Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 3, 17/10/2019

Basic definitions about categories

Recall that a category $\mathcal{C}$ is a collection of "objects" denoted by $\mathrm{ob}(\mathcal{C})$ and a collection of "morphisms" (or arrows) $\mathrm{mor}(\mathcal{C})$ $u : a \to b$ between objects of $\mathcal{C}$ (here, one can think of u as being a map, but it can be just an "abstract arrow" not associated to an actual map), satisfying the following axioms:
– for all morphisms $u : a \to b$ and $v : b \to c$ in $\mathrm{mor}(\mathcal{C})$, there is a composed morphism $v \circ u : a \to c$ (here again, $\circ$ can be an abstract composition law, not necessarily the composition of maps, although this is the most frequent case)
– composition of morphisms is associative
– for each object $a \in \mathrm{ob}(\mathcal{C})$, there is an identity morphism $1_a : a \to a$ which is a left and right unit element for composition.
Examples 1: category $\mathcal{S}$ of sets: $\mathrm{ob}(\mathcal{S})$ = all sets, $\mathrm{mor}(\mathcal{S})$ = all maps between sets; in a similar way, categories of groups, vector spaces, rings, A-modules, $\mathbb{K}$-algebras, where the morphisms are taken to be morphisms of these algebraic structures.
Remark: The collection of all sets is not a set, nor is the collection of all maps between sets !
Examples 2: category of sheaves $\mathcal{S}$ of abelian groups over a given topological space X, together with morphisms of sheaves of abelian groups; category of $\mathbb{C}$-vector bundles E over a given manifold or topological space X, together with continuous (resp. smooth) morphisms of bundles.

A *functor* $F : \mathcal{C} \to \mathcal{C}'$ from a category $\mathcal{C}$ into a category $\mathcal{C}'$ is an association $F : a \mapsto a' = F(a)$ from $\mathrm{ob}(\mathcal{C})$ to $\mathrm{ob}(\mathcal{C}')$ and $F : u \mapsto u' = F(u)$ from $\mathrm{mor}(\mathcal{C})$ to $\mathrm{mor}(\mathcal{C}')$ in such a way that $F(1_a) = 1_{F(a)}$ and $F(v \circ u) = F(v) \circ F(u)$ for all composable morphisms $u, v \in \mathrm{mor}(\mathcal{C})$. An *equivalence* of categories $\mathcal{C}, \mathcal{C}'$ is a functor $F : \mathcal{C} \to \mathcal{C}'$ such that there is a left and right inverse functor $G : \mathcal{C}' \to \mathcal{C}$.

1. Show that a morphism $\varphi : (X, \mathcal{C}^k_X) \to (Y, \mathcal{C}^k_Y)$ of ringed spaces between C^k differential manifolds is the same as a C^k map $X \to Y$. Show that a morphism $\varphi : (X, \mathcal{C}^k_X) \to (Y, \mathcal{C}^\ell_Y)$ of ringed spaces can exist only if $\ell \geq k$ (one agrees that $\omega > \infty > k$ for all $k \in \mathbb{N}$).

2. Let $(X, \mathcal{A})$ be a ringed space. Show that a morphism $\mathcal{A}^{\oplus q} \to \mathcal{A}^{\oplus p}$ of $\mathcal{A}$-modules is given by a $p \times q$ matrix of global sections in $\mathcal{A}(X)$, and that the homomorphism sheaf $\mathcal{H}om(\mathcal{A}^{\oplus q}, \mathcal{A}^{\oplus p})$, whose sections over $U \subset X$ consist of morphisms from $\mathcal{A}^{\oplus q}_{|U}$ to $\mathcal{A}^{\oplus p}_{|U}$, satisfies $\mathcal{H}om(\mathcal{A}^{\oplus q}, \mathcal{A}^{\oplus p}) \simeq \mathcal{A}^{\oplus pq}$.

3. Let $(X, \mathcal{O}_X)$ be a complex manifold. The goal of this exercise is to show that there is an equivalence of categories between locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and holomorphic vector bundles E over X.

(a) Let $\pi : E \to X$ be a holomorphic vector bundle of rank r (in the sense defined by Catriona Maclean). One defines a presheaf $\mathcal{E}$ over X by $\mathcal{E}(U) = \{\text{sections of } E \text{ over } U\}$, i.e. holomorphic maps $s : U \to E$ such that $\pi \circ s = \mathrm{Id}_U$. Show that $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank r.

(b) Conversely, let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-modules of rank r. Observing that $\mathbb{C} = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is an $\mathcal{O}_{X,x}$-module, one defines $E_x = \mathcal{E}_x \otimes \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. Show that $E_x \simeq \mathbb{C}^r$ and that the projection $\pi : E \to X$ which maps every fiber E_x to x defines a holomorphic vector bundles defined by the same cocycle of holomorphic matrices as $\mathcal{E}$.

(c) Observe that one has similar equivalences of categories for smooth topological bundles, or even for topological vector bundles, over the field $\mathbb{K} = \mathbb{R}$ or over the field $\mathbb{K} = \mathbb{C}$.

4. If $E \to X$ is a real C^p-vector bundle or even rank $2r$ over a C^p differential manifold X, one defines an *almost complex structure* J on E to be a global endomorphism $J \in C^p(X, \mathrm{End}_{\mathbb{R}}(E))$ such that $J^2 = -\mathrm{Id}$.

(a) show that E equipped with J can then be considered as a complex C^p-vector bundle of rank r over X. *Hint*. Show that one can extract from any local C^p-frame $(e_1, \ldots, e_{2r})$ over $\mathbb{R}$ a local complex frame, e.g. $(e_1, \ldots, e_r)$ after a permutation of the indices. Using the fact that J is C^p, show that the complex transition matrices obtained by taking such frames over an open covering (U_α) of X are still C^p.

(b) Show that there is a direct sum decomposition of C^p vector subbundles

$$E^{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C} = E^{1,0} \oplus E^{0,1}$$

where $E^{1,0}$ (resp. $E^{0,1}$) is the eigenspace of eigenvalue $+i$ (resp. $-i$) of the complexified endomorphism $J^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(E^{\mathbb{C}})$.

5. Let X be a differential manifold of class C^p, $p \geq 2$.
(a) Given vector fields $\xi = \sum \xi_j \frac{\partial}{\partial x_j}$, $\eta = \sum \eta_k \frac{\partial}{\partial x_k}$ of class C^ℓ, $1 \leq \ell \leq p-1$, on an open set U in X, the Lie bracket $[\xi, \eta]$ of ξ, η is defined to be

$$[\xi, \eta] \cdot f := \xi \cdot (\eta \cdot f) - \eta \cdot (\xi \cdot f), \quad \forall f \in C^2(U, \mathbb{R})$$

(viewing vector fields as derivations acting on functions). Show that this defines again a vector field, compute explicitly $[\xi, \eta]$ in coordinates, and show that $[\xi, \eta]$ is of class $C^{\ell-1}$. For $g \in C^1(U)$, prove that

$$[g\xi, \eta] = g[\xi, \eta] - (\eta \cdot g)\xi, \quad [\xi, g\eta] = g[\xi, \eta] + (\xi \cdot g)\eta.$$

(b) Show that the Lie bracket extends in a natural way to complex vector fields (i.e. sections $\xi = \sum \xi_j(x) \frac{\partial}{\xi_j}$ of $T_X \otimes_{\mathbb{R}} \mathbb{C}$, whose components are complex valued functions insteal of real ones).
(c) An *almost complex structure* on an even dimensional differential manifold $(X, \mathcal{C}^p_X)$ is by definition an almost complex structure J on its tangent bundle T_X, so that one has a decomposition $T_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$. The almost complex structure is said to be *integrable* if Lie brackets of vector fields in $T_X^{1,0}$ remain in $T_X^{1,0}$, i.e. the sheaf of sections $\mathcal{T}_X^{1,0}(U) = C^{p-1}(U, T_X^{1,0})$ is stable by Lie bracket. In case $(X, \mathcal{O}_X)$ is a complex analytic manifold and J is its "natural" almost complex structure, compute the Lie brackets $[f\frac{\partial}{\partial z_j}, g\frac{\partial}{\partial z_k}]$, $[f\frac{\partial}{\partial z_j}, g\frac{\partial}{\partial \overline{z}_k}]$, $[f\frac{\partial}{\partial \overline{z}_j}, g\frac{\partial}{\partial \overline{z}_k}]$, and conclude that $\mathcal{T}_X^{1,0}$ as well as $\mathcal{T}_X^{0,1}$ are stable by Lie bracket.

(d) On $\mathbb{C}^2 \simeq \mathbb{R}^4$, equipped with coordinates (x_1, y_1, x_2, y_2), one defines $J \in C^\infty(\mathbb{R}^4, \mathrm{End}(T_{\mathbb{R}^4}))$ by

$$J\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}, \quad J\frac{\partial}{\partial y_1} = -\frac{\partial}{\partial x_1}, \quad J\frac{\partial}{\partial x_2} = p\,\frac{\partial}{\partial x_2} + q\,\frac{\partial}{\partial y_2}, \quad J\frac{\partial}{\partial y_2} = r\,\frac{\partial}{\partial x_2} + s\,\frac{\partial}{\partial y_2}$$

where p, q, r, s are functions of (x_1, y_1, x_2, y_2). Show that this defines an almost complex structure on $\mathbb{R}^4$ if and only if

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \iff s = -p \text{ and } p^2 + rq = -1.$$

Compute the Lie bracket of

$$\xi = \frac{\partial}{\partial z_1} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - iJ\frac{\partial}{\partial x_1}\Big), \quad \eta = \frac{1}{2}\Big(\frac{\partial}{\partial x_2} - iJ\frac{\partial}{\partial x_2}\Big),$$

and conclude from there that J is integrable if and only if

$$\begin{vmatrix} \dfrac{\partial p}{\partial z_1} & 1 - ip \\ \dfrac{\partial q}{\partial z_1} & -iq \end{vmatrix} = 0 \iff \frac{\partial}{\partial z_1}\Big(\frac{q}{1 - ip}\Big) = 0 \iff \frac{\partial}{\partial \overline{z}_1}\Big(\frac{q}{1 + ip}\Big) = 0,$$

i.e. $\frac{q}{1+ip}$ is holomorphic in $z_1 = x_1 + iy_1$. Show that in case one takes e.g. $p = -s = x_1$, $q = -r = \sqrt{1 + x_1^2}$, then the almost complex structure J cannot derive from a holomorphic structure.

Remark. A deep theorem, known as the Newlander-Nirenberg theorem, originally proved in 1957, states that a C^∞ almost complex structure derives from a holomorphic structure if and only if it is integrable (which is *always* the case in real dimension 2). The result is also true for J of class C^ℓ regularity, $\ell > 1$, even when $\ell = r + \alpha$ is a non integer value, $r \in \mathbb{N}^*$, $0 < \alpha < 1$. The holomorphic coordinate chart can then taken to be of class $C^{r+\alpha+1}$ with respect to the original coordinates (cf. e.g. `https://www.esi.ac.at/static/esiprpr/esi2239.pdf`).