \bigoplus_{p,q\in\mathbb{N}} H^{p,q}(X,\mathbb{C})$ has the structure of a (non commutative) $\mathbb{C}$-algebra.

(c) If $(X,\mathcal{O}_X)$ is a complex manifold, one defines its "Bott-Chern cohomology groups" by

$$H^{p,q}_{\rm BC}(X,\mathbb{C}) := \frac{\{C^\infty \text{ smooth } (p,q)\text{-forms } u \text{ on } X \text{ such that } \partial u = 0 \text{ and } \overline{\partial} u = 0\}}{\{(p,q)\text{-forms } u \text{ on } X \text{ that can be written } u = \partial\overline{\partial} v,\ v\in C^\infty(X,\Lambda^{p-1,q-1}T^*_X)\}}.$$

Show that $H^{\bullet,\bullet}_{\rm BC}(X,\mathbb{C}) = \bigoplus_{p,q\in\mathbb{N}} H^{p,q}_{\rm BC}(X,\mathbb{C})$ is again a (non commutative) $\mathbb{C}$-algebra, and identify the bidegree $(0,0)$ group $H^{0,0}_{\rm BC}(X,\mathbb{C})$.

(d) If $(X,\mathcal{O}_X)$ is a complex manifold, show that there are always well defined algebra morphisms

$$H^{p,q}_{\rm BC}(X,\mathbb{C}) \to H^{p,q}_{\overline{\partial}}(X,\mathbb{C}), \quad H^{p,q}_{\rm BC}(X,\mathbb{C}) \to H^{p+q}_{\rm DR}(X,\mathbb{C})$$

(In other words, for a complex manifold, Bott-Chern cohomology carries "more information" than Dolbeault or De Rham cohomology).

2. Let $(X,\mathcal{O}_X)$ be a complex manifold.

(a) If X is *compact and connected*, use the maximum principle to show that global holomorphic functions in $\mathcal{O}_X(X)$ are constant.

(b) In general, for X compact, describe what is the Dolbeault cohomology group $H^{0,0}(X,\mathbb{C})$.

(c) For $X=\mathbb{C}^n$, show that $H^{p,0}(X,\mathbb{C})$ is infinite dimensional, and that its algebraic dimension is uncountable.
Hint. By Baire's theorem, an infinite dimensional Fréchet space cannot have a countable algebraic dimension.

3. Let $(X,\mathcal{O}_X)$ be a n-dimensional complex manifold and $g_\ell\in\mathcal{O}_X(X)$, $1\le\ell\le N$.

(a) Show that $\varphi(z)=\sum_{1\le\ell\le N}|g_\ell|^2$ has a complex Hessian $H\varphi(z)(\xi)=\sum_{1\le j,k\le n}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}(z)\,\xi_j\xi_k$, $\xi\in T_X$, given by

$$H\varphi(z)(\xi) = \sum_{1\le\ell\le N}|dg_\ell(z)\cdot\xi|^2,$$

and that φ is strictly psh on X as soon as the rank of the system of linear forms $(dg_\ell(z))_{1\le\ell\le N}$ in $T^*_{X,z}$ is equal to n at every point.

(b) Infer from (a) that every complex holomorphic submanifold $X\subset\mathbb{C}^N$ possesses a Kähler metric $\omega = \mathrm{i}\,\partial\overline{\partial}\varphi$.

(c) Show that $\psi(z)=\log\sum_{1\le\ell\le N}|g_\ell|^2$ has a complex Hessian given by

$$H\psi(z)(\xi) = \frac{\sum_\ell |dg_\ell(z)\cdot\xi|^2}{\sum_\ell|g_\ell(z)|^2} - \frac{\big|\sum_\ell \overline{g}_\ell dg_\ell(z)\cdot\xi\big|^2}{\big(\sum_\ell|g_\ell(z)|^2\big)^2},$$

and infer from the Cauchy-Schwarz inequality that ψ is psh outside of the common zero locus of the g_ℓ's. Write explicitly $H\psi(z)(\xi)$ as a sum of squares.
Hint. For $a,b\in\mathbb{C}^N$, recall (and prove) the so called Lagrange identity

$$|a|^2|b|^2 = |\langle a,b\rangle|^2 + |a\wedge b|^2, \ \text{ i.e. } \sum_j |a_j|^2\sum_j|b_j|^2 = \Big|\sum_j a_j\overline{b}_j\Big|^2 + \Big|\sum_{j<k} a_jb_k - b_ja_k\Big|^2.$$

(d) On $X = \mathbb{C}^n \simeq \mathbb{R}^{2n}$, show that the complex Hessian $H\psi$ is related to the real Hessian $H^{\mathbb{R}}\psi(x)(\xi) = \sum_{1\le i,j\le 2n} \frac{\partial^2\psi}{\partial x_j \partial x_k}\,\xi_j\xi_k$, $x, \xi \in \mathbb{R}^{2n}$, by a relation

$$c\,H\psi(z)(\xi) = H^{\mathbb{R}}\psi(z)(\xi) + H^{\mathbb{R}}\psi(z)(J\xi), \quad z, \xi \in \mathbb{C}^n,$$

where $c > 0$ is a constant. Infer from there that the convexity of ψ implies the plurisubharmonicity of ψ.

(e) On $X = \mathbb{C}^n$, show that $\psi(z) = \log(1 + |z|^2)$ (where $|z|^2 = \sum |z_\ell|^2$) is strictly psh, although is is not convex. More generally, on a complex manifold X, show that $\psi(z) = \log(1 + \sum_{1\le\ell\le N} |g_\ell(z)|^2)$ is strictly psh as soon as the rank of the system of linear forms $(dg_\ell(z))_{1\le\ell\le N}$ in $T^*_{X,z}$ is equal to n at every point.

(f) On $X = \mathbb{C}^n$, a C^2 function $z \mapsto \chi(z)$ that depends only on $x = \operatorname{Re} z \in \mathbb{R}^n$ (i.e. $\chi(z) = \chi(x)$) is psh if and only if it is convex.