Introduction to analytic geometry (course by Jean-Pierre Demailly)
Final examination, 09/01/2020, 09:00 – 13:00

Let K be a compact subset of $\mathbb{C}^n$. One defines the polynomial hull of K to be
$$\widehat{K} = \big\{z \in \mathbb{C}^n \,/\, |P(z)| \le \sup_K |P|,\ \forall P \in \mathbb{C}[z_1, \dots, z_n]\big\}.$$

1. (a) Show that the polynomial hull $\widehat{K}$ is compact, coincides with the holomorphic hull
$$\widehat{K}_{\mathcal{O}} = \big\{z \in \mathbb{C}^n \,/\, |f(z)| \le \sup_K |f|,\ \forall f \in \mathcal{O}(\mathbb{C}^n)\big\}$$
and that $\widehat{K} = \widehat{\partial K}$.

By definition $\widehat{K}$ contains K and $\sup_{\widehat{K}} |P| = \sup_K |P|$ for every $P \in \mathbb{C}[z_1, \dots, z_n]$. If we take $P(z) = z_j$, we find $\sup_{\widehat{K}} |z_j| = \sup_K |z_j| < +\infty$, hence $\widehat{K}$ is bounded. On the other hand the continuity of $|P|$ implies that
$$\widehat{K} = \bigcap_{P\in\mathbb{C}[z_1,\dots,z_n]} \{z \in \mathbb{C}^n \,/\, |P(z)| \le M_P\} \quad (\text{where } M_P = \sup_K |P|)$$
is closed. Therefore $\widehat{K}$ is compact. Since $\mathbb{C}[z_1, \dots, z_n] \subset \mathcal{O}(\mathbb{C}^n)$, it is obvious from the definition that $\widehat{K}_{\mathcal{O}} \subset \widehat{K}$. However every entire function $f \in \mathcal{O}(\mathbb{C}^n)$ is a limit, uniformly of every compact set of $\mathbb{C}^n$, of a sequence of polynomials P_ν (just take the truncated Taylor series $P_\nu(z) = \sum_{|\alpha|\le\nu} a_\alpha z^\alpha$ centered at 0). The equality $\sup_{\widehat{K}} |P_\nu| = \sup_K |P_\nu|$ implies in the limit $\sup_{\widehat{K}} |f| = \sup_K |f|$, and we conclude that $\widehat{K} \subset \widehat{K}_{\mathcal{O}}$, whence $\widehat{K} = \widehat{K}_{\mathcal{O}}$. Finally, the maximum principle tells us that $\sup_{\partial K} |f| = \sup_K |f|$ for every $f \in \mathcal{O}(\mathbb{C}^n)$. This implies immediately $\widehat{K} = \widehat{\partial K}$.

Contrary to what I have seen stated in some tests, it is not true in $\mathbb{C}^n$ $(n \ge 2)$ that $\{z \in \mathbb{C}^n \,/\, |P(z)| \le M\}$ is a bounded set. Just take $P(z) = z_1$ to see that this is wrong. In fact, this is wrong for all polynomials!

(b) Prove that $\widehat{K}$ is contained in the convex hull $\widetilde{K}$ of K.
Hint. Consider $f(z) = e^{\ell(z)}$ where ℓ runs over all linear forms on $\mathbb{C}^n$, and recall that the convex hull $\widetilde{K}$ is equal to the intersection of closed half spaces that contain K (this is a consequence of the Hahn-Banach theorem in finite dimension).

In fact $|e^{\ell(z)}| = e^{\mathrm{Re}\,\ell(z)}$, and for $\ell \in (\mathbb{C}^n)^*$, the real parts $z \mapsto \mathrm{Re}\,\ell(z)$ are just the real linear forms on the underlying real vector space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. As $f = e^\ell \in \mathcal{O}(\mathbb{C}^n)$, we infer
$$\widehat{K} = \widehat{K}_{\mathcal{O}} \subset \{z \in \mathbb{C}^n \,/\, \mathrm{Re}\,\ell(z) \le m_\ell\} \quad (\text{where } m_\ell = \sup_K \mathrm{Re}\,\ell).$$
The right hand side is precisely equal to the convex hull $\widetilde{K}$ (= intersection of half spaces containing K), thus $\widehat{K} \subset \widetilde{K}$. As a consequence, every compact convex set is polynomially convex.

(c) A (compact) polynomial polyhedron is a compact set of the form $K = \{z \in \mathbb{C}^n \,/\, |P_j(z)| \le c_j\}$ for suitable polynomials $P_1, \dots, P_m \in \mathbb{C}[z_1, \dots, z_n]$ and constants $c_j \in \mathbb{R}_+$. Show that $\widehat{K} = K$ and give an example where K is non convex.
Hint. In $\mathbb{C}^2$, take one of the polynomials to be $P(z_1, z_2) = z_1 z_2$.

If $K = \{z \in \mathbb{C}^n \,/\, |P_j(z)| \le c_j\}$, then $\sup_{\widehat{K}} |P_j| = \sup_K |P_j| \le c_j$, hence $\widehat{K} \subset K$, thus $\widehat{K} = K$. For the example, since we need a *compact* polynomial polyhedron, we cannot just take one polynomial on $\mathbb{C}^n$ (except for $n = 1$ where this would be possible). In $\mathbb{C}^2$, let us consider the polynomial polyhedron
$$K = \big\{(z_1, z_2) \in \mathbb{C}^2 \,/\, |z_1| \le 1,\ |z_2| \le 1,\ |z_1 z_2| \le \delta\big\}$$
with $\delta \in]0, 1[$, which is trivially compact. It contains the points $(1, \delta)$ and $(\delta, 1)$, but does not contain the middle point $(\frac{1+\delta}{2}, \frac{1+\delta}{2})$ as $(\frac{1+\delta}{2})^2 > \delta \Leftrightarrow (\frac{1-\delta}{2})^2 > 0$. Hence $\widehat{K} = K$ is non convex.

Remark: in $\mathbb{C}$, one can show that the polynomial hull $\widehat{K}$ is the union of K with the bounded connected components of $\mathbb{C} \smallsetminus K$ ("holes"), thus $K \subset \mathbb{C}$ is polynomially convex if and only if $\mathbb{C} \smallsetminus K$ is connected.

2. Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a smooth plurisubharmonic exhaustion. For every value $c \in \mathbb{R}$, one defines
$$\Omega_c = \big\{z \in \mathbb{C}^n \,/\, \psi(z) < c\big\}, \quad K_c = \big\{z \in \mathbb{C}^n \,/\, \psi(z) \le c\big\}.$$

Recall that ψ being an exhaustion means that K_c is compact for every $c \in \mathbb{R}$. The goal of the exercise is to show that

(∗∗) every holomorphic function $f \in \mathcal{O}(\Omega_{c+\delta})$, $\delta > 0$, is the uniform limit on K_c of a sequence of polynomials $P_m \in \mathbb{C}[z_1, \ldots, z_n]$.

Let $\theta \in \mathcal{D}(\Omega_{c+\delta})$ be cut-off function equal to 1 on $K_{c+\delta/2}$ with support in $K_{c+3\delta/4}$. One solves the equation $\overline{\partial} u = v$ on $\mathbb{C}^n$, where $v = \overline{\partial}(\theta f)$ is extended by 0 in the complement of $\Omega_{c+\delta}$. For this, one works in the L^2 space of $(0,q)$-forms with respect to the weight function $\varphi_m(z) = m\psi(z) + |z|^2$ and the standard Hermitian metric $\omega = i\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j$.

(a) Show that $\displaystyle\int_{\mathbb{C}^n} |v|^2 e^{-\varphi_m} d\lambda \le C_1 \exp(-m(c+\delta/2))$

for some constant $C_1 \ge 0$ (where $d\lambda$ = Lebesgue measure).

We have $v = \overline{\partial}(\theta f) = (\overline{\partial}\theta) f + \theta(\overline{\partial} f) = (\overline{\partial}\theta) f$, hence $v = 0$ on $\Omega_{c+\delta/2}$ (where $\theta = 1$) as well as on $\mathbb{C}^n \smallsetminus K_{c+3\delta/4}$ (where $\theta = 0$), i.e. $\mathrm{Supp}(v) \subset K_{c+3\delta/4} \smallsetminus \Omega_{c+\delta/2} = \{c+\delta/2 \le \psi \le c+3\delta/4\}$. On this set $e^{-m\psi} \ge \exp(-m(c+\delta/2))$, thus

$$\int_{\mathbb{C}^n} |v|^2 e^{-\varphi_m} d\lambda = \int_{K_{c+3\delta/4}\smallsetminus\Omega_{c+\delta/2}} |v(z)|^2 e^{-m\psi(z)-|z|^2} d\lambda(z) \le C_1 \exp(-m(c+\delta/2))$$

with $C_1 = \int_{K_{c+3\delta/4}\smallsetminus\Omega_{c+\delta/2}} |v(z)|^2 e^{-|z|^2} d\lambda(z) < +\infty$.

(b) Show that the eigenvalues of $i\partial\overline{\partial}\varphi_m$ with respect to ω are at least equal to 1, and derive from the theory of L^2 estimates for $\overline{\partial}$ that there exist solutions of the equations $\overline{\partial} u_m = v$ such that

$$\int_{\mathbb{C}^n} |u_m|^2 e^{-\varphi_m} d\lambda \le C_2 \exp(-m(c+\delta/2)).$$

Hint. On $\mathbb{C}^n$, solving $\overline{\partial}$ for $(0,q)$-forms is the same as solving $\overline{\partial}$ for (n,q)-forms.

As $i\partial\overline{\partial}\psi \ge 0$, we find $i\partial\overline{\partial}\varphi_m \ge i\partial\overline{\partial}|z|^2 = \omega$. This implies that the eigenvalues $\lambda_1 \le \ldots \le \lambda_n$ of $i\partial\overline{\partial}\varphi_m$ with respect to ω satisfy $\lambda_j \ge 1$. Now, the theory of L^2 estimates imply that the equation $\overline{\partial} u = v$ in the space of (n,q)-forms can be solved with a solution $u = u_m$ of bidegree $(n, q-1)$ on $\mathbb{C}^n$, such that

$$(\dagger) \qquad \int_{\mathbb{C}^n} |u_m|^2 e^{-\varphi_m} d\lambda \le \int_{\mathbb{C}^n} \frac{1}{\lambda_1 + \cdots + \lambda_q} |v|^2 e^{-\varphi_m} d\lambda \le \frac{1}{q} \int_{\mathbb{C}^n} |v|^2 e^{-\varphi_m} d\lambda.$$

Here our $v = \overline{\partial}(\theta f)$ is a $(0,1)$-form. Since the bundle $\Lambda^n T^*_{\mathbb{C}^n}$ is trivial an equipped with the trivial metric ω, we can "convert it" to $(n,1)$-form simply by multiplying by $dz_1 \wedge \ldots \wedge dz_n$ (the L^2 norm being unchanged). Thus, for $q = 1$, we actually get a solution u_m as a $(0,1)$-form satisfying

$$\int_{\mathbb{C}^n} |u_m|^2 e^{-\varphi_m} d\lambda \le \int_{\mathbb{C}^n} |v|^2 e^{-\varphi_m} d\lambda \le C_1 \exp(-m(c+\delta/2)).$$

(c) Infer from the above that $\displaystyle\int_{K_{c+\delta/4}} |u_m|^2 d\lambda \le C_3 \exp(-m\delta/4)$ and that u_m converges uniformly to 0 on the compact set K_c.

Hint. Use the mean value inequality on balls $B(z,r)$, where $r = d(K_c, \complement K_{c+\delta/4})$.

We notice that $\overline{\partial} u_m = v = 0$ on $\Omega_{c+\delta/2}$ and that $\varphi_m(z) \le m(c+\delta/4) + R^2$ on $K_{c+\Delta/4}$, where $R = \sup|z|$ on that compact set. Therefore

$$\begin{aligned}\int_{K_{c+\delta/4}} |u_m|^2 d\lambda &\le \int_{K_{c+\delta/4}} |u_m|^2 e^{-\varphi_m} e^{\varphi_m} d\lambda \le \exp(m(c+\delta/4)+R^2) \int_{K_{c+\delta/4}} |u_m|^2 e^{-\varphi_m} d\lambda \\ &\le \exp(m(c+\delta/4)+R^2) \cdot C_1 \exp(-m(c+\delta/2)) = C_3 \exp(-m\delta/4).\end{aligned}$$

Let us take $r = d(K_c, \complement K_{c+\delta/4})$. Then, for every point $z \in K_c$, we have $B(z,r) \subset K_{c+\delta/4}$, and the mean value inequality for the plurisubharmonic function $|u_m|^2$ implies

$$|u_m(z)|^2 \le \frac{1}{\pi^n r^{2n}/n!} \int_{B(z,r)} |u_m|^2 d\lambda \le C_4 \exp(-m\delta/2).$$

We conclude from this that u_m converges uniformly to 0 on K_c.

(d) Prove the assertion $(**)$.

Using the solution u_m obtained in (b) and (c), we set $f_m = \theta f - u_m$. Then

$$\overline{\partial} f_m = \overline{\partial}(\theta f) - \overline{\partial} u_m = v - \overline{\partial} u_m = 0 \quad \text{on } \mathbb{C}^n,$$

thus $f_m \in \mathcal{O}(\mathbb{C}^n)$. Also, by (c), $f_{m|K_c} = f - u_m$ converges uniformly to f on K_c. If we replace f_m by its truncated Taylor series (centererd at 0) of sufficiently large degree, we obtain a polynomial $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\sup_{K_c} |f_m - P_m| \le 2^{-m}$, and infer that P_m converges uniformly to f on K_c.

3. A compact set $K \subset \mathbb{C}^n$ is said to be polynomially convex if $\widehat{K} = K$. In the remainder of this exercise, K is supposed to be a polynomially convex compact set.

(a) Show that the function $\chi(t) = t\, e^{-1/t}$ for $t > 0$, $\chi(t) = 0$ for $t \le 0$ is a convex function that is strictly convex increasing on $]0, +\infty[$.

On the interval $]0, +\infty[$, one gets

$$\chi'(t) = (1 + t^{-1}) e^{-1/t} > 0, \quad \chi''(t) = t^{-3} e^{-1/t} > 0,$$

and inductively $\chi^{(k)}(t) = P_k(1/t) e^{-1/t}$ where P_k is a polynomial of degree $2k - 1$. All derivatives tend to 0 as $t \to 0_+$, hence χ defines a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ that is strictly convex increasing on $]0, +\infty[$ (and convex non decreasing on $\mathbb{R}$).

(b) Let $a \notin K$. Show that one can choose a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ and real numbers $c \in \mathbb{R}$, $\varepsilon > 0$ such that the plurisubharmonic function $u_a(z) = \chi(|P(z)|^2 + \varepsilon |z|^2 - c)$ is identically 0 on K, and satisfies $u_a(a) > 0$, while $i\partial\overline{\partial} u_a$ is positive definite on an open neighborhood V_a of a, $V_a \subset \mathbb{C}^n \smallsetminus K$.

By definition of the polynomial hull, for $a \notin K$, there exists a polynomial P such that $|P(a)| > \sup_K |P|$. Fix $\delta > 0$ such that $|P(a)|^2 \ge \sup_K |P|^2 + 2\delta$ and $R = \sup_K |z|$. Then for $\varepsilon \le \delta / R^2$ and $z \in K$ we find $\varepsilon |z|^2 \le \varepsilon R^2 \le \delta$, hence

$$|P(a)|^2 + \varepsilon |a|^2 \ge |P(a)|^2 > |P(a)|^2 - \delta \ge \sup_K (|P(z)|^2 + \delta) \ge \sup_K (|P(z)|^2 + \varepsilon |z|^2).$$

After subtracting $c = |P(a)|^2 - \delta$ we obtain

$$|P(a)|^2 + \varepsilon |a|^2 - c \ge \delta > 0 \ge \sup_K (|P(z)|^2 + \varepsilon |z|^2 - c).$$

Therefore the function $u_a(z) = \chi(|P(z)|^2 + \varepsilon |z|^2 - c)$ satisfies $u_a(a) > 0$ and $u_a = 0$ on K. Clearly, $u_a \ge 0$ is plurisubharmonic on $\mathbb{C}^n$ as a composition of a convex non decreasing function χ with a plurisubharmonic function $\tau(z) = |P(z)|^2 + \varepsilon |z|^2 - c$. Also, by the convexity of χ, we get

$$i\partial\overline{\partial} u_a \ge i\partial\overline{\partial} \chi(\tau(z)) = \chi'(\tau(z))\, i\partial\overline{\partial}\tau + \chi''(\tau(z))\, i\partial\tau \wedge \overline{\partial}\tau \ge \chi'(\tau(z))\, i\partial\overline{\partial}\tau \ge \varepsilon \chi'(\tau(z))\, i\partial\overline{\partial} |z|^2,$$

and we see that $u_a > 0$ and $i\partial\overline{\partial} u_a > 0$ on a small open neighborhood V_a of a contained in $\mathbb{C}^n \smallsetminus K$.

(c) Using a covering of $\mathbb{C}^n \smallsetminus K$ by countably many open sets V_{a_p}, $p \in \mathbb{N}$, show that one can produce a smooth plurisubharmonic function given by a convergent series $\psi = \sum_{p \in \mathbb{N}} \eta_p u_{a_p}$, $\eta_p = \mathrm{Const} > 0$, such that $\psi = 0$ on K, $\psi > 0$ strictly plurisubharmonic on $\mathbb{C}^n \smallsetminus K$, and finally $\psi(z) \ge \varepsilon' |z|^2 - C$ on $\mathbb{C}^n$.

$\mathbb{C}^n \smallsetminus K$ is a countable union of compact sets L_ν, and each L_ν is covered by finitely many neighborhoods V_a. This implies that we can cover $\mathbb{C}^n \smallsetminus K$ by countably many open neighborhoods V_{a_p}, $p \in \mathbb{N}$. Now, we can always find a (decreasing) sequence $\eta_p > 0$ such that the series

$$\psi = \sum_{p \in \mathbb{N}} \eta_p u_{a_p}$$

converges smoothly in $C^\infty(\mathbb{C}^n)$. For instance, one can take

$$\eta_p < 2^{-p} \Big(\sup_{|z| \le p, |\beta| \le p} |D^\beta u_{a_p}(z)| \Big)^{-1}$$

to achieve the uniform convergence of all series $\sum \eta_p |D^\beta u_{a_p}|$ on all compact subsets of $\mathbb{C}^n$. By construction, ψ is smooth plurisubharmonic on $\mathbb{C}^n$, while $\psi > 0$ and $i\partial\overline{\partial}\psi > 0$ on $\bigcup V_{a_p} = \mathbb{C}^n \smallsetminus K$. Moreover, we have $\psi \geq \eta_0 u_{a_0}$, and since $\chi(t) \geq \chi(1) + \chi'(1)(t-1) = 2e^{-1}t - C_0$ for all $t \in \mathbb{R}$, we infer

$$\psi(z) \geq \eta_0 \Big(2e^{-1}(|P_0(z)|^2 + \varepsilon_0 |z|^2 - c_0) - C_0\Big) \geq \varepsilon' |z|^2 - C \quad \text{on } \mathbb{C}^n.$$

Therefore ψ is a smooth plurisubharmonic exhaustion of $\mathbb{C}^n$.

(d) Using exercise 2, show that for every holomorphic function $f \in \mathcal{O}(\Omega)$, where Ω is an open neighborhood of K, there exists a sequence of polynomials $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ converging uniformly to f on K.

By construction we have $K = K_c = \{z \in \mathbb{C}^n \,/\, \psi(z) \leq c\}$ for $c = 0$. Since $\bigcap K_{1/\nu} = K_0 = K$, there exists $\delta = 1/\nu$ such that $\Omega_\delta = \Omega_{1/\nu} \subset K_{1/\nu} \subset \Omega$. Exercise 2 implies that f is a uniform limit of polynomials $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ on $K = K_0$.

(e) In dimension 1 (i.e. in $\mathbb{C}$), is it possible to approximate $f(z) = 1/z$ by polynomials on the unit circle ? What is the problem here ?

One cannot have $1/z = \lim P_m(z)$ uniformly on S^1, since $\int_{S^1} P_m(z)dz = 0$, while $\int_{S^1} \frac{1}{z} dz = 2\pi i$. This is of course not a contradiction with (d) since S^1 is not polynomially convex (the polynomial hull of S^1 is the closed unit disc $\overline{D}(0,1)$).

4. Throughout this exercise, X denotes a compact complex manifold. Let Y be a complex submanifold of X and $[Y]$ the current of integration over Y.

(a) If T is another current on X, the product $[Y] \wedge T$ cannot be defined in general, since products of measures do not exist in the calculus of distributions. However, if $T = i\partial\overline{\partial}\varphi$ and φ is a plurisubharmonic function on a coordinate open subset $U \subset X$ such that φ is not identically $-\infty$ on any connected component of $Y \cap U$, show that φ is locally integrable on Y, and infer that one can define

$$[Y] \wedge i\partial\overline{\partial}\varphi := i\partial\overline{\partial}(\varphi[Y]).$$

Moreover, prove that if φ is regularized as a decreasing sequence of smooth plurisubharmonic functions $\varphi_\nu = \varphi * \rho_{1/\nu}$ obtained by convolution, the above definition of current products is compatible with weak limits, in the sense that $[Y] \wedge i\partial\overline{\partial}\varphi = \lim_{\nu \to +\infty} [Y] \wedge i\partial\overline{\partial}\varphi_\nu$ weakly in the space of currents.

Since φ is not identically $-\infty$ on any connected component of $Y \cap U$, we know that $\psi_{|Y\cap U}$ is a plurisuharmonic function on $Y \cap U$, hence it is $L^1_{\rm loc}$ with respect to the Lebesgue measure on Y in any coordinates. This implies that the current $\varphi[Y]$ given by

$$f \mapsto \int_Y f\varphi, \quad \forall f \in \mathcal{D}_{p,p}(Y \cap U),\ p = \dim_{\mathbb{C}} Y,$$

is well defined. Therefore $[Y] \wedge i\partial\overline{\partial}\varphi := i\partial\overline{\partial}(\varphi[Y])$ is also well defined. Actually, we get

$$\langle i\partial\overline{\partial}(\varphi[Y]), f\rangle = \langle \varphi[Y], i\partial\overline{\partial}f\rangle = \int_X \varphi[Y] \wedge i\partial\overline{\partial}f = \int_Y \varphi\, i\partial\overline{\partial}f, \quad \forall f \in \mathcal{D}_{p,p}(Y \cap U).$$

Now, if we take a regularization $\varphi_\nu = \varphi * \rho_{1/\nu}$ by convolution, then φ_ν is monotonically converging to φ and we infer that $\varphi_{\nu|Y\cap U}$ converges to $\varphi_{|Y\cap U}$ in $L^1_{\rm loc}(Y)$. Therefore $\int_Y \varphi_\nu\, i\partial\overline{\partial}f \to \int_Y \varphi\, i\partial\overline{\partial}f$ and we conclude that $i\partial\overline{\partial}(\varphi_\nu[Y])$ converges weakly to $i\partial\overline{\partial}(\varphi[Y])$. In other words, $[Y] \wedge i\partial\overline{\partial}\varphi_\nu$ (which is a “standard” wedge product by a smooth form), converges weakly to $[Y] \wedge i\partial\overline{\partial}\varphi$ (which is possibly “non standard”, as $i\partial\overline{\partial}\varphi$ may have measure coefficients).

(b) Let σ be a holomorphic section of some holomorphic line bundle $L \to X$, equipped with a smooth Hermitian metric h. One says that σ is transverse to Y if at every point $a \in Y \cap \sigma^{-1}(0)$ one has $d\sigma(a) \neq 0$ when the differentiation is made in a trivialization of L near a. Show that the above transversality condition is independent of the choice of the trivialization of L, and assuming transversality, that the product $[Y] \wedge [D]$ of $[Y]$ by the current of integration over the divisor $D = \mathrm{div}(\sigma)$ is well defined. Prove moreover via the Lelong-Poincaré equation that $[Y] \wedge [D]$ and $[Y] \wedge \frac{1}{2\pi}\Theta_{L,h}$ induce the same cohomology classes in De Rham (or Bott-Chern) cohomology.

If we change the trivialization, we get a new expression $\tilde\sigma = g\sigma$ where $g \neq 0$ is the transition automorphism. Then $d\tilde\sigma = (dg)\sigma + g(d\sigma) = g\,d\sigma)$ along $\sigma^{-1}(0)$, thus $d\tilde\sigma(a) \neq 0$ is equivalent to $d\sigma(a) \neq 0$ at any point $a \in Y \cap \sigma^{-1}(0)$. The Lelong-Poincaré equation gives $[D] = \frac{i}{2\pi}\partial\overline\partial \log|\sigma|^2$ (locally) in any local trivialization of $L_{|U}$. By (a), we can define

$$[Y] \cap [D] = [Y] \wedge \frac{i}{2\pi}\partial\overline\partial \log|\sigma|^2 = \frac{i}{2\pi}\partial\overline\partial(\log|\sigma|^2[Y]) : f \mapsto \int_Y \log|\sigma|^2 \frac{i}{2\pi}\partial\overline\partial f,$$

and by the Lelong-Poincaré equation applied again on Y to $\sigma_{|Y}$, we find

$$\int_Y \log|\sigma|^2 \frac{i}{2\pi}\partial\overline\partial f = \int_Y \frac{i}{2\pi}\partial\overline\partial \log|\sigma_{|Y}|^2 \wedge f = \langle [Y \cap D], f\rangle$$

because the zero divisor of $\sigma_{|Y}$ is $Y \cap D$. We see that $[Y] \wedge [D] = [Y \cap D]$. This holds on any trivializing open set U of L, hence this is in fact an equality of currents on the wole of X. Now, if we use a hermitian metric h on L, the generalized Lelong-Poincaré equation gives

$$[D] = \frac{i}{2\pi}\partial\overline\partial \log|\sigma|_h^2 + \frac{i}{2\pi}\Theta_{L,h}.$$

Here $\varphi = \log|\sigma|_h^2$ is not plurisubharmonic, but it differs from the plurisubharmonic function $\log|\sigma|^2$ by the addition of a smooth function, and we can still apply (a). This gives

$$[Y \cap D] = [Y] \wedge [D] = [Y] \wedge \Big(\frac{i}{2\pi}\partial\overline\partial \log|\sigma|_h^2 + \frac{i}{2\pi}\Theta_{L,h}\Big) = \frac{i}{2\pi}\partial\overline\partial(\log|\sigma|_h^2[Y]) + [Y] \wedge \frac{i}{2\pi}\Theta_{L,h}.$$

Since $\frac{i}{2\pi}\partial\overline\partial(\log|\sigma|_h^2[Y])$ is a coboundary, we conclude that the cohomology class of $[Y \cap D]$ is equal to the cohomology class of $[Y] \wedge \frac{i}{2\pi}\Theta_{L,h}$, in the De Rham (or Bott-Chern) cohomology groups computed via complexes of currents.

(c) Let σ_j, $1 \leq j \leq p$, be holomorphic sections of Hermitian line bundles (L_j, h_j) over X. One assumes that the divisors $D_j = \mathrm{div}(\sigma_j)$ are non singular and intersect transversally, in the sense that the differentials $d\sigma_{j_1}(a), \ldots, d\sigma_{j_k}(a)$ are linearly independent at any point $a \in \sigma_{j_1}^{-1}(0) \cap \ldots \cap \sigma_{j_k}^{-1}(0)$. Prove inductively that the wedge product $[D_1] \wedge \ldots \wedge [D_p]$ is well defined, coincides with $[D_1 \cap \ldots \cap D_p]$ and belongs to the cohomology class of $\frac{i}{2\pi}\Theta_{L_1,h_1} \wedge \ldots \wedge \frac{i}{2\pi}\Theta_{L_p,h_p}$.

We proceed by induction on p. For $p = 1$, the result is a direct consequence of the Lelong-Poincaré formula. Assuming the result proved for $p-1$, we consider

$$Y = D_1 \cap \ldots \cap D_{p-1}$$

which is a non singular subvariety of X by the transversality assumption and the implicit function theorem. By the induction hypothesis, we infer that

$$[D_1 \cap \ldots \cap D_p] = [Y \cap D_p] = [Y] \wedge [D_p] = ([D_1] \wedge \ldots \wedge [D_{p-1}]) \wedge [D_p]$$

and that the cohomology class coincides with

$$[Y] \wedge \frac{i}{2\pi}\Theta_{L_p,h_p} = [D_1 \cap \ldots \cap D_{p-1}] \wedge \frac{i}{2\pi}\Theta_{L_p,h_p} = \Big(\frac{i}{2\pi}\Theta_{L_1,h_1} \wedge \ldots \wedge \frac{i}{2\pi}\Theta_{L_{p-1},h_{p-1}}\Big) \wedge \frac{i}{2\pi}\Theta_{L_p,h_p}.$$

(d) (Bézout formula) If D_j, $1 \leq j \leq n$, are non singular algebraic hypersurfaces $\{P_j(z) = 0\}$ of degree $\deg P_j = \delta_j$ intersecting transversally in complex projective space $\mathbb{P}^n$, compute the number of intersection points in $D_1 \cap \ldots \cap D_n$.

Hint. Apply the results of exercise 4 (d,e) in exercise sheet 9 (or reprove them using (c)).

We apply (c) to $L_j = \mathcal{O}_{\mathbb{P}^n}(\delta_j)$ with its standard Fubini-Study metric. Since $\Theta_{L_j,h_j} = \delta_j\omega_{\mathrm{FS}}$, we see that the cohomology class of $[D_1 \cap \ldots \cap D_p]$ equals that of $\delta_1 \ldots \delta_p\, \omega_{\mathrm{FS}}^p$. If we take the $D_1, \ldots, D_n$ to be n coordinate hyperplanes $z_1 = 0, \ldots, z_n = 0$, then $D_1 \cap \ldots \cap D_n$ consists of the single point $a = [1 : 0 : \ldots : 0]$. In that case $\delta_j = 1$ and we get that $\int_{\mathbb{P}^n} \delta_a = 1 = \int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n$. In general, we find

$$\int_{\mathbb{P}^n} [D_1 \cap \ldots \cap D_p] \wedge \omega_{\mathrm{FS}}^{n-p} = \delta_1 \ldots \delta_p \int_{\mathbb{P}^n} \omega_{\mathrm{FS}}^n = \delta_1 \ldots \delta_p,$$

i.e. the "Fubini-Study area" of $D_1\cap\ldots\cap D_p$ is equal to $\delta_1\ldots\delta_p$ (up to a factor $(n-p)!$). In the particular case of the transverse intersection $D_1\cap\ldots\cap D_n$ of n non singular hypersurfaces, we conclude that the number of intersection points is equal to the product $\delta_1\ldots\delta_n$ of the degrees.
Remark: these conclusions are also true in the singular or non transverse case if one simply assumes $\mathrm{codim}\,D_1\cap\ldots\cap D_p=p$ ("complete intersection case"), but it is then necessary to introduce ad hoc multiplicities along the various irreducible components of $D_1\cap\ldots\cap D_p$ to take into account singularities and tangencies of intersection. This actually leads to defining an appropriate "intersection cycle".

5. The goal of this exercise is to investigate the Künneth formula for products $X\times Y$ of compact Kähler manifolds. Results of pure topology imply that for any field $\mathbb{K}$, e.g. $\mathbb{K}=\mathbb{R}$ or $\mathbb{R}=\mathbb{C}$, one has $H^k(X\times Y,\mathbb{K})=\bigoplus_{i+j=k}H^i(X,\mathbb{K})\otimes H^j(Y,\mathbb{K})$ for sheaf cohomology with values in the locally constant sheaf $\mathbb{K}$. In the differential case, $H^k(X,\mathbb{K})$ is known to be isomorphic to De Rham cohomology.

(a) Let (X,g_X) and (Y,g_Y) be Riemannian manifolds and $(X\times Y,g_X\oplus g_Y)$ their product. The exterior derivative obviously splits as $d=d_x+d_y$, and one can observe that d_x anticommutes with d_y and d_y^*. Infer that the Laplace-Beltrami operator Δ of $X\times Y$ is given by $\Delta=\Delta_x+\Delta_y$ where Δ_x, say, means the Laplace-Beltrami operator applied in x to differential forms $u(x,y)$ on $X\times Y$.

A differential form of degree k on $X\times Y$ can be expanded as

$$u(x,y)=\sum_{i+j=k}\ \sum_{|I|=i,\,|J|=j}u_{IJ}(x,y)\,dx_I\wedge dy_J$$

with respect to any systems of coordinates $(x_1,\ldots,x_n)$ on X and $(y_1,\ldots,y_m)$ on Y. Clearly, the exterior differential d can be split as d_x+d_y by grouping together the derivatives in the x_i's (resp. y_j's), i.e.

$$d_xu=\sum_i\Big(dx_i\wedge\frac{\partial}{\partial x_i}\Big)u,\quad d_yu=\sum_j\Big(dy_j\wedge\frac{\partial}{\partial y_j}\Big)u.$$

The rekation $d^2=(d_x+d_y)^2=0$ implies $d_x^2=0$, $d_y^2=0$ and $d_xd_y+d_yd_x=0$ by looking at "bidegrees". Also, the volume element takes the form $\gamma_X(x)\gamma_Y(y)$ where $\gamma_X(x)=(\det g_{X,ij}(x))^{1/2}$, $\gamma_Y(y)=(\det g_{Y,ij}(y))^{1/2}$, and one easily computes that

$$d_x^*u=\Big(\sum_i dx_i\wedge\frac{\partial}{\partial x_j}\Big)^*u=-\gamma_X^{-1}\sum_i\frac{\partial}{\partial x_j}\circ i_{(dx_j)^*}(\gamma_Xu),$$

where $(dx_j)^*$ is the dual vector field to dx_j with respect to g_X, and $i_\bullet$ is the interior product. A similar formula holds for d_y^*. Since these operations do not involve the variable y, it is easy to see that d_x^* commutes with $\partial/\partial y_j$, and since $i_\xi(\alpha\wedge\beta)=i_\xi\alpha\wedge\beta+(-1)^{\deg\alpha}\alpha\wedge i_\xi\beta$, one sees that $i_{(dx_j)^*}$ anticommutes with $dy_j\wedge\bullet$ (because $i_{(dx_j)^*}dy_j=0$). As a result, d_x^* also anticommutes with d_y, i.e. $d_x^*d_y+d_yd_x^*=0$, and likewise $d_xd_y^*+d_y^*d_x=0$. From this we infer that the Laplace-Beltrami operator of $X\times Y$ is

$$\Delta=dd^*+d^*d=(d_x+d_y)(d_x^*+d_y^*)+(d_x^*+d_y^*)(d_x+d_y)=(d_xd_x^*+d_x^*d_x)+(d_yd_y^*+d_y^*d_y)=\Delta_x+\Delta_y.$$

(b) For (X,ω_X) and (Y,ω_Y) compact Kähler, show that

$$H^{p,q}(X\times Y,\mathbb{C})\simeq\bigoplus_{(k,\ell)+(r,s)=(p,q)}H^{k,\ell}(X,\mathbb{C})\otimes H^{r,s}(Y,\mathbb{C}).$$

Hint. Use (a) and Hodge theory to find an injection of $\bigoplus_{(k,\ell)+(r,s)=(p,q)}H^{k,\ell}(X,\mathbb{C})\otimes H^{r,s}(Y,\mathbb{C})$ into $H^{p,q}(X\times Y,\mathbb{C})$, and conclude by Hodge decomposition and comparison of dimensions.

Let $u(x)$, $v(y)$ be differential forms on X and Y. We consider the "tensor product" $u\otimes v$ on $X\times Y$ which, as a form written in coordinates, is just $u(x)\wedge v(y)$. Then

$$\Delta(u(x)\wedge v(y))=(\Delta_x+\Delta_y)(u(x)\wedge v(y))=\Delta_xu(x)\wedge v(y)+u(x)\wedge\Delta_yv(y),$$

since Δ_x (resp Δ_y) only involves differentiations in x (resp. y). Then we see that $u(x)\wedge v(y)$ is harmonic on $X\times Y$ as soon as u is harmonic on X and v is harmonic on Y. By looking at harmonic forms of pure bidegrees, we get a morphism

$$\Phi_{\mathcal{H},p,q}:\bigoplus_{(k,\ell)+(r,s)=(p,q)}\mathcal{H}^{k,\ell}(X,\mathbb{C})\otimes\mathcal{H}^{r,s}(Y,\mathbb{C})\to\mathcal{H}^{p,q}(X\times Y,\mathbb{C}),$$

$$\bigoplus_{(k,\ell)+(r,s)=(p,q)}u^{k,\ell}\otimes v^{r,s}\mapsto\Big((x,y)\mapsto\sum_{(k,\ell)+(r,s)=(p,q)}u^{k,\ell}(x)\wedge v^{r,s}(y)\Big).$$

We claim that $\Phi_{\mathcal{H},p,q}$ is injective – a statement that could appear as "intuitively obvious". This can be checked by an abstract linear algebra argument that has nothing to do with the theory of harmonic forms, and is applicable in general to tensor products of functional spaces. In fact, given vector bundles $E\to X$ and $F\to Y$, we are going to check that

$$\Phi_{\mathcal{F}}:\mathcal{F}(X,E)\otimes\mathcal{F}(Y,F)\to\mathcal{F}(X\times Y,E\boxtimes F)$$

is always injective, where $\mathcal{F}(\bullet)$ denotes the space of all set theoretic maps and $E\boxtimes F\to X\times Y$ is the bundle such that $(E\boxtimes F)_{(x,y)}=E_x\otimes F_y$. Here we need the case where

$$E=\bigoplus_i\Lambda^i_{\mathbb{R}}T^*_X\otimes_{\mathbb{R}}\mathbb{C},\quad F=\bigoplus_j\Lambda^j_{\mathbb{R}}T^*_Y\otimes_{\mathbb{R}}\mathbb{C},\quad E\boxtimes F\simeq\bigoplus_k\Lambda^k_{\mathbb{R}}T^*_{X\times Y}\otimes_{\mathbb{R}}\mathbb{C}$$

are the exterior algebra bundles of $X,Y,X\times Y$. Notice that in general $\Phi_{\mathcal{F}}$ *will not be surjective*, as a consequence of the fact that a tensor product only involves finite combinations of elementary tensors, while X and Y can be infinite. We have to show that a non zero element $w=\sum_{1\le s\le N}u_s\otimes v_s$ is mapped to a non zero element $(x,y)\mapsto\sum_{1\le s\le N}u_s(x)\otimes v_s(y)$ in the image. Observe that $w\in\mathcal{F}_u\otimes\mathcal{F}_v$ where

$$\mathcal{F}_u=\mathrm{Span}(u_1,\ldots,u_N)\subset\mathcal{F}(X,E),\quad\mathcal{F}_v=\mathrm{Span}(v_1,\ldots,v_N)\subset\mathcal{F}(Y,F).$$

We can find finite subsets $A\subset X$, $B\subset Y$ such that the restrictions $(u_{s|A})_{1\le s\le N}$, $(v_{s|B})_{1\le s\le N}$ still have ranks $r_u=\dim\mathcal{F}_u$, $r_v=\dim\mathcal{F}_v$; in other words, if $\rho_A:\mathcal{F}(X,E)\to\mathcal{F}(A,E)$ and $\rho_B:\mathcal{F}(Y,F)\to\mathcal{F}(B,F)$ are the restriction morphisms, then $\rho_{A|\mathcal{F}_u}:\mathcal{F}_u\to\mathcal{F}(A,E)$ and $\rho_{B|\mathcal{F}_v}:\mathcal{F}_v\to\mathcal{F}(B,F)$ are injective. In fact (for A, say), if $\mathrm{Ker}\,\rho_{A|\mathcal{F}_u}\ne0$, we can pick a non zero element $g\in\mathrm{Ker}\,\rho_{A|\mathcal{F}_u}\subset\mathcal{F}_u$ and add to A a point a' at which $g(a')\ne0$ to eliminate g from the kernel; after doing this, the dimension of the kernel has decreased at least by 1, and we can repeat the process until the kernel has become equal to 0. In this circumstance, $\rho_{A|\mathcal{F}_u}\otimes\rho_{B|\mathcal{F}_v}:\mathcal{F}_u\otimes\mathcal{F}_v\to\mathcal{F}(A,E)\otimes\mathcal{F}(B,F)$ is injective, thus $w_{|A\times B}=(\rho_{A|\mathcal{F}_u}\otimes\rho_{B|\mathcal{F}_v})(w)\ne0$. Next, we have an isomorphism

$$\Phi_{\mathcal{F}|A\times B}:\mathcal{F}(A,E)\otimes\mathcal{F}(B,F)\to\mathcal{F}(A\times B,E\boxtimes F),$$

since both sides are isomorphic to $\bigoplus_{x\in A,y\in B}E_x\otimes F_y$, with $\Phi_{\mathcal{F}|A\times B}$ being then the identity map. We infer that $\Phi_{\mathcal{F}}(w)_{|A\times B}\ne0$. Therefore $\Phi_{\mathcal{F}}(w)\ne0$, and we conclude that $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{H},p,q}$ are injective, as desired. However, if we now take the sum of the injections $\Phi_{\mathcal{H},p,q}$ for all pairs (p,q) with $p+q=$ given degree k, we get an equality of dimensions on both sides, by the topological Künneth formula. We conclude that the injections $\Phi_{\mathcal{H},p,q}$ must be isomorphisms, thus each

$$\Phi_{\overline{\partial},p,q}:\bigoplus_{(k,\ell)+(r,s)=(p,q)}H^{k,\ell}(X,\mathbb{C})\otimes H^{r,s}(Y,\mathbb{C})\xrightarrow{\simeq}H^{p,q}(X\times Y,\mathbb{C})$$

is an isomorphism.
Remark: the isomorphism $\Phi_{\overline{\partial},p,q}$ can be seen to be independent of the choice of Kähler metrics, since it is defined by taking wedge products of $\overline{\partial}$-closed forms on X and Y, without further requirements.

(c) Show that there are infinitely many non diffeomorphic connected compact Kähler manifolds in any dimension n.

Let C_g be a curve of genus g, so that $\dim H^1(C_g,\mathbb{C})=2g$. If we take $X_g=C_g\times(\mathbb{P}^1)^{n-1}$, then $H^1(\mathbb{P}^1,\mathbb{C})=0$ gives

$$H^1(X_g,\mathbb{C})\simeq H^1(C_g,\mathbb{C})\simeq\mathbb{C}^{2g}.$$

This implies that the n-dimensional manifolds X_g are pairwise non diffeomorphic. Since $H^1(\bullet,\mathbb{C})$ is topological, they are even non homeomorphic.