Introduction to analytic geometry (course by Jean-Pierre Demailly)
Sheet number 1, 03/10/2019

1. *Standard area and volume calculations in $\mathbb{R}^n$ and $\mathbb{C}^n$. Here, $\mathbb{R}^n$ and $\mathbb{C}^n$ are equipped with their standard Euclidean/Hermitian structures, the Hermitian norm being denoted $z \mapsto |z|$.*

(a) The Lebesgue measure in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ can be written as

$$d\lambda(z) = \frac{i}{2}\, dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \frac{i}{2}\, dz_n \wedge d\overline{z}_n = \frac{i^{n^2}}{2^n}\, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n,$$

where $z_j = x_j + iy_n$ and $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ is oriented (conventionally) by coordinates $(x_1, y_1, \ldots, x_n, y_n)$.

(b) *Surface element of Euclidean spheres*
Show that the degree $n-1$ differential form on $\mathbb{R}^n$ defined by

$$d\sigma(x) = \frac{1}{R}\sum_{j=1}^{n}(-1)^{j-1}\, x_j\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \quad \text{(where } \widehat{*} \text{ means a term } * \text{ omitted)},$$

induces on the sphere $S(0,R) \subset \mathbb{R}^n$ the Euclidean area measure.
In the case of $\mathbb{C}^n$, show that a similar role can be played by

$$d\sigma(z) = \frac{i^{n^2}}{2^n}\,\frac{2}{R}\sum_{j=1}^{n}(-1)^{j-1}\, z_j\, dz_1 \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n.$$

Hint. Compute $dr(x) \wedge d\sigma(x)$ with $r(x) = (\sum |x_j|^2)^{1/2}$ (resp. $r(z) = (\sum z_j \overline{z}_j)^{1/2}$); compare with $d\lambda$.

(c) If $x = ru$ is the usual polar decomposition in $\mathbb{R}^n \smallsetminus \{0\}$ (with $r = |x| \in \mathbb{R}_+$ and $u = \frac{x}{|x|} \in \mathbb{S}^{n-1}$, show that $d\lambda(x) = r^{n-1} dr \wedge d\sigma(u)$ where $d\sigma(u)$ is given as in (b). (Of course, one has a similar formula in $\mathbb{C}^n$ with $z = ru$ and $d\lambda(z) = r^{2n-1} dr \wedge d\sigma(u)$).
Hint. Notice that $du_1 \wedge \ldots \wedge du_n = 0$ since $\dim \mathbb{S}^{n-1} < n$.

(d) Let $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ be the unit sphere, and consider the projection

$$\mathbb{S}^{2n-1} \to \mathbb{C}^{n-1}, \quad z = (z_1, \ldots, z_n) \mapsto z' = (z_1, \ldots, z_{n-1}).$$

Write $z' = \rho u$, $\rho \in \mathbb{R}_+$, $u \in \mathbb{S}^{2n-3} \subset \mathbb{C}^{n-1}$, and $z_n = r_n e^{i\theta_n}$ in polar coordinates on $\mathbb{C}^{n-1}$ (resp. $\mathbb{C}$). If λ' is the Lebesgue measure of $\mathbb{C}^{n-1}$ and σ' the area measure of $\mathbb{S}^{2n-3}$, show that

$$d\sigma(z) = d\lambda'(z') \wedge d\theta_n = \rho^{2n-3}\, d\rho \wedge d\sigma'(u) \wedge d\theta_n$$

in the complement of $\mathbb{S}^{2n-1} \cap \{z_n = 0\} \simeq \mathbb{S}^{2n-3}$ (which is $d\sigma$-negligible) in $\mathbb{S}^{2n-1}$. Derive from there the Fubini type formula

$$\int_{\mathbb{S}^{2n-1}} f(z)\, d\sigma(z) = \int_{\rho\in[0,1]}\int_{\theta_n\in[0,2\pi]}\int_{u\in\mathbb{S}^{2n-3}} f\big(\rho u, (1-\rho^2)^{1/2} e^{i\theta_n}\big)\, \rho^{2n-3}\, d\rho\, d\theta_n\, d\sigma'(u).$$

Hint. Use $d\theta_n = \mathrm{Im}(d\log z_n) = \frac{1}{2i}\big(\frac{dz_n}{z_n} - \frac{d\overline{z}_n}{\overline{z}_n}\big)$ to compute $dr \wedge d\lambda(z') \wedge d\theta_n$.

(e) Show by induction on n that the area of $\mathbb{S}^{2n-1}$ is $\sigma_{2n-1} = \frac{2\,\pi^n}{(n-1)!}$. The area of $S(0,R) \subset \mathbb{C}^n$ is $\frac{2\,\pi^n}{(n-1)!}\, R^{2n-1}$ and the volume of $B(0,R) \subset \mathbb{C}^n$ is $\frac{\pi^n}{n!} R^{2n}$.

2. Let $\Omega \subset \mathbb{C}^n$ be an open set such that

$$\forall z \in \Omega, \quad \forall \lambda \in \mathbb{C}, \quad |\lambda| \le 1 \quad \Rightarrow \quad \lambda z \in \Omega.$$

Show that Ω is a union of polydisks of center 0 (with respect to coordinates $z' = u(z)$ associated with arbitrary unitary matrices $u \in U(n)$) and infer that the space of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ is dense in $\mathcal{O}(\Omega)$

for the topology of uniform convergence on compact subsets. If Ω is bounded, show that $\mathbb{C}[z_1, \ldots, z_n]$ is dense in $\mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$ for the topology of uniform convergence on $\overline{\Omega}$.
Hint: consider the Taylor expansion of a function $f \in \mathcal{O}(\Omega)$ at the origin, writing it as a series of homogeneous polynomials. To deal with the case of $\mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$, first apply a dilation to f.

3. The goal of this exercise is to prove the Cauchy formula for the unit ball in $\mathbb{C}^n$. Let $B \subset \mathbb{C}^n$ be the unit Hermitian ball, $S = \partial B$ and $f \in \mathcal{O}(B) \cap C^0(\overline{B})$. Our goal is to check the following Cauchy formula:

$$f(w) = \frac{1}{\sigma_{2n-1}} \int_S \frac{f(z)}{(1 - \langle w, z \rangle)^n} \, d\sigma(z).$$

(a) By means of a unitary transformation and exercise 2, reduce the question to the case when $w = (0, \ldots, 0, w_n)$ and $f(z)$ is a monomial $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$.

(b) Show that the integral $\int_S z^\alpha \overline{z}_n^k \, d\sigma(z)$ vanishes unless $\alpha = (0, \ldots, 0, k)$. Compute the value of the non zero integral by means of suitable integration by parts.
Hint. Use formula 1 (d), invariance of σ by rotation, and/or Fourier series arguments.

(c) Prove the formula by means of a suitable power series expansion of $(1 - \langle w, z \rangle)^{-n}$.

4. *Montel spaces*
Let E be a topological vector space E over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A subset $S \subset E$ is said to be bounded if for every neighborhood U of zero in E, there exists $\lambda \in \mathbb{K}^*$ such that $A \subset \lambda U$ (or equivalently, $\lambda^{-1} A \subset U$).

(a) Show that if A is compact, then A is bounded.
Hint. Use the continuity of the operations to prove the fact that for any given open neighborhood U of 0, one can find a neighborhood V of 0 and $\delta > 0$ such that $V + \mu V \subset U$ for all $\mu \in \mathbb{K}$, $|\mu| < \delta$, and cover A by translates $x_j + V$.

(b) If $(E, \|\ \ \|)$ is a normed vector space, show that A is bounded if and only if $\sup_{x \in A} \|x\| < +\infty$. More generally, if E is locally convex and the topology of E is defined by a collection of semi-norms $(p_\alpha)_{\alpha \in I}$, then A is bounded if and only if for every $\alpha \in I$ one has $\sup_{x \in A} p_\alpha(x) < +\infty$.

(c) A Fréchet space E will be said to be a *Montel space* if the compact subsets of E are exactly the closed bounded subsets of E. Show that for every open set $\Omega \subset \mathbb{C}^n$, the space $\mathcal{O}(\Omega)$ is a Montel space; likewise, for $\Omega \subset \mathbb{R}^n$, $C^\infty(\Omega)$ is a Montel space.

(d) Show that an infinite dimensional Banach space is never a Montel space. Derive from there that the topology of $\mathcal{O}(\Omega)$ or $C^\infty(\Omega)$ cannot be defined by a single norm.