Since the norm of the linear form is $\|u\|$, we also get $\|u\| \le C^{1/2}$, that is,

$$\int_X |u|^2\,dV_\omega \le C = \int_X \langle (A^{p,q})^{-1}v, v\rangle\,dV_\omega.$$

We have therefore proved the following result.

Theorem (S. Bochner, K. Kodaira, S. Nakano, J. Kohn, A. Andreotti - E. Vesentini, L. Hörmander and continuators)
Let (X,ω) be a complete Kähler manifold and (E,h) a hermitian holomorphic vector bundle over X. Assume that the self-adjoint operator

$$A^{p,q} = A^{p,q}_{X,\omega\,;\,E,h} := [\Theta_{E,h}, \Lambda_\omega]$$

is positive definite on $\Lambda^{p,q}T^*_X \otimes E$. Then for every (p,q) form $v \in L^2(X, \Lambda^{p,q}T^*_X \otimes E)$ such that $\overline{\partial}_E v = 0$, the del-bar equation

$$\text{(a)} \qquad \overline{\partial}_E u = v$$

admits a solution $u \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes E)$ in the sense of distributions, such that

$$\text{(b)} \qquad \int_X |u|^2\,dV_\omega \le \int_X \langle (A^{p,q})^{-1}v, v\rangle\,dV_\omega,$$

provided that the right hand side of (b) *is convergent.*

(c) *The solution of minimal L^2 norm is the one such that $u \in (\mathrm{Ker}\overline{\partial}_E)^\perp = \overline{\mathrm{Im}\overline{\partial}{}^*_E}$. This solution is unique and satisfies the additional property*

$$\overline{\partial}{}^*_E u = 0.$$

(d) *The minimal L^2 solution satisfies $\overline{\Box}_E u = \overline{\partial}{}^*_E v$, therefore by ellipticity, one gets automatically $u \in C^\infty(X, \Lambda^{p,q-1}T^*_X \otimes E)$ if $v \in C^\infty(X, \Lambda^{p,q}T^*_X \otimes E)$.*

Corollary 1. *Let (X,ω) be a Kähler manifold (where ω is not necessarily complete), and let (E,h) be a hermitian holomomorphic line bundle such that $i\Theta_{E,h} > 0$ as a real $(1,1)$-form. Assume additionally that X is weakly pseudoconvex, i.e. that X possesses a smooth psh exhaustion function ψ. Then for every (n,q)-form v in $L^2_{\mathrm{loc}}(X, \Lambda^{p,q}T^*_X \otimes E)$ $(q \ge 1)$, such that $\overline{\partial}_E v = 0$ there exists v in $L^2_{\mathrm{loc}}(X, \Lambda^{p,q-1}T^*_X \otimes E)$ such that $\overline{\partial}_E u = v$ and*

$$\int_X |u|^2\,dV_\omega \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |v|^2\,dV_\omega$$

where $0 < \lambda_1(z) \le \cdots \le \lambda_n(z)$ are the eigenvalues of $i\Theta_{E,h}(z)$ with respect to $\omega(z)$.

Proof. When ω is complete and additionally $v \in L^2$, this is just a special case of the theorem. Otherwise, we can apply the theorem after replacing ω by $\widehat{\omega}_\varepsilon = \omega + \varepsilon i\partial\overline{\partial}(\psi^2)$ which is complete for any $\varepsilon > 0$. The integral involving v and $\widehat{\omega}_\varepsilon$ is then uniformly bounded by the same integral calculated for ω (exercise, see Lemma 6.3 in Chapter VIII of my online book). One then gets a L^2 solution u_ε with respect to $\widehat{\omega}_\varepsilon$. By weak compactness of closed balls in Hilbert spaces, it is easily shown that there is a weakly convergent sequence u_{ε_k}