Introduction to analytic geometry (course by Jean-Pierre Demailly)
Final examination, 09/01/2020, 09:00 – 13:00

Any use of connected electronic devices is forbidden.
Handwritten documents and lecture notes are allowed, printed documents are not (except exercise sheets).
Answers to questions can be developed in French or English.

Let K be a compact subset of $\mathbb{C}^n$. One defines the polynomial hull of K to be

$$\widehat{K} = \big\{z \in \mathbb{C}^n \;/\; |P(z)| \le \sup_K |P|,\ \forall P \in \mathbb{C}[z_1,\ldots,z_n]\big\}.$$

1. (a) Show that the polynomial hull $\widehat{K}$ is compact, coincides with the holomorphic hull

$$\widehat{K}_{\mathcal{O}} = \big\{z \in \mathbb{C}^n \;/\; |f(z)| \le \sup_K |f|,\ \forall f \in \mathcal{O}(\mathbb{C}^n)\big\}$$

and that $\widehat{K} = \widehat{\partial K}$.

(b) Prove that $\widehat{K}$ is contained in the convex hull $\widetilde{K}$ of K.
Hint. Consider $f(z) = e^{\ell(z)}$ where ℓ runs over all linear forms on $\mathbb{C}^n$, and recall that the convex hull $\widetilde{K}$ is equal to the intersection of closed half spaces that contain K (this is a consequence of the Hahn-Banach theorem in finite dimension).

(c) A (compact) polynomial polyhedron is a compact set of the form $K = \{z \in \mathbb{C}^n \;/\; |P_j(z)| \le c_j\}$ for suitable polynomials $P_1, \ldots, P_m \in \mathbb{C}[z_1,\ldots,z_n]$ and constants $c_j \in \mathbb{R}_+$. Show that $\widehat{K} = K$ and give an example where K is non convex.
Hint. In $\mathbb{C}^2$, take one of the polynomials to be $P(z_1,z_2) = z_1 z_2$.

2. Let $\psi : \mathbb{C}^n \to \mathbb{R}$ be a smooth plurisubharmonic exhaustion. For every value $c \in \mathbb{R}$, one defines

$$\Omega_c = \big\{z \in \mathbb{C}^n \;/\; \psi(z) < c\big\}, \quad K_c = \big\{z \in \mathbb{C}^n \;/\; \psi(z) \le c\big\}.$$

Recall that ψ being an exhaustion means that K_c is compact for every $c \in \mathbb{R}$. The goal of the exercise is to show that

(∗∗) every holomorphic function $f \in \mathcal{O}(\Omega_{c+\delta})$, $\delta > 0$, is the uniform limit on K_c of a sequence of polynomials $P_m \in \mathbb{C}[z_1,\ldots,z_n]$.

Let $\theta \in \mathcal{D}(\Omega_{c+\delta})$ be cut-off function equal to 1 on $K_{c+\delta/2}$ with support in $K_{c+3\delta/4}$. One solves the equation $\overline{\partial} u = v$ on $\mathbb{C}^n$, where $v = \overline{\partial}(\theta f)$ is extended by 0 in the complement of $\Omega_{c+\delta}$. For this, one works in the L^2 space of $(0,q)$-forms with respect to the weight function $\varphi_m(z) = m\psi(z) + |z|^2$ and the standard Hermitian metric $\omega = i\sum_{1\le j\le n} dz_j \wedge d\overline{z}_j$.

(a) Show that $\displaystyle\int_{\mathbb{C}^n} |v|^2 e^{-\varphi_m} d\lambda \le C_1 \exp(-m(c+\delta/2))$

for some constant $C_1 \ge 0$ (where $d\lambda$ = Lebesgue measure).

(b) Show that the eigenvalues of $i\partial\overline{\partial}\varphi_m$ with respect to ω are at least equal to 1, and derive from the theory of L^2 estimates for $\overline{\partial}$ that there exist solutions of the equations $\overline{\partial} u_m = v$ such that

$$\int_{\mathbb{C}^n} |u_m|^2 e^{-\varphi_m} d\lambda \le C_2 \exp(-m(c+\delta/2)).$$

Hint. On $\mathbb{C}^n$, solving $\overline{\partial}$ for $(0,q)$-forms is the same as solving $\overline{\partial}$ for (n,q)-forms.

(c) Infer from the above that $\displaystyle\int_{K_{c+\delta/4}} |u_m|^2 d\lambda \le C_3 \exp(-m\delta/4)$ and that u_m converges uniformly to 0 on the compact set K_c.
Hint. Use the mean value inequality on balls $B(z,r)$, where $r = d(K_c, \complement K_{c+\delta/4})$.

(d) Prove the assertion (∗∗).

3. A compact set $K \subset \mathbb{C}^n$ is said to be polynomially convex if $\widehat{K} = K$. In the remainder of this exercise, K is supposed to be a polynomially convex compact set.

(a) Show that the function $\chi(t) = t\,e^{-1/t}$ for $t > 0$, $\chi(t) = 0$ for $t \le 0$ is a convex function that is strictly convex increasing on $]0,+\infty[$.

(b) Let $a \notin K$. Show that one can choose a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ and real numbers $c \in \mathbb{R}$, $\varepsilon > 0$ such that the plurisubharmonic function $u_a(z) = \chi(|P(z)|^2 + \varepsilon|z|^2 - c)$ is identically 0 on K, and satisfies $u_a(a) > 0$, while $i\partial\overline{\partial}u_a$ is positive definite on an open neighborhood V_a of a, $V_a \subset \mathbb{C}^n \smallsetminus K$.

(c) Using a covering of $\mathbb{C}^n \smallsetminus K$ by countably many open sets V_{a_p}, $p \in \mathbb{N}$, show that one can produce a smooth plurisubharmonic function given by a convergent series $\psi = \sum_{p\in\mathbb{N}} \eta_p u_{a_p}$, $\eta_p = \mathrm{Const} > 0$, such that $\psi = 0$ on K, $\psi > 0$ strictly plurisubharmonic on $\mathbb{C}^n \smallsetminus K$, and finally $\psi(z) \geq \varepsilon'|z|^2 - C$ on $\mathbb{C}^n$.

(d) Using exercise 2, show that for every holomorphic function $f \in \mathcal{O}(\Omega)$, where Ω is an open neighborhood of K, there exists a sequence of polynomials $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ converging uniformly to f on K.

(e) In dimension 1 (i.e. in $\mathbb{C}$), is it possible to approximate $f(z) = 1/z$ by polynomials on the unit circle ? What is the problem here ?

4. Throughout this exercise, X denotes a compact complex manifold. Let Y be a complex submanifold of X and $[Y]$ the current of integration over Y.

(a) If T is another current on X, the product $[Y] \wedge T$ cannot be defined in general, since products of measures do not exist in the calculus of distributions. However, if $T = i\partial\overline{\partial}\varphi$ and φ is a plurisubharmonic function on a coordinate open subset $U \subset X$ such that φ is not identically $-\infty$ on any connected component of $Y \cap U$, show that φ is locally integrable on Y, and infer that one can define

$$[Y] \wedge i\partial\overline{\partial}\varphi := i\partial\overline{\partial}(\varphi[Y]).$$

Moreover, prove that if φ is regularized as a decreasing sequence of smooth plurisubharmonic functions $\varphi_\nu = \varphi * \rho_{1/\nu}$ obtained by convolution, the above definition of current products is compatible with weak limits, in the sense that $[Y]\wedge i\partial\overline{\partial}\varphi = \lim_{\nu\to+\infty}[Y]\wedge i\partial\overline{\partial}\varphi_\nu$ weakly in the space of currents.

(b) Let σ be a holomorphic section of some holomorphic line bundle $L \to X$, equipped with a smooth Hermitian metric h. One says that σ is transverse to Y if at every point $a \in Y \cap \sigma^{-1}(0)$ one has $d\sigma(a) \neq 0$ when the differentiation is made in a trivialization of L near a. Show that the above transversality condition is independent of the choice of the trivialization of L, and assuming transversality, that the product $[Y] \wedge [D]$ of $[Y]$ by the current of integration over the divisor $D = \mathrm{div}(\sigma)$ is well defined. Prove moreover via the Lelong-Poincaré equation that $[Y] \wedge [D]$ and $[Y] \wedge \frac{1}{2\pi}\Theta_{L,h}$ induce the same cohomology classes in De Rham (or Bott-Chern) cohomology.

(c) Let σ_j, $1 \leq j \leq p$, be holomorphic sections of Hermitian line bundles (L_j, h_j) over X. One assumes that the divisors $D_j = \mathrm{div}(\sigma_j)$ are non singular and intersect transversally, in the sense that the differentials $d\sigma_{j_1}(a), \ldots, d\sigma_{j_k}(a)$ are linearly independent at any point $a \in \sigma_{j_1}^{-1}(0) \cap \ldots \cap \sigma_{j_k}^{-1}(0)$. Prove inductively that the wedge product $[D_1]\wedge\ldots\wedge[D_p]$ is well defined, coincides with $[D_1\cap\ldots\cap D_p]$ and belongs to the cohomology class of $\frac{i}{2\pi}\Theta_{L_1,h_1} \wedge \ldots \wedge \frac{i}{2\pi}\Theta_{L_p,h_p}$.

(d) (Bézout formula) If D_j, $1 \leq j \leq n$, are non singular algebraic hypersurfaces $\{P_j(z) = 0\}$ of degree $\deg P_j = \delta_j$ intersecting transversally in complex projective space $\mathbb{P}^n$, compute the number of intersection points in $D_1 \cap \ldots \cap D_n$.
Hint. Apply the results of exercise 4 (d,e) in exercise sheet 9 (or reprove them using (c)).

5. The goal of this exercise is to investigate the Künneth formula for products $X \times Y$ of compact Kähler manifolds. Results of pure topology imply that for any field $\mathbb{K}$, e.g. $\mathbb{K} = \mathbb{R}$ or $\mathbb{R} = \mathbb{C}$, one has $H^k(X \times Y, \mathbb{K}) = \bigoplus_{i+j=k} H^i(X, \mathbb{K}) \otimes H^j(Y, \mathbb{K})$ for sheaf cohomology with values in the locally constant sheaf $\mathbb{K}$. In the differential case, $H^k(X, \mathbb{K})$ is known to be isomorphic to De Rham cohomology.

(a) Let (X, g_X) and (Y, g_Y) be Riemannian manifolds and $(X \times Y, g_X \oplus g_Y)$ their product. The exterior derivative obviously splits as $d = d_x + d_y$, and one can observe that d_x anticommutes with d_y and d_y^*. Infer that the Laplace-Beltrami operator Δ of $X \times Y$ is given by $\Delta = \Delta_x + \Delta_y$ where Δ_x, say, means the Laplace-Beltrami operator applied in x to differential forms $u(x, y)$ on $X \times Y$.

(b) For (X, ω_X) and (Y, ω_Y) compact Kähler, show that

$$H^{p,q}(X \times Y, \mathbb{C}) \simeq \bigoplus_{(k,\ell)+(r,s)=(p,q)} H^{k,\ell}(X, \mathbb{C}) \otimes H^{r,s}(Y, \mathbb{C}).$$

Hint. Use (a) and Hodge theory to find an injection of $\bigoplus_{(k,\ell)+(r,s)=(p,q)} H^{k,\ell}(X, \mathbb{C}) \otimes H^{r,s}(Y, \mathbb{C})$ into $H^{p,q}(X \times Y, \mathbb{C})$, and conclude by Hodge decomposition and comparison of dimensions.

(c) Show that there are infinitely many non diffeomorphic connected compact Kähler manifolds in any dimension n.