Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 2, 10/10/2019

1. *Analytic hypersurfaces – very direct application of the course.* Let $\Omega' \subset \mathbb{C}^{n-1}$ be a connected open set, and let

$$P(z', z_n) = z_n^d + a_1(z')z_n^{d-1} + \cdots + a_{d-1}(z')z_n + ad(z')$$

where $a_j \in \mathcal{O}(\Omega')$ et let $A \subset \Omega' \times \mathbb{C}$ be the analytic set defined by $P(z', z_n) = 0$. One assumes that the discriminant $\Delta(z')$ of $P(z', z_n)$ with respect to the polynomial ring $\mathcal{O}(\Omega')[z_n]$ is non identically zero (recall that $\Delta(z') = \mathrm{Res}(P, \partial P/\partial z_n)$).

(a) Show that $\pi : A \to \Omega'$ is an (étale) covering from $A \smallsetminus \pi^{-1}(\Sigma) \to \Omega' \smallsetminus \Sigma$, where $\Sigma = \Delta^{-1}(0) \subset \Omega'$, and conclude from there that $A_{\mathrm{sing}} \subset A \cap \pi^{-1}(\Sigma)$.

(b) In general, show that one can factorize P as $P = P_1 \ldots P_N$, with $P_j \in \mathcal{O}(\Omega')[z_n]$ and the factors P_j are irreducible in $\mathcal{O}(\Omega')[z_n]$, and in one-to-one correspondence with the connected components of $A \smallsetminus \pi^{-1}(\Sigma)$.

(c) In the case $P(z) = P(z_1, z_2, z_3) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ on $\mathbb{C}^3$, with $a_j \in \mathbb{N}^*$, show that $A_{\mathrm{sing}} = \{0\}$ if all $a_j \geq 2$ (while $A_{\mathrm{sing}} = \emptyset$, were one of the a_j's be equal to 1). Is $A_{\mathrm{sing}} = A \cap \pi_{12}^{-1}(\Sigma_{12})$ where π_{12} is the projection $z \mapsto (z_1, z_2)$ and Σ_{12} the corresponding branching locus ? Check that however

$$A_{\mathrm{sing}} = A \cap \pi_{12}^{-1}(\Sigma_{12}) \cap \pi_{23}^{-1}(\Sigma_{23}) \cap \pi_{13}^{-1}(\Sigma_{13}) \quad \text{(with an obvious notation).}$$

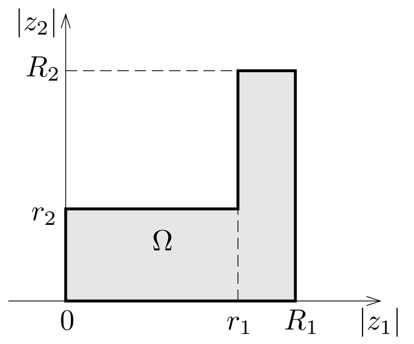
2. (a) Let Ω be an open set in $\mathbb{C}$. Let K_j the exhaustive sequence of compact sets in Ω defined by

$$K_j = \{z \in \Omega / |z| \leq 2^j,\ d(z, \mathbb{C} \smallsetminus \Omega) \geq 2^{-j}\}.$$

Fix $E \subset \partial\Omega$ a finite or countable dense set, and $(w_j)_{j \in \mathbb{N}}$ a sequence of points of E such that each point of E is repeated infinitely many times (this is possible...). Now, take inductively points $z_j \in \Omega$ such that $|z_j - w_j| \leq \min(2^{-j} d(w_j, K_j), \frac{1}{2}|z_k - w_k|)_{k<j}$. Show that $f(z) = \prod_j \frac{z - z_j}{z - w_j}$ defines a holomorphic function on Ω that cannot be extended through any boundary point $a \in \partial\Omega$, i.e. for an arbitrary small neighborhood V of a, f does not extend holomorphically to $\Omega \cup V$.
Hint. $|\frac{z - z_j}{z - w_j} - 1| \leq 2^{-j}$ on K_j, and $\partial E \cap V$ contains a point w_j at which zeroes of f accumulate.

(b) (*Extension from a "Hartogs figure" in* $\mathbb{C}^2$). Let $0 < r_1 < R_1$ and $0 < r_2 < R_2$ and

$$\Omega = (D(0, R_1) \smallsetminus \overline{D}(0, r_1)) \times D(0, R_2) \ \cup \ (D(0, R_1) \times D(0, r_2) \ \subset \ \mathbb{C}^2.$$



Show that every function $f \in \mathcal{O}(\Omega)$ extends holomorphically as $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ to $\widetilde{\Omega} = D(0, R_1) \times D(0, R_2)$.

3. Show that $z \mapsto e^{e^z}$, $\mathbb{C} \to \mathbb{C}^*$ is étale and surjective, but that it is *not* a covering. Show that there exists an open set $\Omega \subset \mathbb{C}$ (actually unique) such that $f_{|\Omega} : \Omega \to \mathbb{C} \smallsetminus \{0, 1\}$ is a covering, but that it is impossible to find $\Omega' \subset \mathbb{C}$ such that $f_{|\Omega'} : \Omega' \to \mathbb{C}^*$ becomes a covering.

4. Let X be a topological space.

(a) Show that $\mathcal{CB}(U) = \{f : U \to \mathbb{R}$ continuous and bounded$\}$ is a presheaf, but not a sheaf in general.

(b) If $\mathcal{P}$ is a presheaf, show that if one defines $S = \coprod_{x \in X} \mathcal{P}_x$ and open sets of S to be union of open sets of the form $\Omega_{U,f} = \{f_x / x \in U\}$ with $f \in \mathcal{P}(U)$, one still obtains an "étalé space" $\pi : S \to X$. Actually, check that we do have a topology – this requires e.g. to show that $\Omega_{U,f} \cap \Omega_{V,g}$ is open, and in fact this intersection coincides with $\Omega_{W,h}$, where W is the open subset of points $x \in U \cap V$ where $f_x = g_x$ and $h = \rho^U_W(f)$ (say).

(c) If $\mathcal{C}$ is the sheaf of continuous functions $U \to \mathbb{R}$, show that $\mathcal{C}$ and $\mathcal{CB}$ have the same corresponding étalé space (with the same topology).

(d) In general, il $\mathcal{P}$ is a presheaf, one defines the associated sheaf $\mathcal{S}$ to be the sheaf of sections of the étalé space S corresponding to $\mathcal{P}$. Show that there is a natural morphism of presheaves $\mathcal{P} \to \mathcal{S}$, i.e. for every open set $U \subset X$, there is a natural map $\mathcal{P}(U) \to \mathcal{S}(U)$, (that will be e.g. a group morphism if we have a presheaf $\mathcal{P}$ of abelian groups).

(e) Give an example where $\mathcal{P}(U) \to \mathcal{S}(U)$ is not surjective (easy from the above!), resp. non injective.
Hint. For the non injectivity, take the following "weird" presheaf: $\mathcal{P}(X) = G$ (some non trivial abelian group), and $\mathcal{P}(U) = \{0\}$ for $U \neq X$.

5. (a) Show that the étalé space S associated with $\mathcal{O}_{\mathbb{C}^n}$ (or any $\mathcal{O}_X$ over any complex manifold X) is Hausdorff.
Hint. It is enough to find disjoint neighborhoods of any two distinct germs f_x, g_x in the same stalk $\mathcal{O}_{X,x}$, because it is easy to separate germs in different stalks $\mathcal{O}_{X,x}$, $\mathcal{O}_{X,y}$.

(b) Show that the étalé space S associated with $\mathcal{S} = \mathcal{C}^\infty_{\mathbb{R}}$ (the sheaf of C^∞ functions $\mathbb{R} \supset U \to \mathbb{R}$) is *not Hausdorff.*
Hint. Try to separate the germs at $x_0 = 0$ of the functions $x \mapsto f(x) = 0$, $x \mapsto g(x) = e^{-1/x}$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$.