Introduction to analytic geometry (course by Jean-Pierre Demailly)
Partial examination, 07/11/2019, 13:00 – 17:00

Any use of electronic devices is forbidden
Handwritten documents and lecture notes are allowed, printed documents are not (except exercise sheets)
Answers to questions can be developed in French or English
Justifications of more or less trivial questions need not be very long !
The four problems are independent.

1. (a) Let $\Omega \subset \mathbb{C}^n$ be an open set, let $E = \{w_k\}_{k\in\mathbb{N}} \subset \Omega$ be an infinite discrete (i.e. locally finite) subset of Ω. By looking at the sets $E_N = \{w_k\}_{k\ge N}$ and the associated ideals, show that the "global" ring $R = \mathcal{O}(\Omega)$ is never Noetherian.

Let I_N be the ideal of functions $f \in \mathcal{O}(\Omega)$ such that f vanishes at every point of E_N. In dimension 1, a well known theorem of Weierstrass states that there exists a holomorphic function vanishing with prescribed multiplicities (say 1) at all points of E_N and nowhere else, hence the sequence of ideals I_N is strictly increasing and Noetherianity is contradicted. In dimension n, we can reduce ourselves to dimension 1 by considering the first projection $\Omega_1 = p_1(\Omega)$ with $p_1 : \mathbb{C}^n \to \mathbb{C}$, and exploiting the idea of the proof of the Weierstrass theorem. If $\Omega_1 = \mathbb{C}$, we can take inductively a sequence $w_k \in \Omega$ with $w_{k,1} = p_1(w_k) \to \infty$, for instance with $|w_{0,1}| \ge 1$ and $|w_{k,1}| \ge 2|w_{k-1,1}|$, so that $|w_{k,1}| \ge 2^k$. Then the infinite product

$$f_N(z) = f_N(z_1) = \prod_{k\ge N} \Big(1 - \frac{z_1}{w_{k,1}}\Big)$$

is uniformly convergent on every compact subset, and we have $f_N \in I_N$ and $f_N \notin I_{N-1}$ for $N \ge 1$. If $\Omega_1 \ne \mathbb{C}$, we can pick a boundary point $a \in \partial\Omega_1$ and a sequence of points $\Omega_1 \ni w_{k,1} = p_1(w_k) \to a$ with $w_k \in \Omega$, e.g. satisfying $|w_{0,1} - a| \le 1$ and $|w_{k,1} - a| \le \frac{1}{2}|w_{k-1,1} - a|$, so that $|w_{k,1} - a| \le 2^{-k}$. Then the infinite product

$$f_N(z) = f_N(z_1) = \prod_{k\ge N} \frac{z_1 - w_{k,1}}{z_1 - a} = \prod_{k\ge N} \Big(1 - \frac{w_{k,1} - a}{z_1 - a}\Big)$$

converges on compact subsets of Ω (as $z_1 - a \ne 0$ there), and so defines a function $f_N \in I_N \smallsetminus I_{N-1}$.

(b) Let S be a complex analytic submanifold of Ω. If S is compact, prove that S is in fact of dimension 0 and consists of a finite set of points.
Hint. Apply the maximum principle to the coordinate functions.

Indeed, S can have only finitely many connected components S_j. As S_j is also compact, any coordinate function z_ℓ reaches the maximum of its absolute value at some point of S_j and must therefore be constant. This implies that S_j is a singleton.

(c) Show by an example that result (b) does not hold if one replaces "complex analytic submanifold" by real analytic (or even real algebraic) submanifold.

In fact, any sphere $\sum(x_j - a_j)^2 = \varepsilon^2$ defines a $(n-1)$-dimensional compact real algebraic manifold, and is contained in Ω if $a \in \Omega$ and $\varepsilon > 0$ is small.

2. Let A be a germ of analytic set of pure dimension k in a neighborhood of 0 in $\mathbb{C}^n$. One assumes that coordinates $z = (z', z'')$, $z' = (z_1, \ldots, z_k)$, $z'' = (z_{k+1}, \ldots, z_n)$ have been chosen, and well as a small polydisk $D = D' \times D''$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$, such that the projection $p : A \cap D \to D'$, $z \mapsto z'$ defines a ramified covering, with ramification locus $\Sigma \subset D'$. One denotes by d the degree of the covering, and by $z'' = w_j(z')$, $1 \le j \le d$, the branches of A in a neighborhood of every point $z'_0 \in D' \smallsetminus \Sigma$.

(a) If $f \in \mathcal{O}(D)$ is a holomorphic function, show that

$$Q_f(z') = \prod_{(z',z'')\in A} f(z', z'') = \prod_{j=1}^{d} f(z', w_j(z'))$$

is holomorphic on $D' \smallsetminus \Sigma$ and actually extends holomorphically to D'.

The holomorphicity comes from the holomorphicity of the w_j's, and by a theorem explained in the course, the extension property comes from the fact that Q_f is locally bounded when z' approaches a point $a' \in \Sigma$.

(b) Let $B \subset A$ be an analytic subset defined as $B = \{z \in A / g_1(z) = \ldots = g_N(z) = 0\}$ with $g_j \in \mathcal{O}(D)$. The goal is to show that $p(B)$ is an analytic subset of D'. For $t = (t_1, \ldots, t_N) \in \mathbb{C}^N$, define

$$h(z', t) = Q_{t_1 g_1 + \cdots + t_N g_N}(z').$$

Show that h defines a holomorphic function on $D' \times \mathbb{C}^N$ that is a homogeneous degree d polynomial in $t = (t_1, \ldots, t_N)$. If $h(z', t) = \sum h_\alpha(z') t^\alpha$, show that $p(B)$ is the common zero set of the functions h_α. *Hint.* For any finite set of non zero points in $\mathbb{C}^N$, one can find a linear form that does not vanish on any of them ...

Clearly, $(z', t) \mapsto h(z', t)$ is holomorphic for $(z', t) \in (D' \smallsetminus \Sigma) \times \mathbb{C}^n$, and is a homogeneous polynomial of degree d in t there, i.e.

$$h(z', t) = \sum h_\alpha(z') t^\alpha, \quad h_\alpha \in \mathcal{O}(D' \smallsetminus \Sigma).$$

As h is locally bounded near every point of $\Sigma \times \mathbb{C}^N$, it must extend to $D' \times \mathbb{C}^N$ and is still a homogeneous polynomial there, by applying a continuity argument for the coefficients h_α (we know that $D' \smallsetminus \Sigma$ is dense in D'). The hint is obtained e.g. by saying that the union of finitely many hyperplanes cannot be the whole space in a complex vector space (e.g. by Baire – this is not true for vector spaces over a finite field!). Now, if $z' \notin p(B)$, the preimages are $(z', w_j(z'))$, $1 \le j \le d$ (where some of the roots may coincide if $z' \in \Sigma$), and for every $j = 1, \ldots, d$, the N-tuple $(g_k(z', w_j(z')))_{1 \le k \le N}$ is not zero. We can therefore find a linear form $\mathbb{C}^N \ni (\xi_k) \mapsto \sum t_k \xi_k$ such that $\sum t_k g_k(z', w_j(z')) \ne 0$ for all j, so that $h(z', t) \ne 0$. This implies that some coefficient $h_\alpha(z')$ is non zero. On the other hand, if $z' \in p(B)$, there exists j such that $(z', w_j(z')) \in B$, and so $\sum_{1 \le k \le N} t_k g_k(z', w_j(z')) = 0$; this implies that $t \mapsto h(z', t)$ vanishes identically, thus $h_\alpha(z') = 0$ for all α. We conclude that $p(B)$ is equal to the common zero set of the h_α's, hence that $p(B)$ is analytic in D'.

(c) Let H be the hyperbola $z_1 z_2 = 1$ in $\mathbb{C}^2$ and $p : \mathbb{C}^2 \to \mathbb{C}$ be the first projection $p : (z_1, z_2) \mapsto z_1$. Is $p(H)$ analytic in $\mathbb{C}$? What happens ?

In that case $p(H) = \mathbb{C}^*$ is not analytic in $\mathbb{C}$ as it is not even closed (it turns out to be the complement of an analytic set!). There is no contradiction, since $p : H \to \mathbb{C}$ is not a ramified covering.

(d) Let $B \subset \mathbb{C}^n$ be a compact complex analytic set. The goal is to show that B is finite and of dimension 0 (thus generalizing the result of 1 b). One argues by induction on n, letting $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection to the first $n-1$ coordinates.
– case $n = 1$.
– for $n \ge 2$, show that the fibers of $p_{|B}$ must be finite.
– then show that $p(B)$ is analytic in $\mathbb{C}^{n-1}$ and conclude.
Hint. For any point $a' \in p(B)$ with $p^{-1}(a') \cap B = \{(a', a''_j) \in \mathbb{C}^{n-1} \times \mathbb{C} \,/\, 1 \le j \le m\}$, construct a neighborhood V' of a' and polynomials $P_{j,k}(z', z_n) \in \mathcal{O}(V')[z_n]$ so that $B_{V'} := B \cap p^{-1}(V')$ is the union of

$$B_{V',j} := B \cap (V' \times D(a''_j, \varepsilon)) = \{(z', z_n) \in V' \times D(a''_j, \varepsilon) \,/\, P_{j,k}(z', z_n) = 0,\ 1 \le k \le N_j\}, \quad 1 \le j \le m$$

and apply (b) with e.g. $B_{V'} \subset A := \{\prod_j P_{j,0}(z', z_n) = 0\}$ to infer that $p(B) \cap V'$ is analytic.

– In dimension 1, an infinite compact set must have an accumulation point, but zeroes of holomorphic functions cannot have accumulation points. This implies that B must be finite.

– Assume now that $n \ge 2$. The fibers of $p_{|B}$ can be seen as an intersection of analytic sets $B \cap (\{z'\} \times \mathbb{C})$ in dimension 1. Their compactness implies that they are finite (possibly empty).

– Let $a' \in p(B)$. Then we now that there are only finitely many points (a', a''_j), $1 \le j \le m$ in the fiber. Let $\varepsilon > 0$ be so small that the disks $\overline{D}(a''_j, \varepsilon)$ are disjoint. There exists $r > 0$ so that

$$B \cap p^{-1}(B(a', r)) \ \subset\ \bigcup_j B(a', r) \times D(a''_j, \varepsilon),$$

otherwise $k \mapsto B \cap p^{-1}(\overline{B}(a', 2^{-k})) \cap (\mathbb{C}^{n-1} \smallsetminus \bigcup_j D(a''_j, \varepsilon))$ would be a decreasing sequence of non empty closed sets with empty intersection. Since we can take ε and r arbitrarily small, we can assume that $B \cap (B(a', r) \times D(a''_j, \varepsilon))$ is defined by a finite collection of holomorphic equations $g_{j,k}(z) = 0$

in $B(a',r)\times D(a''_j,\varepsilon)$, $1\le k\le N_j$, and since the fiber $B\cap p^{-1}(a')$ is finite, we can always arrange that $z_n\mapsto g_{j,k}(a',z_n)$ is not identically zero (eventually by adding one of the functions to those which do not have this property). By the Weierstrass preparation theorem, we can assume that $g_{j,k}$ is a polynomial $P_{j,k}(z',z_n)\in\mathcal{O}(B(a',r))[z_n]$, possibly after shrinking r and ε, all zeroes of $z_n\mapsto P_{j,k}(z',z_n)$ being contained in $D(a''_j,\varepsilon)$. We take $V'=B(a',r)$ and $A:=\{\prod_j P_{j,0}(z',z_n)=0\}$ in $V'\times D(0,R)$ (R very large). Then $B_{V'}$ is defined in A by the finite set of equations $P_{1k_1}P_{2k_2}\ldots P_{mk_m}=0$ where $1\le k_j\le N_j$, and we can apply (b) to conclude that $p(B)\cap V'=p(B_{V'})$ is analytic in V'. This implies that $p(B)$ is analytic in $\mathbb{C}^{n-1}$, and by induction that $p(B)$ is finite. As the fibers are finite, we conclude that B itself is finite.

3. Let $\alpha\in\,]0,1[$. One defines the *Hopf surface* X_α to be the quotient $(\mathbb{C}^2\smallsetminus\{(0,0)\})/\Gamma$ by the discrete group $\Gamma\simeq\mathbb{Z}$ of homotheties h_α^k, $k\in\mathbb{Z}$, with $h_\alpha(z)=\alpha z$.

(a) Prove that X_α is C^∞ (or even C^ω)-diffeomorphic to the product of spheres $S^1\times S^3$ via the map

$$z\mapsto\Big(\exp(2\pi\mathrm{i}\log|z|/\log\alpha)\ ,\ z/|z|\Big)\in S^1\times S^3.$$

We have here $h_\alpha^k(z)=\alpha^k z$. Clearly, the above map f is real analytic on $\mathbb{C}^2\smallsetminus\{(0,0)\}$ as a composition of real analytic functions, and passes to the quotient on X_α. Indeed, if we replace z by αz, then $\log|z|/\log\alpha$ is replaced by $1+\log|z|/\log\alpha$ and $\exp(2\pi\mathrm{i}\log|z|/\log\alpha)$ does not change; the quotient $z/|z|$ is not changed either. Now, one easily sees that f is bijective and has an inverse bijection $g=f^{-1}$ given by

$$g(t,u)=\exp(\mathrm{Arg}(t)\times\log\alpha)/2\pi)\,u\in(\mathbb{C}^2\smallsetminus\{(0,0)\})/\Gamma.$$

This is again well defined since a change of $\mathrm{Arg}(t)$ into $\mathrm{Arg}(t)+2k\pi$ multiplies the image by α^k, so that we still get the same point in $(\mathbb{C}^2\smallsetminus\{(0,0)\})/\Gamma$. The above formula shows that g is real analytic on $S^1\times S^3$.

(b) Show that X_α can be equipped with the structure of a complex analytic surface, and give explicitly an atlas consisting of 2 open sets in $\mathbb{C}^2$.

One can take for instance $U_0=\{z\in\mathbb{C}^2\,/\ \alpha<|z|<1\}$ and $U_1=\{z\in\mathbb{C}^2\,/\ \alpha^{3/2}<|z|<\alpha^{1/2}\}$ and Ω_0, Ω_1 their respective images in X_α. By definition of the quotient topology, these are open sets homeomorphic to U_0, U_1, and the charts $\tau_j:\Omega_j\to U_j$ are defined by assigning to every class $\dot z\in\Omega_j$ its unique representative $z\in U_j$. It is more or less obvious that $\Omega_0\cap\Omega_1$ consists of all points $\dot z$ such that $|z|\notin\{\alpha^k,\alpha^{k+1/2},k\in\mathbb{Z}\}$, with transition maps

$$\tau_{01}:\tau_1(\Omega_0\cap\Omega_1)\to\tau_0(\Omega_0\cap\Omega_1),\qquad\begin{cases}z\mapsto\alpha^{-1}z & \text{on } \alpha^{3/2}<|z|<\alpha\\ z\mapsto z & \text{on } \alpha<|z|<\alpha^{1/2}\end{cases}$$

$$\tau_{10}:\tau_0(\Omega_0\cap\Omega_1)\to\tau_1(\Omega_0\cap\Omega_1),\qquad\begin{cases}z\mapsto z & \text{on } \alpha<|z|<\alpha^{1/2}\\ z\mapsto\alpha z & \text{on } \alpha^{1/2}<|z|<1.\end{cases}$$

This is a holomorphic atlas in complex dimension 2.

(c) Check that

$$\omega(z)=\frac{\mathrm{i}}{|z|^2}\,\partial\overline\partial\,|z|^2$$

defines a hermitian metric on X_α that is not a Kähler metric, but show however that $\partial\overline\partial\,\omega=0$.

Since $h_\alpha^*|z|^2=\alpha^2|z|^2$ and $h_\alpha^*\,\partial\overline\partial\,|z|^2=\alpha^2\,\partial\overline\partial\,|z|^2$, we see that $h_\alpha^*\omega=\omega$. Therefore ω passes to the quotient and defines a $(1,1)$-form on X_α. As $\mathrm{i}\,\partial\overline\partial\,|z|^2=\mathrm{i}(dz_1\wedge d\overline z_1+dz_2\wedge d\overline z_2)$ is positive definite on $\mathbb{C}^2\smallsetminus\{(0,0)\}$, we conclude that ω is a positive definite hermitian metric on X_α. A calculation gives

$$\begin{aligned}\overline\partial\,\omega&=\frac{-\mathrm{i}}{|z|^4}\,\overline\partial\,|z|^2\wedge\partial\overline\partial\,|z|^2=\frac{-\mathrm{i}}{|z|^4}(z_1d\overline z_1+z_2d\overline z_2)\wedge(dz_1\wedge d\overline z_1+dz_2\wedge d\overline z_2)\\&=\frac{-\mathrm{i}}{|z|^4}\big(z_1d\overline z_1\wedge dz_2\wedge d\overline z_2+z_2d\overline z_2\wedge dz_1\wedge d\overline z_1\big)\neq0,\quad\text{thus }\omega\text{ is not Kähler,}\\ \partial\overline\partial\,\omega&=-\frac{\mathrm{i}}{|z|^4}\partial\big(z_1d\overline z_1\wedge dz_2\wedge d\overline z_2+z_2d\overline z_2\wedge dz_1\wedge d\overline z_1\big)+\frac{2\mathrm{i}}{|z|^6}\partial|z|^2\wedge\big(z_1d\overline z_1\wedge dz_2\wedge d\overline z_2+z_2d\overline z_2\wedge dz_1\wedge d\overline z_1\big)\\&=-\frac{\mathrm{i}}{|z|^4}\cdot2dz_1\wedge d\overline z_1\wedge dz_2\wedge d\overline z_2+\frac{2\mathrm{i}}{|z|^6}(\overline z_1dz_1+\overline z_2dz_2)\wedge\big(z_1d\overline z_1\wedge dz_2\wedge d\overline z_2+z_2d\overline z_2\wedge dz_1\wedge d\overline z_1\big)\\&=-\frac{\mathrm{i}}{|z|^4}\cdot2dz_1\wedge d\overline z_1\wedge dz_2\wedge d\overline z_2+\frac{2\mathrm{i}}{|z|^6}(|z_1|^2+|z_2|^2)\,dz_1\wedge d\overline z_1\wedge dz_2\wedge d\overline z_2=0.\end{aligned}$$

(d) If $P(z), Q(z)$ are homogeneous polynomials of degree d on $\mathbb{C}^2$ without common zeroes, show that $\Phi(z) = (P(z), Q(z))$ defines a holomorphic morphism $\varphi : X_\alpha \to X_\beta$ for certain values of β (which ones ?).

If $\Phi : \mathbb{C}^2 \smallsetminus \{(0,0)\} \to \mathbb{C}^2 \smallsetminus \{(0,0)\}$ descends to a map $\varphi : X_\alpha \to X_\beta$, we must have by definition $\Phi(\alpha z) = \beta^{k(z)}\Phi(z)$ for some $k(z) \in \mathbb{Z}$. In fact, the connectedness of $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ and the continuity of $k(z) = (\log\beta)^{-1}\log|\Phi(\alpha z)|/\log|\Phi(z)|$ imply that $k(z)$ is a constant $k \in \mathbb{Z}$. Now, if $\Phi = (P, Q)$ with P, Q homogeneous of degree d, we have $\Phi(\alpha z) = \alpha^d\Phi(z)$ for every $z \in \mathbb{C}^2 \smallsetminus \{(0,0)\}$, i.e. $\alpha^d = \beta^k$. If $d = 0$, Φ is a constant, we have $k = 0$ and there is no constraint on β. If $d > 0$, then $\alpha^d < 1$ and thus $k > 0$ as well. Therefore β must be of the form $\alpha^{d/k}$, i.e. $\log\beta/\log\alpha = d/k \in \mathbb{Q}_+^*$. Conversely, if $\log\beta/\log\alpha = d/k \in \mathbb{Q}_+^*$, such polynomial maps exist, e.g. $\Phi(z_1, z_2) = (z_1^d, z_2^d)$.

(e) Observing that the universal cover of X_α is $\mathbb{C}^2 \smallsetminus \{0\}$, conclude that any non constant holomorphic morphism $\varphi : X_\alpha \to X_\beta$ lifts to a holomorphic map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Phi(\alpha z) = \beta^p\Phi(z)$ for some $p \in \mathbb{N}^*$. Infer that a necessary and sufficient condition for the existence of such morphisms is $\log\beta/\log\alpha \in \mathbb{Q}_+^*$, and that Φ must be homogeneous of some degree.

As $\mathbb{C}^2 \smallsetminus \{0\}$ is simply connected, the composition

$$\mathbb{C}^2 \smallsetminus \{0\} \longrightarrow X_\alpha \xrightarrow{\varphi} X_\beta = (\mathbb{C}^2 \smallsetminus \{0\})/\Gamma$$

lifts to a holomorphic map $\Phi : \mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{C}^2 \smallsetminus \{0\}$. The Riemann-Hartogs extension theorem implies that Φ extends to a holomorphic map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$. Also, the discussion made in (d) implies that $\Phi(\alpha z) = \beta^k\Phi(z)$ for some $k \in \mathbb{Z}$. Let $\Phi(z) = \sum_{p\in\mathbb{N}^2} a_p z^p$ ($a_p \in \mathbb{C}^2$) be the Taylor expansion of Φ. An identification of coefficients in the relation $\Phi(\alpha z) = \beta^k\Phi(z)$ leads to the conclusion that we can only have monomials of degree $d = |p| = p_1 + p_2$ such that $\alpha^d = \beta^k$, i.e. $d = k\log\beta/\log\alpha$ and Φ must be of the form $\Phi = (P, Q)$ with homogeneous polynomials of degree d. Since φ is assumed to be non constant, this is possible only with $d, k > 0$ and $\log\beta/\log\alpha = d/k \in \mathbb{Q}_+^*$.

(f) Give an example of a pair of non homeomorphic *compact* complex surfaces, resp. of a pair of homeomorphic (and even diffeomorphic) but non biholomorphic ones.

Examples on non homeomorphic compact complex surfaces are for instance $X_{1/2}$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$ (Y is simply connected and $X_{1/2}$ is not). Now, we have seen that $X_{1/2}$ and $X_{1/3}$ are both C^ω-diffeomorphic to $S^1 \times S^3$, but non biholomorphic since $\log 3/\log 2 \notin \mathbb{Q}_+^*$ (there are no relations of the form $3^k = 2^d$ with $d, k > 0$).

4. Except for (g) below, let $X = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus, considered as a C^∞ manifold.

(a) Show that C^∞ differential forms of degree p on X can be interpreted as forms $u(x) = \sum_{|I|=p} u_I(x)dx_I$ on $\mathbb{R}^n$ (the summation is over length p increasing multi-indices), where each function u_I satisfies a certain periodicity condition.

Pulling back any form u on X by the quotient maps $\mathbb{R}^n \to X = \mathbb{R}^n/\mathbb{Z}^n$ leads to interpreting u as a differential form on $\mathbb{R}^n$ with coefficients u_I that are $\mathbb{Z}^n$-periodic functions.

(b) Infer from (a) that forms with constant coefficients give rise to a natural ring morphism $\varphi : \Lambda^p(\mathbb{R}^n)^* \to H^p_{\mathrm{DR}}(X, \mathbb{R})$.

If we take the coefficients u_I to be constant, then one trivially gets $du = 0$, and thus we get a morphism

$$\varphi : \Lambda^p(\mathbb{R}^n)^* \to H^p_{\mathrm{DR}}(X, \mathbb{R}), \quad u \mapsto \{u\}$$

where $\{u\}$ is seen as the cohomology class of the corresponding constant-coefficient form on $\mathbb{R}^n/\mathbb{Z}^n$. By definition, this is trivially a ring morphism.

(c) For $p + q = n$, one defines a bilinear map

$$H^p_{\mathrm{DR}}(X, \mathbb{R}) \times H^q_{\mathrm{DR}}(X, \mathbb{R}) \longrightarrow \mathbb{R}, \quad (\{u\}, \{v\}) \mapsto \int_X u \wedge v$$

where X is given the usual orientation of $\mathbb{R}^n$, and $\{u\}$ denotes the cohomology class of a d-closed form u. Show that the above bilinear form is well defined.

We have to show that $\int_X u \wedge v$ does not change when the representatives u, v of our classes $\{u\}$, $\{v\}$ are changed. By definition of cohomology classes, we have to take $du = 0$, $dv = 0$. Now if u is changed to $u + dw$, then

$$\int_X (u + dw) \wedge v - \int_X u \wedge v = \int_X dw \wedge v = \int_X d(w \wedge v) = 0$$

by Stokes'formula. Similarly, $\int_X u \wedge v$ is left unchanged if v is replaced by $v + dw$. The $\mathbb{R}$-bilinearity of the map is trivial.

(d) Derive from (c) that the cohomology class of a constant-coefficient form $u = \sum u_I dx_I$ is equal to 0 if and only if $u = 0$, in other words that φ is injective.

Hint. Use $v = dx_{\complement I}$ where $\complement I$ means the complement of I in $\{1, 2, \ldots, n\}$.

If $u = dw$, then

$$\int_X u \wedge dx_{\complement I} = \int_X dw \wedge dx_{\complement I} = \int_X d(w \wedge dx_{\complement I}) = 0$$

by Stokes, and on the other hand

$$\int_X u \wedge dx_{\complement I} = \int_X u_I \, dx_I \wedge dx_{\complement I} = \pm \int_X u_I \, dx_1 \wedge \ldots \wedge dx_n = \pm u_I,$$

thus $u_I = 0$ for every I. This means that φ is injective.

(e) Given $a \in \mathbb{R}^n$, one defines operators L_a, G_a and M acting on smooth p-forms u by $L_a(u) = v$ (resp. $G_a u = w$, $Mu = \widetilde{u}$) where

$$v_I(x) = D_a u_I(x), \quad w_I(x) = \int_0^1 u_I(x + ta) \, dt, \quad \widetilde{u}_I(x) = \int_{a \in [0,1]^n} u_I(x + a) \, d\lambda(a),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^n$ and D_a the derivative in direction a. Show that L_a, G_a and M commute with the exterior derivative d, and that $Mu = \widetilde{u}$ always has constant coefficients (i.e. independent of x). Compute explicitly $G_a \circ L_a$ and $d \circ M$.

The commutation of L_a, G_a and M with d comes from the commutation of D_a with partial differentiations $\partial/\partial x_k$ (Schwarz' theorem for smooth functions), resp. from differentiation with respect to parameters x_k under the integral sign, for the integral expressions of $w_I(x)$ and $\widetilde{u}_I(x)$. The periodicity of u_I tells us that $\widetilde{u}_I$ is just the average value of u_I when seen as a function on $\mathbb{R}^n/\mathbb{Z}^n$, hence it is a constant. Therefore $d \circ M(u) = d\widetilde{u} = 0$ and we already conclude from this that $d \circ M = 0$. By definition

$$L_a(u) = \sum_I D_a u_I(x) \, dx_I, \quad G_a(u) = \sum_I \Big(\int_0^1 u_I(x + ta) \, dt \Big) dx_I,$$

thus

$$G_a \circ L_a(u)(x) = \sum_I \Big(\int_0^1 D_a u_I(x + ta) \, dt \Big) dx_I = \sum_I (u_I(x + a) - u_I(x)) \, dx_I = u(x + a) - u(x).$$

(f) The "interior product" $i_a u$ of p-form u by a is defined to be the alternate $(p-1)$-form such that $i_a u(x)(\xi_2, \ldots, \xi_p) = u(x)(a, \xi_2, \ldots, \xi_p)$. The well known "Lie derivative formula" (that can be admitted here) states that $d(i_a u) + i_a(du) = L_a u$. Infer from this that for the torus, the operator

$$h_a : C^\infty(X, \Lambda^p T_X^*) \to C^\infty(X, \Lambda^{p-1} T_X^*)$$

defined by $h_a = G_a \circ i_a$ satisfies the so called "homotopy formula" $d(h_a(u)) + h_a(du) = v_a$ with $v_a(x) = u(x + a) - u(x)$, and that for every closed form u, v_a is cohomologous to zero. Finally, conclude from all the above results that $\widetilde{u} - u$ is cohomologous to zero and that φ is an isomorphism.

By the commutation of G_a with d and the Lie derivative formula, we find

$$d(h_a(u)) + h_a(du) = d(G_a \circ i_a(u)) + G_a \circ i_a(du) = G_a \circ (d(i_a(u)) + i_a(du)) = G_a \circ L_a u = \big(x \mapsto u(x+a) - u(x)\big).$$

When $du = 0$, this implies that

$$u(x+a) - u(x) = dh_a(u)(x),$$

i.e. $x \mapsto u(x+a) - u(x)$ is cohomologous to 0. It we integrate over $a \in \mathbb{R}^n/\mathbb{Z}^n$, this gives

$$\widetilde{u}(x) - u(x) = \int_{a\in\mathbb{R}^n/\mathbb{Z}^n} dh_a(u)(x)\,d\lambda(a) = d\int_{a\in\mathbb{R}^n/\mathbb{Z}^n} h_a(u)(x)\,d\lambda(a)$$

(again, by commutation of $\partial/\partial x_k$ and $\int$ with parameters x_k), that is

$$\widetilde{u} - u = dK(u) \quad \text{where} \quad K(u)(x) = \int_{a\in\mathbb{R}^n/\mathbb{Z}^n} h_a(u)(x)\,d\lambda(a) = \int_{a\in\mathbb{R}^n/\mathbb{Z}^n} \Big(\int_0^1 i_a u(x+ta)\,dt\Big) d\lambda(a).$$

This means that every d-closed form u is cohomologous to its average $\widetilde{u}$. We infer from this that φ is surjective, hence an isomorphism.

(g) if $X = \mathbb{C}^n/\Lambda$ is a compact complex torus (where Λ is a lattice in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$), use a similar technique to show that there is an *injective morphism* given by constant-coefficient forms

$$\psi : \Lambda^{p,q}(\mathbb{C}^*)^n \to H^{p,q}_{\overline{\partial}}(X,\mathbb{C}).$$

Let us compute $\int_X u\wedge v$ with $u = \sum_{|I|=p,|J|=q} u_{I,J} dz_I \wedge d\overline{z}_J$ and $v = dz_{\complement I} \wedge d\overline{z}_{\complement J}$. If $u = \overline{\partial}\, w$ for a certain $(p, q-1)$-form w, then

$$\int_X u \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J} = \int_X \overline{\partial}\, w \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J} = \int_X dw \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J} = \int_X d(w \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J}) = 0.$$

Here, we have used the fact that $\partial w \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J} = 0$, since the bidegree of this wedge product is $(p+1, q-1) + (n-p, n-q) = (n+1, n-1)$. On the other hand, if u has constant coefficients, then

$$\int_X u \wedge dz_{\complement I} \wedge d\overline{z}_{\complement J} = \pm u_{I,J} \int_X dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n = \pm(2\mathrm{i})^n\, u_{I,J},$$

therefore $u_{I,J} = 0$ and ψ is injective.

Note. One can show that ψ is actually an isomorphism, but this is a bit harder than for De Rham cohomology (and the required technology has not yet been explained in the course!).

The result was probably (somehow) already known to Riemann, but the "modern technology" is to use Hodge theory: on a compact Kähler manifold, $H^{p,q}_{\overline{\partial}}(X,\mathbb{C})$ is isomorphic to the space of harmonic (p,q)-forms. On a torus, this means that $\Delta u_{I,J} = 0$, but by the maximum principle, harmonic functions are constant on a compact manifold. Thus, for a torus, we get that $H^{p,q}_{\overline{\partial}}(X,\mathbb{C})$ is isomorphic to the space of constant-coefficient forms. (This turns out to be also true for Bott-Chern cohomology groups).