Analytical Geometry 412 — Chapter 1: Holomorphic functions of several variables — part 1

I - Basic definitions

Let $\Omega \subset \mathbb{C}^n$ be an open subset

Theorem and definition:

Let $f : \Omega \longrightarrow \mathbb{C}$ then the following are equivalent (TFAE):

(i) f is $\mathbb{C}$-analytic ie $\forall z_0 \in \Omega$, $\exists$ convergent power series representing f in a small open polydisc

$$D(z_0, r) = \prod_{j=1}^{n} D(z_{0,j}, r_j) \subset \Omega$$

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$$

$z_0 = (z_{01}, \dots, z_{0n})$
$r = (r_0, \dots, r_n)$
$\alpha = (\alpha_1, \dots, \alpha_n)$
$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$
a_α depends on z_0

(ii) f is $\mathbb{C}$-differentiable on Ω ie $\forall z_0 \in \Omega$ $\exists V \ni z_0$ neighbourhood which $f(z) = f(z_0) + l\cdot(z - z_0) + o(|z - z_0|)$
$l \in \mathcal{L}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ $\mathbb{C}$-linear

(iii) f continuous on Ω and $\forall j$ $z_j \longrightarrow f(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$ is holomorphic
(f is said "separately holomorphic").

(iv) Cauchy formula holds: $\forall \overline{D}(z, r) \subset \Omega$

$$f(z) = \frac{1}{(2i\pi)^n} \int_{\prod \partial D(z_j, r_j)} \frac{f(w)}{\prod (w_j - z_j)} \, dw_1 \cdots dw_n$$

$w_j = z_{0,j} + r_j e^{i\theta_j}$
$dw_j = i r_j e^{i\theta_j} d\theta_j$
$\theta_j \in [0, 2\pi]$

Definition: One then says that f is holomorphic on Ω

notation: $\mathcal{O}(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ holomorphic} \}$, this is a $\mathbb{C}$-algebra

proove: Assume (i)

- Convergence $\Rightarrow$ $|a_\alpha| (r - \varepsilon)^\alpha \leq M$ some bound

then $\sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ converge uniformly on $D(0, r - 2\varepsilon)$

$$\left| \frac{(z - z_0)^\alpha}{(r - \varepsilon)^\alpha} \right| \leq (1 - \delta)^{|\alpha|} \qquad |\alpha| = \alpha_1 + \dots + \alpha_n$$

$$\frac{\partial f}{\partial z_j}(z) = \sum_{\alpha \in \mathbb{N}^n} \alpha_j a_\alpha (z - z_0)^{\alpha - \delta_j} \qquad \delta_j = (0, \dots, 1, 0, \dots 0)$$

$\Rightarrow$ $f \in \mathcal{C}^\infty$ and one has complexe derivatives at any order

$\Rightarrow$ (i) $\Rightarrow$ (2)

- (ii) $\Rightarrow$ (iii) is clear

- (iii) $\Rightarrow$ (iv) Induction on n

known for $n = 1$

assume known for $n - 1$

$$(z_1, \dots, z_{n-1}) \longrightarrow f(z_1, \dots, z_{n-1}, \underset{\text{fixed}}{w_n})$$

$$f(z_1, \dots, z_{n-1}, w_n) = \frac{1}{(2i\pi)^{n-1}} \int_{(w_1, \dots, w_{n-1}) \in \prod_{j=1} \partial D(z_{0,j}, r_j)} \frac{f(w_1, \dots, w_n)}{\prod_{j=1}^{n-1} (w_j - z_j)} dw_1 \dots dw_{n-1}$$

holomorphic with respect to w_n on both sides

↳ just apply Cauchy formula w.r.t variables w_n and use Fubini for continuous functions

• (iv) $\Rightarrow$ (i)

$$\frac{1}{w_j - z_j} = \frac{1}{w_j} \frac{1}{1 - \frac{z_j}{w_j}} = \sum_{\alpha_j \in \mathbb{N}} \frac{z_j^{\alpha_j}}{w_j^{\alpha_j + 1}}$$ absolutely convergent

$|w_j| < r_j$ $|z_j| < r_j$

so $$\prod_{j=1}^{n} \frac{1}{w_j - z_j} = \sum_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} \frac{z^\alpha}{w^{\alpha + \mathbb{1}}}$$ $\mathbb{1} = (1, \dots, 1)$

this gives $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ where $a_\alpha = \frac{1}{(2i\pi)^n} \int_{w \in \prod \partial D(z_{0,j}, r_j)} \frac{f(w)}{w^{\alpha + \mathbb{1}}} dw_1 \dots dw_n$

hence f is $\mathbb{C}$-analytic ✓

Hartog's theorem :

f separate holomorphic (even without assuming f continuous) implies f is holomorphic

Topology of $\mathcal{O}(\Omega)$:

Let $K \subset \Omega$ be a compact subset.

$p_K(f) = \sup_{z \in K} |f(z)| = \max_{z \in K} |f(z)|$ this is a semi-norm

Definition : E a $\mathbb{K}$-vector space $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

a semi-norm is a map $p : E \longrightarrow \mathbb{R}_+$, $x \longrightarrow p(x)$ such that

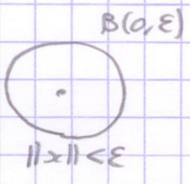
- $\forall \lambda \in \mathbb{K}$ $p(\lambda x) = |\lambda| p(x)$
- $\forall x, y \in E$ $p(x+y) \leq p(x) + p(y)$

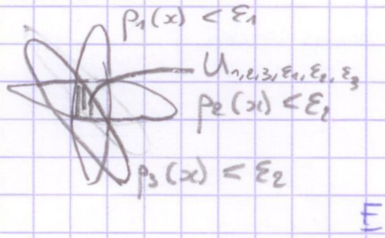
(don't assume the separation : $p(x) ~~= 0 \Rightarrow x = 0~~$)

Suppose you are given an arbitrary family $(p_\alpha)_{\alpha \in I}$ of semi-norms.

"Fundamental system of neighborhood of 0" will be :

$$U_{\alpha_1, \dots, \alpha_N, \varepsilon_1, \dots, \varepsilon_N} = \left\{ x \in E \;/\; p_{\alpha_j}(x) < \varepsilon_j \quad \{\alpha_1, \dots, \alpha_N\} \underset{\text{finite}}{\subset} I \right\}$$





$U \subset E$ is open if $\forall x \in U$ $\exists$ fundamental neighborhood (voisinage) of 0, say V such that $x + V \subset U$

E is a topological vector space !

E Hausdorff $\Leftrightarrow$ $\left(x = 0 \Leftarrow \forall \alpha \in I \; p_\alpha(x) = 0 \right)$

Algebraic weak topology of vector space E -

↳ E^* dual → $\varphi \in E^*$ $\quad p_\varphi(x) = |\varphi(x)|$ (bad, not good enough)

Weak topology :

If E already a topological vector space : E' = topological dual of continuous linear forms $\subset E^*$

get family of semi-norms $p_\varphi(x) = |\varphi(x)|$ for $\varphi \in E'$

Locally convex topological vector space : $\left(E, \text{ with topology defined by a collection of semi-norms } (p_\alpha)_{\alpha \in I} \right)$

$(x_k)_{k\in\mathbb{N}}$ is called a Cauchy sequence if :

↳ $\forall \alpha \in I \quad \forall \varepsilon > 0, \quad \exists N \quad \forall n, m \geq N \quad p_\alpha(x_m - x_n) < \varepsilon$

Definition : E is sequentially complete if every Cauchy sequence is convergent

Observation : If the topology is Hausdorff and can be defined by a countable family of semi-norms, then the topology is metrisable (definable by a distance).

↳ proof : $(p_\alpha)_{\alpha\in\mathbb{N}}$ a countable family of semi-norms, E Hausdorff

$$d(x,y) = \sum_{\alpha\in\mathbb{N}} 2^{-\alpha} \min(1, p_\alpha(x-y)) \quad \text{is a distance (exercice)}$$

. d is translation invariant $\quad d(x+a, y+a) = d(x,y) \quad \forall x, y, a \in E$

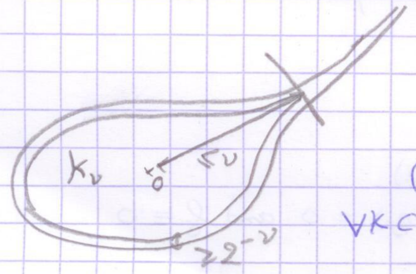
. neighborhood of 0 $\quad d(x,0) < \varepsilon \iff \sum_{\alpha\in\mathbb{N}} 2^{-\alpha} \min(1, p_\alpha(x)) < \varepsilon$

$\Leftarrow \quad p_\alpha(x) < 2^{\alpha} \varepsilon$ means good for a finite subfamily -

Definition : A Fréchet space is a topological vector space E where topology is defined by a countable family of semi-norms, that is Hausdorff and complete -

$\mathcal{O}(\Omega)$, $\quad p_K \quad K \subset \Omega$

$K_\nu = \{ x \in \Omega \ / \ |x| \leq \nu, \ d(x, \complement\Omega) \geq 2^{-\nu} \}$ $\qquad \complement\Omega$ complémentaire de Ω



bounded and closed hence compact

(K_ν) is an "exhausting sequence" :

$\forall K \subset \Omega$ compact, $\exists \nu$ such that $K \subset K_\nu \subset \mathring{K}_{\nu+1}$

$\mathring{K}_\nu \supset \{x \in \Omega \,/\, |x| < \nu \,,\, d(x, \complement\Omega) > 2^{-\nu}\}$

$\Omega = \bigcup \mathring{K}_\nu$

$K \subset K_\nu \qquad p_K = \sup_{z \in K} |f(z)| \leq p_{K_\nu}$

$\{f \,/\, p_{K_\nu}(f) < \varepsilon\} \subset \{f \,/\, p_K(f) < \varepsilon\}$

} Topology of $\mathcal{O}(\Omega)$ can be defined by the $(p_{K_\nu})_{\nu \in \mathbb{N}}$

Theorem: $(\mathcal{O}(\Omega), p_{K_\nu})_{\nu \in \mathbb{N}}$ is a Fréchet space

Remark: $C^q(\Omega) = \{C^q \text{ functions}\}$ for $\Omega \subset \mathbb{R}^n$

$$p_{K,q} = \sup_{x \in K} \max_{|\alpha| \leq q} |D^\alpha f(x)|$$

$(C^q(\Omega), p_{K,q})_{\nu \in \mathbb{N}}$ is a Fréchet space as well

$(C^\infty(\Omega), p_{K,q})_{\nu \in \mathbb{N}, q \in \mathbb{N}}$ is a Fréchet space

$D^\alpha = \dfrac{\partial^{\alpha_1 + \dots + \alpha_m}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_m^{\alpha_m}} : \mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\Omega)$ is continuous for the Fréchet topology

$\hookrightarrow f(z) = \dfrac{1}{(2i\pi)^n} \displaystyle\int_{\prod \partial D(z_j, r_j)} \dfrac{f(w)}{\prod (w_j - z_j)} dw_1 \dots dw_n$

$$D^\alpha f(z) = \frac{\alpha!}{(2i\pi)^n} \int_{\prod \partial D(z_j, r_j)} \frac{f(w)}{(w - z)^{\alpha + 1}} dw_1 \dots dw_n \qquad \alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$$

Cauchy inequalities: $|D^\alpha(f z_0)| \leq \dfrac{\alpha!}{(2\pi)^n} \displaystyle\int_{\theta_1 \dots \theta_n} \dfrac{|f(w)|}{r^{\alpha+1}} r_1 d\theta_1 \dots r_n d\theta_n$

$$\leq \frac{\alpha!}{r^\alpha} \sup_{w \in \prod \partial D(z_j, r_j)} |f(w)|$$

$p_K(D^\alpha f) \leq \dfrac{\alpha!}{\varepsilon^{|\alpha|}} p_{K_{\nu,\varepsilon}}(f)$ where $K_\delta = \{z \in \Omega \,/\, d(z, K) \leq \delta\}$

Montel's Theorem:

Take $\Omega \subset \mathbb{C}^n$ an open subset. If $f_\nu \in \mathcal{O}(\Omega)$ is a sequence that is uniformly bounded on every compact subset (ie: $\forall K \subset\subset \Omega$), $\exists C_K$ such that $\forall \nu \quad \sup_K |f_\nu| \leq C_K$

Then $\exists$ convergent subsequence $(f_{\nu_p})_{p \in \mathbb{N}}$ in $\mathcal{O}(\Omega)$ $\quad \nu_0 < \nu_1 < \dots < \nu_p < \dots$

proof: Ascoli's theorem + diagonal subsequence technique (TD 1 exercice 4)

Reformulation: $\mathcal{O}(\Omega)$ is a Montel space

Remark: $C^\infty(\Omega)$ is a Montel space but not $C^q(\Omega)$ for $q < +\infty$

Analytic continuation theorem:

Lemma: Let $\Omega \subset \mathbb{C}^n$ be a connected open set and $f \in \mathcal{O}(\Omega)$.
Then if $\exists z_0 \in \Omega$ such that $\forall \alpha \in \mathbb{N}^n$, $D^\alpha f(z_0) = 0$
$\hookrightarrow$ then $f \equiv 0$ in Ω

proof: $E = \{ z \in \Omega \ / \ \forall \alpha \in \mathbb{N}^n \ \ D^\alpha f(z) = 0 \}$

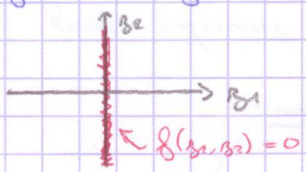
- E closed
- $E \neq \emptyset$ since $z_0 \in E$
- E is open :
  ↳ pick $\overline{D}(z_1, r) \subset \Omega$, then $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_1)^\alpha$ for $z_1 \in E$
  $a_\alpha = \frac{1}{\alpha!} D^\alpha f(z_1) = 0$
  $\Rightarrow f \equiv 0$ in $D(z_1, r)$
  $\Rightarrow D(z_1, r) \subset E$
  $\Rightarrow E$ is open

Ω connected + E open and closed $\Rightarrow E = \Omega$
$\Rightarrow f(z) = 0 \quad \forall z \in \Omega \quad \Rightarrow f \equiv 0$ in Ω

Consequence: If $f \in \mathcal{O}(\Omega)$ vanishes on a neighborhood of a point $z_0 \in \Omega$, then $f \equiv 0$ on the connected component of z_0 in Ω

⚠ for $n \geq 2$, zeros of a holomorphic functions f are never isolated !

$\Omega = \mathbb{C}^2 \qquad f(z_1, z_2) = z_1$



$\Leftarrow$ zeros aren't isolated

II - Algebraic properties of the rings of holomorphic functions

Germs $\quad \Omega \subset \mathbb{C}^n, \ x \in \Omega$

$$\mathcal{O}_{\Omega,x} = \left\{ \begin{array}{l} \text{germs of holomorphic function } f : V \to \mathbb{C} \\ V \text{ small neighborhood of } x \end{array} \right\} \Big/ \sim$$

where
$f_1 : V_1 \to \mathbb{C}$
$f_2 : V_2 \to \mathbb{C}$ $\qquad f_1 \sim f_2 \iff \exists W$ neighborhood of x such that $f_1|_W = f_2|_W$

A germ is an equivalent class w.r.t $\sim$

$\mathcal{O}_{\Omega,x}$ is a $\mathbb{C}$-algebra

notation: $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n, 0} \cong \mathbb{C}\{z_1, \dots, z_n\}$ braces

where $\mathbb{C}\{z_1, \dots, z_n\} = \{ \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ that are converging on some polydisc $D(0,r) \}$

$\mathbb{C}[z_1, \dots, z_n] \subset \mathbb{C}\{z_1, \dots, z_n\}$ a sub-algebra

$\boxed{\mathbb{C}[z_1, \dots, z_n] \text{ is a noetherian and a factorial ring !}}$

↳ show that $\mathbb{C}[z_1, \dots, z_n]$ is noetherian and factorial by induction on n :

$n=1$: $\mathbb{C}[z_1]$ is an euclidian ring $\Rightarrow$ principal ring
$\Rightarrow$ noetherian ✓

$\mathbb{C}\{z\}$ one variable . $f \in \mathcal{O}_{\mathbb{C},0}$

$f(0) \neq 0 \iff f$ invertible in $\mathbb{C}\{z\}$

proof

$\forall f \in \mathbb{C}\{z\}$, $f \neq 0$ $\quad f(z) = \sum a_k z^k$

$\exists!$ $f(z) = z^p g(z)$ which $g(0) \neq 0$

$a_0 = \dots = a_{p-1} = 0 \quad a_p \neq 0 \quad g(z) = \sum_{k=p}^{+\infty} a_k z^{k-p}$

$g(0) = a_p$

unique maximal ideal $m = (z)$

other ideals $m^p = (z^p)_{p \in \mathbb{N}^*}$ and $(0), (1)$

$P(z) = a_d \prod (z - z_j)^{p_j} \quad m_w = (z - w) \quad w \in \mathbb{C}$

infinitely many ideals

$n \geq 2$ $\left.\begin{array}{l}\mathbb{C}[z_1, z_2] \\ \mathbb{C}\{z_1, z_2\}\end{array}\right\}$ not principal

$m_{(0,0)} = (z_1, z_2)$ is not a principal ideal (exercise)

<u>Weierstrass factorisation theorem</u> :

$g \in \mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}$

$g \neq 0$

$g(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$

take $t \in \mathbb{C}$ $\quad g(tz) = \sum_{k=0}^{\infty} t^k \underbrace{\sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} a_\alpha z^\alpha}_{\text{homogenious polynomial of degree } k}$

order of vanishing of g at $0 = m = \mathrm{ord}_0(g) = \min\{k \in \mathbb{N} \ / \ \exists \alpha, |\alpha| = k \text{ with } a_\alpha \neq 0\}$

$g_k(z) = \sum_{|\alpha| = k} a_\alpha z^\alpha \qquad g_0 = \dots = g_{m-1} = 0$

$g_m \neq 0$

$\exists w \in \mathbb{C}^n$ such that $g_m(w) \neq 0$

$t \longrightarrow g(tw) = t^m g_m(w) + \sum_{k>m} t^k g_k(w)$ vanishes exactly at order m at 0

By picking a rotation of coordinate (writ $U(n)$) one can assume $w = e_n = (0, \dots, 0, 1)$

<u>Then</u> : $g(0, \dots, 0, z_n) \neq 0$

"Weierstrass preparation theorem":

Take $g \in \mathcal{O}_n$, $g \not\equiv 0$ such that $g(0,\dots,0,z_n) \not\equiv 0$

let $s = \operatorname{ord}_{z_n=0}(g(0,\dots,0,z_n)) \geq n$

Then one can write $g(z) = u(z)\, P(z', z_n)$

where
- $u \in \mathcal{O}_n$ is invertible (ie $u(0) \neq 0$)
- P is a "Weierstrass polynomial" of degree s

$z' = (z_1,\dots,z_{n-1}) \in \mathbb{C}^{n-1}$ $\longrightarrow$ $P(z', z_n) = \sum_{k=0}^{s} a_k(z')\, z_n^{s-k}$

with $a_k \in \mathcal{O}_{n-1}$

$a_0 = 1$

$a_k(0) = 0$ for $0 < k \leq s$

example: $\mathbb{C}\{z_1, z_2, z_3\}$

$$z_3^4 + (z_1 + z_2 e^{z_2^2}) z_3^3 + (3z_1^2 - z_2) z_3 + z_1 e^{z_2 z_2} = P(z', z_3)$$

$P(z', z_3) \in \mathcal{O}_{3-1}[z_3]$

prove: $z_n \longrightarrow g(0,\dots,0,z_n)$ has an isolated zero due to the isolated zeros theorem of one variable holomorphic functions.

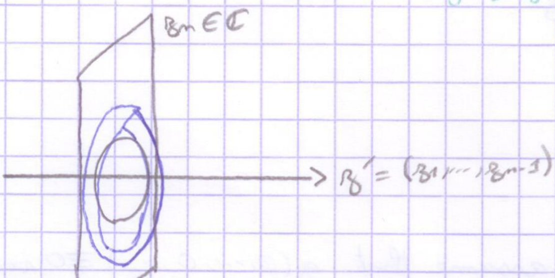
↳ $z_n \to g(0,\dots,0,z_n)$ vanishes at order s at $z_n = 0$

$\exists r_n > 0$ small such that $g(0,\dots,0,z_n) \neq 0$ on circle $|z_n| = r_n$

Continuity of g in z $\Rightarrow$ $z = (z', z_n)$

$\Rightarrow \exists r' = (r_1,\dots,r_{n-1})$ $r_j > 0$ and $\varepsilon > 0$ such that

$g(z', z_n) \neq 0$ for $\begin{cases} |z_j| \leq r_j & \text{for } 1 \leq j \leq n-1 \\ r_n - \varepsilon \leq |z_n| \leq r_n + \varepsilon \end{cases}$

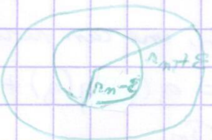


Put $S_k(z', z_n) = \frac{1}{2i\pi} \int_{|w_n| = r_n+\varepsilon} \frac{\frac{\partial}{\partial z_n} g(z', z_n)}{g(z', w_n)} w_n^k \, dw_n$

$S_k \in \mathcal{O}(D'(0, r'))$ $\qquad r' = (r_1,\dots,r_{n-1})$ $\qquad k \in \mathbb{N}$

look at zeros of $w_n \longrightarrow g(z', w_n)$

these zeros are noted $w_j(z')$ with multiplicities $m_j(z')$



$z_n \in \mathbb{C}$

$S_0(z') = \sum m_j(z') \in \mathbb{N}$

$S_0(0) = s$ because only $z_n = 0$ with multiplicity s at $z' = 0$

Therefore $S_0(z') \equiv s$ $\qquad S_k(z') = \sum_j m_j(z')\, w_j(z')^k = \sum$ of k^{th} power of roots of $w_n \to g(z', w_n)$ taking into account their multiplicity

$\sigma_k(z') = k^{th}$ elementary symmetric function in the roots

$\sigma_1(z') = S_1(z')$

$\sigma_2(z') = \sum_{j<k} m_j(z') m_k(z') w_j(z') w_k(z')$

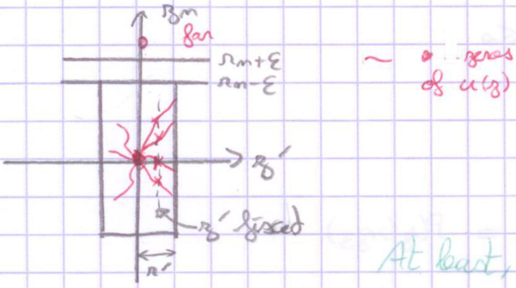
$\sigma_k = \underset{\text{polynomial}}{Q_k}(S_1, \dots, S_k)$ (Newton)

$\sigma_k \in \mathcal{O}(D'(0,r'))$

$\hookrightarrow P(z', z_n) = z_n^{s} - \sigma_1(z') z_n^{s-1} + \dots + (-1)^s \sigma_s(z')$

for $z' = 0$, one root $w_n = 0$ of multiplicity s
hence $\sigma_k(0) = 0$ for $k \geq 1$

We have now to show that $\dfrac{g(z', z_n)}{P(z', z_n)} = u(z)$ is an invertible holomorphic function in $D(0,r)$.



By construction $g(z', z_n)$ and $P(z', z_n)$ have the same 0 in z_n for z' fixed
so I have to show that $f = \frac{g}{P}$ and $f = \frac{P}{g}$ are both holomorphic on the polydisc $D(0,r)$

At last, they are holomorphic w.r.t. to z_n

$$f(z', z_n) = \int_{|w_n| = r_n + \varepsilon} \frac{1}{2\pi i} \frac{f(z', w_n)}{w_n - z_n} \, dw_n$$

neither P nor g vanishes near $|w_n| = r_n + \varepsilon$
$\Rightarrow f(z', z_n)$ is holomorphic in (z', z_n) on $D(0,r)$ ✓

<u>Gauss Theorem</u>:

A factorial ring $\Rightarrow A[T]$ factorial
$\Rightarrow A[T_1, \dots, T_n]$ factorial ring

<u>strategy</u>: $\mathcal{O}_n$ is factorial by induction on n
$\mathcal{O}_{n-1}$ factorial $\Rightarrow \mathcal{O}_{n-1}[z_n]$ factorial

<u>Weierstrass division theorem</u>: Take $g \in \mathcal{O}_n$, $g \neq 0$, assume that $g(0, \dots, 0, z_n) \not\equiv 0$ vanishes at order s at $z_n = 0$

Then, $\forall f \in \mathcal{O}_n$, one can write

$$(\circledast) \begin{cases} f(z) = g(z)\, q(z) + R(z', z_n) \\ q \in \mathcal{O}_n \\ R \in \mathcal{O}_{n-1}[z_n] \quad \deg_{z_n} R < s \end{cases}$$

and the division is unique if $\circledast$ is met

Additional facts:

- The division works on any polydisc $D(0,r)$ where f, g are defined and chosen in the preparation theorem
- $f \longrightarrow (q, R)$ is a linear map

$$\sup_{D(0,r)} |q| \leq C \sup_{D(0,r)} |f|$$

$$\sup_{D(0,r)} |R| \leq C' \sup_{D(0,r)} |f|$$

proof: We can assume that $g(z) = P(z', z_n)$

$$q(z', z_n) = \frac{1}{2i\pi} \int_{|w_n| = r_n - \varepsilon} \frac{f(z', w_n)}{P(z', w_n)(w_n - z_n)} \, dw_n$$

$$R(z', z_n) = \underbrace{f(z)}_{*} - \underbrace{P(z', z_n) q(z)}_{**}$$

$$= \frac{1}{2i\pi} \int_{|w_n| = r_n - \varepsilon} \frac{f(z', w_n) [\overbrace{P(z', w_n)}^{*} - \overbrace{P(z', z_n)}^{**}]}{P(z', w_n) \, (w_n - z_n)} \, dw_n$$

$\dfrac{P(z', w_n) - P(z', z_n)}{w_n - z_n}$ is a polynomial in w_n and z_n of degree $\leq s-1$

$\Rightarrow R(z', z_n) \in \mathcal{O}(D(0, r'))[z_n]$ of degree $\leq s-1$

- existence is proved ✓
- uniqueness $\rightarrow$ $f = P q_1 + R_1$
  $f = P q_2 + R_2$

$$f - f = \underset{\substack{s \text{ zeros including} \\ \text{multiplicity}}}{P(q_1 - q_2)} + \underbrace{(R_1 - R_2)}_{\deg_{z_n} \leq s-1} = 0$$

$$\Rightarrow \begin{cases} R_1 - R_2 = 0 \\ q_1 - q_2 = 0 \end{cases} \qquad \Rightarrow \text{uniqueness} ✓$$

Lemma 1: $P, F \in \mathcal{O}_{n-1}[z_n]$ where P is a Weierstrass polynomial and P divides F in $\mathcal{O}_n$. Then P divides F in $\mathcal{O}_{n-1}[z_n]$.

proof: $F = PQ + R$ $\qquad \deg_{z_n} R < s = \deg_{z_n} P$

$F = Ph$ $\qquad h \in \mathcal{O}_n$

by uniqueness $Q = h$ and $h \in \mathcal{O}_n$

Lemma 2: Let P be a Weierstrass a polynomial.

a: if $P = P_1 \cdots P_N$ with $P_j \in \mathcal{O}_{n-1}[z_n]$
then $P_j = u_j \tilde{P}_j$ where $\tilde{P}_j$ Weierstrass and $u_j \in \mathcal{O}_{n-1}$ invertible

b: $P(z', z_n)$ irreductible in $\mathcal{O}_n \iff P(z', z_n)$ irreductible in $\mathcal{O}_{n-1}[z_n]$

proof: a: Let u_j = leading coefficient of P_j, $u_j \in \mathcal{O}_{n-1}$
then $u_1 \cdots u_N = 1$ = leading coefficient of P
so the $u_j \in \mathcal{O}_{n-1}$ are invertible and $P = \tilde{P}_1 \cdots \tilde{P}_N$

look at zeros for $z' = 0$ $\rightarrow$ $P(0,\dots,0,z_n) = z_n^s$
$\Rightarrow$ only zeros of the $\tilde{P}_j$'s are $z_n = 0$
$\Rightarrow$ $\tilde{P}_j$ are Weierstrass polynomial

b: Exercise

Look at $\mathcal{O}_n$
- Invertible elements = $u \in \mathcal{O}_n$ with $u(0) \neq 0$
- Irreductible elements of $\mathcal{O}_n$ = they are, up to change of coordinates, the irreductible Weierstrass polynomials.

Theorem: For every n, $\mathcal{O}_n = \mathbb{C}\{z_1,\dots,z_n\}$ is a factorial ring!

proof: By induction on n:

a: $\rightarrow$ every non-zero $f \in \mathcal{O}_n$ decompose in irreductible elements (up to invertible elements)

Rotate coordinates, so that $f(0,\dots,0,z_n) \not\equiv 0$

$\Rightarrow f = uP$ $\begin{cases} u \text{ invertible} \\ P \text{ Weierstrass polynomial} \end{cases}$

$P \in \mathcal{O}_{n-1}[z_n]$ and $\mathcal{O}_{n-1}$ factorial $\Rightarrow$ $\mathcal{O}_{n-1}[z_n]$ factorial

$P = \underbrace{P_1 \cdots P_N}_{\substack{\text{irreductible in} \\ \mathcal{O}_{n-1}[z_n]}} = \underbrace{\tilde{P}_1 \cdots \tilde{P}_N}_{\substack{\text{irreductible Weierstrass} \\ \text{polynomials}}}$

in fact (Lemma 1) $\rightarrow$ $\tilde{P}_j$ is irreductible in $\mathcal{O}_n$

✓

b: $\Rightarrow$ the uniqueness (in exercice!)

Theorem: $\mathcal{O}_n = \mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian (every ideal is finitly generated)

proof: Induction on n

- $n = 0$ $\quad \mathcal{O}_0 = \mathbb{C}$
  $n = 1$ $\quad \mathcal{O}_1 = \mathbb{C}\{z_1\}$ is principal $\Rightarrow$ is noetherian ✓

- $n \geq 2$, let $I \subset \mathcal{O}_n$ be an ideal $I \neq \{0\}$
  Pick $g \in I$, $g \neq 0$

Can assume in fact $g(z) = P(z', z_n)$ Weierstrass polynomial
take any $f \in I$
divide $f = gq + R$ $\qquad R \in \mathcal{O}_{n-1}[z_n]$, $\deg_{z_n} R \leq s-1$

$f, g \in I \Rightarrow R = f - gq \in I$
$\Rightarrow R \in \underbrace{I \cap \mathcal{O}_{n-1}[z_n]_{\deg \leq s-1}}_{\text{submodule of } \mathcal{O}_{n-1}[z_n]_{\deg \leq s-1}} \simeq \mathcal{O}_{n-1}^{\oplus s}$

let $\Sigma \subset \mathcal{O}_{n-1}^{\oplus s}$ be the submodule
As $\mathcal{O}_{n-1}$ is noetherian by hypothesis, then Σ must be finitely generated
↳ let $\sigma_1, \dots, \sigma_N \in \Sigma$ be generators
$\Rightarrow (g, \sigma_1, \dots, \sigma_N)$ are generators of I ✓ $f = gq = \sum h_j \sigma_j$

$m = (z_1, z_2)$ $\qquad \dim_{\mathbb{C}} m^p / m^{p+1} = p+1$
$m^p = (z_1^p, z_1^{p-1} z_2, \dots, z_2^p)$ $\qquad m^p$ cannot have less than $p+1$ generators

---

<u>Question:</u> A a factorial ring - When is a monic polynomial $P(T) \in A[T]$ irreductible ?

$P(T) = a_0 + a_1 T + a_2 T^2 + \dots + a_d T^d$
$Q(T) = b_0 + b_1 T + b_2 T^2 + \dots + b_\delta T^\delta$
↳ <u>Resultant of P, Q</u> :

$$\mathrm{Res}(P,Q) = \begin{vmatrix} a_0 & a_1 & \dots & a_d & \overbrace{0 \ \dots \ \dots \ 0}^{\delta} & & \\ 0 & a_1 & \dots & a_{d-1} & a_d & 0 \ \dots & 0 \\ 0 & 0 & a_1 & \dots & a_{d-1} & a_d \ 0 \ \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & \dots & 0 & a_1 & \dots & a_d \\ b_1 & b_2 & \dots & b_\delta & 0 & \dots & 0 \\ 0 & b_1 & b_2 & \dots & b_\delta & 0 \ \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & \dots & \dots & 0 \ b_1 & \dots & b_\delta \end{vmatrix}$$

K = field of quotients of $A \supseteq A$ ($\overline{K}$ algebra closure)

$$\mathrm{Res}(P,Q) = \pm a_d^\delta b_\delta^d \prod_{j,k} (\underset{\substack{\uparrow \\ \text{roots of } P \\ \text{in } \overline{K}}}{w_j} - \underset{\substack{\uparrow \\ \text{roots of } Q \\ \text{in } \overline{K}}}{t_k})$$

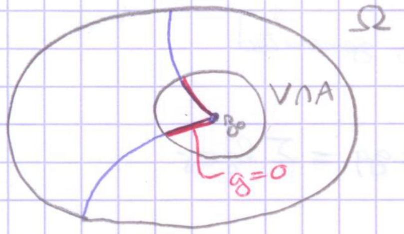
$\mathrm{Res}(P,Q) \in A$ vanishes $\Leftrightarrow$ P, Q have a common root in $\overline{K}$

for $\frac{1}{a_d} \mathrm{Res}(P, \frac{dP}{dT}) = \mathrm{Disc}(P) = \Delta(P)$ $\qquad \Delta(P)$ vanishes $\Leftrightarrow$ P has a multiple root !
$\qquad b_{d-1} = d a_d$

P irreductible $\Rightarrow \mathrm{PGCD}(P, \frac{dP}{dT}) = 1$
$\Rightarrow \Delta(P) \neq 0$

III - Analytic sets

Let $\Omega \subset \mathbb{C}^n$ be an open subset. A subset $A \subset \Omega$ is said to be complex analytic if:
- A is closed
- $\forall z_0 \in A$, $\exists$ open V neighborhood such that $A \cap V = \{ g_1 = \dots = g_N = 0 \quad g_j \in \mathcal{O}(V) \}$



Look at the ring $\mathcal{O}_{\Omega, z_0} \cong \mathcal{O}_n$ (a noetherian ring). Let $I \subset \mathcal{O}_{\Omega, z_0}$ be an ideal.

$I = (g_1, \dots, g_N)$ because of noetherianity.

zero variety of I = $V(I) = \{ z \in V$ small neighborhood of z_0 / $g_1(z) = \dots = g_N(z) = 0 \}$

One looks only at the germs of $V(I)$ at z_0 (identify 2 sets in their intersection with small W coincide).

If A is a germ of closed set in a small neighborhood of z_0

$A \rightsquigarrow I_A \subset \mathcal{O}_{\Omega, z_0}$

$I_A = \{ f \in \mathcal{O}_{\Omega, z_0} \ / \ f|_{A \cap V} = 0$ for small $V \}$

$I \rightsquigarrow V(I) = A$ a germ of analytic set

$I_A \leftarrow A$ any germ of closed set

so $\Rightarrow$ $I \rightsquigarrow A = V(I) \longrightarrow \mathcal{J} = I_A$ which $I \subset I_A$ (Nullstellensatz $\rightarrow \mathcal{J} = \sqrt{I}$)

$A \rightsquigarrow I_A \longrightarrow V(I_A)$ which $A \subset V(I_A)$

if A is analytic, then $V(I_A) = A$

$\hookrightarrow$ let us consider germs of analytic sets!

Definition: A irreductible if A cannot be decomposed as $A = A' \cup A''$ where $A' \subsetneq A$, $A'' \subsetneq A$ as germs

Fact: Every germ of analytic set A can be decomposed in finitly many irreductible germs of analytic sets

$A = A_1 \cup A_1'$
$= A_2 \cup A_2' \cup A_1'$
$= A_3 \cup A_3' \cup A_2' \cup A_1'$

if wrong, construct a infinite decreasing sequence $A \supsetneq A_1 \supsetneq A_2 \supsetneq \dots$ A_j germ of analytic sets non irreductible

$I_A \subsetneq I_{A_1} \subsetneq I_{A_2} \subsetneq \dots$

By the fact of $\mathcal{O}_n$ is noetherian, the sequence $I_A \subsetneq I_{A_1} \subset I_{A_2} \subsetneq \dots$ is stationnar.

What is a "germ of analytic hypersurface" looking like?

↳ $f \in \mathcal{O}_{\mathbb{C}^n,0}$, $f \neq 0$
$A = \{z \in V \,/\, f(z) = 0\}$
can rotate coordinates, and write $f(z) = u(z)P(z', z_n)$ where P is a Weierstrass polynomial.
$V = D(0,r)$ polydisc, u invertible on V

$A = \{z \in D(0,r) \,/\, P(z', z_n) = 0\}$

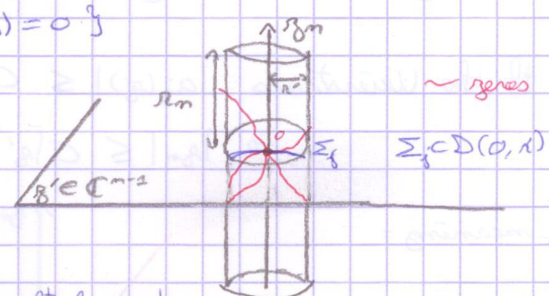
$P = P_1 \dots P_N$ where P_j irreductible Weierstrass polynomials

$A = \bigcup A_j$ where $A_j = \{z \in D(0,r) \,/\, P_j(z', z_n) = 0\}$

$P_j(z', z_n) \in \mathcal{O}_{n-1}[z_n]$

$\Delta_{\mathcal{O}_{n-1}}(P_j(z', z_n)) = \Delta_j(z') \neq 0$

$$\Sigma = \{z' \in D(0,r') \,/\, \Delta_j(z') = 0\}$$



If $z' \in D(0,r') \setminus \Sigma_j$ then, there are no multiple roots.

⇒ $P(z', z_n)$ has simple roots

⇒ $\frac{\partial P_j}{\partial z_n}(z', z_n)$ has no roots → does not vanishes

⇒ $\frac{\partial P_j}{\partial z_n}(z', z_n) \neq 0$ if z_n is a root

⇒ there are exacthy s_j roots z_n where $s_j = \deg_{z_n} P_j$

$\pi = A_j \longrightarrow D(0,r')$ a projection on $D(0,r')$
$z = (z', z_n) \longrightarrow z'$

so $\pi : A_j \setminus \pi^{-1}(\Sigma_j) \longrightarrow D(0,r') \setminus \Sigma_j$ is an etal covering with s_j sheets.
and $A_j \setminus \pi^{-1}(\Sigma_j)$ is a smooth complexe analytic hypersurface

$\Sigma = \bigcup \Sigma_j \bigcup_{j \neq k} V(\mathrm{Res}(P_j, P_k))$ $\qquad R_{jk}(z') = \mathrm{Res}(P_j(z', z_n), P_k(z', z_n)) \neq 0$

⇒ $\pi : A \setminus \pi^{-1}(\Sigma) \longrightarrow D(0,r') \setminus \Sigma$ is an etal covering with $\sum s_j$ sheets

Hypersurface $f(z) = 0$ with $f \in \mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$ $f \neq 0$ and $m = \mathrm{ord}_0(f)$

We can take coordinates $(z_1, \dots, z_n)$ so that $f(0, \dots, 0, z_n)$ vanishes at order m.

Up to invertible element in $\mathcal{O}_n$, we can assume for $z' = (z_1, \dots, z_{n-1})$ that

$$f(z) = z_n^m + a_1(z') z_n^{m-1} + \dots + a_m(z')$$

$$\mathrm{ord}_0\, a_j(z') \geq j$$

<u>Lemma</u>: $f(z) = z^m + a_1 z^{m-1} + \dots + a_m$

$$\max_{\substack{z_j \\ \mathbb{C} \text{ roots}}} |z_j| \leq 2 \max_{j=1,\dots,m} |a_j|^{1/j}$$

<u>proof</u>: Assume z root such that $|z| > 2 \max |a_j|^{1/j}$

$$\frac{f(z)}{z^m} : 1 = -\frac{a_1}{z} - \frac{a_2}{z^2} - \dots - \frac{a_m}{z^m}$$

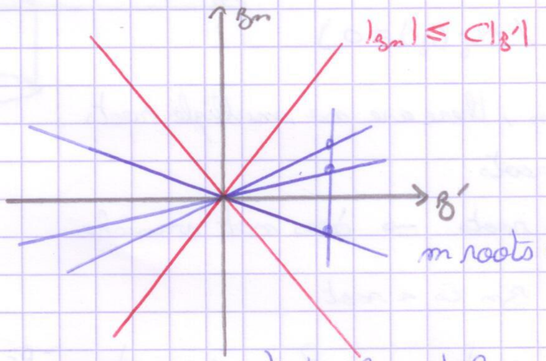
$$\Rightarrow \quad 1 \leq \frac{|a_1|}{|z|} + \frac{|a_2|}{|z|^2} + \dots + \frac{|a_m|}{|z|^m}$$

$$1 \leq \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^m} < 1 \quad \text{contradiction} \checkmark$$

Apply this to the Weierstrass $|a_j(z')| \leq C_j |z'|^j$ (with Taylor expansion)

$$\Rightarrow |z_n| \leq C|z'| \text{ for the roots}$$

geometric meaning:



<u>Definition</u>: a map $\varphi : X \longrightarrow Y$ is said to be étale if

$\forall x_0 \in X$ $\exists V$ neighborhood of x_0 in X
$\exists W$ neighborhood of $\varphi(x_0)$ in Y such that

- $\varphi : V \longrightarrow W$ homeomorphism ($\mathcal{C} = \mathcal{C}_{top}$)
- $\varphi : V \longrightarrow W$ $\mathcal{C}^k$ diffeomorphism ($\mathcal{C}^k$ manifolds)
- $\varphi : V \longrightarrow W$ biholomorphism (hol manifolds)

<u>Theorem</u>: Let A be a germ of a $\mathbb{C}$-analytic set in a neighborhood $V \ni 0$ in $\mathbb{C}^n$

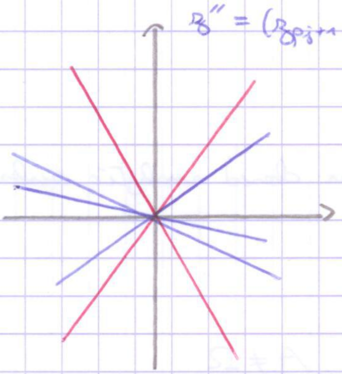
$$A = \{ z \in V \,/\, g_1(z) = \dots = g_N(z) = 0 \quad g_j \in \mathcal{O}(V) \}$$

<u>i</u>: $A = \bigcup_{1 \leq j \leq s} A_j$ with A_j irreducible germs

existence: consequence of noetherianity
uniqueness: exercice

<u>ii</u>: $\dim A_j = p_j$

$\exists$ coordinates $(z_1, \dots, z_n)$ (the same for all A_j) such that A_j looks like this



$$\begin{cases} \underset{\mathbb{C}^{p_j}}{D(0,r')} \times \underset{\mathbb{C}^{n-p_j}}{D(0,r'')} = \Delta \\ r' << r'' \end{cases}$$

$\exists$ analytic set $\Sigma_j \subset D(0,r')$ such that the projection

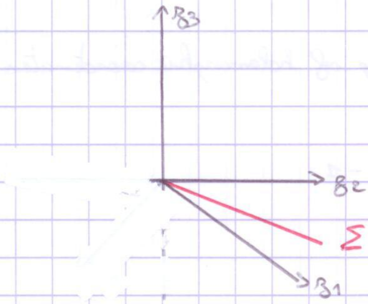
$\pi : \Delta \longrightarrow D(0,r')$ étale cover of given degree d_j

$z = (z', z'') \longrightarrow z'$ on $D(0,r') \setminus \Sigma_j$

Examples:

1: $f(z) = z_3^2 + z_1^4 + z_2^5$

$$z_3 = \pm i \sqrt{z_1^4 + z_2^5}$$



$\Sigma = \{(z_1, z_2) \in \mathbb{C}^2 \,/\, z_1^4 + z_2^5 = 0\}$

2: $\mathbb{C}^3$ $\quad g_1(z) = z_1 z_2$
$\quad g_2(z) = z_1 z_3$

$A = \{z \in \mathbb{C}^3 \,/\, g_1(z) = g_2(z) = 0\}$
$= A_1 \cup A_2$

$A_1 = \{z_1 = 0\}$ a plane
$A_2 = \{z_2 = z_3 = 0\}$ a line

(need "oblique coordinates" to realize as branched covering)

Definition: If A is an analytic set

a: A_{reg} = set of regular points $= \{z_0 \in A \,/\, \exists V \ni z_0$ neigh such that $A \cap V$ smooth subvariety of any dimension $p\}$

in that case $A \cap V = \{g_1(z) = \dots = g_{n-p}(z) = 0\}$ $\quad dg_1, \dots, dg_{n-p}$ are independent

b: $A_{sing} = A \setminus A_{reg}$

Lemma: • If A is regular at $0 \in \mathbb{C}^n$ then it is irreducible as a germ.
• If $\dim A = p$, as a complex submanifold, A is biholomorphic to $(\mathbb{C}^p, 0)$.

proof: Have to show that $(\mathbb{C}^n, 0)$ is irreducible.

$(\mathbb{C}^n, 0) = A' \cup A''$ $\quad A' \subset \{f' = 0\}$ $f' \neq 0$
$\quad A'' \subset \{f'' = 0\}$ $f'' \neq 0$

then $f'f'' = 0$ contradiction because $\mathcal{O}_n$ is an integral ring $\Rightarrow (\mathbb{C}^n, 0)$ irreducible

Assume $A = \bigcup A_j$ ← irreducibles

$A_{sing} = \bigcup_{j \neq k} A_j \cap A_k \cup \left(\bigcup_j (A_j)_{sing} \right)$

$A_{reg} = \bigcup_j \left(A_{j\,reg} \setminus \bigcup_{k \neq j} A_k \right)$

Theorem: If $A \subset \underset{open}{\Omega} \subset \mathbb{C}^n$ is an analytic set, the A_{sing} is a closed analytic subset of A.

↳ consequence: $A_{reg} = A \setminus A_{sing}$ is open in A

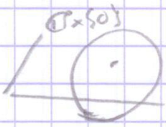
Theorem: $\Omega \subset \mathbb{C}^n$ a connected open set. $A \subset \Omega$ analytic subset, $A \neq \Omega$

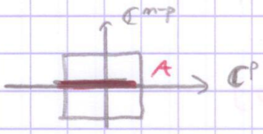
Take $f \in \mathcal{O}(\Omega \setminus A)$

(i) if f is bounded near every $z_0 \in A$, then f extends to $\mathcal{O}(\Omega)$

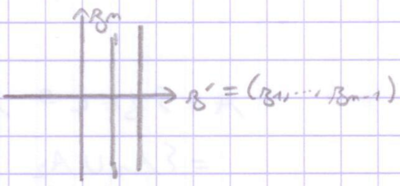
(ii) Riemann - Hartogs
If codim $A \geq 2$, then $f \in \mathcal{O}(\Omega \setminus A)$ always extends to $\mathcal{O}(\Omega)$

main point: for A is a smooth $\mathbb{C}$ p-dim submanifold. After a change of holomorphic coordinates, can assume (in a neighborhood of 0)





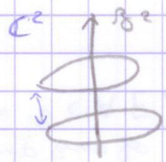
- codim $A = 1$ $\rightarrow p = n-1$
  assuming f bounded
  apply 1-variable know result



$z_n \longrightarrow f(z', z_n)$ extends holomorphically at $z_n = 0$

Cauchy formula in variable z_n $\quad f(z', z_n) = \frac{1}{2i\pi} \int_{|w_n| = \varepsilon_n} \frac{f(z', w_n)}{w_n - z_n} \, dw_n$ holomorphic in (z', z_n) for $|z_n| < \varepsilon_n$

- codim $A \geq 2$
  don't even assume f bounded.



$D(0,\varepsilon_1) \times D(0,\varepsilon_2) \setminus \{(0,0)\}$

$D(0, r_1) \times \{z_2\}$ $\quad z_2 \neq 0$

$$\sup_{D(0,r_1) \times \{z_2\}} |f| \overset{\text{maximum principle}}{=} \sup_{\partial D(0,r_1) \times \{z_2\}} |f|$$

$$\Rightarrow \sup_{\overline{D}(0,r_1) \times \overline{D}(0,r_2) \setminus \{(0,0)\}} |f| = \sup_{\partial D(0,r_1) \times \overline{D}(0,r_2)} |f|$$

$\Rightarrow$ f is automatically bounded

observation: A irreducible analytic set $\Rightarrow \dim A_{sing} < \dim A$

$A \subset \Omega$

$\underset{A_{reg}}{\underset{\|}{A \setminus A_{sing}}} \subset \Omega \setminus A_{sing} = \Omega' \in \mathbb{C}^n$

A_{reg} is a smooth disjoint union of closed submanifolds in Ω'

$\Rightarrow f$ extends to $\Omega' = \Omega \setminus A_{sing}$ $\qquad$ apply induction on $\dim A$.

Proposition : Every analytic set $A \subset \Omega$ admits a "complex analytic stratification", ie one can find

$$A^{(0)} \subset A^{(1)} \subset \cdots \subset A^{(p)} = A$$

$A^{(j)}$ is a closed analytic set in Ω

$A^{(j)} \setminus A^{(j-1)}$ smooth submanifold of $\Omega \setminus A^{(j-1)}$ of dimension j.

$p = \dim A$ = max of dimension of various components.

proof :
$A^{(p)} = A$
$A^{(p-1)} = A_{sing}$
$A^{(p-2)} = (p-2)$ components of $A_{sing} \cup (A_{sing})_{sing}$
$A^{(p-3)} = (p-3)$ " " $A_{sing} \cup (A_{sing})_{sing} \cup ((A_{sing})_{sing})_{sing}$
$\vdots$

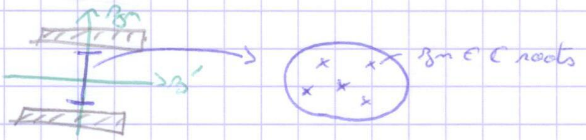
Theorem :

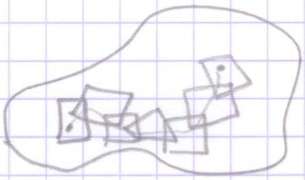
(1) $\Omega \subset \mathbb{C}^n$ connected, $A \subsetneq \Omega$ analytic $\Rightarrow \Omega \setminus A$ is connected

(2) A irreducible germ at z_0 - $\exists$ fundamental system of neighborhoods V of z_0 such that $A_{reg} \cap V$ is a connected manifold

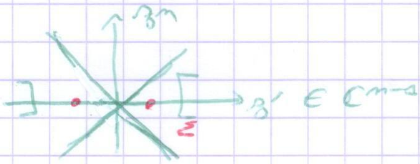
proof :

1 : long to write





2 : case $\operatorname{codim} A = 1$, A irreducible



$z \longrightarrow z'$ branched covering
$D(0,r) \longrightarrow D(0,r')$

$D(0,r') \setminus \Sigma$ is connected
$\in \mathbb{C}^{n-1}$; Σ : branching locus

covering has a finite number of connected components that provide coverings $(\rho_j)_{1 \le j \le N}$ over $D(0,r) \setminus \Sigma$

$$f(z', z_n) = \prod (z_n - w_\ell(z'))$$
$w_\ell(z')$: roots

$$f_j(z', z_n) = \prod_{\substack{\text{roots } w_\ell(z') \\ \text{that belong to} \\ \rho_j \text{ covering}}} (z_n - w_\ell(z'))$$

f_j holomorphic on $(D(0,r') \setminus \Sigma) \times D(0,r_n)$ and bounded $\Rightarrow$ it extends !

$f = f_1 \cdots f_N$ $\quad \{f = 0\} = \bigcup \{f_j = 0\}$ not irreducible unless 1 component

I - Concept of sheaf

Sheaf = faisceau, invented by Jean Leray in a camp in Germany during WW2. Cartan, Serre and Grothendick also worked on this subject.

X a topological space

Definition: $\mathcal{S}$ a presheaf of sets is:

- a collection of sets $\mathcal{S}(U)$ attached to each $U \neq \emptyset$ open in X
- Whenever $V \subset U$ open sets in X one has a "restriction map"

$$\rho^U_V : \mathcal{S}(U) \longrightarrow \mathcal{S}(V)$$
$$f \longrightarrow \rho^U_V(f)$$

- Axiom $W \subset V \subset U$

$$\begin{tikzcd} \mathcal{S}(U) \arrow[r, "\rho^U_V"] \arrow[rr, bend right, "\rho^U_W"'] & \mathcal{S}(V) \arrow[r, "\rho^V_W"] & \mathcal{S}(W) \end{tikzcd}$$

and $\rho^U_W = \rho^V_W \circ \rho^U_V$

- Presheaf of groups $\begin{cases} \mathcal{S}(U) \text{ group} \\ \rho^U_V \text{ homomorphisms of groups} \end{cases}$

- $\mathbb{K}$ vector space ρ^U_V a $\mathbb{K}$ linear map

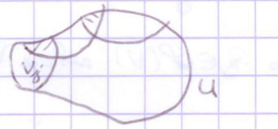
Category: Collection $\mathcal{C}$ of objects X and of arrows $f : X \longrightarrow Y$ which f preserve the structure of the object

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "h = g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

categories of sets, groups, A-modules, rings, $\mathbb{K}$-algebra ...

Definition: A sheaf is a presheaf satisfying the following additional axiom:

assume
- $U = \bigcup_{j \in I} V_j$



- given $f_j \in \mathcal{S}(V_j)$ such that

$$\rho^{V_j}_{V_j \cap V_k}(f_j) = \rho^{V_k}_{V_j \cap V_k}(f_k)$$

then exists a unique $f \in \mathcal{S}(U)$ such that $\rho_{V_j}(f) = f_j$

Examples:

1: $\mathcal{C}(U) = \{ U \xrightarrow{f} \mathbb{R} \text{ continuous} \}$

$\rho^U_V(f) = f|_V$ the restriction

$\mathcal{C}$ is a sheaf

2: $\mathcal{B}(U) = \{ U \xrightarrow{f} \mathbb{R} \text{ bounded} \}$

$\rho_V^U(f) = f|_V$

$\mathcal{B}$ is a presheaf but not a sheaf

indeed: take $X = \mathbb{R}$, $U = \mathbb{R}$, $V_j =]j-1, j+1[$ $j \in \mathbb{Z}$

we have $U = \bigcup_{j \in \mathbb{Z}} V_j$

$f_j(x) = x$ on V_j and f_j is bounded on V_j

we have $\rho^{V_j}_{V_i \cap V_j}(f_j) = \rho^{V_i}_{V_i \cap V_j}(f_i)$ $\forall i,j \in \mathbb{Z}$ but

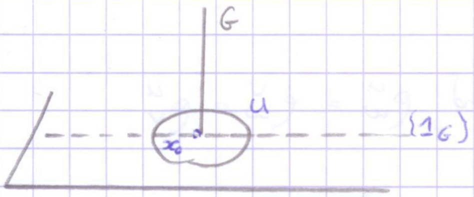
$\rightarrow f(x) = x$ $\forall x \in U$, but in this case f is not bounded $\rightarrow f \notin \mathcal{B}$

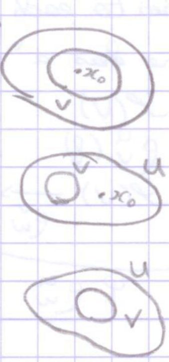
so $\mathcal{B}$ is not a sheaf ✓

3: Skyscraper sheaf at a point $x_0 \in X$, Let G be a group

$\mathcal{S}(U) = G$ if $x_0 \in U$

$\mathcal{S}(U) = \{1_G\}$ if $x_0 \notin U$

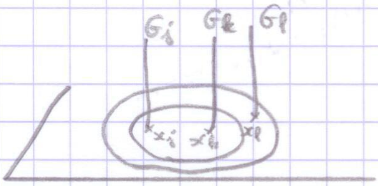




$\rho_V^U = Id_G : G \to G$

$\rho_V^U = G \longrightarrow \{1_G\}$

$\rho_V^U = \{1_G\} \longrightarrow \{1_G\}$



$(x_j)_{j \in \mathbb{N}}$ a discret sequence

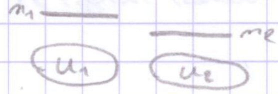
G_i abelian group

$\mathcal{S}(U) = \prod_{x_j \in U} G_j$

4: $\underline{\mathbb{Z}}$ = sheaf of locally constant functions

$\underline{\mathbb{Z}}(U) = \{ U \xrightarrow{f} \mathbb{Z} \text{ locally constant} \}$

$\{$constant function $U \xrightarrow{f} \mathbb{Z}\}$ is only a presheaf



$U = U_1 \cup U_2$

$f = \begin{cases} n_1 \text{ on } U_1 \\ n_2 \text{ on } U_2 \end{cases}$ is not constant

Definition: Let $\mathcal{S}$ be a sheaf.

$\mathcal{S}_x = \{$germs of sections $f \in \mathcal{S}(V)$ where V is an arbitrary small neighborhood of $x\}$

$= \coprod_{\substack{V \ni x \\ \text{open}}} \mathcal{S}(V) \big/ \sim$ "stalk of $\mathcal{S}$ at x"

(en fr: = la fibre en x du fibré
⚠ pas le même type de fibré qu'en géométrie diff)

$f_1 \in \mathcal{S}(V_1)$
$f_2 \in \mathcal{S}(V_2)$

$f_1 \sim f_2 \iff \exists W \subset V_1 \cap V_2$ such that $\rho_W^{V_1}(f_1) = \rho_W^{V_2}(f_2)$

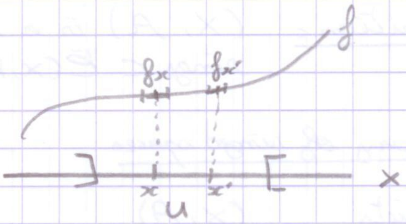
disjoint

$$\begin{array}{ccc} S & = & \coprod_{x \in X} \mathcal{S}_x \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & & x \end{array}$$

S = "étalé space associated with $\mathcal{S}$"

Take U an open set in X

$$f \in \mathcal{S}(U) \rightsquigarrow f_x = \text{germ of } f \text{ at } X$$



$$\Omega_{U,f} = \{ f_x \ / x \in U \} \subset S$$
($f_x \in \mathcal{S}_x$)

so: an open set of the étale space associated with $\mathcal{S}$ (S) is any $W = \bigcup_{j \in I} \Omega_{U_j, f_j}$ for $f_j \in \mathcal{S}(U_j)$, or $W = S$ or $W = \emptyset$.

$p : S \longrightarrow X$ is an étale map

$$\begin{array}{ccc} S & \longrightarrow & X \\ \cup & & \cup \\ \Omega_{U,f} & \xrightarrow{\text{homeo}} & U \\ \cup\!\!\!\!\!\!\!\!\!\;\;\,\in & & \in \\ f_x & \longrightarrow & x \end{array}$$

$V \subset U$
$$p^{-1}(V) = \Omega_{V, \rho_U^V(f)}$$

Definition: An étale space S over X is just a topological space S together with an étale map $p : S \longrightarrow X$

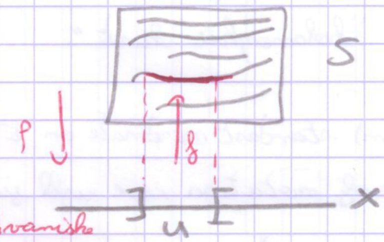
$S_x = p^{-1}(x)$ object in a given category | set, groups, ring, K-vector space, ⋮

so we have associated to a sheaf $\mathcal{S}$ an étale space S. Now, we are going to associate to an étale space a sheaf.

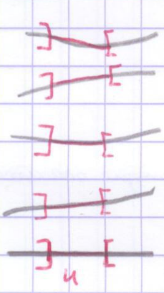
Fix $p : S \longrightarrow X$ an étale space.
Define a sheaf $\mathcal{S}$ by:

$$\mathcal{S}(U) \underset{\text{def}}{=} \left\{ f : \begin{array}{l} U \longrightarrow S \\ x \longrightarrow S_x = p^{-1}(x) \end{array} \quad \text{continuous} \right\}$$

$U \subset X$ open $\qquad p \circ f = Id_U$

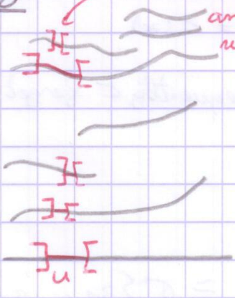


Different between étale space and covering



covering space
↳ "uniform shape"

can be reduce to a point, vanish and became longer if we modify U.



étale space
↳ non uniformity

II - Ringed spaces

Definition: (X, A) is a ringed space for X a topological space and A a sheaf of rings $\subset \mathcal{F}(X,R)$ with R a given ring.

Category of ringed spaces

objects (X, A)

morphism? $(X, A) \xrightarrow{\varphi} (Y, \mathcal{B})$

morphism:

(i) $\varphi : X \longrightarrow Y$ continuous

(ii) $\varphi^* : \mathcal{B}_{\varphi(x)} \longrightarrow A_x$

$g \longrightarrow g \circ \varphi \in A_x$

isomorphism $(X, A) \xrightarrow{\varphi} (Y, \mathcal{B})$

$(X, A) \xleftarrow{\varphi^{-1}} (Y, \mathcal{B})$

examples:

- $X = \mathbb{R}^n$ $\quad \mathcal{C}^k_{X,\mathbb{R}} = \{U \xrightarrow{f} \mathbb{R} \quad \mathcal{C}^k \text{ functions}\}$ $\quad U \subset X = \mathbb{R}^n$

$\rightarrow$ ringed space $(\mathbb{R}^n, \mathcal{C}^k_{X,\mathbb{R}})$ $\quad k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$

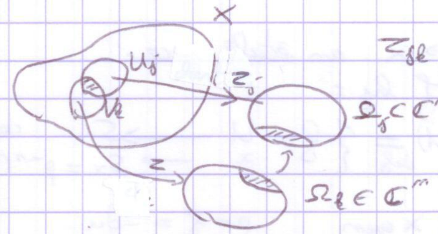
- $X = \mathbb{C}^n$ $\quad \mathcal{O}_X = \{U \xrightarrow{f} \mathbb{C} \text{ holomorphic}\}$

Definition: A complex holomorphic manifold of dimension n is a ringed space $(X, \mathcal{O}_X)$ that is
- locally isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$
- X Hausdorff, countable union of open sets

means $\forall x_0 \in X \quad \exists U \ni x_0 \quad \exists V \subset \mathbb{C}^n$

$\exists \tau \quad (U, \mathcal{O}_{X|U}) \xrightarrow{\tau} (\Omega, \mathcal{O}_{\mathbb{C}^n|V})$

"holomorphic chart".



$\tau_{jk} = \tau_k(U_j \cap V_k) \to \tau_j(U_j \cap V_k)$ biholomorphic

$(z_1, \dots, z_n)$ standard coordinate on $\mathbb{C}^n \rightsquigarrow (z_1 \circ \tau, \dots, z_n \circ \tau)$ functions on $\mathcal{O}_X(U)$

By abuse of notation, we will just consider $(z_1, \dots, z_n)$ to be functions on U and don't note τ.

$\mathcal{O}_{X,x} \overset{\tau^*}{\cong} \mathcal{O}_{\mathbb{C}^n, p}$ $\quad$ with $p = \tau(x)$

$f \circ \tau \longleftarrow f$ $\quad$ (frequently τ forget)

$\mathcal{O}_{\mathbb{C}^n, p}$ is a noetherian local ring $\quad$ local ring = has a unique maximal ideal

$\hookrightarrow$ can assume $p = 0$

unique maximal ideal $m_{X,x} \in \mathcal{O}_{X,x} = \mathbb{C}\{z_1, \dots, z_n\}$

so $\quad m_{X,x} = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$

$= \{f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \quad (a_0 = 0)\}$

$= (z_1, \dots, z_n)$ factor of $z_1, z_2, \dots z_n$

$m_{X,x}^k = (z^\alpha)_{|\alpha|=k}$

- $\mathcal{O}_{X,x}/m_{X,x} \simeq \mathbb{C}$
  $f \to f(0)$
- $\mathcal{O}_{X,x}/m_{X,x}^{k+1} = \mathbb{C}[z_1, \dots, z_n]_{\leq k}$
- $m_{X,x}^k / m_{X,x}^{k+1} = \mathbb{C}[z_1, \dots, z_n]_{=k}$ homogenous of degree k

$\dim(m_{X,x}/m_{X,x}^2) = n$

Vector fields, tangent sheaf

Definition: R a ring, a derivation of R is a map $D : R \longrightarrow R$ such that

- $D(a+b) = D(a) + D(b)$
- $D(ab) = D(a).b + a.D(b)$

In case $R = A$ a $\mathbb{K}$-algebra (commutative) over a field $\mathbb{K}$, one assume implicitly D to be $\mathbb{K}$-linear

$\hookrightarrow \forall \lambda, \mu \in \mathbb{K} \quad \forall a, b \in A$
$D(\lambda a + \mu b) = \lambda D(a) + \mu D(b)$

Fact: $D(1) = 0$ and in a $\mathbb{K}$-algebra A, $D(\lambda) = 0$

$\hookrightarrow D(1) = D(1.1) = D(1).1 + 1.D(1) = 2D(1) \Rightarrow D(1) = 0$

$D(\underbrace{\lambda}_{\text{constant}}) = D(\lambda.1) = \lambda.D(1) = 0$

goal: understand what are the derivations of the sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras $U \rightsquigarrow \mathcal{O}_X(U)$

A derivation means that for every U open in X, one has $D_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)$ a $\mathbb{C}$-linear derivation such that:

$\forall U' \subset U \quad (D_U f)_{|U'} = D_{U'}(f_{|U'})$ for $f \in \mathcal{O}_X(U)$

(this definition will works for any ringed space (X, A)).

fix $x \in X \xrightarrow{z} p = z(x)$ and can assume that $p = 0$

$f(x) = a_0 + a_1 z_1 + \dots + a_n z_n + g(x)$ where $g \in m_{X,x}^2$

so $Df(x) = \sum_{j=1}^{n} a_j D(z_j) + \underbrace{Dg(x)}_{=0}$

$Dg(x) = 0$ because $g \in m_{X,x}^2$ so $g = \sum_\ell u_\ell v_\ell$ for $u_\ell, v_\ell \in m_{X,x}$

↳ $Dg(x) = \sum_\ell Du_\ell(x) \cdot v_\ell(x) + u_\ell(x) D v_\ell(x)$
$= 0$

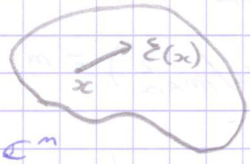
let us introduce $\xi_j(x) = Dz_j(x)$ and suppose $(z_1, \dots, z_n)$ defined on $U \subset X$

↳ then $\xi_j \in O_X(U)$

$Df(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(x) \cdot \xi_j(x)$

<u>Theorem</u>: Given a coordinate chart $U \xrightarrow{z} \Omega \subset \mathbb{C}^n$ on X, the derivations D of $O_X|_U$ are given by vector fields:

$\xi(x) = \sum_{j=1}^{n} \xi_j(x) \frac{\partial}{\partial z_j}$



$Df = D_\xi f = \sum_{j=1}^{n} \xi_j(x) \frac{\partial f}{\partial z_j}(x)$

$U \xrightarrow{z} \Omega \subset \mathbb{C}^n$

$\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$ canonical basis of $\mathbb{C}^n$

<u>Tangent sheaf</u>: $T_X(U) = \{\text{derivations of } O_X(U)\}$

↳ sheaf of O_X-modules

$\xi \in T_X(U)$
$g \in O_X(U)$ $\Rightarrow$ $g \cdot \xi = \sum_{j=1}^{n} g(x)\, \xi_j(x) \frac{\partial}{\partial z_j}$

<u>Definition</u>: Given a ringed space (X, A), an A-module $\mathcal{M}$ is a sheaf

$U \longrightarrow \mathcal{M}(U)$ of $A(U)$-modules

(with compatibility with restriction)

<u>Example</u>: • Free module $\mathcal{M}$ of rank r over A

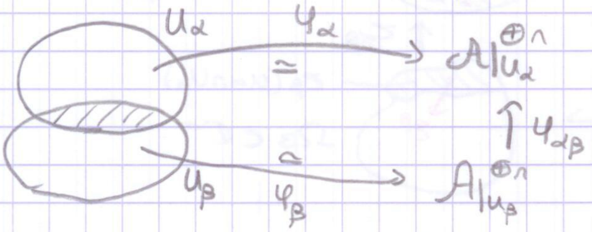
$\mathcal{M} = A^{\oplus r}$

• $\mathcal{M} \simeq A^{\oplus r}$ free A-module

Definition : $\mathcal{M}$ is said to be a locally free $\mathcal{A}$-module of rank r if

$\forall x_0 \in X$, $\exists U$ a neighborhood of x_0 such that $\mathcal{M}|_U \underset{\varphi_U}{\simeq} \mathcal{A}|_U^{\oplus r}$

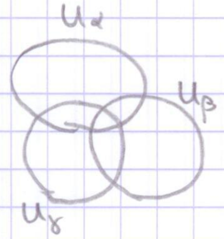
then $\exists$ open covering $(U_\alpha)_{\alpha\in I}$ of X and an isomorphism $\mathcal{M}|_{U_\alpha} \xrightarrow[\varphi_\alpha]{\sim} \mathcal{A}|_{U_\alpha}^{\oplus r}$



on $U_\alpha \cap U_\beta$ we have the transition isomorphism

$$\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \mathcal{A}|_{U_\alpha\cap U_\beta}^{\oplus r} \longrightarrow \mathcal{A}|_{U_\alpha\cap U_\beta}^{\oplus r}$$

- $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \end{pmatrix} \in \mathcal{A}|_{U_\alpha\cap U_\beta}^{\oplus r}$ $\quad \xi \longrightarrow g_{\alpha\beta}\cdot \xi$
- $g_{\alpha\beta}$ a $r\times r$ matrix with coefficients in $\mathcal{A}|_{U_\alpha\cap U_\beta}$



On $U_\alpha \cap U_\beta \cap U_\gamma \Rightarrow \begin{cases} \varphi_{\alpha\gamma} = \varphi_\alpha \circ \varphi_\gamma^{-1} = \varphi_{\alpha\beta}\circ\varphi_{\beta\gamma} \\ g_{\alpha\gamma} = g_{\alpha\beta}\cdot g_{\beta\gamma} \end{cases}$ cocycle condition

in particular $\varphi_{\alpha\alpha} = Id$ on U_α

$\Rightarrow \varphi_{\alpha\beta}\circ\varphi_{\beta\alpha} = Id$ on $U_\alpha\cap U_\beta$

$\Rightarrow g_{\alpha\beta}\cdot g_{\beta\alpha} = Id$ on $U_\alpha \cap U_\beta$

$\Rightarrow g_{\alpha\beta}$ is an invertible matrix with coefficient in $\mathcal{A}|_{U_\alpha\cap U_\beta}$

$\Rightarrow g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

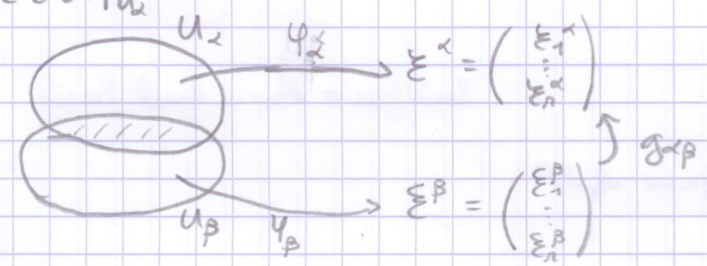
Conversely : Take open covering (U_α) of X and a collection of "transition matrices" $g_{\alpha\beta}$ of size $r\times r$ with coefficients in $\mathcal{A}|_{U_\alpha\cap U_\beta}$ satisfying the cocycle conditions.

- $g_{\alpha\alpha} = I$ on each U_α
- $g_{\alpha\gamma} = g_{\alpha\beta}\circ g_{\beta\gamma}$ on $U_\alpha\cap U_\beta\cap U_\gamma$ for all indices $\alpha,\beta,\gamma\in I$

Then one can construct a locally free rank r $\mathcal{A}$-module $\mathcal{M}$ such that

$\mathcal{M}|_{U_\alpha} \simeq \mathcal{A}|_{U_\alpha}^{\oplus r}$ glued by the $g_{\alpha\beta}$ on $U_\alpha\cap U_\beta$

$\xi \in \mathcal{M}|_{U_\alpha}$



gluing condition : $\xi^\alpha = g_{\alpha\beta}\,\xi^\beta$ on $U_\alpha \cap U_\beta$

reason for this : take $\xi \in \mathcal{M}|_{U_\alpha\cap U_\beta}$

$\xi^\alpha = \varphi_\alpha(\xi)$ $\quad$ $\xi^\beta = \varphi_\beta(\xi)$

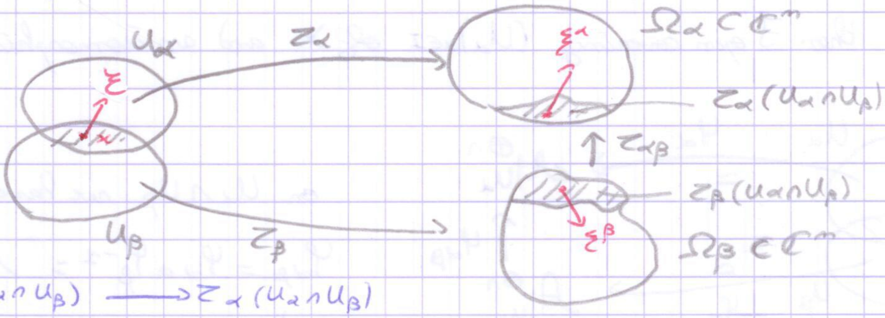
$\xi^\alpha = \varphi_{\alpha\beta}(\xi^\beta) = g_{\alpha\beta} \cdot \xi^\beta$

Let us come back to $\mathcal{T}_X$ which is a locally free $\mathcal{O}_X$-module of rank n.

↳ given a chart $U \xrightarrow{z} \Omega \subset \mathbb{C}^n$

A basis of $\mathcal{T}_X|_U$ is $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$

what are the transition matrices?



$z_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} = z_\beta(U_\alpha \cap U_\beta) \longrightarrow z_\alpha(U_\alpha \cap U_\beta)$

is a biholomorphism

$\xi^\alpha = d z_{\alpha\beta}(z)(\xi^\beta)$ this is the "chain rule" of derivative of $f_\alpha \circ z_{\alpha\beta}$

↳ for $f \in \mathcal{O}_X(U_\alpha \cap U_\beta)$.

$f_\alpha = f \circ z_\alpha^{-1}$

$f_\beta = f \circ z_\beta^{-1}$

$$\begin{aligned} D f &= D_\xi \cdot f \\ &= D_{\xi^\alpha}(f_\alpha) \circ z_\alpha^{-1} \\ &= D_{\xi^\beta}(f_\beta) \circ z_\beta^{-1} \end{aligned}$$

$$g_{\alpha\beta}(z) = \left(\frac{\partial z_{\alpha\beta}(z)_i}{\partial z_j} \right)_{1 \le i \le n}$$ $n \times n$ matrix

$z_{\alpha\beta} =$ open set $\subset \mathbb{C}^n \longrightarrow$ open set $\subset \mathbb{C}^n$

$z \longrightarrow (z_{\alpha\beta}(z)_i)_{1 \le i \le n}$

satisfying the cocycl condition

⊛ Dual modules

$\mathcal{M}$ an A-module

$\downarrow$

$\mathcal{M}^* = \mathrm{Hom}_A(\mathcal{M}, A)$

$\mathcal{M}^*(U) = \mathrm{Hom}_{A(U)}(\mathcal{M}(U), A(U))$

fact: $\mathcal{M}$ locally free $\Rightarrow$ $\mathcal{M}^*$ locally free

local free $\equiv$ has a local basis

$(e_1, \dots, e_r)$ local basis $\Rightarrow$ $(e_1^*, \dots, e_r^*)$ dual frame of $\mathcal{M}^*$

"local frame"

$e_j^*(e_k) = \delta_{jk}$

$\xi \in \mathcal{T}_X(U)$ a vector field on U
$f \in \mathcal{O}_X(U)$

notation $\xi . f = D_\xi f = \sum_{j=1}^{n} \xi_j(x) \frac{\partial f}{\partial z_j} = df(\xi)$

this defines $df \in \mathcal{T}_X^*(U)$ called the differential of f.

Given local holomorphic coordinates $(z_1, \dots, z_n)$ on $U \subset X$, so $(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})$ defines a holomorphic frame (basis) of $\mathcal{T}_X(U)$ and $(dz_1, \dots, dz_n)$ the dual frame of $\mathcal{T}_X^*(U)$.

$$dz_j\left(\frac{\partial}{\partial z_k}\right) = \frac{\partial}{\partial z_k}(z_j) = \delta_{jk} \text{ matrix}$$

Definition: $\mathcal{T}_X^*$ is the cotangent sheaf

Fact: there is an "isomorphism of categories" between locally free sheaves of rank n on $(X, \mathcal{O}_X)$ and holomorphic vector bundles $E \to X$

$\mathcal{F}, \mathcal{G}$ $\mathcal{A}$-modules on $(X, \mathcal{A})$ and a morphism $\mathcal{F} \to \mathcal{G}$ on X
$\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is an $\mathcal{A}(U)$-homomorphism for $U \subset X$
commuting with restrictions

morphism $\mathcal{F}|_U \longrightarrow \mathcal{G}|_U$ for $V \subset U$
$\mathcal{F}(V) \longrightarrow \mathcal{G}(V)$

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) = \{ \underset{V \subset U}{\mathcal{F}(V) \longrightarrow \mathcal{G}(V)} \}$$

Smooth manifold of class $\mathcal{E}^k$ $\to$ $(X, \mathcal{E}_X^k)$ ringed space on Hausdorff topological space, that is a countable $\cup$ compact sets locally isomorphic to $(\mathbb{R}^n, \mathcal{E}_{\mathbb{R}^n}^k)$

- differential atlas
- locally free $\mathcal{E}_X^k$-module

$k \in \mathbb{N} \cup \{\infty, \omega\}$

Definition: $A \subset B$ subring.

$$\mathrm{Der}(A, B) = \{ D : A \longrightarrow B \text{ derivation} \}$$

$\hookrightarrow$ $D(a+b) = D(a) + D(b)$
$D(ab) = D(a)b + aD(b)$
$\mathbb{K}$-linear for $\mathbb{K}$-algebra

$\mathcal{E}_X^k \subset \mathcal{E}_X^{k-1}$

$\mathcal{T}_X$ sheaf of derivations $\mathcal{C}^k_X \longrightarrow \mathcal{C}^{k-1}_X$

<u>Lemma:</u> $f \in \mathcal{C}^k_{X,x}$ $\quad U \xrightarrow{\tau} \Omega \subset \mathbb{R}^n$ $\quad (x_1, \dots, x_n)$ local coordinate

$x \longrightarrow p = 0$

maximal ideal $\quad m_{X,x} = \{ f \,/\, f(x) = 0 \}$

$$\mathcal{C}^k_{X,x} / m_{X,x} \cong \mathbb{R}$$

for $f(x) = O(|x|^2)$

$f(x) = \sum x_i x_j u_{ij}(x)$

Taylor's formula $\longrightarrow \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tx)\, dt = u_{ij}(x) \qquad u_{ij} \in \mathcal{C}^{k-2}$

If $f \in \mathcal{C}^{k+2}$, then $u_{ij} \in \mathcal{C}^k$

$\Rightarrow$ for any derivation D, $Df(x) = 0$

Additional condition: derivation D to be considered should be continuous operators

$$\mathcal{C}^k_X(U) \xrightarrow{D} \mathcal{C}^{k-1}_X(U)$$

By density of $\mathcal{C}^{k+2}$ in $\mathcal{C}^k$ (or even $\mathcal{C}^\infty$ in $\mathcal{C}^2$), we can conclude that any derivation is of the form:

$$\hookrightarrow \quad \xi(x) = \sum_{j=1}^{n} \xi_j(x) \frac{\partial}{\partial x_j}$$

$$\xi_j(x) = \xi \cdot x_j \in \mathcal{C}^{k-1}$$

cotangent sheaf $\mathcal{T}^*_X$, $\mathcal{T}_X$ and $\mathcal{T}_X^*$ are $\mathcal{C}^{k-1}_X$-modules

Let $(X, \mathcal{O}_X)$ be a n-dimensional complex manifold locally isomorphic to $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$

$$\mathcal{O}_X \subset \mathcal{C}^\infty_{X,\mathbb{C}} = \mathcal{C}^\infty_X \otimes_{\mathbb{R}} \mathbb{C}$$

Therefore this defines a morphism of ringed spaces

$$(X, \mathcal{C}^\infty_{X,\mathbb{C}}) \xrightarrow{\varphi = Id} (X, \mathcal{O}_X)$$

$$f \circ \varphi \longleftarrow f$$

<u>Definition:</u> $(X, \mathcal{C}^\infty_X)$ is called the underlying $\mathcal{C}^\infty$ manifold of the complex analytic manifold

$\dim_{\mathbb{R}} X = n$

$\dim_{\mathbb{C}} X = 2n$

- complex holomorphic tangent sheaf $\mathcal{T}_X$ locally generated by $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)$

as a real C^∞ manifold, we use real coordinates $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ where $z_j = x_j + i y_j$

- real tangent sheaf $\mathcal{T}_{X\mathbb{R}}$ is a $\mathcal{C}^\infty_{X,\mathbb{R}}$-locally free sheaf locally generated by $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right)$

$\mathrm{Der}_{\mathbb{C}}(\mathcal{C}^\infty_{X,\mathbb{C}}) = \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{X,\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$ correspond to the derivations of the form:

$$\sum_{j=1}^{m} \alpha_j(x,y)\frac{\partial}{\partial x_j} + \beta_j(x,y)\frac{\partial}{\partial y_j}$$ α_j, β_j complex valued functions

$T_X^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Der}(\mathcal{E}^{\infty}_{X,\mathbb{C}})$

In case of using $\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right)$, we will frequently use instead

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad dz_j = dx_j + i\, dy_j$$

$$\frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \qquad d\bar z_j = dx_j - i\, dy_j$$

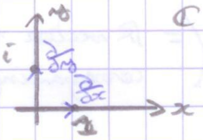
$f(x,y)$ has a differential $\displaystyle df = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(x,y)\, dx_j + \frac{\partial f}{\partial y_j}(x,y)\, dy_j$

$f \in \mathcal{E}^1_{X,\mathbb{C}}(U)$ $\displaystyle \quad = \sum_{j=1}^{m} \frac{\partial f}{\partial z_j}(z)\, dz_j + \frac{\partial f}{\partial \bar z_j}(z)\, d\bar z_j$

Almost complex structure :

$$J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$$

$$J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$$



$J \in \mathrm{End}_{\mathbb{R}}(T_X^{\mathbb{R}})$ where $J^2 = -\mathrm{Id}$ $\rightsquigarrow$ $J^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(T_X^{\mathbb{R}} \otimes \mathbb{C})$ where $(J^{\mathbb{C}})^2 = -\mathrm{Id}$

$$J^{\mathbb{C}} \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial y_j} + i\frac{\partial}{\partial x_j}\right) = i\frac{\partial}{\partial z_j}$$

$$J^{\mathbb{C}} \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial y_j} - i\frac{\partial}{\partial x_j}\right) = -i\frac{\partial}{\partial \bar z_j}$$

Span $\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}\right)$ = eigenspace of eigenvalue $+i$ of $J^{\mathbb{C}}$

Span $\left(\frac{\partial}{\partial \bar z_1}, \dots, \frac{\partial}{\partial \bar z_m}\right)$ = " " $-i$ " "

usual notation :

$$\hookrightarrow T_X^{\mathbb{R}} \otimes \mathbb{C} = \underbrace{T_X^{1,0}}_{\substack{+i \\ \text{eigenvalue}}} \oplus \underbrace{T_X^{0,1}}_{\substack{-i \\ \text{eigenvalue}}}$$

- $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_m}$
- $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}, \frac{\partial}{\partial \bar z_1}, \dots, \frac{\partial}{\partial \bar z_m}$

For cotangent vector $\rightarrow$ $dz_j(J^{\mathbb{C}}\xi) = +i\, dz_j(\xi)$

$$d\bar z_j(J^{\mathbb{C}}\xi) = -i\, d\bar z_j(\xi)$$

Fact: $\mathcal{T}_X^{1,0} \simeq \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty$

remark: • $\mathcal{M}$ a A-module, $A \subset B$ subring

$B \otimes_A \mathcal{M}$ is a B-module

$\beta \in B \quad \beta \cdot (\underset{\in B}{\lambda} \otimes \underset{\in \mathcal{M}}{x}) \underset{\text{def}}{=} (\beta\lambda) \otimes_A x$

• $\mathcal{T}_X$ = derivations of $\mathcal{O}_X$ (of the form $\sum \xi_j(x) \frac{\partial}{\partial z_j}$ where ξ_j is holomorph)

• $\mathcal{T}_X^{1,0} \subset \mathcal{T}_{X\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Der}_{\mathbb{C}}(\mathcal{C}^\infty_{X,\mathbb{C}})$

$\sum \xi_j(x) \frac{\partial}{\partial z_j} \qquad \sum \xi_j(z) \frac{\partial}{\partial z_j} + \eta_j(z) \frac{\partial}{\partial \bar z_j}$

$\xi_j \in \mathcal{C}^\infty_{X,\mathbb{C}}(U) \qquad \xi_j, \eta_j \in \mathcal{C}^\infty_{X,\mathbb{C}}(U)$

Given $f \in \mathcal{C}^1_{X,\mathbb{C}}$

↳ notation $\quad \partial f = \sum_{j=1}^m \frac{\partial f}{\partial z_j}(z)\, dz_j \qquad$ is $J^{\mathbb{C}}$-linear

$\bar\partial f = \sum_{j=1}^m \frac{\partial f}{\partial \bar z_j}(z)\, d\bar z_j \qquad$ is $J^{\mathbb{C}}$-conjugate linear

$$\boxed{df = \partial f + \bar\partial f}$$

linear algebra fact:

E a $\mathbb{C}$-vector space ($= \mathbb{R}$ vector space together with $J \in \mathrm{End}_{\mathbb{R}}(E)$ such that $J^2 = -\mathrm{Id}$, corresponding to multiplication by $+i$)

$$\underset{\substack{\wr\wr \\ \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C})}}{\mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})} = \underset{\mathbb{C}\text{ linear}}{\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})} \oplus \underset{\bar{\mathbb{C}}\text{ linear}}{\mathrm{Hom}_{\bar{\mathbb{C}}}(E, \mathbb{C})}$$

↳ $\varphi \in \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$ $\mathbb{R}$-linear

$\varphi^{1,0}(\xi) = \frac{1}{2}(\varphi(\xi) - i\,\varphi(J\xi)) \qquad \varphi = \varphi^{0,1} + \varphi^{1,0}$

$\varphi^{0,1}(\xi) = \frac{1}{2}(\varphi(\xi) + i\,\varphi(J\xi))$

$\varphi^{1,0}(J\xi) = \frac{1}{2}(\varphi(J\xi) + i\varphi(\xi)) = i\,\varphi^{1,0}(\xi) \qquad \varphi^{1,0}$ is $\mathbb{C}$ linear

$\varphi^{0,1}(J\xi) = \frac{1}{2}(\varphi(J\xi) - i\varphi(\xi)) = -i\,\varphi^{0,1}(\xi) \qquad \varphi^{0,1}$ is $\bar{\mathbb{C}}$-linear.

$\varphi^{0,1}(\lambda\xi) = \bar\lambda\, \varphi^{0,1}(\xi)$

so ∂f is $\mathbb{C}$-linear part of df

$\bar\partial f$ is $\bar{\mathbb{C}}$-linear part of df

↳ $\partial f(\xi) = \frac{1}{2}(df(\xi) - i\, df(J\xi))$

$\bar\partial f(\xi) = \frac{1}{2}(df(\xi) + i\, df(J\xi))$

I - Formalism of (p,q)-forms and complex exterior differential calculus

Let $(X, \mathcal{O}_X)$ be a complex n dimensional manifold.
Take a chart $X \supset U \xrightarrow{z} \Omega \subset \mathbb{C}^n$
Take local coordinates $(z_1, \dots, z_n)$ associated with z $\quad z_j = x_j + i y_j$

In real terms, one looks at differential forms

$$u(x,y) = \sum{}' u_{IJ}(x,y)\, dx_I \wedge dy_J$$

with multi-index notation
$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_p}$$
$$dy_J = dy_{j_1} \wedge \dots \wedge dy_{j_q}$$

$$I = (i_1, \dots, i_p) \qquad 1 \leq i_1 < i_2 < \dots < i_p \leq n$$
$$J = (j_1, \dots, j_q) \qquad 1 \leq j_1 < j_2 < \dots < j_q \leq n$$

$\sum'$ summing on increasing indices
(If I is not in increasing order, just make dx_I (and u_{IJ}) alternate)

Here $u_{IJ} \in \mathcal{E}^k(\Omega, \mathbb{C})$
$\hookrightarrow u \in \mathcal{E}^k(U, \Lambda^s\, {}^{\mathbb{R}}T^*_X \otimes \mathbb{C})$

Instead, one would like to express u in terms of dz_j and $d\bar{z}_j$

$$dx_j = \frac{1}{2}(dz_j + d\bar{z}_j)$$
$$dy_j = \frac{1}{2}(dz_j - d\bar{z}_j)$$

$$\Rightarrow u(z) = \sum_{I,J}{}' \tilde{u}_{IJ}(z)\, dz_I \wedge d\bar{z}_J$$

$|I| = p$
$|J| = q$ $\qquad p+q = s$ "degree of u"

$${}^{\mathbb{R}}T_X \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1} \qquad \text{complexified tangent bundle}$$

$${}^{\mathbb{R}}T^*_X \otimes_{\mathbb{R}} \mathbb{C} = \underset{\substack{(dz_j)\\ \mathbb{C}\text{ linear}}}{T_X^{*1,0}} \oplus \underset{\substack{(d\bar{z}_j)\\ \mathbb{C}\text{ conjugate linear}}}{T_X^{*0,1}} \qquad \text{complexified cotangent bundle}$$

so $\left(\Lambda^s_{\mathbb{R}}\, {}^{\mathbb{R}}T^*_X\right) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^s_{\mathbb{C}}\left({}^{\mathbb{R}}T^*_X \otimes_{\mathbb{R}} \mathbb{C}\right)$

V, W $\mathbb{K}$-vector-spaces over a field $\mathbb{K}$
$$\hookrightarrow \Lambda^s(V \oplus W) \simeq \bigoplus_{p+q=s} (\Lambda^p V) \otimes (\Lambda^q W)$$

(e_i) basis of V $\qquad i \in [\![1, n]\!]$
(ε_j) basis of W $\qquad j \in [\![1, m]\!]$
$\alpha_l = (e_1, \dots, e_n, \varepsilon_1, \dots, \varepsilon_m)$ basis of $V \oplus W$

basis of $\Lambda^p V$ given by $e_{i_1} \wedge \dots \wedge e_{i_p} = e_I$
basis of $\Lambda^q W$ given by $\varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_q} = \varepsilon_J$

$\Lambda^s(V \oplus W)$ has the basis $a_{\ell_1} \wedge \dots \wedge a_{\ell_s}$ with a_ℓ either e_i or ε_j

$$\pm\, \underbrace{e_{i_1} \wedge \dots \wedge e_{i_p} \wedge \varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_q}}_{} \qquad p+q=s$$

↳ replace by $\otimes$

If (V, J) is a $\mathbb{C}$-vector space J multiplication by $+i$
then $(\overline{V}, -J)$ is a $\mathbb{C}$-vector space

in $\overline{V}$ $\lambda_{\overline{V}} \cdot v = \overline{\lambda}_V \cdot v$

$$\Lambda^s(V \oplus \overline{V}) = \bigoplus_{p+q=s} (\Lambda^p V) \otimes (\Lambda^q \overline{V})$$

↳ notation $\Lambda^{p,q} V = (\Lambda^p V) \otimes (\Lambda^q \overline{V})$

so $\Lambda^{p,q} V \subset \Lambda^s(V \oplus \overline{V})$ "bidegree (p,q) component"

Apply this to $V = T_x^{*1,0} = T_x^*$ (with its J almost complex structure)

$$(\Lambda^s\, {}^{\mathbb{R}}T_x^*) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=s} \Lambda^{p,q}(T_x^*)$$

$$\Rightarrow u(z) = \sum_{\substack{|I|=p \\ |J|=q}}' u_{IJ}(z)\, dz_I \wedge d\overline{z}_J \qquad \text{is a bidegree } (p,q) \text{ form}$$

This decomposition is independent of the choice of complex coordinates, and depends only on the almost complex structure J.

$T_x^{1,0} = +i$ eigenspace of $J^{\mathbb{C}}$ on ${}^{\mathbb{R}}T_x \otimes \mathbb{C}$

$T_x^{0,1} = -i$ " " "

↳ ${}^{\mathbb{R}}V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} \simeq V \oplus \overline{V}$

$J \in \mathrm{End}_{\mathbb{R}}(V)$ such that $J^2 = -Id$

$J^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}({}^{\mathbb{R}}V \otimes_{\mathbb{R}} \mathbb{C})$ (same matrix seen in complex)

Exterior derivative

- On a real manifold, with coordinates $(x_1, \dots, x_m)$

$$u(x) = \sum_{|I|=p}' u_I(x)\, dx_I$$

$$du = \sum_{|I|=p} du_I(x) \wedge dx_I$$

$$= \sum_{|I|=p}' \sum_{j=1}^{m} \frac{\partial u_I}{\partial x_j}(x)\, \underbrace{dx_j \wedge dx_I}_{0 \text{ if } j \in I} \qquad du \text{ of degree } p+1$$

fundamental properties

- $d(u \wedge v) = (du) \wedge v + (-1)^{\deg u} u \wedge (dv)$ (Leibniz)
- $d^2 = 0$

so. for $u(z) = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ}(z)\, dz_I \wedge d\overline{z_J}$

$$\Rightarrow du(z) = \sum_{\substack{|I|=p \\ |J|=q}} du_{IJ}(z) \wedge dz_I \wedge d\overline{z_J}$$

$$du_{IJ}(z) = \sum_{k=1}^{n} \frac{\partial u_{IJ}}{\partial z_k}(z)\, dz_k + \sum_{k=1}^{n} \frac{\partial u_{IJ}}{\partial \overline{z_k}}\, d\overline{z_k}$$

One gets $du = \underbrace{\partial u}_{(p+1,q)} + \underbrace{\overline{\partial} u}_{(p,q+1)}$

with $\partial u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^{n} \frac{\partial u_{IJ}}{\partial z_k}\, dz_k \wedge dz_I \wedge d\overline{z_J}$

$\overline{\partial} u = \sum_{\substack{|I|=p \\ |J|=q}} \sum_{k=1}^{n} \frac{\partial u_{IJ}}{\partial \overline{z_k}}\, d\overline{z_k} \wedge dz_I \wedge d\overline{z_J}$

- from Leibniz rule :

$$\partial(u \wedge v) = (\partial u) \wedge v + (-1)^{p+q} u \wedge (\partial v)$$
$$\overline{\partial}(u \wedge v) = (\overline{\partial} u) \wedge v + (-1)^{p+q} u \wedge (\overline{\partial} v)$$

- $d^2 u = (\partial + \overline{\partial})^2 u$

$= \underbrace{\partial^2 u}_{(p+2,q)} + \underbrace{(\partial\overline{\partial} + \overline{\partial}\partial) u}_{(p+1,q+1)} + \underbrace{\overline{\partial}^2 u}_{(p,q+2)}$

↳ need $\begin{cases} \partial^2 = 0 \\ \overline{\partial} = 0 \\ \partial\overline{\partial} + \overline{\partial}\partial = 0 \end{cases} \Rightarrow \begin{cases} \partial^2 = 0 \\ \overline{\partial}^2 = 0 \\ \overline{\partial}\partial = -\partial\overline{\partial} \end{cases}$

- ∂ and $\overline{\partial}$ are conjugate

$u = (p,q)$ $\qquad \overline{\partial u} = \overline{\partial}\,\overline{u}$

$\overline{u} = (q,p)$ $\qquad \overline{\overline{\partial} u} = \partial \overline{u}$

Definition : A differential module M over a ring R is (M, d) where :

- M an R-module
- $d \in \mathrm{End}_R(M)$ such that $d^2 = 0$

frequently M is written like this $M = \bigoplus_{p \in \mathbb{Z}} M_p$

One then speaks of graded differential module if

$$d: M_p \longrightarrow M_{p-1} \qquad \text{"homology"}$$
$$d: M^p \longrightarrow M^{p+1} \qquad \text{"cohomology"}$$

$$M^p \xrightarrow{d} M^{p+1} \xrightarrow{d} M^{p+2} \quad \text{; yields 0 for every } p$$

$(M, d) \rightsquigarrow \ker(d) \subset M$ "cycles"

$\operatorname{Im}(d) \subset M$ "boundaries"

$d^2 = 0 \iff \operatorname{Im}(d) \subset \ker(d)$

Homology $\boxed{H(M,d) = \ker(d) / \operatorname{Im}(d)}$

in the graded case (cohomology) $\boxed{H^p(M^\bullet) = \ker(d: M^p \to M^{p+1}) / \operatorname{Im}(d: M^{p-1} \to M^p)}$

<u>Fundamental situations:</u>

- <u>De Rham cohomology</u>

↳ $(X, \mathcal{C}^\infty_X)$ real manifold

<u>1</u>: $M^p = C^\infty(X, \Lambda^p T^*_X)$ $\mathbb{R}$-module

d exterior derivative

$\Rightarrow H^p_{DR}(M, \mathbb{R})$ is the cohomology of $(M^\bullet, d)$

<u>2</u>: $M^p = C^\infty(X, \Lambda^p T^*_X \otimes_{\mathbb{R}} \mathbb{C})$

$\Rightarrow H^p_{DR}(M, \mathbb{C}) = H^p_{DR}(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$

- <u>Dolbeault cohomology</u>

↳ $(X, \mathcal{O}_X)$ a complex manifold

$M^{p,q} = C^\infty(X, \Lambda^{p,q} T^*_X)$

$\bar\partial: M^{p,q} \longrightarrow M^{p,q+1}$

for each p, get $\quad 0 \longrightarrow M^{p,0} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} M^{p,m} \longrightarrow 0$

$H^{p,q}(X, \mathbb{C})$ is the cohomology of $(M^{p,\bullet}, \bar\partial)$

<u>remark</u>: ∂ would define an anti-isomorphic spaces via conjugation

Examples: Degree 0 cohomology

1: De Rham

$$0 \longrightarrow \mathcal{M}^0 \xrightarrow{d} \mathcal{M}^1$$

$$H^0_{DR}(X,\mathbb{R}) = \frac{\ker(d : \mathcal{M}^0 \to \mathcal{M}^1)}{\operatorname{Im}(d : \mathcal{M}^{-1} \to \mathcal{M}^0)}$$

$\operatorname{Im}(d : \mathcal{M}^{-1} \to \mathcal{M}^0) = \{0\}$

$= \ker(d : \mathcal{M}^0 \longrightarrow \mathcal{M}^1)$

$= \{$functions f with $df = 0\}$

$= \{$ functions that are constant on each connected component $\}$

for X connected $\quad H^0_{DR}(X,\mathbb{R}) \cong \mathbb{R}$ in

in general $\quad H^0_{DR}(X,\mathbb{R}) = \mathbb{R}^S \quad S =$ set of components

2: Dolbeault

$$H^{p,0}(X,\mathbb{C}) = \ker(\bar\partial : \mathcal{M}^{p,0} \longrightarrow \mathcal{M}^{p,1})$$

a $(p,0)$ form is of the type $u(z) = \sum_{|I|=p} u_I(z)\, dz_I$

so $\bar\partial u(z) = \sum_{|I|=p} \frac{\partial u_I}{\partial \bar z_k} d\bar z_k \wedge du_I$

$\bar\partial u = 0 \iff \frac{\partial u_I}{\partial \bar z_k} = 0 \quad \forall k$

$\iff$ all coefficients of u_I are holomorphic

$\Rightarrow H^{p,0}(X,\mathbb{C}) = \ker(\bar\partial)$

$= \{u \ / \ u_I$ are holomorphic $\}$

Usual notation:

- $\Lambda^p T^*_X = \Omega^*_X$ of rank $\binom{n}{p}$
- $p = n = \dim_{\mathbb{C}} X$

  $\Lambda^n T^*_X = \Omega^n_X = K_X$ of rank 1 the canonical bundle of X

so $H^{p,0}(X,\mathbb{C}) = \mathcal{O}(\Omega^p_X)(X)$

with notation E holomorphic vector bundle

$\mathcal{O}(E) =$ sheaf of it's holomorphic sections)

Euclidian geometry $dx_1^2 + \dots + dx_n^2$ metric in $\mathbb{R}^n$

$\downarrow$

Riemannian geometry X $(x_1, \dots, x_n) \to$ flat space

$g = \sum g_{ij}(x)\, dx_i \otimes dx_j$ the metric depends of the position

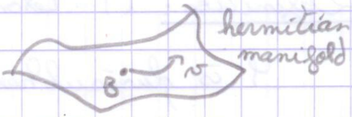
$|v|_g^2 = \sum g_{ij}(x)\, v_i v_j$

$\downarrow$

Hermitian geometry $\to$ flat : $|dz_1|^2 + \dots + |dz_n|^2$ $\mathbb{C}^n$

$\to$ not flat $h(z) = \sum h_{j\bar{k}}(z)\, dz_j \otimes d\bar{z}_k$ looks like a 1-form

$|v|_h^2 = \sum h_{j\bar{k}}(z)\, v_j \overline{v_k}$



remark : Lorentzian metric $dx^2 + dy^2 + dz^2 - c^2 dt^2$

(x, y, z, t) the space-time for special relativity theory (relativité restreinte)

for the general relativity we have $g = \sum_{1 \le i,j \le 4} g_{ij}(x)\, dx_i \otimes dx_j$ signature $(3/1)$

$\downarrow$

Symplectic geometry $\to$ flat $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$ a 2 form on $\mathbb{R}^{2n}$

$\to$ not flat $\omega(x) \to d\omega = 0$ ω non degenerated 2-form $(\omega^n \neq 0)$

(important in mecanic for example)

Erich Kähler (1933) create a new type of geometry at the intersection of hermitian and symplectic geometry

Calabi - Yau manifolds (zero Ricci curvature)

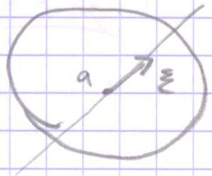
Kähler geometry is really useful for physics for making the string theory for example because it concerns objects of great energy or at a very little scale $(R \sim 10^{-34}\ m)$.

I - Plurisubharmonic functions

$\Omega \subset \mathbb{R}^n$ a convex open set

$f : \Omega \longrightarrow \mathbb{R}$ of class C^2

f convex $\Leftrightarrow$ $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1 \le i \le n}$ semi-positive and symmetric $\forall x \in \Omega$



$\varphi(t) = f(a + t\xi)$ convex in t

$$\varphi''(t) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(a + t\xi)\, \xi_i \xi_j \geq 0$$

X a complex manifold and $U \subset X$ an open.

$\varphi : U \longrightarrow \mathbb{R}$ of class C^2 we have $\quad i\partial\bar\partial\varphi(z) = \sum_{1 \leq j,k \leq n} i \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k$

$$\overline{i\partial\bar\partial\varphi} = -i\bar\partial\partial\varphi = i\partial\bar\partial\varphi$$

$\Rightarrow$ $i\partial\bar\partial\varphi$ is a <u>real</u> 2-form

<u>Definition:</u> (P. Lelong / K. Oka 1942 ~ 1936)

φ is plurisubharmonic (psh) if $\left(\frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k}\right)_{j,k}$ is an hermitian symmetric and positive $\forall z \in \Omega = \mathrm{Dom}(\varphi)$

$\overline{H_{jk}} = H_{kj}$ $\rightarrow$ hermitian symmetric $\quad \bar H = {}^t\bar H$

<u>Observation:</u> $\partial, \bar\partial$ commutes with holomorphic maps

$X \xrightarrow[\text{holo}]{F} Y$ $\quad$ $\dim X = n$, $\dim Y = m$ $\quad$ $F : X \longrightarrow Y$, $z \longrightarrow F(z) = w = (w_1, \dots, w_m)$

for $u \in C^k(Y, \Lambda^{p,q} T_Y^*)$ we have the pull-back $F^* u \in C^k(X, \Lambda^{p,q} T_X^*)$

$u(w) = \sum u_{IJ}\, dw_I \wedge d\bar w_J$

substitute $w = F(z)$

$w_j = F_j(z) \quad 1 \leq j \leq m$

$$dw_j = \sum_{l=1}^{n} \frac{\partial F_j}{\partial z_l} dz_l \qquad (\text{no } d\bar z_l)$$

$\Rightarrow$ so the pull-back $F^* u$ is of bidegree (p,q)

<u>Properties:</u>

- $d(F^* u) = F^*(du)$
- $\partial(F^* u) = F^*(\partial u)$
- $\bar\partial(F^* u) = F^*(\bar\partial u)$

<u>note:</u> if F is anti-holomorphic we have

$$dF_j = \sum \frac{\partial F}{\partial \bar z_l} d\bar z_l$$

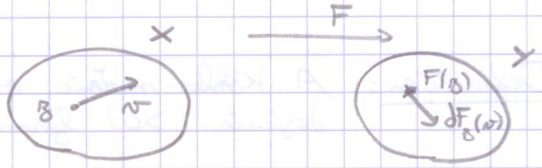
$$\Rightarrow \begin{cases} \partial(F^* u) = F^*(\bar\partial u) \\ \bar\partial(F^* u) = F^*(\partial u) \end{cases}$$

The "complex Hessian operation" $i\partial\bar\partial$ commutes with holomorphic maps

$\hookrightarrow F^*(i\partial\bar\partial\varphi) = i\partial\bar\partial(\varphi \circ F)$

$$i\partial\bar\partial\,\varphi(v) = \sum \frac{\partial^2\varphi}{\partial z_j\,\partial \bar z_k}\, v_j \overline{v_k}$$

$$(F^* i\partial\bar\partial\varphi)_z(v) = i\partial\bar\partial\,\varphi_{F(z)}(dF_z(v))$$



__Corollary:__ φ psh and F holomorphic $\Rightarrow$ $\varphi\circ F$ psh

⊗ Correspondance (1,1)-forms $\xleftrightarrow{\text{isomorphic}}$ hermitian forms

$h = \sum\limits_{1\le j,k\le n} h_{jk}\, dz_j \otimes d\bar z_k$ an hermitian form

$h(v) = \sum\limits_{j,k} h_{jk}\, v_j \overline{v_k}$

define $\tilde h(v,w) = \sum\limits_{j,k} h_{jk}\, v_j \overline{w_k}$

we have $\operatorname{Im} \tilde h(v,w) = \dfrac{1}{2i} \sum\limits_{j,k} \left(h_{jk} v_j \overline{w_k} - \overline{h_{jk}}\, \overline{v_j} w_k \right)$

$= \dfrac{1}{2i} \sum\limits_{j,k} h_{jk} (v_j \overline{w_k} - w_j \overline{v_k})$ ← $\overline{h_{jk}} = h_{kj}$

$= \dfrac{-i}{2} \sum\limits_{j,k} h_{jk}\, dz_j \wedge d\bar z_k (v,w)$

so $-\operatorname{Im}\tilde h = \dfrac{i}{2} \sum\limits_{j,k} h_{jk}\, dz_j \wedge d\bar z_k$

this is the isomorphism we want for making the correspondance between the hermitian forms and the (1,1)-forms.

Hermitian forms $\longleftrightarrow$ real (1,1)-forms

$h = \sum h_{jk}\, dz_j \otimes d\bar z_k \longleftarrow \begin{cases} \omega = \frac{i}{2} \sum h_{jk}\, dz_j \wedge d\bar z_k \\ \omega = -\operatorname{Im}\tilde h \end{cases}$

So $\underline{\omega \text{ real} \Leftrightarrow (h_{jk}) \text{ is hermitian}}$

__example:__ φ: $\omega = \dfrac{i}{2}\partial\bar\partial\varphi = \dfrac{i}{2}\sum \dfrac{\partial^2\varphi}{\partial z_j\,\partial\bar z_k}\, dz_j \wedge d\bar z_k$

$\updownarrow$

$h = \sum \dfrac{\partial^2\varphi}{\partial z_j\,\partial\bar z_k}\, dz_j \otimes d\bar z_k$

__Definition:__ φ is strictly psh $\Leftrightarrow$ $\left(\dfrac{\partial^2\varphi}{\partial z_j\,\partial\bar z_k}\right)_{j,k} > 0$ positive definite at every point.

$X = \mathbb{C}^n \qquad \varphi(z) = |z|^2 = \sum_{j=1}^{n} |z_j|^2$

$\frac{i}{2}\partial\bar\partial\varphi = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar z_j$ corresponds to standard hermitian form on $\mathbb{C}^n$.

<u>Definition:</u> A Kähler metric on a complex manifold X is a C^∞ hermitian positive definite (>0) form:

$$h = \sum_{j,k} h_{jk}\, dz_j \otimes d\bar z_k$$

$$\Updownarrow$$

$$\begin{cases} \omega = \frac{i}{2}\sum_{j,k} \omega_{jk}(z)\, dz_j \wedge d\bar z_k \\ \omega_{jk} = h_{jk} \\ \boxed{d\omega = 0} \end{cases}$$

for $(X, \mathcal{O}_X)$ a complex manifold X said to be Kähler $\iff$ $\exists\, \omega$ Kähler metric $\omega > 0$ $d\omega = 0 \iff \partial\omega = 0 \iff \bar\partial\omega = 0$

ω is a symplectic 2-form

<u>example:</u> $\omega = i\partial\bar\partial\varphi$ for φ strictly psh

$$\begin{aligned} d\omega &= (\partial + \bar\partial)(i\partial\bar\partial\varphi) \\ &= i\cdot\partial^2(\bar\partial\varphi) + i\partial(\bar\partial^2\varphi) \\ &= i\times 0 + i\partial\times 0 \\ &= 0 \end{aligned}$$

<u>Remark:</u> $\bar\partial\omega = \overline{\partial\omega}$ because ω is real $(\omega = \bar\omega)$

$$d\omega = 0 \iff \partial\omega = 0 \iff \bar\partial\omega = 0$$

$\hookrightarrow \partial\omega = \frac{i}{2}\sum_{j,k,l} \frac{\partial\omega_{jk}}{\partial z_l}\, dz_l \wedge dz_j \wedge d\bar z_k = 0$

by Kähler condition $\frac{\partial\omega_{jk}}{\partial z_l} = \frac{\partial\omega_{lk}}{\partial z_j}$

$$\Rightarrow \partial\omega = \frac{i}{2}\sum_{j,k,l} \frac{\partial\omega_{lk}}{\partial z_j}\, \underbrace{dz_j \wedge dz_l}_{=0} \wedge d\bar z_k = 0$$

$\bar\partial\omega = \frac{i}{2}\sum_{j,k,l} \frac{\partial\omega_{jk}}{\partial \bar z_l}\, d\bar z_l \wedge dz_j \wedge d\bar z_k$

by Kähler condition $\frac{\partial\omega_{jk}}{\partial \bar z_l} = \frac{\partial\omega_{jl}}{\partial \bar z_k}$

$$\Rightarrow \bar\partial\omega = \frac{i}{2}\sum_{j,k,l} \frac{\partial\omega_{jl}}{\partial \bar z_k}\, \underbrace{d\bar z_k \wedge dz_j \wedge d\bar z_l}_{=0} = 0$$

<u>Properties:</u>

<u>1:</u> φ_j psh, $\lambda_j \in \mathbb{R}_+$ constants $\Rightarrow \sum_{j=1}^{n} \lambda_j \varphi_j$ is psh

<u>2:</u> $\varphi: X \longrightarrow \mathbb{R}$
$\chi: \mathbb{R} \longrightarrow \mathbb{R}$

$i\partial\bar\partial(\chi\circ\varphi) = i(\chi'\circ\varphi)\times\partial\bar\partial\varphi + i(\chi''\circ\varphi)\partial\varphi\wedge\bar\partial\varphi$

$H(\chi\circ\varphi)(v) = (\chi'\circ\varphi).H\varphi(v) + (\chi''\circ\varphi)\,|\partial\varphi(v)|^2$ hessian form

$\hookrightarrow d(\chi\circ\varphi) = (\chi'\circ\varphi)\times d\varphi$

$\partial(\chi\circ\varphi) = (\chi'\circ\varphi)\times\partial\varphi$

$\bar\partial(\chi\circ\varphi) = (\chi'\circ\varphi)\times\bar\partial\varphi$

$\longrightarrow i\partial\bar\partial(\chi\circ\varphi) = i\partial\big((\chi'\circ\varphi)\bar\partial\varphi\big)$
$= i(\chi'\circ\varphi)\times\partial\bar\partial\varphi + i(\chi''\circ\varphi)\partial\varphi\wedge\bar\partial\varphi$

conclusion : If φ is psh and $\chi : \mathbb{R} \to \mathbb{R}$ convex increasing then $\chi \circ \varphi$ is psh

More generally $\to$ $\Psi = \chi(\varphi_1, \dots, \varphi_p)$ for φ_j psh and $\chi : \mathbb{R}^p \to \mathbb{R}$, $(t_1, \dots, t_p) \to \chi(t_1, \dots, t_p)$

$$\Rightarrow \quad H\Psi(v) = \sum_{j=1}^{p} \frac{\partial \chi}{\partial t_j}(\varphi_1, \dots, \varphi_p)\, H\varphi_j(v) + \sum_{j,k} \frac{\partial^2 \chi}{\partial t_j \partial t_k}\, \partial\varphi_j \otimes \overline{\partial \varphi_k}(v)$$

Conclusion : If $\chi(t_1, \dots, t_p)$ is increasing for each variable t_j and convex in $(t_1, \dots, t_p) \in \mathbb{R}^p$ and φ_j psh, then :

$$\Rightarrow \chi(\varphi_1, \dots, \varphi_p) \text{ is psh}$$

example : $\chi(t_1, \dots, t_p) = \log(e^{t_1} + e^{t_2} + \dots + e^{t_p})$

$\chi : \mathbb{R}^p \to \mathbb{R}$ is increasing on each variable $\left(\frac{\partial \chi}{\partial t_i} \geq 0\right)$ and χ is convex $\left(\frac{\partial^2 \chi}{\partial t_j \partial t_k}\right) \geq 0$

for $p = 2$ $\log(e^{t_1} + e^{t_2}) = t_1 + \log(1 + e^{t_2 - t_1})$

$t \to \log(1 + e^t)$ is convex on $\mathbb{R}$

for $p = 3$ $\log(e^{t_1} + e^{t_2} + e^{t_3}) = \log\left(e^{t_1} + e^{\log(e^{t_2} + e^{t_3})}\right)$ ...

example : for F an holomorphic function $F : X \to \mathbb{C}$ $\quad \varphi(z) = \log(|F(z)|^2)$ on $X \setminus F^{-1}(0)$

$$\begin{aligned}
\partial \varphi &= \partial \log(F\overline{F}) \\
&= \frac{\partial(F\overline{F})}{F\overline{F}} \\
&= \frac{\partial F \times \overline{F}}{F\overline{F}} + \frac{\overbrace{\partial \overline{F}}^{=0} \times F}{F\overline{F}} \\
&= \frac{\partial F}{F} \quad \text{holomorphic}
\end{aligned}$$

$\overline{\partial}\partial \varphi = 0$ on $X \setminus F^{-1}(0)$

example : Take $F_1, \dots, F_p \in \mathcal{O}(X)$

$\varphi_j(z) = \alpha_j \log(|F_1(z)|)$ $\quad \alpha_j \in \mathbb{R}_+$

$\hookrightarrow$ is psh of $F_j = 0$

$\chi(t_1, \dots, t_p) = \log(e^{t_1} + \dots + e^{t_p})$

Corollary : $\forall F_1, \dots, F_p \in \mathcal{O}(X)$, $\alpha_j \in \mathbb{R}_+$ $\quad \Psi(z) = \log(|F_1(z)|^{\alpha_1} + \dots + |F_p(z)|^{\alpha_p})$ is C^∞ psh on $X \setminus \bigcup_j F_j^{-1}(0)$

to define Ψ psh on X with the $\bigcup_j F_j^{-1}(0)$ we need to use distributions

Chapter 3 : Kähler geometry

III - Prerequisites on distribution theory

We want to deal with $L^p_{loc}(\Omega, d\lambda)$ $\qquad \Omega \subset \mathbb{R}^n$

$d\lambda$ = Lebesgue measure

$p \geq 1$

$$L^p_{loc}(\Omega, d\lambda) = \left\{ f \;/\; \int_K |f|^p d\lambda < +\infty \quad \forall K \text{ compact in } \Omega \right\}$$

Goal: Define $P(D)(f)$ for any $f \in L^p_{loc}(\Omega, d\lambda)$ and any $P(D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$

$a_\alpha \in C^\infty(\Omega, \mathbb{C})$

$$D^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

Definition:

- $\mathcal{D}(\Omega)$ = space of compactly supported C^∞ functions on Ω
- $\mathcal{D}_K(\Omega) = \{ f \in \mathcal{D}(\Omega) \;/\; \mathrm{Supp}(f) \subset K \text{ for } K \subset\subset \Omega \}$

$\mathcal{D}_K(\Omega)$ is a Fréchet space with the semi-norms $p_{K,m}(f) = \sup_{\substack{x \in K \\ |\alpha| \leq m}} |D^\alpha f(x)|$

we have $\mathcal{D}(\Omega) = \bigcup_K \mathcal{D}_K(\Omega)$ (not a Fréchet space)

Definition: The space of distributions on Ω is

$$\mathcal{D}'(\Omega) = \left\{ \begin{array}{l} u \text{ linear form } f \longmapsto u(f) \text{ on } \mathcal{D}(\Omega) \\ u \text{ is continuous on each } \mathcal{D}_K(\Omega) \end{array} \right\}$$

examples:

- $g \in L^p_{loc}(\Omega) \subset L^1_{loc}(\Omega)$

$$\begin{array}{ccc} L^p_{loc}(\Omega) & \longrightarrow & \mathcal{D}'(\Omega) \\ g & \longmapsto & u_g(f) = \int_\Omega f(x) g(x) \, d\lambda(x) \end{array} \qquad f \in \mathcal{D}_K(\Omega)$$

$$|u_g(f)| \leq \sup_K |f(x)| \int_K |g(x)| d\lambda(x) \leq C \sup_* |f(x)| \left(\int_K |g(x)|^p d\lambda(x) \right)^{1/p}$$

Definition: An order m distribution is a distribution that is continuous wrt $p_{K,m}$ $\forall K$

$\forall g \in L^p_{loc}(\Omega)$, u_g is a distribution of order 0

Lemma:
$$\begin{array}{ccc} L^p_{loc}(\Omega) & \hookrightarrow & \mathcal{D}'(\Omega) \\ g & \longmapsto & u_g(f) = \int f(x) g(x) d\lambda(x) \end{array}$$
is an injection

taking quotient by negligible functions in $L^p_{loc}(\Omega)$

proof: $\mathcal{D}_K(\Omega)$ is dense in $L^q(K, d\lambda)$

$$u_g(f) = 0 \quad \forall f \Rightarrow g = 0$$

- $\mu \geq 0$ or even real, or complex measure on Ω

$$f \longrightarrow \mu(f) \qquad f \in \mathcal{D}(\Omega)$$

Radon measures $= \mathcal{C}_c(\Omega)'$

$=$ Borel measures that are finite on compact sets

real measures $\quad \mu = \mu_+ - \mu_-$

complex measures $\quad \mu = \mu_1 + i\mu_2$

If K has smooth boundary, then $\mathcal{D}_K(\Omega)$ is dense in $\mathcal{C}_K(\Omega)$

$\Rightarrow$ one gets an inclusion of Radon measures into $\mathcal{D}'(\Omega)$

$$L^p_{loc}(\Omega) \hookrightarrow \mathcal{M}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$$

$g \qquad d\mu = g(x)\, d\lambda(x)$

absolutely continuous with L^p density

Differential

$$\mathcal{C}^1(\Omega) \subset \mathcal{C}^0(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$$

$g \in \mathcal{C}^1(\Omega)$

$$u_g(f) = \int_\Omega f(x) g(x)\, d\lambda(x)$$

f is a test function $f \in \mathcal{D}_K(\Omega)$

$$u_{\frac{\partial g}{\partial x_k}}(f) = \int_\Omega f(x) \frac{\partial g}{\partial x_k}(x)\, d\lambda(x)$$

$$\overset{IPP}{=} -\int_\Omega \frac{\partial f}{\partial x_k}(x)\, g(x)\, d\lambda(x)$$

Definition: For $u \in \mathcal{D}'(\Omega)$ define

$$\frac{\partial u}{\partial x_k}(f) = -u\left(\frac{\partial f}{\partial x_k}\right)$$

$$f \longrightarrow \frac{\partial f}{\partial x_k} \xrightarrow{-u} -u\left(\frac{\partial f}{\partial x_k}\right)$$

, u order $m \rightsquigarrow \frac{\partial u}{\partial x_k}$ is of order $m+1$

notation: $u(f) = \int_\Omega u(x) f(x)\, d\lambda(x)$

although formally $u(x)$ doesn't make sens as a pointwise value

example: $\Omega = \mathbb{R}$

Heaviside function $H(x)$

$$\frac{\partial H}{\partial x}(f) = -H\left(\frac{\partial f}{\partial x}\right)$$
$$= -\int_0^{+\infty} f'(x)\,dx$$
$$= -[f(x)]_0^{+\infty}$$
$$= f(0)$$
$$= \delta_0(f)$$

so $\frac{\partial H}{\partial x} = \delta_0$

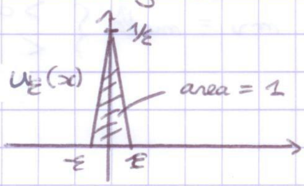
Weak convergence

Definition : $u_\nu \longrightarrow u \in \mathcal{D}'(\Omega)$ is a weakly convergence if $\forall f \in \mathcal{D}(\Omega)$
$u_\nu(f) \longrightarrow u(f)$
$\hookrightarrow \int_\Omega u_\nu(x) f(x)\,d\lambda(x) \longrightarrow \int_\Omega u(x) f(x)\,d\lambda(x)$

$D^\alpha u(f) = (-1)^{|\alpha|} u(D^\alpha f)$ and $u_\nu \xrightarrow[\text{weakly}]{} u \Rightarrow D^\alpha u_\nu \xrightarrow[\text{weakly}]{} D^\alpha u$

weak approximation of δ_0

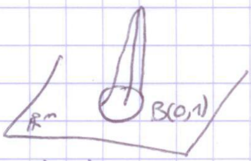
- $\Omega = \mathbb{R}$



$\int u_\varepsilon(x) f(x)\,dx \longrightarrow f(0)$

so $u_\varepsilon \longrightarrow \delta_0$ ✓

- $\Omega = \mathbb{R}^n$



$\rho \in \mathbb{R}^n$ $\quad$ $\mathrm{Supp}(\rho) \subset\subset B(0,1)$
$\rho \geq 0$

$\int_{\mathbb{R}^n} \rho(x)\,d\lambda(x) = 1$

$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ $\quad$ $\mathrm{Supp}\,\rho_\varepsilon \subset B(0,1)$ $\quad$ so $\int_{\mathbb{R}^n} \rho_\varepsilon(x)\,d\lambda(x) = 1$

By a theorem $\rho_\varepsilon \xrightarrow[\text{weakly}]{} \delta_0$

for $g \in C^2(\Omega)$ $\quad$ $\Omega \subset \mathbb{C}$

- g harmonic $\Leftrightarrow \Delta g = 0 \Leftrightarrow \frac{\partial^2 g}{\partial z \partial \bar{z}} = 0$
- g subharmonic $\Leftrightarrow$ psh (in dim 1) ie $\Delta g \geq 0$

$g_\varepsilon(z) = \log(\varepsilon^2 + |z|^2) = \log(\varepsilon^2 + z\bar{z})$

what is $i\partial\bar{\partial} g_\varepsilon$?

$\hookrightarrow \bar{\partial} g_\varepsilon = \bar{\partial}\log(\varepsilon^2 + z\bar{z}) = \frac{z\,d\bar{z}}{\varepsilon^2 + z\bar{z}}$

$$\partial\bar{\partial} g_\varepsilon = \frac{dz \wedge d\bar{z}}{\varepsilon^2 + z\bar{z}} - \frac{\bar{z}dz \wedge z d\bar{z}}{(\varepsilon^2 + z\bar{z})^2} = \frac{(\varepsilon^2 + |z|^2 - |z|^2)}{(\varepsilon^2 + |z|^2)^2} dz \wedge d\bar{z} = \frac{\varepsilon^2\,dz \wedge d\bar{z}}{(\varepsilon^2 + |z|^2)^2}$$

$$\frac{i}{2\pi}\partial\bar\partial g_\varepsilon = \frac{\varepsilon^2}{2\pi(\varepsilon^2+|z|^2)^2}\, i\,dz\wedge d\bar z = \frac{\varepsilon^2}{\pi(\varepsilon^2+|z|^2)^2}\, dx\wedge dy \xrightarrow[\text{weakly}]{\varepsilon\to 0} \delta_0$$

so $\displaystyle\int_{\mathbb{C}} \frac{\varepsilon^2}{\pi(\varepsilon^2+|z|^2)^2}\, dx\wedge dy = \int_0^{2\pi}\int_0^{+\infty} \frac{\varepsilon^2}{\pi(\varepsilon^2+|z|^2)^2}\, r\,dr\,d\theta = 1$ ✓

but we can check that $g_\varepsilon(z) = \log(\varepsilon^2+|z|^2) \xrightarrow[\varepsilon\to 0]{\text{hence weakly}} \log(|z|^2)$ in $L^p(K)$ $\forall p\in[0,+\infty[$

$$\hookrightarrow \begin{cases} \frac{i}{2\pi}\partial\bar\partial \log|z|^2 = \delta_0 \\ \frac{i}{\pi}\partial\bar\partial \log|z| = \delta_0 \end{cases}$$ Poincaré equations

IV – Poincaré Equation

$f\in\mathcal{O}(\Omega)$ $\quad\Omega\subset\mathbb{C}$

zeroes (p_ν) of order m_ν form a locally finite sequence

$\operatorname{div}(f) = \sum m_\nu [p_\nu]$

$\mathbb{Z}$-divisors $\quad\sum m_\nu[p_\nu]\quad m_\nu\in\mathbb{Z}\quad$ the $\mathbb{Z}$-divisors form a group

$\mathbb{R}$-divisors $\quad\sum m_\nu[p_\nu]\quad m_\nu\in\mathbb{R}\quad$ the $\mathbb{R}$-divisors form a real vector space

- $h=\frac{f}{g}$ meromorphic, then $\operatorname{div}(h) = \operatorname{div}(f)-\operatorname{div}(g)$ is a $\mathbb{Z}$-divisor
  $= \sum m_\nu[p_\nu]$ $\quad p_\nu$ either zeroes or poles
  $m_\nu = \text{mult}\begin{cases} >0 & \text{for a zero}\\ < & \text{for a pole}\end{cases}$

- $\operatorname{div}(h_1h_2) = \operatorname{div}(h_1)+\operatorname{div}(h_2)$
  $\operatorname{div}\left(\frac{h_1}{h_2}\right) = \operatorname{div}(h_1)-\operatorname{div}(h_2)$

$\mathbb{R}$-divisors $\longrightarrow \mathcal{D}'(\Omega)$

$\sum m_\nu[p_\nu] \longrightarrow \sum m_\nu\delta_{p_\nu}$

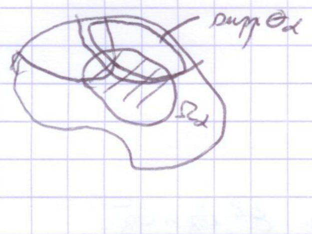
Poincaré equation $\quad\forall f\in\mathcal{M}^*(\Omega)\quad$ $\mathcal{M}^*$ is the sheaf of non zero meromorphic functions

$$\hookrightarrow \boxed{\frac{i}{\pi}\partial\bar\partial\log|f| = [\operatorname{div}(f)] = \sum m_\nu\delta_{p_\nu}}$$

Remark: to test $u=v$ in $\mathcal{D}'(\Omega)$ we have only need to check that on a certain open covering (Ω_α) of Ω, and $\theta_\alpha\in\mathcal{D}(\Omega)$ a partition of unity

$\theta_\alpha\geq 0$ and $\sum\theta_\alpha = 1$ locally finite

then, for $f\in\mathcal{D}_k(\Omega)$ $\quad u(f) = u\left(\left(\sum_\alpha\theta_\alpha\right)f\right) = \sum_\alpha u(\theta_\alpha f)\quad$ $\operatorname{supp}(\theta_\alpha f)\subset\Omega_\alpha$



proof of Poincaré formula :

need only in a small neighborhood of any given point $z_0 \in \Omega$

- $z_0 \notin \{p_\nu\} =$ zeros $\cup$ poles
  then $\exists\, V \ni z_0$ a neighborhood on which f is holomorphic and non vanishing.
  $$\frac{i}{2\pi}\partial\bar\partial \log|f|^2 = 0 \text{ on } V$$

- $z_0 = p_\nu$

$$f(z) = (z-p_\nu)^{m_\nu} \cdot \underbrace{g(z)}_{\text{non vanishing holomorphic function}}$$

$$\log(f(z)) = m_\nu \log|z-p_\nu| + \log|g(z)|$$

$$\frac{i}{\pi}\partial\bar\partial \log(f(z)) = m_\nu \frac{i}{\pi}\partial\bar\partial\log|z-p_\nu| + 0 \quad \text{case before}$$
$$= m_\nu \delta_{p_\nu}$$

V - Currents

For functions L^1_{loc} we have define a generalization with the distributions. We will do the same for the currents which are a generalisation of differential forms.

Definition :

real case : $\Omega \subset \mathbb{R}^n$ open

$$\mathcal{D}_k(\Omega) = \left\{ \begin{array}{l} \Omega \longrightarrow \Lambda^k T^*_{\mathbb{R}^n} \\ x \longrightarrow u(x) = \sum_{|I|=k} u_I(x)\, dx_I \end{array} \middle/ u_I \in \mathcal{D}(\Omega), \text{ smooth compactly supported} \right\}$$

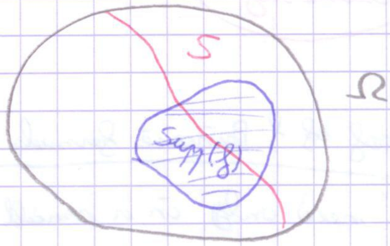
The space of currents of dimension k is the topological dual $\mathcal{D}_k'(\Omega)$

$\Leftrightarrow$ linear forms that are continuous with respect to semi-norms
$$p_{m,K} = \sup_{x\in K}\ \sup_{|\alpha|\le m}\ \sup_{|I|=k} |D^\alpha u_I(x)|$$
for every Fréchet subspace of forms of support in K (any $K \subset\subset \Omega$).

$$\mathcal{D}_k(\Omega) = \mathcal{D}(\Omega)^{\oplus N} \qquad \text{for } N = \binom{n}{k}$$
$$\mathcal{D}_k'(\Omega) = \mathcal{D}'(\Omega)^{\oplus}$$

example 1: Let $S \subset \Omega$ a smooth real submanifold of dimension k, oriented

$$\langle [S], f \rangle = \int_S f|_S \qquad \text{for } f \in \mathcal{D}_k(\Omega)$$



$\deg(f)$ has to be $= k$ then $f|_S$ is a volume form on S.

$[S]$ is called the "current of integration" on a submanifold S.

example 2: $u \in L^p_{loc}(\Omega, \Lambda^{n-k} T^*_{\mathbb{R}^n})$ $\quad p \geq 1$

$$u(x) = \sum_{|I| = n-k} u_I(x)\, dx_I \qquad , u_I \in L^p_{loc}(\Omega)$$

we can associate to u a current $T_u \in \mathcal{D}'_k(\Omega)$

$$T_u(f) = \int_\Omega \underbrace{u(x)}_{\deg\, n-k} \wedge \underbrace{f(x)}_{\deg\, k} \qquad \text{for } f \in \mathcal{D}_k(\Omega)$$

deg n

so $u \wedge f$ is a n-form on Ω with compact support and $L^p \subset L^1$ coefficients (so we can integrate).

T_u is a current of dimension k, of order 0 (continuous with respect to $p_{0,K}$ semi-norms).

↳ so we get an injective map $L^1_{loc}(\Omega, \Lambda^{n-k} T^*_{\mathbb{R}^n}) \longrightarrow \mathcal{D}'_k(\Omega)$
$\qquad u \longrightarrow T_u$

$\Rightarrow \mathcal{D}'_k(\Omega)$ is called the space of currents of dimension k and degree $n-k$

Notation: for a current $T \in \mathcal{D}'_k(\Omega)$

$$f \longrightarrow \int_\Omega T(x) \wedge f(x)$$

Remarks:

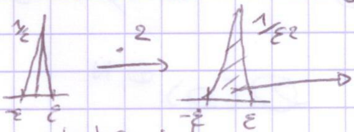
- $\Omega \longrightarrow \mathcal{D}'_k(\Omega)$ is a sheaf (using partitions of unity)
- One can extend the concept of currents to (oriented) manifolds, just by using charts.

Operations on currents:

⊗ Wedge product:

For the fact ⊗ at the following page, we have that for $T \in \mathcal{D}'_k(\Omega)$, $g \in C^\infty(\Omega, \Lambda^p T^*_{\mathbb{R}^n})$ then $g \wedge T$ make sens

note: we can't define the product of two distributions, for $\delta_0 \times \delta_0$ no sens



$\frac{2}{3}\varepsilon \times \frac{1}{\varepsilon^2} \longrightarrow +\infty$

not defined!

It exists a generalisation of distributions for that

- If T is of order m, we only need $g \in C^m(\Omega, \Lambda^\bullet T^*_{\mathbb{R}^n})$
- $g \wedge [S]$ is defined for any continuous form g.

however $u \in \mathcal{D}'(\Omega)$ $g \in C^\infty(\Omega)$
$gu(f) = u(gf)$
$f \longrightarrow gf$
$h \longrightarrow u(h)$
if u is of order m, gu is well defined for $g \in C^m(\Omega)$.
Especially, order 0 distributions can be multiplied by continuous functions

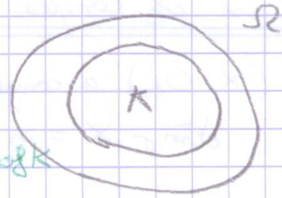
⊛ Exterior derivative:

$$\mathcal{D}'_k(\Omega) \xrightarrow{d} \mathcal{D}'_{k-1}(\Omega)$$
degree $n-k$ — degree $n-k+1$

$$dT(f) = \int_\Omega dT(x) \wedge f(x)$$
$$= \int_\Omega \underbrace{d(T\wedge f)}_{=0 \text{ by Stockes theorem}} - (-1)^k T \wedge df$$

$$\Rightarrow \boxed{dT(f) = (-1)^{k-1} \int_\Omega T \wedge df = (-1)^{k-1} T(df)}$$

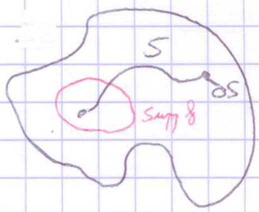
if T is of dimension 1 (degree $n-1$) and has a compact support K
$\hookrightarrow \int_\Omega dT = 0$



$\theta \in \mathcal{D}(\Omega)$
$\theta = 1$ on a neighborhood of K

proof: $\int_\Omega dT = \int_\Omega \theta . dT$
$$= \pm \int_\Omega \underbrace{d\theta}_{\substack{= 0 \\ \text{on a neigh of } K}} \wedge \underbrace{T}_{\text{supp } K} = 0$$ ✓

example:



X a n-dimensional C^∞ manifold, X is oriented
S is a C^1 submanifold of X with boundary ∂S, S is oriented (so we have the orientation on ∂S).

$\dim_{\mathbb{R}} S = k \longrightarrow [S] \in \mathcal{D}'_k(X)$

$$d[S](f) = (-1)^{k-1} [S](df)$$
$$= (-1)^{k-1} \int_S df|_S$$
$$\overset{\text{Stockes}}{=} (-1)^{k-1} \int_{\partial S} f|_{\partial S}$$
$$= (-1)^{k-1} [\partial S](f) \qquad \forall f \in \mathcal{D}_k(X)$$

so $\boxed{d[S] = (-1)^{k-1}[\partial S]}$

fundamental consequence $\to$ if $\partial S = \emptyset$ then $d[S] = 0$

Notation: $\underset{\substack{\uparrow \\ \text{degree}}}{\mathcal{D}^k}(X)' = \underset{\substack{\uparrow \\ \text{dimension}}}{\mathcal{D}'_{n-k}}(X)$

$$\begin{array}{ccccccccc} 0 \to \mathcal{D}^0(X)' & \xrightarrow{d} & \mathcal{D}^1(X)' & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathcal{D}^n(X)' & \longrightarrow & 0 \\ \cup & & \cup & & & & \cup & & \\ 0 \to C^\infty(X,\mathbb{R}) & \xrightarrow{d} & C^\infty(X,\Lambda^1 T^*_X) & \xrightarrow{d} & \cdots & \longrightarrow & C^\infty(X,\Lambda^n T^*_X) & \longrightarrow & 0 \end{array}$$

$\Leftarrow$ This forms a cohomology of currents!

$\Leftarrow$ this is De Rham cohomology!

One get a commutative diagramm

Theorem: The inclusion of forms into currents induce an isomorphism in cohomology

$$H^q(C^\infty(X,\Lambda^k T^*_X), d) \xrightarrow{\simeq} H^q(\mathcal{D}^k(X)', d)$$

proof: not yet we need more theory on sheaves, in particulary what a cohomology of sheaves.
the main ingredient is the Poincaré Lemma holds for C^∞ forms as well as for currents.

Case of complex manifolds:

$(X, \mathcal{O}_X)$ a complex holomorphic manifold
$\dim_{\mathbb{C}} X = n$

charts $\Omega \subset \mathbb{C}^n$
$\hookrightarrow$ $\mathcal{D}_{p,q}(\Omega) = \left\{ f(z) = \sum_{\substack{|I|=p \\ |J|=q}} f_{IJ}(z)\, dz_I \wedge d\overline{z_J} \right\}$

$\mathcal{D}_{p,q}(X)$ for a manifold

Definition: $\mathcal{D}'_{p,q}(X)$ is the space of currents of bidimension (p,q) and bidegree $(n-p, n-q)$.

$$L^1_{loc}(X, \Lambda^{a,b} T^*_X) \hookrightarrow \mathcal{D}'_{n-a,n-b}(\Omega)$$
$$u \longrightarrow \Big(f \to \int_X \underbrace{u(x) \wedge f(x)}_{\text{bidegree }(n,n)} \Big)$$

Let $S \subset X$ be a complex holomorphic submanifold with boundary, of $\dim_{\mathbb{C}} S = k$
($\dim_{\mathbb{R}} S = 2k$, $\dim_{\mathbb{R}} \partial S = 2k-1$ $\Rightarrow$ ∂S can't be holomorphic submanifold because its dimension is odd.)

Then $[S] \in \mathcal{D}'_{k,k}(X)$ of bidimension (k,k)

↳ $\int_S f_{|S}$ makes sens only if $\dim f = 2k$, only possible for $(p,q) = (k,k)$

Proposition : $d[S] = -[\partial S]$

proof : $d[S] = (-1)^{\dim_{\mathbb{R}} S - 1}[\partial S] = (-1)^{2k-1}[\partial S] = -[\partial S]$

$$\underbrace{d[S]}_{\text{real current}} = \underbrace{\partial[S]}_{\substack{\text{bidegree}\\(n-k+1,\,n-k)}} + \underbrace{\bar\partial[S]}_{(n-k,\,n-k+1)}$$

conjugate

$[\partial S]$ is not of pure bidegree !

If $\partial S = \emptyset$, then $d[S] = 0$

Dolbeault complex

$$\begin{tikzcd} C^\infty(X, \Lambda^{p,\bullet} T^*_X) \arrow[r, "\bar\partial"] \arrow[d, hook] & {} \arrow[d, hook] \\ \mathcal{D}^{p,\bullet}(X)' \arrow[r, "\bar\partial"] & {} \end{tikzcd}$$

Dolbeault - Grothendick lemma : If v is a C^∞ (resp. $\mathcal{D}'$) (p,q)-form with $\bar\partial v = 0$ in a neighborhood of a point $z_0 \in X$

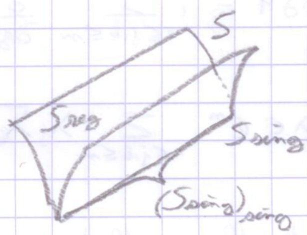
then $\exists u$ a $(p,q-1)$-form such that $\bar\partial u = v$ near z_0

note : In fact, if v is defined on a ball or a polydisk, the solution of $\bar\partial u = v$ can be obtained on the same ball or polydisk.

In both cases, one gets the same cohomology groups

$$H^{p,q}(X, \mathbb{C}) = H^{p,q}_{\bar\partial}(X, \mathbb{C})$$

Case of analytic subsets :

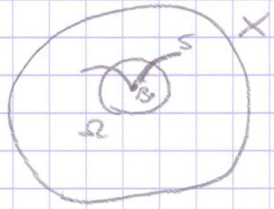


$S \subset X$ an analytic subset of pure dimension k

$S_{reg} = S \setminus S_{sing}$ is a k-dimensional holomorphic (non closed) submanifold of X

$\dim_{\mathbb{C}} S_{sing} \leq k-1$

Basic observation:



Ω coordinates charts $(z_1, \dots, z_n)$

$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ standard hermitian metric

Theorem: (i) Area $(S_{reg} \cap V(z_0))$ is finite

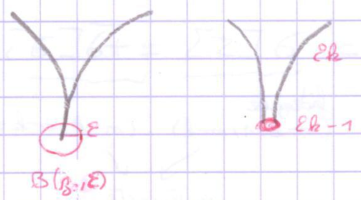
$\Rightarrow [S_{reg}] \in \mathcal{D}'_{k,k}(X)$

$\int_{S_{reg}} f$ always convergent if $f \in \mathcal{D}_{k,k}(\Omega)$

$\Rightarrow [S] = [S_{reg}]$

(ii) $d[S] = 0$ (P. Lelong 1957)

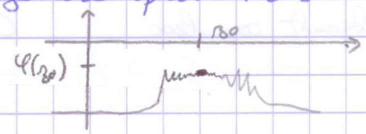
we don't need to avoid the singular points to be able to compute the integrale becaus the area near these points remains finite (not the case in real case).



Non-smooth plurisubharmonic functions

Theorem and definition: Let $\Omega \subset \mathbb{C}^n$ open, the following things are equivalent:

① $\varphi : \Omega \longrightarrow [-\infty, +\infty[$ usc (upper semi-continuous)



ie
- $\forall z_0 \in \Omega \ \limsup_{z \to z_0} \varphi(z) \leq \varphi(z_0)$

  $\forall z_0 \in \Omega, \forall \alpha > \varphi(z_0) \ \exists V$ neighb $\ni z_0$ such that $\forall z \in V \ \varphi(z) < \alpha$
- (more importantly) mean value inequality

  $\forall z_0 \in \Omega \ \forall a \in \mathbb{C}^n$ such that $z_0 + \bar{D}(0,1)a \subset \Omega$ $\frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + a e^{i\theta})\, d\theta \geq \varphi(z_0)$

  $\hookrightarrow$ always OK if $\varphi(z_0) = -\infty$
- $\varphi \not\equiv -\infty$ on any of the connected components of Ω

② $\exists$ increasing sequence of open subsets $\Omega_\nu \subset\subset \Omega$ with $\Omega = \bigcup \Omega_\nu$ and $\varphi_\nu \in PSH(\Omega_\nu) \cap C^\infty(\Omega_\nu)$ non increasing

and $\varphi = \lim_{\nu \to +\infty} \downarrow \varphi_\nu$ pointwise (or in L^1_{loc} topology)

③ $\varphi \in L^1_{loc}(\Omega)$ and when computing $i\partial\bar{\partial}\varphi$ in the sens of currents, one get

$$i\partial\bar{\partial}\varphi = i \sum_{1 \leq j,k \leq n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \geq 0 \quad \text{as a current}$$

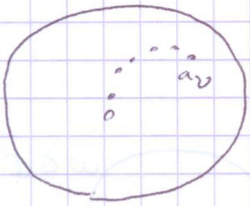
ie $\forall \lambda \in \mathbb{C}^n \quad \sum_{1 \leq j,k \leq n} \lambda_j \bar{\lambda}_k \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \geq 0$

Actually in that case, one can attribute at every point z_0 a unique value $\varphi(z_0)$ where

$\varphi(z_0) = \lim_{r \to 0} \fint_{B(z_0,r)} \varphi(z)\, d\lambda(z)$ with $\fint_{B(z_0,r)} \varphi(z)\, d\lambda(z) = \frac{n!}{\pi^n r^{2n}} \int_{B(z_0,r)} \varphi(z)\, d\lambda(z)$

is an non increasing function of r on $]0,R[$ such that $B(z_0,R)\subset\Omega$

Example: $n=1$ $\quad \Omega = D(0,1/3)$



$(a_\nu)_{\nu\in\mathbb{N}}$ is a sequence of points in Ω

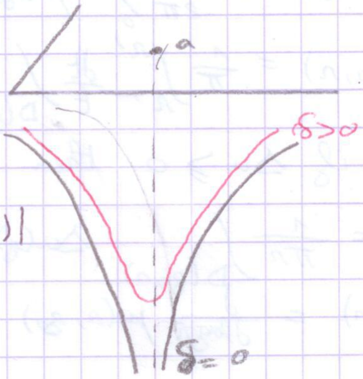
$$\sum_{\nu\in\mathbb{N}} \varepsilon_\nu \log(|z-a_\nu|^2+\delta_\nu^2) \qquad \text{with } \varepsilon_\nu>0,\ \delta_\nu \geq 0 \text{ and } \delta_\nu \leq 1/2$$

$$\Rightarrow |z-a_\nu|^2+\delta_\nu^2 \leq \frac{4}{9}+\frac{1}{2} < 1 \qquad (|z-a_\nu|<2/3)$$

$$\Rightarrow \log(|z-a_\nu|^2+\delta_\nu^2) \leq 0$$

norm in $L^1(\Omega)$ of $|\log(|z-a_i|^2+\delta_i)|$

the function is smooth if $\delta>0$ has a log pole at a if $\delta=0$



$\Rightarrow L^1$ norm $\leq L^1$ norm of $z\longrightarrow|\log(|z-a|^2)|$

$\log|z|\in L^1_{loc}(\mathbb{C})$

all these L^1 norms are $\leq C$ constant

One gets L^1 convergence on Ω as soon as $\sum_{\nu\in\mathbb{N}}\varepsilon_\nu<+\infty$ and then $\varphi\in L^1(\Omega)$ ✓

Then $\varphi(z)=\lim\downarrow S_N(z)$ for $S_N(z)=\sum_{\nu=0}^{N}\cdots$

$S_N:\Omega\longrightarrow[-\infty,+\infty[$ is continuous $\qquad$ (we did ②)

$\Rightarrow \varphi$ is usc ✓

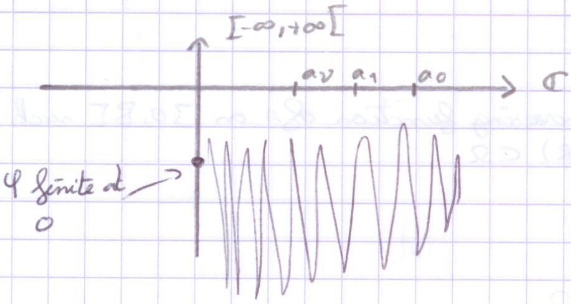
- Can take $a_\nu\in\Omega$ to be a dense sequence and $\delta_\nu=0$
  ↳ then $\varphi(a_\nu)=-\infty$ on a countable dense subset
- Take $a_\nu=2^{-\nu}$

$$\varphi(0)=\sum_{\nu=0}^{+\infty}\varepsilon_\nu\log(|a_\nu|^2+\delta_\nu^2)$$
$$\geq \sum_{\nu=0}^{+\infty}\varepsilon_\nu\log 2^{-2\nu}$$
$$= -2\log 2\sum_{\nu=0}^{+\infty}\nu\,\varepsilon_\nu$$

take $\varepsilon_\nu=\dfrac{1}{(\nu+1)^3}$ so that $\varphi(0)>-\infty$

$\varphi(a_\nu)\leq\varepsilon_\nu\log(\delta_\nu^2)$ $\qquad$ take $\delta_\nu=e^{-(\nu+1)^5}$ then $\varphi(a_\nu)\longrightarrow-\infty$

Exercice : Check that $\varphi \in C^\infty(\Omega \setminus \{0\})$ (φ same as just before)

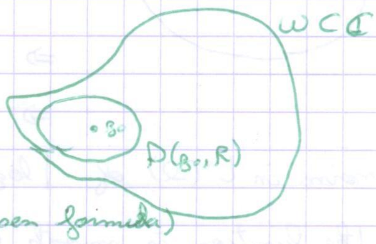


proof of the theorem :

- Generalized Jensen formula

$\hookrightarrow$ $u \in C^2(\omega)$ $\quad \omega \subset \mathbb{C}$ open $\quad r \in]0,R[$

$$\mu(z_0,r) = \frac{1}{2\pi}\int_0^{2\pi} \mu(z_0 + re^{i\theta})\,d\theta$$



$$\mu(z_0,r') - \mu(z_0,r) = \frac{1}{\pi}\int_r^{r'} \frac{dt}{t}\int_{D(z_0,t)} \Delta u(z)\,d\lambda(z) \qquad \text{(Jensen formula)}$$

consequence $\Rightarrow$ if $\Delta u \geq 0$ then $r \to \mu(z_0,r)$ is a non decreasing function

$$\frac{d}{dr}\mu(z_0,r) = \frac{1}{\pi r}\int_{D(z_0,r)} \Delta u(z)\,d\lambda(z)$$

$$\Rightarrow r\frac{d}{dr}\mu(z_0,r) = \frac{d}{d\log r}\mu(r,z_0) = \frac{1}{\pi}\int_{D(z_0,r)} \Delta u(z)\,d\lambda(z) \quad \text{is non decreasing}$$

$t = \log r \Rightarrow \mu(z_0,r) = \mu(z_0,e^t)$ is a convex function of t ✓

- for $\Omega \subset \mathbb{C}^n$ and $\varphi \in C^2(\Omega)$

$$\fint_{B(z_0,r)} \varphi(z)\,d\lambda(z) = \fint_{a \in S(0,1)} \left(\frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + re^{i\theta}a)\,d\theta\right) d\mu(a)$$

conclusion : $r \longrightarrow \mu_\varphi(z_0,r) = \fint_{B(z_0,r)} \varphi(z)\,d\lambda(z)$ is convexe and non decreasing of $\log r$ ✓

- (3) $\Rightarrow$ (2)

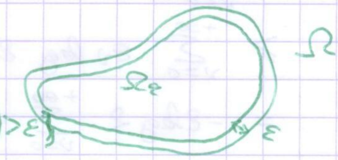
$\varphi \in L^1_{loc}(\Omega)$ such that $i\partial\bar\partial\varphi \geq 0$ in sens of distributions

(ρ_ε) a familly of regularizing kernels

$$\varphi * \rho_\varepsilon(z) = \int_{B(0,\varepsilon)} \varphi(z-w)\rho_\varepsilon(w)\,d\lambda(w)$$

defined on Ω_ε

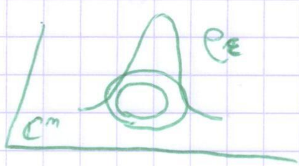
$\Omega_\varepsilon = \{z \in \Omega \,/\, d(z,\complement\Omega) > \varepsilon\}$



so $\varphi * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$

$i\partial\bar\partial(\varphi * \rho_\varepsilon) = (i\partial\bar\partial\varphi) * \rho_\varepsilon \geq 0$

$\Rightarrow \varphi * \rho_\varepsilon \in PSH(\Omega_\varepsilon)$



take $\rho_\varepsilon(w) = \frac{1}{\varepsilon^{2n}}\rho\left(\frac{|w|}{\varepsilon}\right)$ so that :

$$\fint_{B(0,\varepsilon)} \varphi(z-\omega)\rho(\omega)\,d\lambda(\omega) \underset{\text{exercice}}{=} \int_0^1 t^{2n-1}\underbrace{\left(\fint_{\xi\in S(0,t\varepsilon)}\varphi(z-\xi)\,d\mu(\xi)\right)}_{\text{expectation : this is non decreasing in } \varepsilon}\rho(t)\,dt$$

clever trick : take $(\underbrace{\varphi * \rho_\varepsilon}_{\text{smooth and psh}}) * \rho_\eta$ defined on $\Omega_{\varepsilon+\eta}$

$\downarrow$ non increasing and convex in $\log \eta$

let $\varepsilon \to 0$ so $\varphi * \rho_\varepsilon \xrightarrow[L^1_{loc}]{} \varphi$

so $\varphi * \rho_\eta = \lim_{\varepsilon\to 0} (\varphi * \rho_\varepsilon) * \rho_\eta$ is convex and non decreasing in η

$\Rightarrow \varphi = \lim_{\eta \to +\infty} \varphi * \rho_\eta$ exists in L^1_{loc} and also pointwise everywhere ✓

- (3) $\Rightarrow$ (1) as well

φ psh and smooth $\Rightarrow$ φ satisfies the mean value inequality

$$\mu\varphi(z_0,r) - \underbrace{\mu\varphi(z_0,r')}_{\downarrow\, r'\to 0,\ \varphi(z_0)} \geq 0$$

$$\Rightarrow \mu\varphi(z_0,r) \geq \varphi(z_0)$$

if φ is just psh then $\varphi * \rho_\eta$ satisfies the mean value

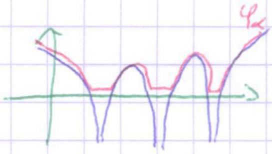
$\underset{\text{Lebesgue measure CV}}{\Rightarrow}$ φ " " " ✓

- (1) $\Rightarrow$ (2) and (3)

φ usc and $< +\infty$ $\Rightarrow$ $\sup_K \varphi < +\infty$ $\forall K$ compact

φ usc $\Rightarrow$ φ is measurable

so $\fint_{B(z_0,r)} \varphi(z)\,d\lambda(z)$ $\qquad \varphi_\alpha(z) = \max(\varphi(z), \alpha)$ for $\alpha \in \mathbb{N}$



observation $\rightarrow$ for φ, ψ satisfies the mean value inequality

$\Rightarrow \max(\varphi,\psi)$ " " " "

$\varphi_\alpha * \rho_\varepsilon \in C^\infty(\Omega_\varepsilon)$

Fubini $\Rightarrow$ $\varphi_\alpha * \rho_\varepsilon$ satisfies the mean value inequality

$\underset{(\text{exercice})}{\Rightarrow}$ $\varphi_\alpha * \rho_\varepsilon$ is psh

$\Rightarrow \varphi_\alpha * \rho_\varepsilon$ satisfies (3)

$\Rightarrow \varphi_\alpha$ satisfies (3)

$\tilde{\varphi}_\nu = \varphi_\nu * \rho_{1/\nu}$ is a non decreasing sequence of smooth psh functions converging pointwise to φ (2) ✓

- (2) $\Rightarrow$ (1) trivial

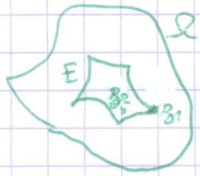
Last thing to prove:

(1) $\Rightarrow \varphi \in L^1_{loc}$

- φ usc
- $\varphi(z_0) \leq \fint_{B(z_0,r)} \varphi(z)\, d\lambda(z)$ $\Rightarrow \varphi \in L^1_{loc}$ — one then says that φ is subharmonic
- $\varphi \not\equiv -\infty$ on any connected component of Ω

proof: $E \underset{\text{def}}{=} \{ z_0 \in \Omega \mid \exists V$ neighb. which $z_0 \in V$ such that $\varphi \in L^1(V)\}$ assume Ω connected

- E is open
- $E \neq \emptyset$ $\to$ indeed $\exists z_0$ such that $\varphi(z_0) \neq -\infty$ $\quad -\infty < \varphi(z_0) \leq \fint_{B(z_0,r)} \varphi(z) d\lambda(z) \leq \sup_{\overline{B}(z_0,r)} \varphi < +\infty$ whenever $\overline{B}(z_0,r) \subset \Omega$, so $B(z_0,r) \subset E$
- E is closed $\to$ take $z_1 \in \overline{E}$, $\exists z_2 \in E$ such that $|z_1 - z_2| < \varepsilon = \frac{1}{3} d(z_1, \complement\Omega)$



$\exists V_2 \ni z_2$ such that $\varphi \in L^1(V_2) \Rightarrow \varphi \not\equiv -\infty$ on V_2

we can take $z_0 \in V_2$ closed to z_2 such that $|z_1 - z_0| < \varepsilon$ where $\varphi(z_0) > -\infty$

$r = \varepsilon \quad \overline{B}(z_0, \varepsilon) \subset \overline{B}(z_1, 2\varepsilon) \subset \Omega$

mean value inequality $\Rightarrow \varphi \in L^1(B(z_0,\varepsilon))$

$B(z_0, \varepsilon) \ni z_1 \Rightarrow z_1 \in E$

$\Rightarrow E$ closed

Conclusion $E = \Omega$ by connectedness ✓

$\Rightarrow \varphi \in L^1_{loc}$

plurisubharmonicity $\quad \{$ mean value inequality on circles

$\Downarrow$

subharmonicity $\quad \{$ mean value inequality on spheres

$\Downarrow$

" " " balls

Properties of psh functions

- $\varphi_1, \dots, \varphi_p \in PSH(\Omega) \Rightarrow \sum_{j=1}^{p} \lambda_j \varphi_j \in PSH(\Omega)$ for $\lambda_j \geq 0$
- $\chi : \mathbb{R}^p \to \mathbb{R}$ convex and non decreasing on each variable

  we can extend uniquely as $\chi = [-\infty, +\infty[^p \to [-\infty, +\infty[$ such that $\chi(\varphi_1, \dots, \varphi_p) \in PSH(\Omega)$
- $f \in \mathcal{O}(\Omega)$, $\log|f| \in PSH(\Omega)$

  $f_1, \dots, f_p \in \mathcal{O}(\Omega)$, $\log(\sum |f_j|^2) \in PSH(\Omega)$

$\varphi_\varepsilon := \log(|f|^2+\varepsilon^2)$ and $\varphi_\varepsilon = \log(|f_1|^2+\dots+|f_p|^2+\varepsilon^2)$ are smooth and psh
take the limite as $\varepsilon \to 0$

- $\Omega \xrightarrow[\text{holom}]{F} \Omega'$ $\quad \Omega, \Omega'$ connected

$\varphi \in PSH(\Omega') \Rightarrow \varphi \circ F \in PSH(\Omega)$ or $\varphi \circ F \equiv -\infty$

φ smooth $\quad i\partial\bar\partial(\varphi \circ F) = F^*\underbrace{(i\partial\bar\partial\varphi)}_{\geq 0}$ ok

$\varphi = \lim \downarrow \varphi_\nu \Rightarrow \varphi \circ F = \lim_{\nu \to +\infty} \downarrow \varphi_\nu \circ F$

example $\quad F = \mathbb{C} \to \mathbb{C}$, $z \to 0$ $\qquad \varphi : \mathbb{C} \to [-\infty, +\infty[$, $z \to \log|z|$ $\quad \Rightarrow \varphi \circ F \equiv -\infty$

Analytic cycles

X a n-dimensional manifold
Z_α a familly of irreducible q-dimensional analytic subsets

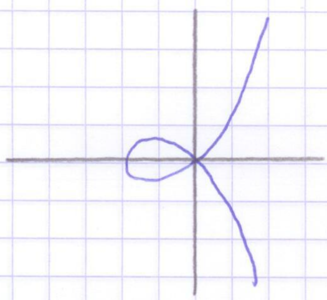
Definition : Z is (globally) irreducible if Z cannot be decompose as $Z = Z' \cup Z''$ for $Z' \subsetneq Z$, $Z'' \subsetneq Z$ analytics

Theorem :
- Z irreducible $\Leftrightarrow$ Z_{reg} connected
- If $Z_{reg,j}$ are the connected components of the regular points, then $\overline{Z_{reg,j}}$ are the irreducible components of Z.

example : $X = \mathbb{C}^2$

cubic curve with a nod $\quad y^2 = x^2(1+x)$

near $(0,0)$ we have 2 branches $y = \pm x\sqrt{1+x}$ where $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} \dots$ Newton expansion

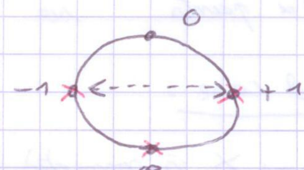


$x \to +\infty \quad y^2 \sim x^3$
$x \to -\infty \quad y^2 \sim x^{3/2}$

$t = \frac{y}{x}$, $1+x = t^2$ $\Rightarrow \begin{cases} x = t^2 - 1 \\ y = t(t^2-1) \end{cases}$ singular rational curve

$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \{\text{line at } \infty\}$
$t \longrightarrow (x,y) = (t^2-1, t(t^2-1))$

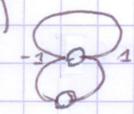
$t = \pm 1 \longrightarrow (0,0)$



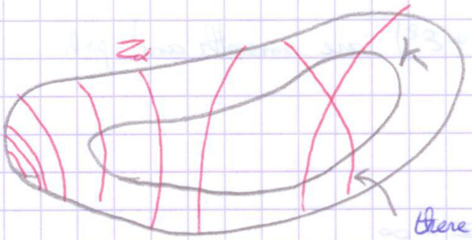
(± 1 are singular points
∞ is a singular point because $\infty \notin Z$)

Z = affine curve $y^2 = x^2(1+x)$
$Z_{reg} \simeq \mathbb{P}^1 \setminus \{-1, 1, \infty\}$

$\Rightarrow Z$ globally irreducible but the germ $Z_{(0,0)}$ has two branches analytically



Definition: A q-cycle in X is a locally finite sum $\sum_\alpha \lambda_\alpha Z_\alpha$ for
$\begin{cases} \lambda_\alpha \in \mathbb{R} \\ Z_\alpha \text{ irreducible } q\text{-dimensional analytical sets} \end{cases}$



K compact

note: if X is compact, the condition is that the sum is finite

there is a finite number of Z_α crossing the compact K

Definition: $Z = \sum_\alpha \lambda_\alpha Z_\alpha$ is effective ($Z \geq 0$) if $\lambda_\alpha \geq 0$

$Z = Z_+ - Z_-$ $\qquad \lambda_\alpha = (\lambda_\alpha)_+ - (\lambda_\alpha)_-$

$\dim Z = q$

$\operatorname{codim} Z = n-q$

Definition: A divisor D is the same as a codim 1 cycle

$$D = \sum_{\substack{\text{locally}\\ \text{finite}}} \lambda_\alpha D_\alpha \qquad D_\alpha \text{ hypersurface}$$

- Current of integration on a q-cycle:

$\hookrightarrow Z = \sum \lambda_\alpha Z_\alpha \longrightarrow T = [Z]$

$$T(f) = \sum \lambda_\alpha \int_{Z_\alpha} f \qquad \forall f \in \mathcal{D}_{q,q}(X)$$
$$= \sum \lambda_\alpha \int_{Z_{\alpha,reg}} f \longleftarrow \begin{cases} \text{bidim } (q,q) \\ \text{bidegree } (n-q, n-q) \end{cases}$$

special case $\quad D$ is a divisor $\longrightarrow [D]$ is a $(1,1)$-current

$D \geq 0 \Leftrightarrow [D] \geq 0$

$(Z \geq 0 \Leftrightarrow [Z] \geq 0)$

- $T = i \sum T_{jk} \, dz_j \, d\bar{z}_k \qquad (1,1)$ - current

$T \geq 0 \Leftrightarrow \forall \lambda \in \mathbb{C}^n \quad \sum \lambda_j \bar{\lambda}_k T_{jk} \geq 0$ measure

- $T = i^{p^2} \sum_{|J|=|K|=p} T_{JK} \, dz_J \wedge d\bar{z}_K \qquad (p,p)$-current

$T \geq 0 \underset{\text{def}}{\Leftrightarrow} \forall (\lambda_J) \in \mathbb{C}^{\binom{n}{p}} \quad \sum \lambda_J \bar{\lambda}_K \, T_{JK} \geq 0$ measure

special case: $n = 1$

$D_\alpha = p_\alpha$ points $\quad$ so $\quad \sum \lambda_\alpha [p_\alpha] = \sum \lambda_\alpha \delta_{p_\alpha}$

- General case:

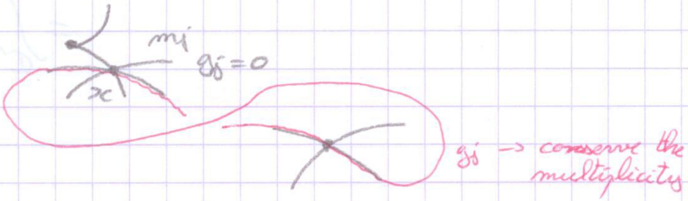
Assume X is connected (not compact) and $f \in \mathcal{O}(X)$, $f \not\equiv 0$

$Z_f \underset{\text{def}}{=} \sum m_j Z_j \qquad \begin{cases} \text{where } Z_j \text{ are the irreducibles components of } f^{-1}(0) \\ m_j = \text{multiplicity of } f \text{ along } Z_j \end{cases}$

$\mathcal{O}_{X,x}$ = germs of holomorphic functions at x

$\uparrow$ factorial ring

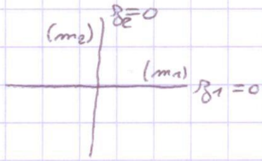
so $f_x = \prod g_j^{m_j}$



$\mathrm{Supp}\, Z_f = f^{-1}(0) = \bigcup Z_j$

Example : $X = \mathbb{C}^n$

- $f(z) = z_1^{m_1} z_2^{m_2} \cdots z_p^{m_p}$



$f^{-1}(0) = \bigcup_{1 \le j \le p} \{z_j = 0\}$

$$\frac{\partial^k f}{\partial z_j^k} = \begin{cases} = 0 & k < m_j \\ \neq 0 & k = m_j \text{ if other coordinates } \neq 0 \end{cases}$$

at a regular point $x \in f^{-1}(0)$ $\exists (z_1, \dots, z_n)$ holomorphic cordinates such that $f(z) = z_1^m g(z)$ for $g(x) \neq 0$

Lelong - Poincaré formula :

$f \in \mathcal{O}(X)$, $f \not\equiv 0$ on any component

$\otimes$ $\quad \frac{i}{\pi} \partial\bar\partial \log |f| = [Z_f]$ current of integration on the zero divisors of f

proof : $Y = \mathrm{Supp}\, Z_f = f^{-1}(0)$ is of codim 1 (dim $n-1$)

$W = Y_{sing}$ $\quad$ dim $W = n-2$

- 1st step : $\otimes$ holds on $X \setminus W$

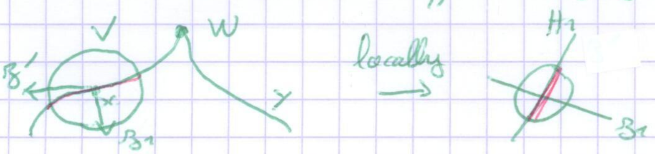
Lemma : $x \in Y \setminus W = Y_{reg}$
locally near x, we can find coordinates such that on $V \ni x$ $\quad f^{-1}(0) \cap V = Y \cap V = \{z_1 = 0\}$
we have $f(z) = z_1^{m_1} g(z)$ $\quad g \in \mathcal{O}(V)$ doesn't vanishe

$\log |f(z)| = m_1 \log |z_1| + \log |g(z)|$

$i\partial\bar\partial \log |g| = 0$ on V because g doesn't vanishe on z_1

so $\frac{i}{\pi} \partial\bar\partial \log |f| = m_1 \frac{i}{\pi} \partial\bar\partial \log |z_1|$

we have to check that $\frac{i}{\pi} \partial\bar\partial \log |z_1| = [H_1]$ for $H_1 = \{z_1 = 0\}$



$\frac{i}{\pi} \partial\bar\partial \log |z_1| \, (f) = \int_{H_1} f \qquad f \in \mathcal{D}_{n-1,n-1}(\mathbb{C}^n)$

but $\quad \frac{i}{\pi}\partial\bar\partial \log|z_1| = \int_{\mathbb{C}^n} \log|z_1| \, \frac{i}{\pi}\partial\bar\partial f$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{C}^{n-1}\times\mathbb{C}} \log|z_1| \left(\frac{i}{\pi}\frac{\partial^2 f}{\partial z_1 \partial \bar z_1} dz_1 \wedge d\bar z_1 + \underbrace{\cancel{\frac{i}{\pi}\sum_{(j,k)\neq(1,1)} \frac{\partial^2 f}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k}}_{0 \text{ contribution}} \right.$$

$$= \int_{\mathbb{C}^{n-1}} f|_{z_1=0}$$

Chapter 4 : Operators in Hermitian and Kähler geometry

I - Basic hermitian linear algebra

(E, h_E), (F, h_F) are hermitian vector spaces

- $(E \otimes F, h_E \otimes h_F)$

  ↳ we want that if $(e_\alpha)_{1\le\alpha\le s}$ is a orthonormal basis of E for h_E and $(f_\beta)_{1\le\beta\le t}$ is one for F for h_F, then we want $(e_\alpha \otimes e_\beta)_{\substack{1\le\alpha\le s\\ 1\le\beta\le t}}$ is a orthonormal basis of $E\otimes F$.

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{h_E \otimes h_F} = \langle u_1, u_2\rangle_{h_E} \langle v_1, v_2\rangle_{h_F}$$

$\langle u_1, u_2 \rangle_{h_E}$

u_1: $\mathbb{C}$-linear, u_2: $\mathbb{C}$-conjugate linear

- $(E \oplus F, h_E \oplus h_F)$

  ↳ $\langle u_1 \oplus v_1, u_2 \oplus v_2 \rangle = \langle u_1, u_2\rangle_{h_E} + \langle v_1, v_2 \rangle_{h_F}$

  $E \oplus \{0\}$ and $\{0\} \oplus F$ become orthogonal

- (E^*, h_E^*) dual

  (e_α) ONB in E $\longrightarrow$ (e_α^*) dual basis ON in E^*

  $$\|u^*\| = \sup_{x\in E,\ \|x\|_{h_E}\le 1} |u(x)|$$

- $(\Lambda^p E, \Lambda^p h_E)$

  (e_α) ONB of E $\leadsto$ (e_I) for $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ is a ON basis for $\Lambda^p E$

  ↳ $\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p \rangle_{\Lambda^p h_E} = \det\left(\langle u_\alpha, v_\beta\rangle_{h_E}\right)_{1\le\alpha,\beta\le p}$

- $(\overline{E}, \overline{h_E})$

  ↳ $\overline{J} = -J$ $\qquad \lambda_{\overline{E}} \cdot u = \overline{\lambda}_E \cdot u$ $\qquad E \xrightarrow{Id} \overline{E}$, $u \longrightarrow \overline{u}$

$$\langle \overline{u}, \overline{v} \rangle_{\overline{h_E}} = \langle v, u\rangle_{h_E} = \overline{\langle u, v\rangle_{h_E}}$$

- $S^p E = \underbrace{E \otimes E \otimes \cdots \otimes E}_{p \text{ times}} / S$ with $S = \mathrm{Span}\left(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} - x_1 \otimes \cdots \otimes x_p \ / \ \sigma \in \mathfrak{S}_p\right)$

  In finite dimension $E = E^{**} = \{\text{linear forms on } E^*\}$
  $= \{\text{homogeneous polynomials of degree } 1 \text{ on } E^*\}$

  $\Rightarrow S^p E \cong \{\text{homogeneous polynomials of degree } p \text{ on } E^*\}$

$(e_j)_{1\le j\le n}$ a basis of E $\qquad e^\alpha = e_1^{\alpha_1} \cdot e_2^{\alpha_2} \cdots e_n^{\alpha_n}$ $\qquad |\alpha| = \alpha_1 + \cdots + \alpha_n = p$

$\underbrace{e_1 \otimes \cdots \otimes e_1}_{\alpha_1 \text{ times}} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{\alpha_n \text{ times}}$ mod S is a basis for $S^p E$

- $S^1E = E$ $\quad E \times E^* \xrightarrow{\text{bilinear}} \mathbb{K}$

$\quad (x,u) \longrightarrow u(x) = \langle x, u\rangle_{E\times E^*}$

- $S^pE \otimes S^pE^*$

$\hookrightarrow \quad f = \sum_{|\alpha|=p} f_\alpha e^\alpha \in S^pE$

$u = \sum_{1\le j\le m} u_j e_j^* \in E^*$

$u^p = \sum_{|\alpha|=p} u^\alpha (e^*)^\alpha \in S^pE^*$ $\quad$ because $\bigoplus S^pE$ is a graded algebra

$\quad u^\alpha = u_1^{\alpha_1} \cdots u_m^{\alpha_m}$

$f \cdot u^p = \sum_{|\alpha|=p} f_\alpha u^\alpha$

$(e_j)_{1\le j\le m}$ orthonormal basis of E

$\Downarrow$

$\left(\sqrt{\frac{p!}{\alpha_1!\cdots\alpha_m!}}\, e^\alpha\right)$ is an orthonormal basis of S^pE

$$\|f\|^2_{S^pE} = c \int_{\substack{u\in E^* \\ \|u\|_{h_E^*}=1}} |f\cdot u^p|^2 \, d\mu(u) \qquad c \text{ is a constant (see TD 1)}$$

II - L^2 spaces of forms

- Assume (M,g) is a Riemann manifold $\quad g = \sum_{1\le i,j\le m} g_{ij}(x)\, dx_i \otimes dx_j$ (metric on T_M)

Riemann volume element $\quad dV_g = \sqrt{\det g_{ij}(x)}\, |dx_1 \wedge \cdots \wedge dx_m|$

Take E a hermitian vector bundle

$\downarrow$

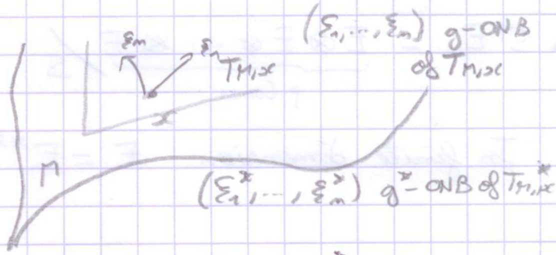
M $\qquad$ p-forms with values in E :: sections of $\Lambda^p T_M^* \otimes E$

$\qquad \Lambda^p g^* \otimes h_E^*$

One gets a hermitian metric on all fibers of the bundle $(\Lambda^p T_M^* \otimes E)_x$ depending smoothly on x.

$f \in L^2(M, \Lambda^p T_M^* \otimes E) \leftarrow$ is a Hilbert space

$$\|f\|^2 = \int_M |f(x)|^2_{\Lambda^p g^* \otimes h_E} \, dV_g$$



$$\langle\langle f, g\rangle\rangle = \int_M \langle f(x), g(x)\rangle_{\Lambda^p g^* \otimes h_E} \, dV_g$$

$\xi_I^* \otimes e_\alpha$ with $\xi_I^* = \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^*$ is a ONB of $\Lambda^p T_M^* \otimes_{\mathbb{R}} E$.

$$f(x) = \sum_{I,\alpha} f_{I,\alpha}(x)\, \xi_I^* \otimes e_\alpha$$

$$|f(x)|^2_{\Lambda^p g^* \otimes h_E} = \sum_{\substack{|I|=p \\ \alpha}} |f_{I,\alpha}(x)|^2$$

$dV_g(x) = |\xi_1^* \wedge \cdots \wedge \xi_m^*|$ at x

So the L^2 spaces requires coefficients $f_{I,\alpha}(x)$ to be measurable and in L^2_{loc} when express in smooth orthonormal frames.

Remark : smooth normal orthonormal frames always exists (just apply Gram-Schmidt algorithm)

Proposition : $L^2(M, \Lambda^p T^*_M \otimes E)$ is a Hilbert space

- X a complex n-dimensional manifold

$$\omega = \sum_{1 \leq j,k \leq n} \omega_{jk}(z)\, dz_j \otimes d\overline{z_k} \qquad \text{an Hermitian structure on } X$$

$$\updownarrow$$

$$\omega = \frac{i}{2} \sum_{1 \leq j,k \leq n} \omega_{jk}(z)\, dz_j \wedge d\overline{z_k} \qquad \text{real } (1,1)\text{-form}$$

$$dV_\omega = \frac{\omega^n}{n!} = \underbrace{\det(\omega_{jk}(z))}_{\text{density of the hermitian volume form}} \underbrace{\frac{i}{2} dz_1 \wedge d\overline{z_1} \wedge \dots \wedge \frac{i}{2} dz_n \wedge d\overline{z_n}}_{d\lambda(z)}$$

Recall : $u \in End_{\mathbb{C}}(E) \subset End_{\mathbb{R}}(E)$ then $\det_{\mathbb{R}}(u) = |\det_{\mathbb{C}}(u)|^2$

One gets a Hilbert space $L^2(X, \Lambda^{p,q} T^*_X \otimes E)$ with

$$\hookrightarrow f(z) = \sum_{\substack{|I|=p \\ |J|=q \\ 1 \leq \alpha \leq r}} f_{IJ\alpha}(z)\, dz_I \wedge d\overline{z_J} \otimes e_\alpha \qquad \text{with global } L^2 \text{ norm } \int_X |f(z)|^2 dV_\omega$$

III - Differential operators

Take (M, g) a Riemannian manifold, $\underset{M}{\overset{E}{\downarrow}}$, $\underset{M}{\overset{F}{\downarrow}}$ two hermitian bundles over M

Definition : A differential operator $P(D) = C^\infty(M,E) \longrightarrow C^\infty(M,F)$ of order m is such that :

for $f \in C^\infty(M,E)$ given by $f(x) = \sum_{1 \leq j \leq r_E} f_j(x)\, e_j$

$$P(D)f(x) = \sum_{|\alpha| \leq m} a_\alpha(x)\, D^\alpha f(x) \qquad \text{locally}$$

$x = (x_1, \dots, x_m)$ local coordinates on M.
$D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}$

because $F_{|U} \cong U \times \mathbb{K}^{r_F}$ locally, we have

$$(P(D)f(x))_i = \sum_{\substack{|\alpha| \leq m \\ j}} a_{\alpha, ij}(x)\, D^\alpha f_j(x)$$

$a_\alpha(x) = (a_{\alpha ij}(x))_{\substack{1 \leq i \leq r_F \\ 1 \leq j \leq r_E}}$

$a_\alpha \in C^\infty(U, r_F \times r_E \text{ matrix})$

By distribution theory, one gets automatically an exterior

$$P(D) = \mathcal{D}'(M,E) \longrightarrow \mathcal{D}'(M,F) \quad \text{weakly continuous.}$$

we have $L^2(M,E) \subset \mathcal{D}'(M,E)$ so $\underset{L^2(M,E)}{\overset{\cap}{\mathrm{Dom}\,(P(D))}} \xrightarrow{P(E)} L^2(M,F)$

with $\mathrm{Dom}\,(P(D)) = \{ f \in L^2(M,E) \,/\, P(D)(f) \in L^2(M,F) \}$

$\mathrm{Graph}\,(P(D)) = \{ (f, P(D)(f)) \in L^2(M,E) \times L^2(M,F) \,/\, f \in \mathrm{Dom}\,(P(D)) \}$

<u>Proposition:</u> $\mathrm{Graph}\,(P(D))$ is a closed subspace of the Hilbert space $L^2(M,E) \times L^2(M,F)$.

<u>proof:</u> take a sequence $(f_\nu, P(D)(f_\nu)) \xrightarrow[L^2]{} (f, g)$

$f_\nu \xrightarrow[L^2]{} f \Rightarrow f_\nu \xrightarrow[\text{weakly}]{} f$

$\Rightarrow P(D)(f_\nu) \xrightarrow[\text{weakly}]{} P(D)(f)$ because $P(D)$ is weakly continuous

$P(D)(f_\nu) \xrightarrow[L^2]{} g \Rightarrow P(D)(f_\nu) \xrightarrow[\text{weakly}]{} g$

by uniquiness of a limite $\Rightarrow P(D)(f) = g$ and $f \in \mathrm{Dom}\,(P(D))$

$\Rightarrow (f, g) \in \mathrm{Graph}\,(P(D))$

$\Rightarrow \mathrm{Graph}\,(P(D))$ is closed ✓

<u>Proposition:</u>

- (M, g) Riemannian manifold
- $P(D) = C^\infty(M,E) \longrightarrow C^\infty(M,F)$ differential operator of order m between hermitian vector bundles E, F.

Then $\exists !$ "formal adjoint" $P(D)^* = C^\infty(M,F) \longrightarrow C^\infty(M,E)$ of the same order m

$$\langle\langle P(D) f, g \rangle\rangle_{L^2(M,F)} = \langle\langle f, P(D)^* g \rangle\rangle_{L^2(M,E)}$$

$\forall f \in \mathcal{D}'(M,E)$ and $\forall g \in \mathcal{D}(M,F)$ smooth with compact support.

or $\forall f \in \mathcal{D}(M,E)$ and $\forall g \in \mathcal{D}'(M,F)$

<u>proof:</u> (In the first case) $\mathrm{Supp}\,(g)$ is compact

Take a covering (U_α) of M such that

- U_α are coordinates charts, $U_\alpha \subset\subset M$
- $E|_{U_\alpha}$, $F|_{U_\alpha}$ are trivial

take (θ_α) a smooth partition of unity, $\mathrm{supp}\,\theta_\alpha \subset U_\alpha$

so $g = \sum_\alpha \theta_\alpha g$ $\quad \mathrm{supp}\,(\theta_\alpha g) \subset \mathrm{supp}\,(\theta_\alpha) \cap \mathrm{supp}\,(g) \subset U_\alpha$

replacing g by $\theta_\alpha g$, one can assume $\mathrm{supp}\,(g) \subset U_\alpha$

in U_α, $P(D)(f) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha f(x)$

the riemannian volume element $dV_g(x) = \gamma(x)\, dx_1 \wedge \dots \wedge dx_m$ with $(x_1, \dots, x_m)$ coordinates on U_α (oriented).

$$\langle\langle P(D)f, g \rangle\rangle = \int_{U_\alpha} \left\langle \sum_\alpha a_\alpha(x) D^\alpha f(x), g(x) \right\rangle dV_g(x)$$

$$= \int_{U_\alpha} \sum_\alpha a_{\alpha,ij}(x) D^\alpha f_j(x) \cdot \overline{g_i(x)}\ \gamma(x)\, dx_1 \wedge \dots \wedge dx_m$$

$$\overset{IPP}{=} \int_{U_\alpha} \sum_{\alpha,i,j} (-1)^{|\alpha|} f_j(x)\ \overline{D^\alpha\left(\gamma(x) \overline{a_{\alpha,ij}(x)}\, g_i(x)\right)}\, dx_1 \wedge \dots \wedge dx_m$$

$$= \int_{U_\alpha} \langle f(x), P(D)^* g(x) \rangle\, dV_g$$

$$\Rightarrow: \quad P(D)^* g(x) = \gamma(x)^{-1} \sum_{\alpha,i,j} (-1)^{|\alpha|} D^\alpha\left(\gamma(x) \overline{a_{\alpha,ij}(x)}\, g_i(x)\right) e_j(x)$$

where $(e_j)_{1 \leq j \leq r_E}$ is the C^∞ orthonormal frame used for $E|U_\alpha$ ✓

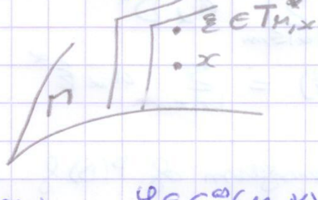
trivial fact $(P(D))^{**} = P(D)$

Symbol of a differential operator :

$$\hookrightarrow \sigma(P(D))(x, \xi) = \sum_{|\alpha| = m} \underbrace{a_\alpha(x)}_{Hom(E,F)} \underbrace{\xi^\alpha}_{S^m T_M}$$

ξ should be seen as an element of the fiber of the cotangent bundle $\rightarrow \xi \in T^*_{M,x}$

so $\sigma(P(D))$ is a C^∞ function on T^*_M into the bundle $Hom_{\mathbb{K}}(E, F)$ that is a homogeneous polynomial of degree m in ξ



$$P(D)(e^{t\varphi} f) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha_x \left(e^{t\varphi(x)} f(x)\right)$$

$$= t^m \left(\sum_{|\alpha| = m} (d\varphi(x))^\alpha\ a_\alpha(x) \cdot f(x) \right) e^{t\varphi(x)} \qquad \begin{array}{l} \varphi \in C^\infty(M, \mathbb{K}) \\ t \in \mathbb{K} \end{array}$$

$\hookrightarrow \left(\left(\frac{\partial\varphi}{\partial x_1}\right)^{\alpha_1}, \dots, \left(\frac{\partial\varphi}{\partial x_m}\right)^{\alpha_m} \right)$ + lower degree of t

$$\Rightarrow \sigma(P(D)(x, d\varphi(x))f(x) = \text{coeff of } t^m \text{ in } e^{-t\varphi} P(D)(e^{t\varphi} f)$$

$$= \sum_{|\alpha| = m} a_\alpha(x) f(x) (d\varphi(x))^\alpha$$

so $\sigma(P(D)) \in C^\infty(M, S^m T_X \otimes_{\mathbb{R}} Hom_{\mathbb{K}}(E, F))$

Definition: $P(D)$ is an elliptic operator if $\forall \xi \in T_M^*$, $\xi \neq 0$

$$\sigma(P(D))(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in Hom(E_x, F_x)$$

is injective

$$\underset{\substack{\uparrow \\ Hom(F_x;E_x)}}{\sigma(P(D)^*)}(x,\xi) = (-1)^m \underbrace{\sigma(P(D))(x,\xi)^*}_{\text{adjoint of an element of } Hom(E_x, F_x)}$$

Consequence: because the adjoin of a surjective application is injective (and the adjoin of an injective application is surjective), if rank E = rank F

$$P(D) \text{ elliptic} \iff P(D)^* \text{ elliptic}$$

Sobolev spaces

$$W^s_{loc}(M,E) = \left\{ f \,/\, \int_K \sum_{|\alpha|\leq s} |D^\alpha f|^2 \, dV_g < +\infty \quad \forall K \subset\subset M \right\}$$

$$W^s_{loc}(\mathbb{R}^n,E) = \left\{ f \,/\, \int_{\mathbb{R}^n} \sum_{|\alpha|\leq s} (1+|\xi^\alpha|^2)\, |\hat{f}(\xi)|^2 \, d\xi < +\infty \right\}$$

Basic ellipticity results:

- $P(D)$ is of order m, $f \in W^s_{loc}(M,E)$

$\Rightarrow P(D)f \in W^{s-m}_{loc}(M,F)$

- If $P(D)$ is elliptic and $P(D)f \in W^s_{loc}(M,F)$

$\Rightarrow f \in W^{s+m}_{loc}(M,E)$

proof: just for the case of constant coefficients

$$P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$$

$$\sigma(P(D))(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha \in Hom(E,F) \text{ is injective because } P(D) \text{ is elliptic}$$

the Fourier transform of $P(D)f$ is $\widehat{P(D)f}(\xi) = (2i\pi)\sum a_\alpha \xi^\alpha \hat{f}(\xi)$

$$\|P(D)f\|^2_{W^s} = \int (1+|\xi|^2)^s \left|\sum_\alpha a_\alpha \xi^\alpha \cdot \hat{f}(\xi)\right|^2$$

$$\text{injectivity} \Rightarrow \; = \int c(1+|\xi|^2)^s \left(|\xi|^m \cdot |\hat{f}(\xi)|\right)^2$$

pseudo-differential theory

Consequence: Every elliptic operator is hypoelliptic

When solving $P(D)f = g$ with $g \in C^\infty$, then $f \in C^\infty$

proof: Sobolev theorem says $C^\infty(M,E) = \bigcap_{s>0} W^s_{loc}(M,E)$

$\mathcal{D}'(M,E) = \bigcup_{s<0} W^s_{loc}(M,E)$ ...

I - Laplace - Beltrami operator

(M, g) a riemannian manifold, we want to find a geometric way of computing $H^p_{DR}(M, \mathbb{R})$.

exterior derivative $d_p : C^\infty(M, \Lambda^p T^*_M) \longrightarrow C^\infty(M, \Lambda^{p+1} T^*_M)$ differential operator of degree 1

$$H^p_{DR}(M, \mathbb{R}^m) = \frac{\ker d_p}{\operatorname{Im} d_{p-1}}$$

We have a riemannian structure but also a L^2 structure given by the metric g: $L^2(M, \Lambda T^*_M)$

we have also the adjoint $d_p^* : C^\infty(M, \Lambda^{p+1} T^*_M) \longrightarrow C^\infty(M, \Lambda^p T^*_M)$ is of order 1 and depends on g

Definition: The Laplace - Beltrami operator associated with g is:

$$\Delta_g = dd^* + d^*d$$

in degree p $\Delta_{g,p} = d_{p-1} d^*_{p-1} + d^*_p d_p$

Calculation of symbols:

for $P(D) : C^\infty(M, E) \longrightarrow C^\infty(M, F)$ $\quad P(D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$

$Q(D) : C^\infty(M, F) \longrightarrow C^\infty(M, G)$ $\quad Q(D) = \sum_{|\beta| \leq m'} b_\beta(x) D^\beta$

$$\sigma(\underbrace{Q(D) \circ P(D)}_{S^{m+m'} T_M \otimes Hom(E,G)}) = \sigma(\underbrace{Q(D)}_{S^{m'} T_M \otimes Hom(F,G)}) \cdot \sigma(\underbrace{P(D)}_{S^m T_M \otimes Hom(E,F)}) \quad \text{(by Leibniz rule)}$$

- symbol of d on $\Lambda^p T^*_M$:

$\hookrightarrow$ $e^{-t\varphi} d(e^{t\varphi} f)$ where $\varphi \in C^\infty(M)$ and $f \in C^\infty(M, \Lambda^p T^*_M)$

$$e^{-t\varphi} d(e^{t\varphi} f) = e^{-t\varphi}\left(t e^{t\varphi} d\varphi \wedge f + e^{t\varphi} df\right)$$
$$= t \overset{\xi}{d\varphi} \wedge f + df$$

so $\sigma(d)_{x,\xi}(f) = \xi \wedge f$ = the degree of t^1 in $e^{-t\varphi} d(e^{t\varphi} f)$

$$\xi \in T^*_M \qquad \begin{array}{ccc} \Lambda^p T^*_M & \longrightarrow & \Lambda^{p+1} T^*_M \\ f & \longrightarrow & \xi \wedge f \end{array}$$

- symbol of d^* on ΛT^*_M:

$\hookrightarrow$ recall that $\sigma(P(D)^*) = (-1)^{\deg P(D)} \cdot \sigma(P(D))^*$

$* : H(E,F) \longrightarrow H(F,E)$ the adjunction

Interior product in the exterior algebra:

V vector space over $\mathbb{K}$, V^* its dual and $\Lambda^p V^*$ the space of the alternate p-multilinear forms $V^p \to \mathbb{K}$.

Given $v \in V$, one defines the interior product by v:

$$i_v : \Lambda^p V^* \longrightarrow \Lambda^{p-1} V^*$$
$$\alpha \longrightarrow i_v\alpha \quad \text{, defined by } i_v\alpha(u_1,\dots,u_{p-1}) = \alpha(v, u_1, \dots, u_{p-1})$$

$(v, \alpha) \longrightarrow i_v\alpha$ is trivially linear in v and in α.

choose $(e_1, \dots, e_n)$ a basis of V $\leadsto$ $(e_I^*)_{|I|=p}$ is a basis of $\Lambda^p V^*$

by definition $e_I^*(u_1,\dots,u_p) = \det\left(e_{i_s}^*(u_t)\right)_{1\le s,t\le p}$

$$\hookrightarrow \quad i_{e_j} e_I^* = e_I^*(e_j, u_1, \dots, u_{p-1})$$
$$= \begin{cases} 0 & \text{if } j \notin I \\ \pm e^*_{I \setminus \{j\}} \end{cases}$$

the signe is $(-1)^{s-1}$

$e_I^* = e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_p}^*$

if $j = i_1$ $\det\begin{pmatrix} 1 & \\ 0 & \\ 0 & \\ 0 & \end{pmatrix}$

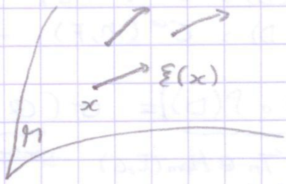
if $j = i_2$ $\det\begin{pmatrix} 0 & \\ 1 & \\ 0 & \\ 0 & \\ 0 & \end{pmatrix}$ ---

Property: $\boxed{i_v(\alpha \wedge \beta) = i_v(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (i_v \beta)}$

Application to differential forms

$\hookrightarrow$ $\xi \in C^\infty(M, T_M)$
$\alpha \in C^\infty(M, \Lambda^p T^*_M)$
$V = T_{M,x}$

Lie derivative formula: take a vector field $\xi \in C^\infty(M, T_M)$



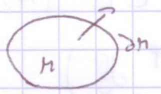
the differential equation is $\frac{dx}{dt} = \xi(x)$ we compute the trajectories of $x(t)$.

$\phi : \mathbb{R} \times M \longrightarrow M$
$(t, x_0) \longrightarrow \phi_t(x_0)$

the flow
the trajectory of a "particule" starting at position x_0 at time $t=0$ (solution of our differential equation such that $x(0) = x_0$).

If M is compact without boundary, ϕ is indeed defined on $\mathbb{R} \times M$

If M has a boundary



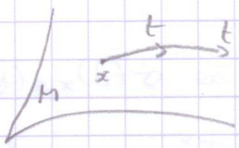
ϕ go out and is not defined anymore, or the particule acquire an infinite speed.

$\forall M$, ϕ exists on a neighborhood of $\{0\} \times M$

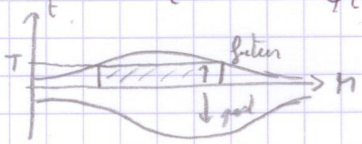
$\phi_t : M \longrightarrow M$
$x \longrightarrow \phi(t,x)$

if M is compact $\mathbb{R} \longrightarrow \mathrm{Diff}^\infty(M)$
$t \longrightarrow \phi_t$

$\phi_{t'} \circ \phi_t = \phi_{t+t'}$



if M non compact



at a time $T \neq 0$, ϕ is defined only on a compact set of M, at $T=0$ ϕ might be defined everywhere

Lie derivative of a p-form α wrt vector field ξ

$$\mathcal{L}_\xi \alpha = \frac{d}{dt}\left(\phi_t^* \alpha\right)_{t=0}$$

since ϕ_t needed only in a neighborhood of any given point, this is always well defined

Analytical Geometry 412 Chapter :

Formula $\longrightarrow$ $\boxed{\mathcal{L}_\xi = d\circ i_\xi + i_\xi \circ d}$ ✓

proof: • if ξ is constant on $\Omega \subset \mathbb{R}^n$ $\phi_t(x) = x + t\xi$ $x \nearrow x+t\xi$

$\alpha = \sum_{|I|=p} \alpha_I(x)\, dx_I \implies \phi_t^* \alpha = \sum_{|I|=p} \alpha_I(x+t\xi)\, dx_I$

so $\mathcal{L}_\xi \alpha = \sum_{|I|=p} D_\xi \alpha_I(x)\, dx_I$

it remains to show that $\mathcal{L}_\xi \alpha = (d\circ i_\xi + i_\xi \circ d)\alpha$ which is not difficult

• for the general case, ... exercise.

for p-forms: $\xi \in T^*_{M,x}$ $\sigma(d) : \Lambda^p T^*_{M,x} \longrightarrow \Lambda^{p+1} T^*_{M,x}$

$f \longrightarrow \xi \wedge f$

g gives an isomorphism between the space and its dual $T_{M,x} \xrightarrow{\sim} T^*_{M,x}$

$v \longrightarrow \langle \cdot, v\rangle_g = \xi$

$\xi^* \in T_{M,x}$

Proposition: The adjoint of $f \longrightarrow \xi \wedge f$ is i_{ξ^*}

proof: take orthonormal basis and compute

so $\sigma(d^*) = -i_{\xi^*}$

in degree p $i_{\xi^*} : \Lambda^p T^*_M \longrightarrow \Lambda^{p-1} T^*_M$

• symbol of Δ_g:

$$\sigma(\Delta_g) = \sigma(d\circ d^* + d^*\circ d)$$
$$= \sigma(d)\circ\sigma(d^*) + \sigma(d^*)\circ\sigma(d)$$

so
$$\sigma(\Delta_g)_{x,\xi}(f) = -(\xi\wedge i_{\xi^*}(f) + i_{\xi^*}(\xi\wedge f))$$
$$= -(\cancel{\xi\wedge i_{\xi^*}(f)} + i_{\xi^*}(\xi).f - \cancel{\xi\wedge i_{\xi^*}(f)})$$
$$= -i_{\xi^*}(\xi).f$$
$$= -\xi(\xi^*).f$$
$$= -\langle \xi^*, \xi^*\rangle_g . f$$
$$= -|\xi|^2_{g^*}.f$$

so $\boxed{\sigma(\Delta_g)_{(x,\xi)} = -|\xi|^2_{g^*}.\mathrm{Id}_{\Lambda^p T^*_M}}$ ✓

Consequence: Δ_g is an elliptic operator!

example: $\Omega = \mathbb{R}^n$, $g(x) = \sum_{i=1}^{n} dx_i^2$

d and d^* are constant coefficient operators with only D^α $|\alpha| = 1$, so Δ_g (in this case) only has derivatives D^α with $|\alpha| = 2$

$\hookrightarrow \Delta_g = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

Definition: A p-form $u \in C^\infty(M, \Lambda^p T^*_M)$ is called a harmonic form of (M, g) if $\Delta_g u = 0$

Remark: By ellipticity, if $u \in \mathcal{D}'(M, \Lambda^p T^*_M)$ satisfies $\Delta_g u = 0$, then $u \in C^\infty(M, \Lambda^p T^*_M)$

Assume M is compact without boundary:
$$\langle\langle \Delta_g u, u \rangle\rangle = \langle\langle dd^* u + d^* du, u \rangle\rangle$$
$$= \langle\langle d^* u, d^* u \rangle\rangle + \langle\langle du, du \rangle\rangle$$
$$= \|d^* u\|^2 + \|du\|^2$$

Consequence: If M is compact

$$\Delta_g u = 0 \iff du = 0 \text{ and } d_g^* u = 0$$
"closed" "co-closed"

Remarks:

1: $(d^*)^2 = 0$

2: If M is not compact, the previous result (in the consequence) is completely false!

$\hookrightarrow$ exemple: take $M = \mathbb{R}^n$ $g = \sum dx_i^2$ $p = 0$
$u = f$ a function

$\Delta f = 0$ has many solutions $x_1^2 - x_2^2 \dots$
$df = 0$ only gives constants functions $\rightarrow$ don't works!

Theorem: Define $\mathcal{H}^p(M, \mathbb{R}) = \{ \Delta_g - \text{harmonic } p\text{-forms on } M \}$

If M is compact, then
i: $\dim \mathcal{H}^p(M, \mathbb{R}) < +\infty$
ii: $C^\infty(M, \Lambda^p T^*_M) = \mathcal{H}^p(M, \mathbb{R}) \oplus \operatorname{Im} \Delta_{g,p}$ orthogonal direct sum

proof:

i: All Sobolev topologies W^s coincide on $\mathcal{H}^p(M, \mathbb{R})$, even C^∞ and L^2 coincide.

$C^1(M, \Lambda^p T^*_M) \hookrightarrow C^0(M, \Lambda^p T^*_M)$ is compact (by Ascoli theorem)

so the unit ball is compact $\Rightarrow$ $\dim < +\infty$

ii: $\langle\langle h, \Delta_g u \rangle\rangle = \langle\langle \Delta_g h, u \rangle\rangle \underset{h \text{ harmonic}}{=} \langle\langle 0, u \rangle\rangle = 0$

$\Delta_g^* = (dd^* + d^* d)^* = d^* d + dd^* = \Delta_g$

Analytical Geometry M2 Chapter :

$$(\operatorname{Im}\Delta_g)^\perp = \{v \,/\, \langle\langle v, \Delta u\rangle\rangle = 0 \quad \forall u\}$$
$$= \{v \,/\, \langle\langle \Delta_g v, u\rangle\rangle = 0 \quad \forall u\}$$
$$= \mathcal{H}^p(M,\mathbb{R}).$$
as a distribution $\Rightarrow \Delta_g v = 0 \quad \forall u$

calculation in L^2 :

$$L^2(M, \Lambda^p T^*_M) = \mathcal{H}^p(M,\mathbb{R}) \oplus \overline{\operatorname{Im}\Delta_g}$$
$$= \mathcal{H}^p(M,\mathbb{R}) \oplus \mathcal{H}^p(M,\mathbb{R})^\perp$$
$$\mathcal{H}^p(M,\mathbb{R})^\perp = (\operatorname{Im}\Delta_g)^{\perp\perp} = \overline{\operatorname{Im}\Delta_g}$$

so $\operatorname{Im}\Delta_g$ is closed in $L^2(M, \Lambda^p T^*_M)$ (by ellipticity theory or pseudo-differential calculus)

$\Rightarrow L^2(M, \Lambda^p T^*_M) = \mathcal{H}^p(M,\mathbb{R}) \oplus \underbrace{\operatorname{Im}\Delta_g}$

image of the maximal closed extension in L^2

$u = \underset{\substack{\uparrow \\ C^\infty}}{h} + \Delta v$ if $u \in C^\infty$ then, by ellipticity, $v \in C^\infty$

Consequence: If M is compact

① . $\operatorname{Im}(\Delta_{g,p}) = \operatorname{Im}(d_{p-1}) \oplus \operatorname{Im}(d^*_p)$ and the sum is orthogonal in C^∞ (or in L^2)

② . $C^\infty(M, \Lambda^p T^*_M) = \mathcal{H}^p(M,\mathbb{R}) \oplus \operatorname{Im}(d_{p-1}) \oplus \operatorname{Im}(d^*_p)$

③ . $\ker(d_p) = \mathcal{H}^p(M,\mathbb{R}) \oplus \operatorname{Im}(d_{p-1})$

proof :

① $\Delta_{g,p} = d_{p-1} d^*_{p-1} + d_p^* d_p$

$\Rightarrow \operatorname{Im}(\Delta_{g,p}) \subset \operatorname{Im}(d_{p-1}) + \operatorname{Im}(d_p^*)$

$\operatorname{Im}(d_{p-1}) \perp \operatorname{Im}(d_p^*)$ because $\langle\langle du, d^* v\rangle\rangle = \langle\langle d^2 u, v\rangle\rangle = 0$

we also have $\operatorname{Im}(d_{p-1}) \perp \operatorname{Im}(d_p^*) \perp \mathcal{H}^p(M,\mathbb{R})$ because (h harmonic

$\langle\langle h, du\rangle\rangle = \langle\langle \underset{0}{d^* h}, u\rangle\rangle = 0$ (because M is compact)

$\langle\langle h, d^* v\rangle\rangle = \langle\langle \underset{0}{dh}, v\rangle\rangle = 0$

$\operatorname{Im}(d_{p-1}), \operatorname{Im}(d_p^*) \subset \mathcal{H}^p(M,\mathbb{R})^\perp = \operatorname{Im}(\Delta_{g,p})$

$\Rightarrow \operatorname{Im}(\Delta_{g,p}) = \operatorname{Im}(d_{p-1}) \oplus \operatorname{Im}(d_p^*)$ ✓

② immediate consequence of the previous theorem and ①

③ Take $u \in C^\infty(M, \Lambda^p T^*_M)$, by ② we have $u = \underset{\text{harmonic}}{h} + dv + d^* w$

$$du = dh + d^2 v + dd^* \omega = dh + dd^* \omega = dd^* \omega$$

$u \in \ker d$ if $du = dd^* \omega = 0$

$$\Rightarrow \quad 0 = \langle\langle dd^* \omega, \omega \rangle\rangle = \| d^* \omega \|^2$$

$$\Rightarrow \quad d^* \omega = 0$$

so $u \in \ker d$ is of the form $u = h + dv$

$$\Rightarrow \ker d_p = \mathcal{H}^p(M, \mathbb{R}) \oplus \operatorname{Im}(d_{p-1}) \quad \checkmark$$

<u>Theorem</u>: $\boxed{H^p_{DR}(M, \mathbb{R}) = \dfrac{\ker d_p}{\operatorname{Im} d_{p-1}} \cong \mathcal{H}^p(M, \mathbb{R})}$ W. V. D. Hodge $\simeq 1940$

So we can compute cohomology groups by harmonic forms! That is the aim of the Hodge theory.

<u>Hodge $*$ operator</u>:

Take (M, g) a riemannian manifold with $\dim_{\mathbb{R}} M = m$, $dV_g = \sqrt{\det(g_{ij}(x))}\, dx_1 \wedge \dots \wedge dx_n$ oriented

$$* : \Lambda^p T^*_M \longrightarrow \Lambda^{m-p} T^*_M$$
$$\alpha \longmapsto * \alpha$$

it is the Hodge operator

$\forall \alpha, \beta \in \Lambda^p T^*_M$, $\underset{p}{\alpha} \wedge \underset{m-p}{*\beta} = \langle \alpha, \beta \rangle_{\Lambda^p g^*} dV_g$

$\dim_{\mathbb{R}} \Lambda^p T^*_M = \dim_{\mathbb{R}} \Lambda^{m-p} T^*_M = \binom{m}{p}$.

Fix $a \in X$, $(x_1, \dots, x_n)$ coordinates centered at a. By a linear combination we can assume that $(dx_1, \dots, dx_n)$ defines an orthonormal basis of $T^*_{M,a}$.

$\hookrightarrow$ $\langle \alpha, \beta \rangle_g = \sum \alpha_I \beta_I$ since the (dx_I) form an ONB

$dV_g(a) = dx_1 \wedge \dots \wedge dx_n$ $\quad \det(g_{ij}(a)) = 1$ because we have an ONB

$*\beta = \sum \varepsilon_I \beta_I \, dx_{C_I}$

$$\Rightarrow \sum_{|I|=p} \alpha_I \, dx_I \wedge \sum_{|J|=p} \varepsilon_J \beta_J \, dx_{C_J} = \sum_{|I|=p} \alpha_I \beta_I \, \varepsilon_I \, dx_I \wedge dx_{C_I} \qquad (0 \text{ if } I \neq J)$$

$\varepsilon_I = \pm 1 =$ signature of the permutation $(I, {}^C I)$

$** \beta = \sum \varepsilon_I \varepsilon_I \beta_I \, dx_I$ $\qquad \operatorname{sign}(I, {}^C I) \times \operatorname{sign}({}^C I, I) = (-1)^{p(m-p)}$

$$\Rightarrow \boxed{** = (-1)^{p(m-p)} \operatorname{Id} \text{ in } \Lambda^p T^*_M}$$

<u>We want a formula for the formal adjoint d^*</u>:

$\hookrightarrow$ by definition $\langle\langle d^* u, v \rangle\rangle = \langle\langle u, dv \rangle\rangle$

$\forall u \in \mathcal{D}_p(M)$, $v \in \mathcal{D}_{p-1}(M)$

$$\int_M \langle u, dv\rangle_g \, dV_g = \int_M \langle *u, *dv\rangle_g \, dV_g$$

$$= \int_M (*u)\wedge(**dv)$$

$$= (-1)^{p(m-p)} \int_M *u \wedge dv$$

$$= -(-1)^{(p+1)(m-p)} \int_M d(*u)\wedge v$$

$$= -(-1)^{(p+1)(m-p)} (-1)^{(p-1)(m-p+1)} \int_M v \wedge d(*u)$$

$$= (-1)^{p} (-1)^{(p-1)(m-p+1)} \int_M v \wedge **d(*u)$$

$$= -(-1)^{(p-1)(m-p)} \int_M \langle v, *d(*u)\rangle$$

$$= -(-1)^{(p-1)(m-p)} \int_M \langle *d(*u), v\rangle$$

take $\alpha = *u$
$\beta = *v$

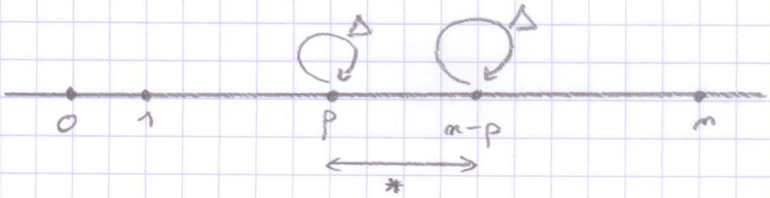
v is of compact support so we can use the Stokes theorem

$$*u\wedge dv = (-1)^{m-p}\big(\underbrace{d(*u\wedge v)}_{\text{will gives } 0} - d(*u)\wedge v\big)$$

so we obtain $\boxed{d^* u = -(-1)^{(p-1)(m-p)} *d(*u)}$

Claim: $\Delta = \pm * \Delta *$

$\Delta = dd^* + d^*d \Rightarrow \Delta^* = \Delta$ self adjoint



Consequence: • For $u \in C^\infty(M, \Lambda^p T^*_M)$

u Δ_g-harmonic $\Leftrightarrow$ $*u$ is Δ_g-harmonic

• $* : \mathcal{H}^p(M,\mathbb{R}) \xrightarrow{\simeq} \mathcal{H}^{m-p}(M,\mathbb{R})$ isometry for M compact.

• Poincaré duality (M compact)

$$H^p_{DR}(M,\mathbb{R}) \times H^{m-p}_{DR}(M,\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{pairing}$$

$$(\{\alpha\}, \{\beta\}) \longrightarrow \int_M \alpha\wedge\beta$$

this is well defined, because if we take $\alpha \longrightarrow \alpha + d\gamma$

$$\int_M (\alpha + d\gamma)\wedge\beta = \int_M \alpha\wedge\beta + \underbrace{\int_M \cancel{d(\gamma\wedge\beta)}}_{=0 \text{ By Stokes}} - \int_M \cancel{\gamma}\wedge \underset{=0}{d\beta}$$

so $(\{\alpha\}, \{*\alpha\}) \longrightarrow \int_M \alpha \wedge * \alpha = \int_M |\alpha|^2 dV_g = \|\alpha\|^2$

$\Rightarrow$ pairing is non degenerated ($\neq 0$ if $\alpha \neq 0$)

↳ One gets an isomorphism independant of g:

$$\boxed{H^{n-p}_{DR}(M,\mathbb{R}) \cong H^{p}_{DR}(M,\mathbb{R})^* \qquad \text{Poincaré duality}}$$

Corollary: Assume that M is compact, connected and oriented of $\dim_{\mathbb{R}} M = n$ then $H^n_{DR}(M,\mathbb{R}) \cong \mathbb{R}$

the isomorphisme is $H^n_{DR}(M,\mathbb{R}) \to \mathbb{R}$
$\{\alpha\} \to \int_M \alpha$

remark: if M is non oriented $H^n_{DR}(M,\mathbb{R}) = 0$

VI - Hodge theory for Hermitian and Kähler manifolds

1 - Hermitian case

Take (X,ω) hermitian with the hermitian metric $\omega = \frac{i}{2} \sum_{1 \le j,k \le n} \omega_{jk}(z) dz_j \wedge d\bar{z}_k$ on T_X

the volume form is $dV_\omega = \frac{1}{n!}\omega^n = \det(\omega_{jk}(z))\, dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$ induces induced metrics on $\Lambda^{p,q} T^*_X = \Lambda^p T^*_X \otimes \overline{\Lambda^q T^*_X}$

$d = \partial + \bar{\partial}$ $\qquad$ $\partial = (p,q) \longrightarrow (p+1,q)$
$\qquad\qquad\qquad\quad \bar{\partial} = (p,q) \longrightarrow (p,q+1)$

- Associated complex Laplace - Beltrami operators

$$\Box = \partial\partial^* + \partial^*\partial$$
$$\bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

we have $\Box^* = \Box$ and $\bar{\Box}^* = \bar{\Box}$

Proposition: $\mathrm{Symbol}(\Box) = \mathrm{Symbol}(\bar{\Box}) = \frac{1}{2}\mathrm{Symbol}(\Delta_\omega)$

proof: $\mathrm{Symbol}(\bar{\partial}) = ?$

↳ $e^{-t\varphi}\bar{\partial}(e^{t\varphi}u) = t\bar{\partial}\varphi \wedge u + \bar{\partial}u$
$= t(d\varphi)^{0,1} \wedge u + \text{order}(0)$

$\sigma_{\bar{\partial}}(z,\xi) = ?$, $\xi \in {}^{\mathbb{R}}T^*_X \subset \mathbb{C} \otimes {}^{\mathbb{R}}T^*_X = (T^*_X)^{1,0} \oplus (T^*_X)^{0,1}$

$\xi = \xi^{1,0} + \xi^{0,1}$

$\xi^{1,0} = \frac{1}{2}(\xi - i J\xi)$
$\xi^{0,1} = \frac{1}{2}(\xi + iJ\xi)$ conjugated with respect to conjugation on $\mathbb{C}$

so $\sigma_{\bar{\partial}}(z,\xi) = \xi^{0,1} \wedge u \quad : \Lambda^{p,q}T^*_X \longrightarrow \Lambda^{p,q+1}T^*_M$

for $\sigma_{\bar\partial^*}(z, \xi) = ?$

$\xi^* \in {}^{\mathbb{R}}T_x$ the ω-dual of ξ $\quad Re\langle v, \xi^*\rangle = \xi(v)$

$\Rightarrow \quad \sigma_{\bar\partial^*}(z, \xi)(u) = -i_{(\xi^*)^{0,1}}(u)$

$\Rightarrow \quad \sigma_{\bar\Box}(u) = -\xi^{0,1} \wedge i_{(\xi^*)^{0,1}}(u) - i_{(\xi^*)^{0,1}}(\xi^{0,1} \wedge u)$

$= -|\xi^{0,1}|^2_\omega \, u$

and $\sigma_{\Box}(u) = -|\xi^{1,0}|^2_\omega u$

so $\sigma_{\Delta_\omega}(u) = -|\xi|^2_\omega u$

$\xi^{0,1} = \overline{\xi^{1,0}} \Rightarrow |\xi^{0,1}|^2_\omega = |\xi^{1,0}|^2_\omega$ with $\xi = \xi^{1,0} + \xi^{0,1}$

In $\bigoplus_{p,q} \Lambda^{p,q} T^*_x$, components are orthogonal

↳ $|\xi|^2_\omega = |\xi^{1,0}|^2 + |\xi^{0,1}|^2 = 2|\xi^{1,0}|^2 = 2|\xi^{0,1}|^2$ ✓

$\langle\langle \bar\Box u, u\rangle\rangle = \|\bar\partial u\|^2 + \|\bar\partial^* u\|^2$ if X is compact

u is $\bar\Box$-harmonic $\iff \begin{cases} \bar\partial u = 0 \\ \bar\partial^* u = 0 \end{cases}$

For X compact, there is an orthogonal decomposition

$$C^\infty(X, \Lambda^{p,q} T^*_X) = \mathcal{H}^{p,q}_{\bar\Box}(X, \mathbb{C}) \oplus \underbrace{Im\,\bar\partial \oplus Im\,\bar\partial^*}_{= Im\,\bar\Box}$$

$$\ker \bar\partial = \mathcal{H}^{p,q}_{\bar\Box}(X, \mathbb{C}) \oplus Im\,\bar\partial$$ $\Rightarrow$ We can define the Dolbeault cohomology

<u>Theorem</u>: $\boxed{H^{p,q}_{\bar\partial}(X, \mathbb{C}) \cong \mathcal{H}^{p,q}_{\bar\Box}(X, \mathbb{C})}$ for X compact and is a finite dimensional space

<u>Hodge $\star$ operator in the complex setting</u>

↳ $\alpha \wedge \overline{\star\beta} = \langle \alpha, \beta\rangle_{\omega(z)} \, dV_\omega(z)$

whenever $\alpha, \beta \in \Lambda^{p,q} T^*_{X,z}$

In an orthonormal basis we have $\beta = \sum_{|I|=p, |J|=q} \beta_{IJ} \, dz_I \wedge d\bar z_J$

$$\star\beta = \sum_{|I|=p, |J|=q} \pm \beta_{IJ} \, dz_{C_J} \wedge d\bar z_{C_I}$$

$* = \Lambda^{p,q} T_X^* \longrightarrow \Lambda^{n-q, n-p} T_X^*$ $\mathbb{C}$-linear operator

$\# \beta = \overline{* \beta}$

$\hookrightarrow \alpha \wedge \# \beta = \langle \alpha, \overline{\beta} \rangle_\omega \, dV_\omega$

$$\# \beta = \sum_{|I|=p, |J|=q} \pm \overline{\beta_{IJ}} \, dz_{C_I} \wedge d\overline{z_{C_J}}$$

Serre duality for X a compact $\mathbb{C}$-manifold

$$H^{p,q}_{\overline{\partial}}(X, \mathbb{C}) \times H^{n-p, n-q}_{\overline{\partial}}(X, \mathbb{C}) \longrightarrow \mathbb{C}$$

$$(\{\alpha\}, \{\beta\}) \longrightarrow \int_X \alpha \wedge \beta$$

is a non degenerate pairing

(n,n)-form

$\hookrightarrow (\{\alpha\}, \{\# \alpha\}) \longrightarrow \int_X \alpha \wedge \# \alpha = \int_X |\alpha|^2 \, dV_\omega = |\alpha|^2_\omega$

Theorem: $\boxed{H^{n-p, n-q}_{\overline{\partial}}(X, \mathbb{C}) \underset{\text{canonically}}{\simeq} H^{p,q}_{\overline{\partial}}(X, \mathbb{C})^*}$ for X compact

Corollary: X compact and oriented, then $H^{n,n}_{\overline{\partial}}(X, \mathbb{C}) \cong \mathbb{C}$

$$\{\alpha\} \longrightarrow \int_X \alpha$$

proof: $\alpha = \overline{\partial} \beta$ for β a $(n, n-1)$-form, $\alpha = \overline{\partial} \beta = d\beta$ because $\underset{(\cancel{n+1}, n-1)}{\partial \beta = 0}$

2 - Kähler case

Examples:

- Compact Riemann surfaces
  $\omega = \frac{i}{2} \omega_{11}(z) \, dz \wedge d\overline{z}$
  $d\omega = 0$ automatically
- $X = \mathbb{C}^n / \Lambda$ for Λ a lattice
- Projective algebraic geometry
  $X \subset \mathbb{P}^N$ an algebraic submanifold
  $\omega_{FS} = \frac{i}{2\pi} \partial \overline{\partial} \log(|z_0|^2 + \dots + |z_N|^2)$

for $\pi = \mathbb{C}^{N+1} \setminus \{0\} \longrightarrow \mathbb{P}^N$ we have $\pi^* = \omega_{FS}$ the Pull-Back

$U_0 = \mathbb{C}^N \subset \mathbb{P}^N \to [1 : \xi_1 : \dots : \xi_N]$ $\xi_j = \frac{z_j}{z_0}$

$$\omega_{FS}|_{U_0} = \frac{i}{2\pi} \partial \overline{\partial} \log(1 + |\xi_1|^2 + \dots |\xi_N|^2)$$

$\omega = \omega_{FS}|_X$ is a Kähler metric on X

Lemma : Take a positive $(1,1)$-form ω on X, then

ω is kähler ① $\iff$ ② $\forall p \in X$ $\exists$ holomorphic coordinates $(z_1, \dots, z_n)$ centered at p in which

$$\omega(z) = \frac{i}{2} \sum_{1 \leq j,k \leq n} \omega_{jk}(z)\, dz_j \wedge d\bar{z}_k$$

with $\omega_{jk}(z) = \delta_{jk} + O(|z|^2)$

ie : a Kahler metric ω is such that ω_{jk} is "tangent at order 2" to the flat metric $\frac{i}{2} \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j$

supplementary fact : When ω is Kähler, we can achieve

$$\omega_{jk}(z) = \delta_{jk} + \sum_{1 \leq l,m \leq n} c_{jklm}\, z_l \bar{z}_m + O(|z|^3)$$

with (c_{jklm}) is the curvature tensor of (T_X, ω)

proof : ② $\Rightarrow$ ① is trivial

because ② $\Rightarrow$ $\frac{\partial}{\partial z_l} \omega_{jk}(z)\Big|_{z=0} = 0$ ie $\partial\omega(p) = 0$

$\Rightarrow$ $\partial\omega = 0$ as p arbitrary so ω is Kähler ✓

① $\Rightarrow$ ② start with the holomorphic coordinates $(\xi_1, \dots, \xi_n)$.
By a linear change, we can assume that $\left(\frac{\partial}{\partial \xi_j}\right)$ form an orthonormal basis of $T_{X,p}$

In the (ξ_j) $\omega_{jk}(\xi) = \left< \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial \xi_k} \right>_\omega$

$\omega_{jk}(0) = \delta_{jk}$

$\omega_{jk}(\xi)$ is a C^∞-function

By a Taylor expansion at the order 1 we have :

$$\hookrightarrow \omega_{jk}(\xi) = \omega_{jk}(0) + \sum_l a_{jkl}\, \xi_l + a'_{jkl}\, \bar{\xi}_l + O(|\xi|^2)$$

$$= \delta_{jk} + \sum_l a_{jkl}\, \xi_l + a'_{jkl}\, \bar{\xi}_l + O(|\xi|^2)$$

because $\overline{\omega_{jk}} = \omega_{kj}$ $\Rightarrow$ $a'_{jkl} = \overline{a_{kjl}}$

The Kähler condition on ω gives $\partial\omega = 0$, so :

$$0 = \frac{\partial}{\partial \xi_l} \omega_{jk} = \frac{\partial}{\partial \xi_j} \omega_{lk} \quad \forall j,k,l$$

$\Rightarrow$ $a_{jkl} = a_{lkj}$

$\Rightarrow$ $z_k = \xi_k - \frac{1}{2} \sum_{j,l} a_{jkl}\, \xi_j \xi_l$

$(\xi_k) \longrightarrow (z_k)$ Jacobian matrix is I_d at $\xi = 0$

so $dz_k = d\xi_k + \frac{1}{2}\sum_{j,l} a_{jkl}(\xi_j\, d\xi_l + \xi_l\, d\xi_j)$ $(\times 1)$ $\hookrightarrow = \det \mathrm{Jac}$

$= d\xi_k + \sum_{j,l} a_{jkl}\, \xi_l\, d\xi_j$

$\Rightarrow \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z_k} =$ Taylor expansion mod $O(|\xi|^2)$ terms

$= \frac{i}{2}\left[\sum_{k=1}^{n} d\xi_k \wedge d\overline{\xi_k} + \sum_{j,k,l} a_{jkl}\, \xi_l\, d\xi_j \wedge d\overline{\xi_k} + \sum_{j,k,l} \overline{a_{jkl}}\, \overline{\xi_l}\, d\xi_k \wedge d\overline{\xi_j} + O(|\xi|^2)\right]$

$= \omega(\xi) + O(|\xi|^2)$ terms

$\lim \frac{|z|^2}{|\xi|^2} = 1$ so $|z|^2 \sim |\xi|^2$

this means that $\omega = \frac{i}{2}\sum dz_k \wedge d\overline{z_k} + O(|z|^2)$ terms ✓

$\omega_{jk}(z) = \delta_{jk} + \sum (a_{jklm}\, z_l z_m + b_{jklm}\, z_l \overline{z_m} + a'_{jklm}\, \overline{z_l}\,\overline{z_m}) + O(|z|^3)$

hermitian condition $\Rightarrow$ $a'_{jklm} = \overline{a_{kjlm}}$

additionnal requirement $\Rightarrow$ $a_{jklm} = a_{jkml}$ (quadratic)

Kähler condition $\Rightarrow$ $a_{lkjm} = a_{jklm}$

so a_{jklm} is symetric in j, l, m

new coordinates $w_k = z_k + \frac{1}{8}\sum_{j,l,m} a_{jklm}\, z_j z_l z_m$

hermitian condition $\Rightarrow$ $\overline{c_{jklm}} = c_{kjml}$

Kähler condition $\Rightarrow$ $c_{jklm} = c_{lkjm} = c_{jmlk}$ ✓

VII - Additional operators of Kähler geometry

Lefschetz operator $L_\omega = \Lambda^{p,q} T^*_X \longrightarrow \Lambda^{p+1,q+1} T^*_X$

$u \longrightarrow \omega \wedge u$

$\Lambda_\omega = L_\omega^* = \Lambda^{p,q} T^*_X \longrightarrow \Lambda^{p-1,q-1} T^*_X$

Commutation relations:

- $P = C^\infty(X, \Lambda^{p,q} T^*_X) \longrightarrow C^\infty(X, \Lambda^{p+a,q+b} T^*_X)$

  operator of type (a,b), total type $a+b = tt(P)$

- $\partial, \bar\partial$ are of total type 1, L_ω of total type 2
- $\partial^*, \bar\partial^*$ are of total type -1, Λ_ω is of total type -2

Commutation bracket : $\boxed{\begin{array}{l}[P,Q] = PQ - (-1)^{\mathrm{dd}(P)\times \mathrm{dd}(Q)} QP \\ [P,Q]^* = [Q^*, P^*]\end{array}}$

so $[\bar\partial, \bar\partial^*] = \bar\partial\bar\partial^* - (-1)^{(-1)\times(-1)} \bar\partial^*\bar\partial = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \bar\Box$

Identity of Jacobi $\longrightarrow$ $\mathrm{dd}(P) = p$, $\mathrm{dd}(Q) = q$, $\mathrm{dd}(R) = r$

we have $[P,[Q,R]] + (-1)^{pq}[Q,[R,P]] + (-1)^{qr}[R,[P,Q]] = 0$

Fundamental commutation relations of Kähler geometry :

(X,ω) Kähler space

$[L_\omega, \partial^*] = \pm i\bar\partial$

$[L_\omega, \bar\partial^*] = i\partial$

$[\partial, \Lambda_\omega] = i\bar\partial^*$

$[\bar\partial, \Lambda_\omega] = i\partial^*$

we have more complicated formulas in only Hermitian case

proof : we have first to check the first case $[L_\omega, \partial^*] = \pm i\bar\partial$, the three others relations will follow without difficulties.

$X = \mathbb{C}^n$ $\qquad \omega = i \sum_{j=1}^{n} dz_j \wedge d\bar z_j$

$\partial u = \sum_{j=1}^{n} dz_j \wedge \frac{\partial u}{\partial z_j}$ $\qquad u \xrightarrow{\frac{\partial}{\partial z_j}} \frac{\partial u}{\partial z_j}$ they commute $\quad v \xrightarrow{dz_j \wedge \cdot} dz_j \wedge v$

$\partial^* = \sum_{j=1}^{n} - i_{\frac{\partial}{\partial z_j}} \left(\frac{\partial u}{\partial \bar z_j}\right)$

$\int_{\mathbb{C}^n} \frac{\partial u}{\partial z_j} \wedge \bar v = - \int_{\mathbb{C}^n} u \wedge \overline{\frac{\partial v}{\partial \bar z_j}}$ $\qquad \Lambda^{p-1,q-1} T^*_X$

the adjoint of $\frac{\partial}{\partial z_j}$ is $-\frac{\partial}{\partial \bar z_j}$

$\Rightarrow [L_\omega, \partial^*](u) = L_\omega \partial^* u - \partial^* L_\omega u$

$= \omega \wedge \partial^* u - \partial^*(\omega \wedge u)$

$\partial^*(\omega\wedge u) = -\sum_{j=1}^{n} i_{\frac{\partial}{\partial z_j}} \left(\frac{\partial}{\partial \bar z_j}(\omega \wedge u)\right)$

$= -\sum_{j=1}^{n} i_{\frac{\partial}{\partial z_j}} \left(\omega \wedge \frac{\partial u}{\partial \bar z_j}\right)$ $\qquad$ because $\frac{\partial}{\partial \bar z_j}\omega = \text{"}\bar\partial\omega\text{"} = 0$ Kähler

$= -\sum_{j=1}^{n} \left[\left(i_{\frac{\partial}{\partial z_j}} \omega\right) \wedge \frac{\partial u}{\partial \bar z_j} + \omega \wedge \left(i_{\frac{\partial}{\partial z_j}}\left(\frac{\partial u}{\partial \bar z_j}\right)\right)\right]$

$$= -\left[\sum_{j=1}^{n} \left(i\, d\bar{z}_j \wedge \frac{\partial u}{\partial \bar{z}_j} \right) - \omega \wedge \partial^* u \right]$$

so $[L_\omega, \partial^*](u) = \cancel{\omega \wedge \partial^* u} + \underbrace{\sum_{j=1}^{n} i\, d\bar{z}_j \wedge \frac{\partial u}{\partial \bar{z}_j}}_{i\bar{\partial} u} - \cancel{\omega \wedge \partial^* u}$

$$= i\bar{\partial} u$$

for the general Kähler case which $X \subseteq \mathbb{C}^n$

↳ we have $\omega_{jk}(z) = \delta_{jk} + O(|z|^2)$ by the previous lemma

so, because we have only order 1 operators, the calculations (only at centered point p) are identical to the calculation in $\mathbb{C}^n$ at 0.

Consequencies: If (X, ω) is a Kähler space, then $\bar{\square} = \square = \frac{1}{2}\Delta_\omega$

proof: $\bar{\square} = [\bar{\partial}, \bar{\partial}^*]$

$$= [\bar{\partial}, i[\partial, \Lambda_\omega]]$$

the Jacobi identity gives us the relation:

$[\bar{\partial}, [\partial, \Lambda_\omega]] - [\partial, [\Lambda_\omega, \bar{\partial}]] + \cancel{[\Lambda_\omega, [\bar{\partial}, \partial]]} = 0$

but $[\bar{\partial}, \partial] = \bar{\partial}\partial + \partial\bar{\partial} = 0$ so $=0$

let us multiply by $i \Rightarrow \bar{\square} - [\partial, i[\Lambda_\omega, \bar{\partial}]] = 0$

$\Leftrightarrow \bar{\square} - [\partial, \partial^*] = 0$ } commutation relation in Kähler geometry

$\Leftrightarrow \bar{\square} - \square = 0$

so $\bar{\square} = \square$

$\Delta_\omega = [d, d^*]$

$= [\partial + \bar{\partial}, \partial^* + \bar{\partial}^*]$

$= [\partial, \partial^*] + [\bar{\partial}, \bar{\partial}^*] + [\bar{\partial}, \partial^*] + [\partial, \bar{\partial}^*]$

$= \square + \bar{\square} + [\bar{\partial}, \partial^*] + [\partial, \bar{\partial}^*]$

$[\bar{\partial}, \partial^*] = [\bar{\partial}, -i[\bar{\partial}, \Lambda_\omega]]$

Jacobi identity

$[\bar{\partial}, [\bar{\partial}, \Lambda_\omega]] - [\bar{\partial}, [\Lambda_\omega, \bar{\partial}]] + [\Lambda_\omega, \underbrace{\cancel{[\bar{\partial}, \bar{\partial}]}}_{0}] = 0$

so $2[\bar{\partial}, [\bar{\partial}, \Lambda_\omega]] = 0$

$\Rightarrow [\bar{\partial}, \partial^*] = 0$ and $[\partial, \bar{\partial}^*] = 0$ by the same way

so $\Delta_\omega = \square + \bar{\square}$
$= 2\square$

$\Rightarrow \square = \bar{\square} = \frac{1}{2}\Delta_\omega$ ✓

Assume X is a Kähler space :

$$H^k_{DR}(X,\mathbb{C}) \cong \mathcal{H}^k_{\Delta_\omega}(X,\mathbb{C}) \quad \text{by Hodge theory}$$

take $u \in C^\infty(X, \mathbb{C} \otimes \Lambda^k {}^{\mathbb{R}}T^*_X)$

$u = \sum_{p+q=k} u_{p,q}$ with $u_{p,q}$ is of type (p,q)

$\Delta_\omega = 2\square$ preserves the bidegrees (not true in hermitian case).

so $\Delta_\omega u = \sum_{p+q=k} \underbrace{\Delta_\omega u_{p,q}}_{\text{type }(p,q)}$

Consequence : u harmonic $\iff$ each $u_{p,q}$ is harmonic

Hodge decomposition theorem :

$$\boxed{\begin{array}{ccc} \mathcal{H}^k_{\Delta_\omega}(X,\mathbb{C}) & = \bigoplus_{p+q=k} & \mathcal{H}^{p,q}_{\square}(X,\mathbb{C}) \\ \wr\! \mid & & \wr\! \mid \\ H^k_{DR}(X,\mathbb{C}) & \simeq \bigoplus_{p+q=k} & H^{p,q}_{\bar\partial}(X,\mathbb{C}) \end{array}}$$

Additional fact : the last line $\simeq$ does not depend on the choice of the Kähler metric ω

TD : Check this by the Bott-chern cohomology

idea : $H^{p,q}_{BC}(X,\mathbb{C}) \xrightarrow{\sim} H^{p,q}_{\bar\partial}(X,\mathbb{C})$ more precise cohomology

$\bigoplus_{p+q=k} H^{p,q}_{BC}(X,\mathbb{C}) \xrightarrow{\sim} H^k_{DR}(X,\mathbb{C})$ this isomorphism doesn't depend of ω and provide the result.

I - Locally free sheaves

$(X, \mathcal{O}_X)$ is a complex manifold with $n = \dim_{\mathbb{C}} X$.

$E \longrightarrow X$ holomorphic vector bundle of rank r

$\Updownarrow$

$\mathcal{E} = \mathcal{O}(E)$ locally free $\mathcal{O}_X$-module of rank r (locally $\cong \mathcal{O}_X^{\oplus r}$)

$\Updownarrow$

$E_x = \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \underbrace{\mathcal{O}_{X,x}/m_x}_{\cong\ \mathbb{C}}$

Basics examples:

1: T_X, T_X^*

$\Lambda^p T_X$, $\Lambda^p T_X^*$

- $\Omega_X^p = \mathcal{O}(\Lambda^p T_X^*)$ is the sheaf of holomorphic p-forms
- $K_X = \Omega_X^n = \mathcal{O}(\Lambda^n T_X^*)$ is called the canonical sheaf

$\operatorname{rank} \Omega_X^p = \binom{n}{p}$

$\operatorname{rank} K_X = 1$

2: line bundles $L \longrightarrow X$ (holomorphic) vector bundle of rank 1

for L_1, L_2 line bundles we have $L_1 \otimes_{\mathbb{C}} L_2$ which is also a line bundle

the trivial line bundle is $X \times \mathbb{C}$

We can associate the sheaf of line bundle $\mathcal{L} = \mathcal{O}(L)$

$\hookrightarrow$ sheaf of $L_1 \otimes_{\mathbb{C}} L_2$ is $\mathcal{L}_1 \otimes \mathcal{L}_2$ with $\mathcal{L}_1 = \mathcal{O}(L_1)$ and $\mathcal{L}_2 = \mathcal{O}(L_2)$

$\hookrightarrow$ sheaf of the trivial line bundle is note $\mathcal{O}_X$ $\quad (\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X = \mathcal{L})$.

Recall $\operatorname{Hom}(V, W) \cong V^* \otimes W$ via $V^* \otimes W \xrightarrow{\sim} \operatorname{Hom}(V, W)$

$\varphi \otimes w \longrightarrow (x \longrightarrow \varphi(x) w)$

$f : V \longrightarrow W$

f is of matrix (a_{ij}) $\rightarrow$

$$\begin{array}{c} \overbrace{e_1 \ \cdots \ e_n}^{\text{basis of } V} \\ \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \end{array} \begin{array}{c} \varepsilon_1 \\ \\ \varepsilon_m \end{array} \Big\} \text{basis of } W$$

$f = \sum_{i,j} a_{ij} e_j^*(x) \varepsilon_i$

then $f = \operatorname{Im}\left(\underbrace{\sum a_{ij} e_j^* \otimes \varepsilon_i}_{\in V^* \otimes W} \right)$

so $L =$ line$/\mathbb{K}$ $\quad \operatorname{Hom}_{\mathbb{K}}(L, L) \cong \mathbb{K}$. $\quad f(x) = \lambda x \quad \lambda \in \mathbb{K}$

$\Rightarrow L^* \otimes_{\mathbb{K}} L \cong \mathbb{K}$

$\mathbb{K}$ is the unit element for $\otimes_{\mathbb{K}}$ and for a line L^* is the inverse of L for $\otimes_{\mathbb{K}}$ $\quad (L^* = L^{-1})$

Picard Group : $Pic(X) = \left\{ \begin{array}{l} \text{holomorphic isomorphic classes of line bundles} \\ \text{together with } \otimes_K \end{array} \right\}$

terminology : "invertible sheaf" = locally free O_X - module $\mathcal{L}$ of rank 1

$$\mathcal{L}^{-1} = \mathcal{L}^* = Hom_{O_X}(\mathcal{L}, O_X)$$

Invertible sheaves $O_X(D)$ for D an integral divisor :

$D = \sum_{\text{divisor}} m_j D_j \qquad m_j \in \mathbb{Z}$

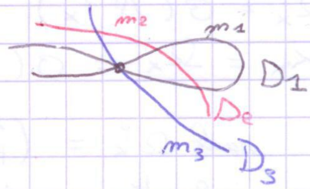
$\mathcal{M}_X$ = sheaf of meromorphic function on X

$\mathcal{M}_{X,x} = \{ f/g \ / \ f \in O_{X,x}, g \in O_{X,x} \quad g \not\equiv 0 \}$

$div(f/g) = div f - div g$

Definition :

- $O_X(D) \subset \mathcal{M}_X$
- $O_X(D)_x = \{ h \in \mathcal{M}_{X,x} \ / \ div(h) + D_{|V} \geq 0$, V is a small neighborhood of $x \}$



By the chapter 1, we now that O_X is a factorial ring

locally near x, D is the divisor of $\prod_j p_j^{\mu_j}$

$p_j \in O_{X,x}$ irreducibles and $\mu_j \in \mathbb{Z}$

for $h \in \mathcal{M}_{X,x}$, $h \not\equiv 0$ $\qquad h = \underbrace{\prod p_j^{\alpha_j}}_{\text{irreducible germs coming from } D} \underbrace{\prod q_k^{\beta_k}}_{\text{other irreducible elements}} \qquad \alpha_j, \beta_k \in \mathbb{Z}$

$$div(h) + D = div(h \cdot \prod p_j^{\mu_j})$$
$$= div\left(\prod p_j^{\alpha_j + \mu_j} \prod q_k^{\beta_k}\right) \geq 0$$

we want that to be positive

$$\Rightarrow \left\{ \begin{array}{l} \alpha_j + \mu_j \geq 0 \\ \beta_k \geq 0 \end{array} \right.$$

So $div(h) + D \geq 0$ near x means $h \overbrace{\prod p_j^{\mu_j}}^{}_{div\,D} \in O_{X,x}$

allowed $h = (\prod p_j^{-\mu_j}) \cdot f$ where $f \in O_{X,x}$

Conclusion : $O_X(D)$ is just the locally free rank 1 O_X-module generated by $\prod p_j^{-\mu_j}$, on a suffiently small neighborhood of x

Proposition : $D \longrightarrow O_X(D)$ is a group homomorphism

$D + D' \longrightarrow O_X(D+D') = O_X(D) \otimes O_X(D')$

II - Dolbeault cohomology

$E \to X$ an holomorphic vector bundle

$C^\infty(X, \Lambda^{p,q} T_X^* \otimes E) \subset \mathcal{D}'(X, \Lambda^{p,q} T_X^* \otimes E)$

$$u(z) = \sum_{\substack{|I|=p \\ |J|=q \\ 1\le\alpha\le r}} u_{I,J,\alpha}(z)\, dz_I \wedge d\bar{z}_J \otimes e_\alpha$$

- $(z_1, \dots, z_n)$ local holomorphic coordinates on X
- $(e_\alpha)_{1\le\alpha\le r}$ local holomorphic frame of E

we have a generalized $\bar\partial$ operator (we want $\bar\partial e_\alpha = 0$)

$$\hookrightarrow \bar\partial u(z) = \sum_{\substack{|I|=p \\ |J|=q \\ 1\le\alpha\le r \\ 1\le k\le n}} \frac{\partial u_{IJ\alpha}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J \otimes e_\alpha$$

we have to check that doesn't depend of the choice of the coordinates $(z_1, \dots, z_n)$ and of the choice of (e_α)

(take $\tilde u$ with $(\tilde e_\alpha)$, we have $\tilde u = g u$ for g a holomorphic $r\times r$ metric $\bar\partial \tilde u = g \bar\partial u$ ...)

One gets the Dolbeault cohomology groups $= H^{p,q}(X, E)$

Attention: ∂ does not exist !!!

$\tilde u = g u \qquad g = (g_{\alpha\beta})_{1\le\alpha,\beta\le r} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$

$$\partial \tilde u = \underbrace{\partial g \wedge u}_{\text{not independant of the choice of the frame } (e_\alpha)} + g \cdot \partial u$$

Assume (E, h) hermitian structure on E

$(E_x, h(x))$, $(h_{\alpha\beta}(x))_{1\le\alpha,\beta\le r}$ C^∞ positive definite

$$\langle\langle u, v \rangle\rangle = \int_X \langle u(x), v(x) \rangle_{\omega, h}\, dV_\omega$$

$$u(x) \in \underbrace{\Lambda^{p,q} T_x^*}_{\omega} \otimes \underbrace{E}_{h}$$

$\bar\square = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$ elliptic

Theorem: If (X, ω) compact hermitian, (E, h) holomorphic hermitian

$$\boxed{H^{p,q}(X, E) \cong \mathcal{H}^{p,q}_{\bar\square}(X, E)}$$ finite dimension space of harmonic forms

Serre duality:
$$H^{p,q}(X, E) \times H^{n-p, n-q}(X, E^*) \longrightarrow \mathbb{C}$$
$$(\{u\}, \{v\}) \longmapsto \int_X u \wedge v$$

$u = \sum u_{IJ\alpha}(z)\, dz_I \wedge d\bar z_J \otimes e_\alpha$

$v = \sum v_{KL\beta}(z)\, dz_K \wedge d\bar z_L \otimes e_\beta^*$ dual frame

$$u \wedge v = \sum u_{IJ\alpha}(z)\, v_{KL\beta}(z)\, \underbrace{dz_I \wedge d\bar z_J \wedge dz_K \wedge d\bar z_L}_{(n,n)}$$

$\# = (\Lambda^{p,q} T_X^* \otimes E)_x \longrightarrow (\Lambda^{n-p,n-q} T_X^* \otimes E^*)_x$ conjugate $\mathbb{C}$-linear

$e_\alpha \longrightarrow e_\alpha^*$

adjusting the frame to be orthonormal at point x

so $\boxed{\bar\square_{E^*}(\# u) = \# \bar\square_E u}$

Dolbeault complex of sheaves on X:

$$\mathcal{K}^q = C^\infty_X(\Lambda^{p,q} T_X^* \otimes E), \bar\partial$$

$$0 \to \mathcal{K}^0 \xrightarrow{\bar\partial} \mathcal{K}^1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{K}^n \longrightarrow 0$$ exact sequence of sheaves except in degree 0.

locally $E_{|V} \cong \mathcal{O}_V^{\oplus r}$

$\ker(\mathcal{K}^0 \xrightarrow{\bar\partial} \mathcal{K}^1) = \mathcal{O}(\Lambda^p T_X^* \otimes E) = \Omega_X^p \otimes E$

"Dolbeault isomorphism"

$$\underset{\text{sheaf cohomology}}{H^q(X, \Omega_q^p \otimes E)} \cong H^{p,q}(X, E)$$

we have $H^{p,q}(X,E) = H^{0,q}(X, \Omega_X^p \otimes E)$

$\underset{(p,q)}{dz_I \wedge d\bar z_J} \otimes \underset{E}{e_\alpha}$ or $\pm \underset{(0,q)}{d\bar z_J} \otimes \underset{(\Omega_X^p \otimes E)}{(dz_I \otimes e_\alpha)}$

III - Connections

$E \longrightarrow M$ a $\mathbb{K}$-vector bundle, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Definition: A connection on E is a differential operator of order 1

$$D : C^\infty(M, \Lambda^p T_M^* \otimes E) \longrightarrow C^\infty(M, \Lambda^{p+1} T_M^* \otimes E)$$
$$u \longrightarrow Du$$

such that D satisfies the Leibniz rule:

- $D(fu) = df \wedge u + f Du$ for $f \in C^\infty(M, \mathbb{K})$
- $D(f \wedge u) = df \wedge u + (-1)^m f \wedge Du$ for $f \in C^\infty(M, \Lambda^m T_M^* \otimes_{\mathbb{R}} \mathbb{K})$

example: $E = M \times \mathbb{K}$ trivial

$D = d$ ✓

General form of a connection :

$(e_1, \dots, e_r)$ C^∞ frame of E, $r = \operatorname{rank}_{\mathbb{K}} E$ on some open set $\Omega \subset X$

$$u = \sum_{\alpha=1}^{r} u_\alpha \wedge e_\alpha \longrightarrow u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} \qquad \begin{array}{l} u_j \in C^\infty(\Omega, \Lambda^p T^*_M \otimes \mathbb{K}) \\ e_\alpha \in C^\infty(\Omega, E) \end{array}$$

p-form with values in E

$$Du = \sum_{\beta=1}^{r} du_\beta \wedge e_\beta + (-1)^p u_\beta \wedge De_\beta \qquad De_\beta \in C^\infty(\Omega, \Lambda^1 T^*_M \otimes E)$$

$$\circledast \quad \boxed{De_\beta = \sum_{\alpha=1}^{r} \Gamma_{\alpha\beta}\, e_\alpha} \qquad \Gamma_{\alpha\beta} \in C^\infty(\Omega, \Lambda^1 T^*_M)$$

so $\Gamma = (\Gamma_{\alpha\beta})_{1 \le \alpha, \beta \le r} \in C^\infty(\Omega, \Lambda^1 T^*_M \otimes \mathrm{Mat}_{r\times r}(\mathbb{K}))$ $\longleftrightarrow$ $\mathrm{Hom}(\mathbb{K}^r, \mathbb{K}^r)$

$$\text{so } Du = \sum_{\alpha=1}^{r} du_\alpha \wedge e_\alpha + (-1)^p \sum_{\beta=1}^{r} u_\beta \wedge \sum_{\alpha=1}^{r} \Gamma_{\alpha\beta}\, e_\alpha$$

$$= \sum_{\alpha=1}^{r} \left(du_\alpha + \sum_{\beta=1}^{r} \Gamma_{\alpha\beta} \wedge u_\beta \right) e_\alpha$$

$$\cong d\begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} + (\Gamma_{\alpha\beta}) \wedge \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$$

$$\text{so } \boxed{\underbrace{Du}_{\text{thought as } E\text{-valued}} \cong \underbrace{du + \Gamma \wedge u}_{\text{calculation as column vectors in } \mathbb{K}^r}}$$

depends of the frames e_α because Γ the matrix of the connection with respect to the frame (e_α) given by $\circledast$

Conversely, any such formula defines a connection (over the open set Ω where the frame is defined).

Complex situation :

for $(X, \mathcal{O}_X)$ a complex manifold
$E \longrightarrow X$ C^∞ $\mathbb{C}$-vector bundle (not necessary holomorphic)

D a connection on E

$Du \simeq du + \Gamma \wedge u$ locally with respect to $E_{|\Omega} \cong \Omega \times \mathbb{K}^r$

$d = \underset{(1,0)}{\partial} + \underset{(0,1)}{\bar\partial}$ and $\Gamma = \Gamma^{1,0} + \Gamma^{0,1}$

$\hookrightarrow$ we have $\begin{cases} D = D^{1,0} + D^{0,1} \\ \text{with } D^{1,0} u = \partial u + \Gamma^{1,0} \wedge u \\ \phantom{\text{with }} D^{0,1} u = \bar\partial u + \Gamma^{0,1} \wedge u \end{cases}$ unique decomposition

$$D^{1,0}(f \wedge u) = \partial f \wedge u + (-1)^{\deg f} f \wedge D^{1,0} u$$
$$D^{0,1}(f \wedge u) = \bar\partial f \wedge u + (-1)^{\deg f} f \wedge D^{0,1} u$$

Now take E, with an hermitian structure (E,h).

$\Rightarrow$ Hermitian pairing

$$C^\infty(X, \Lambda^{p,q} T_X^* \otimes E) \times C^\infty(X, \Lambda^{s,t} T_X^* \otimes E) \longrightarrow C^\infty(X, \Lambda^{p+t, q+s} T_X^*)$$
$$(u, v) \longrightarrow \{u,v\}_h$$

$u = \sum u_{IJ\alpha}\, dz_I \wedge d\bar{z}_J \otimes e_\alpha$

$v = \sum v_{KL\beta}\, dz_K \wedge d\bar{z}_L \otimes e_\beta$

$\{u,v\}_h = \sum u_{IJ\alpha} \overline{v_{KL\beta}}\, dz_I \wedge d\bar{z}_J \wedge d\bar{z}_K \wedge dz_L \langle e_\alpha, e_\beta\rangle_h$

<u>Compatibility condition of the connection D with the hermitian pairing:</u>

$\hookrightarrow \quad d\{u,v\}_h = \{Du, v\}_h + (-1)^{\deg u}\{u, Dv\}_h$

<u>exercice:</u> for (e_α) orthonormal, check

D compatible with hermitian pairing associated with h $\iff \Gamma^* \;``="\; (\overline{\Gamma_{\beta\alpha}}) = -\Gamma$

$$\Rightarrow \begin{cases} (\Gamma^{1,0})^* = -\Gamma^{0,1} \\ (\Gamma^{0,1})^* = -\Gamma^{1,0} \end{cases}$$

<u>Conclusion:</u> If D is h-compatible and $D^{0,1}$ is given, then $D^{1,0}$ is fixed

Take $E \longrightarrow X$ to be holomorphic and equipped with a boundary structure h.

$\bar\partial = C^\infty(X, \Lambda^{p,q} T_X^* \otimes E) \longrightarrow C^\infty(X, \Lambda^{p,q+1} T_X^* \otimes E)$ exists by our previous work

this is the $(0,1)$-connection that has $\Gamma^{0,1} = 0$ in a holomorphic frame (e_α)

<u>Theorem:</u> There exist a unique connection D that is h-compatible and such that $D^{0,1} = \bar\partial$

$D = D_{E,h}$ Chern connection of (E,h).

<u>Computation of the Chern connection:</u>

Take (e_α) a holomorphic frame of $E|_\Omega$

$H = (h_{\alpha\beta})$ is given by $h_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle_h$

$$\{u,v\}_h = {}^t u \wedge H \bar{v}$$
$$= (u_1, \dots, u_r) \wedge \begin{pmatrix} h_{11} & \cdots & h_{1r} \\ \vdots & \ddots & \vdots \\ h_{r1} & \cdots & h_{rr} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}$$

so $d\{u,v\}_h = {}^t(du) \wedge H\bar{v} + (-1)^{\deg u}\left({}^t u \wedge \overbrace{dH}^{\partial H + \bar\partial H} \wedge \bar{v} + {}^t u \wedge H\, d\bar{v}\right)$

In a holomorphic frame $D = D^{1,0} + D^{0,1}$

with $D^{1,0} u = \partial u + \Gamma^{1,0} \wedge u$
$D^{0,1} u = \bar\partial u$

$${}^t\overline{H} = H \qquad {}^tH = \overline{H}$$
$$\overline{H^{-1}\partial \overline{H}} = \partial H \cdot H^{-1}$$

$$\Rightarrow d\{u,v\}_h = {}^t(du + \overline{H}^{-1}\partial\overline{H}\wedge u)\wedge H\bar{v} + {}^tu\wedge h\wedge \overline{(dv + H^{-1}\partial \overline{H}\wedge v)}$$

$$\Rightarrow \Gamma^{1,0} = \overline{H}^{-1}\partial\overline{H} = {}^tH^{-1}\partial {}^tH = (h_{\beta\alpha})^{-1}(\partial h_{\beta\alpha})$$

$$\Rightarrow \begin{cases} D^{1,0}u = \partial u + \overline{H}^{-1}\partial\overline{H}\wedge u = \overline{H}^{-1}\partial(\overline{H}u) \\ D^{0,1}u = \bar\partial u \end{cases}$$

Curvature of a connection:

D a connection on E
$Du \simeq du + \Gamma\wedge u$

$$\begin{aligned} D^2 u &\simeq D(du + \Gamma\wedge u) \\ &\simeq d(du + \Gamma\wedge u) + \Gamma\wedge(du + \Gamma\wedge u) \\ &= \cancel{d^2u} + d(\Gamma\wedge u) + \Gamma\wedge du + \Gamma\wedge\Gamma\wedge u \\ &= d(\Gamma)\wedge u - \cancel{\Gamma\wedge du} + \cancel{\Gamma\wedge du} + \Gamma\wedge\Gamma\wedge u \end{aligned}$$

$$\Rightarrow D^2u \simeq \underbrace{(d\Gamma + \Gamma\wedge\Gamma)}_{\text{2-form } (r\times r \text{ matrix})}\wedge u$$

we differentiate twice and we have no more differentiations
$\Rightarrow d\Gamma + \Gamma\wedge\Gamma$ is a 2-form

Theoreme: $\exists$ a global 2-form $\Theta_{E,D} \in C^\infty(X, \Lambda^2 T_X^* \otimes \mathrm{Hom}(E,E))$
such that $\boxed{D^2 u = \Theta_{E,D}\wedge u}$
with respect to a trivialisation $\boxed{\Theta_{E,D} \simeq d\Gamma + \Gamma\wedge\Gamma}$

$\Theta_{E,D}$ is the curvature tensor of (E,D)

note: if rank $E = 1$ then $\Gamma\wedge\Gamma = 0 \Rightarrow \Theta_{E,D} \simeq d\Gamma$ is a closed form

Case of an hermitian holomorphic bundle (E,h)
↳ $D_{E,h}$ Chern connection
$\Theta_{E,h}$ curvature of the chern connection

$$\begin{aligned} D^2 &= (D^{1,0} + D^{0,1})^2 \\ &= (D^{1,0})^2 + D^{1,0}D^{0,1} + D^{0,1}D^{1,0} + (D^{0,1})^2 \end{aligned}$$

we have $\bullet\ (D^{0,1})^2 = \bar\partial^2 = 0$

$\bullet\ (D^{1,0})^2 u = \bar H^{-1}\partial \bar H . \bar H^{-1}\partial \bar H\, u = \bar H^{-1}\partial^2 \bar H u = 0$

so $D^2 = D^{1,0}D^{0,1} + D^{0,1}D^{1,0}$

with some litte more calculus we check that $\boxed{\Theta_{E,h} \simeq \bar\partial(\bar H^{-1}\partial \bar H)}$ (1,1)-form

Rank 1 bundle:

$L \longrightarrow X$ holomorphic line bundle
h hermitian structure

$L|_\Omega \simeq \Omega \times \mathbb{C}$
$e \simeq 1$

$0 < |e|_h^2 = e^{-\varphi}$
so $\varphi(z) = -\log |e|_h^2$

e any non vanishing holomorphic section

$H = (e^{-\varphi})$

$$\Theta_{E,h} = \bar\partial(e^{\varphi}\partial e^{-\varphi}) = -\bar\partial\partial\varphi = \partial\bar\partial\varphi$$

IV - L^2-estimate for the $\bar\partial$-operator

Take X a complex manifold with $\dim_{\mathbb{C}} X = n$
Let (E,h) be a hermitian holomorphic vector bundle on X (h is C^∞).

Recall: Chern connection $\Delta_{E,h} = D_E = \partial_E + \bar\partial_E$
where ∂_E is (1,0)-part and $\bar\partial_E$ the (0,1)-part

$\hookrightarrow \bar\partial_E^2 = \partial_E^2 = 0$

$D_E^2 = \partial_E\bar\partial_E + \bar\partial_E\partial_E = \Theta_{E,h}$

where $\Theta_{E,h} \in C^\infty(X, \Lambda^{1,1}T_X^* \otimes \mathrm{Hom}(E,E))$ Chern curvature form (acts by multiplication)

let $\omega(z) = i \sum_{1\le j,k\le n} \omega_{jk}(z)\, dz_j \wedge d\bar z_k$ to be a smooth Kähler metric

Then, one get L^2 spaces: $L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ of (p,q)-forms

Given $u(z) = \sum_{\substack{|I|=p\\|J|=q\\1\le\alpha\le r}} \omega_{IJ\alpha}(z)\, dz_I \wedge d\bar z_J \otimes e_\alpha(z)$ where $(e_\alpha)_{\alpha\in[1,r]}$ local frames of E

pointwise norms: $|u|^2 = |u|^2_{\omega,h}$ at $a\in X$ with coordinates $(z_1,\dots,z_n)$ such that

- $\left(\frac{\partial}{\partial z_j}\right)_{1\le j\le n}$ is a orthonormal basis of $T_{X,a}$ with respect to $\omega(a)$
- $(e_\alpha(z))_{1\le\alpha\le r}$ is an orthonormal basis of E_a

Then, at $z = a$

$\hookrightarrow \quad |u|^2_{\omega,h} = \sum_{\substack{|J|=p\\|J|=q\\1\le\alpha\le r}} |u_{IJ\alpha}|^2$

the Kähler volume element is $dV_\omega = \frac{1}{n!}\omega^n$ $\qquad dV_\omega(z) = 1 \cdot d\lambda(z)$

(in fact $dV_\omega(z) = \gamma(z)\, d\lambda(z)$
where $\gamma(z) = \det(\omega_{jk}(z))$
at $z = a$ $\quad \omega_{jk}(a) = \delta_{jk}$, hence $\gamma(a) = 1$)

Lemma 1: One can choose coordinates centered at point a such that

$$\omega_{jk}(z) = \delta_{jk} + O(|z|^2) \qquad \Rightarrow \gamma(z) = 1 + O(|z|^2)$$

Lemma 2: One can choose $(e_\alpha(z))$ to be a holomorphic frame such that

$$\begin{aligned} \langle e_\alpha(z), e_\beta(z)\rangle_h &= h_{\alpha,\beta}(z) \\ &= \delta_{\alpha,\beta} - \sum_{1\le j,k\le n} c_{jk\alpha\beta}\, z_j \overline{z_k} \end{aligned}$$

where $(c_{jk\alpha\beta})$ are the coefficients of $\Theta_{E,h}(a)$

$$\Theta_{E,h}(a) = \sum_{\substack{1\le j,k\le n\\1\le\alpha,\beta\le r}} c_{jk\alpha\beta}\, \underbrace{dz_j \wedge d\overline{z_k}}_{\Lambda^{1,1}T^*_x} \otimes \underbrace{e^*_\alpha \otimes e_\beta}_{Hom(E,E)}$$

proof (for lemma 2)

- start from any holomorphic frame $(\varepsilon_\alpha(z))_{1\le\alpha\le r}$
  Use Gram-Schmidt on "constant coefficients" (ε_α) to get $(\tilde{\varepsilon}_\alpha)$ orthonormal at $z=a$.
- One can assume $(\varepsilon_\alpha(z))$ is orthonormal at a

$$\begin{aligned} \langle \varepsilon_\alpha(z), \varepsilon_\beta(z)\rangle_h &= \delta_{\alpha,\beta} + O(|z|) \\ &= \delta_{\alpha,\beta} + \sum_{1\le j\le n} a_{j\alpha\beta}\, z_j + a'_{j\alpha\beta}\, \overline{z_j} + O(|z|^2) \end{aligned}$$

replace $(\varepsilon_\alpha(z))$ by $\tilde{\varepsilon}_\alpha(z) = \varepsilon_\alpha(z) - \sum_{\substack{1\le j\le n\\1\le\beta\le r}} a_{j\alpha\beta}\, z_j\, \varepsilon_\beta(z)$

- hermitian property $\overline{\langle \varepsilon_\alpha(z), \varepsilon_\beta(z)\rangle_h} = \langle \varepsilon_\beta(z), \varepsilon_\alpha(z)\rangle$

$$\Rightarrow a'_{j\alpha\beta} = \overline{a_{j\beta\alpha}}$$

$$\Rightarrow \langle \tilde{\varepsilon}_\alpha(z), \tilde{\varepsilon}_\beta(z)\rangle_h = \delta_{\alpha,\beta} + O(|z|^2)$$

- restart from $(\varepsilon_\alpha(z))$ such that $\langle \varepsilon_\alpha(z), \varepsilon_\beta(z)\rangle_h = \delta_{\alpha,\beta} + O(|z|^2)$

$$< \mathcal{E}_\alpha(z), \mathcal{E}_\beta(z) >_h = \delta_{\alpha,\beta} + \sum a_{jk\alpha\beta} z_j z_k + \sum a'_{jk\alpha\beta} \overline{z_j} \overline{z_k} + \sum b_{jk\alpha\beta} z_j \overline{z_k} + o(|z|^3)$$

cannot be killed

hermitian $\to$ $\begin{cases} a'_{jk\alpha\beta} = \overline{a_{jk\beta\alpha}} \\ b_{jk\alpha\beta} = \overline{b_{kj\beta\alpha}} \end{cases}$ we can assume that $a_{jk\alpha\beta} = a_{kj\alpha\beta}$ (quadratic form)

$$\Rightarrow \widetilde{\mathcal{E}}_\alpha(z) = \mathcal{E}_\alpha(z) - \sum_{j,k,\beta} a_{jk\alpha\beta} z_j z_k \mathcal{E}_\beta(z)$$

- Take $(e_\alpha(z))$ to be this $(\widetilde{\mathcal{E}}_\alpha(z))$, then

$$< e_\alpha(z), e_\beta(z) >_h = \delta_{\alpha,\beta} - \sum_{j,k} c_{jk\alpha\beta} z_j \overline{z_k} + o(|z|^3)$$

where $c_{jk\alpha\beta} = - b_{jk\alpha\beta}$

- We show that $(c_{jk\alpha\beta})$ are the curvature coefficients

$\hookrightarrow$ $d(< e_\alpha(z), e_\beta(z) >_h) = - \sum_{j,k} c_{jk\alpha\beta} \overline{z_k} dz_j - \sum_{j,k} c_{jk\alpha\beta} z_j d\overline{z_k} + o(|z|^2)$

Leibniz want this $\longrightarrow$ $= \{ D_E e_\alpha, e_\beta \} + \{ e_\alpha, D_E e_\beta \}$

since $\overline{\partial}_E e_\alpha = 0$ because we have holomorphic frames $\longrightarrow$ $= \{ \partial_E e_\alpha, e_\beta \} + \{ e_\alpha, \partial_E e_\beta \}$

$$\Rightarrow \{ \partial_E e_\alpha, e_\beta \} = - \sum_{j,k} c_{jk\alpha\beta} \overline{z_k} dz_j + o(|z|^2)$$

$$\Rightarrow \partial_E e_\alpha = D_E e_\alpha = - \sum_{j,k} c_{jk\alpha\beta} \overline{z_k} dz_j \otimes e_\beta + o(|z|^2)$$

$$\Rightarrow \Theta_{E,h} \cdot e_\alpha = D_E (D_E e_\alpha)$$
$$= - \sum_{j,k,\beta} c_{jk\alpha\beta} d\overline{z_k} \wedge dz_j \otimes e_\beta$$
$$= + \sum_{j,k,\beta} c_{jk\alpha\beta} dz_j \wedge d\overline{z_k} \otimes e_\beta \quad \text{at } z = a$$

$$\Rightarrow \Theta_{E,h} = \sum_{j,k,\beta,\alpha} c_{jk\alpha\beta} dz_j \wedge d\overline{z_k} \otimes e_\beta \otimes e_\alpha^*$$

Global L^2 norms : $\| u \|^2_{\omega,h} = \int_X |u(z)|^2_{\omega,h} \, dV_\omega$

Operators : $D_E = \partial_E + \overline{\partial}_E$ $\quad (1,0) \oplus (0,1)$

Formal adjoint : $D_E^* = \partial_E^* + \overline{\partial}_E^*$ $\quad (-1,0) \oplus (0,-1)$

Lefschitz : $L_\omega u = \omega \wedge u$

$\Lambda_\omega = L_\omega^*$

Commutation relations : (Assuming ω is a Kähler metric)

$$[L_\omega, \partial_E^*] = i\bar\partial_E \quad ; \quad [L_\omega, \bar\partial_E^*] = -i\partial_E$$

$$[\partial_E, \Lambda_\omega] = -i\bar\partial_E^* \quad ; \quad [\bar\partial_E, \Lambda_\omega] = i\partial_E^*$$

proof : Take $U \subset X$ coordinates charts

$\omega(z) = \frac{i}{2} \sum \omega_{jk}(z)\, dz_j \wedge d\bar z_k$ Kähler tangent to $\omega_0(z) = \frac{i}{2}\sum dz_j \wedge d\bar z_j$ modulo $o(|z|^2)$

$h(z) = (h_{\alpha,\beta}(z))$ with $h_{\alpha,\beta}(z) = \delta_{\alpha,\beta}$ modulo $o(|z|^2)$.

$E|_U \cong U \times \mathbb{C}^r$ equipped with "trivial metric" : $h^0_{\alpha\beta}(z) = \delta_{\alpha,\beta}$

↳ Compare (ω, h) and (ω_0, h^0) :

$\partial_E, \bar\partial_E$ and $\partial_E^*, \bar\partial_E^*$ are differentials from the trivial ones by $o(|z|)$ terms

$dV_\omega = (1 + o(|z|^2))\, dV_{\omega_0}$ Then just write $u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$ and reduce to the trivial bundle $X \times \mathbb{C}$.

Let's introduce the Laplace - Beltrami operator

↳ $\square_E = [\partial_E, \partial_E^*] = \partial_E\partial_E^* + \partial_E^*\partial_E$

$\quad \bar\square_E = [\bar\partial_E, \bar\partial_E^*] = \bar\partial_E\bar\partial_E^* + \bar\partial_E^*\bar\partial_E$

$\square_E, \bar\square_E : C^\infty(X, \Lambda^{p,q}T_X^* \otimes E) \longrightarrow C^\infty(X, \Lambda^{p,q}T_X^* \otimes E)$ are of order 2

$$\bar\square_E = [\bar\partial_E, \bar\partial_E^*]$$
$$= [\bar\partial_E, i[\partial_E, \Lambda_\omega]]$$

Jacobi identity $\Rightarrow [\bar\partial_E, [\partial_E, \Lambda_\omega]] - [\partial_E, [\Lambda_\omega, \bar\partial_E]] + [\Lambda_\omega, [\bar\partial_E, \partial_E]] = 0$

$\Leftrightarrow \; -i\bar\square_E + [\partial_E, i\partial_E^*] + [\Lambda_\omega, \partial_E\bar\partial_E + \bar\partial_E\partial_E] = 0$

$\Leftrightarrow \; -i\bar\square_E + i\square_E + [\Lambda_\omega, \Theta_{E,h}] = 0$

$$\Rightarrow \boxed{\bar\square_E = \square_E + [i\Theta_{E,h}, \Lambda_\omega]} \quad \text{Bochner - Kodaira - Nakano identity}$$

Exercise : Show that $\Delta_E = [D_E, D_E^*] = \square_E + \bar\square_E$

Fact : $\square_E, \bar\square_E$ and Δ_E are still elliptic operators (even true in non - Kähler case).

Theorem: If (X,ω) is compact

$$\boxed{H^{p,q}_{\bar\partial_E}(X,E) \cong \mathcal{H}^{p,q}_{\bar\Box_E}(X,E)}$$

Assume X compact $\Rightarrow$ $\langle\langle \bar\Box_E u, u\rangle\rangle = \|\bar\partial_E u\|^2 + \|\bar\partial_E^* u\|^2$

$\langle\langle \Box_E u, u\rangle\rangle = \|\partial_E u\|^2 + \|\partial_E^* u\|^2 \geq 0$

Assume X compact Kähler $\Rightarrow$ $\langle\langle \bar\Box_E u, u\rangle\rangle = \langle\langle \Box_E u, u\rangle\rangle + \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle dV_\omega$

$\Leftrightarrow$ $\boxed{\langle\langle \bar\Box_E u, u\rangle\rangle \geq \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle dV_\omega}$

B-K-N inequality

Observation: $[\underset{(1,1)}{i\Theta_{E,h}}, \underset{(-1,-1)}{\Lambda_\omega}]$ is a pointwise of order 0 hermitian operator acting on $\Lambda^{p,q}T^*_X \otimes E$

Consequence: If (X,ω) is a compact Kähler space and $[i\Theta_{E,h}, \Lambda_\omega] > 0$ in bidegree (p,q).

then $H^{p,q}_{\bar\partial}(X,E) \cong \mathcal{H}^{p,q}_{\bar\Box_E}(X,E) = 0$

proof: Take u $\bar\Box_E$-harmonic ($\bar\Box_E u = 0$)

by B-K-N inequality $\Rightarrow$ $0 \geq \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]u, u\rangle dV_\omega$

$\geq c\int_X |u|^2 dV_\omega \geq 0$

where $c = \inf_{z\in X}$ eigenvalue of the operator $A^{p,q}_{X,\omega,E,h}(z) > 0$

$$A^{p,q}_{X,\omega,E,h} = [i\Theta_{E,h}, \Lambda_\omega]$$

$\Rightarrow u \equiv 0$

$\Rightarrow \mathcal{H}^{p,q}_{\bar\Box_E}(X,E) = 0$ ✓

V – Case of rank 1 bundle E (line bundle)

Because rank $E = 1$, we have $\mathrm{Hom}(E,E) = \mathbb{C}$ trivial, so it's simpler!

$h(z) = (e^{-\varphi(z)})$

$i\Theta_{E,h} = i\partial\bar\partial\varphi$

we can look at the eigenvalues of $i\Theta_{E,h}(z)$ with respect to $\omega(z)$

↳ we have $\lambda_1(z) \leq \dots \leq \lambda_n(z)$ the continuous functions of the eigenvalues at point z (in general non smooth)

$\lambda_i(z) \in \mathbb{R}$ because $i\partial\bar\partial\varphi$ is self adjoin $\Rightarrow$ real form

Lemma: for $u(z) = \sum_{\substack{|I|=p \\ |J|=q}} u_{IJ}(z)\, dz_I \wedge d\bar{z}_J \otimes e(z)$

then $[i\Theta_{E,h}, \Lambda_\omega] u = \sum_{\substack{|I|=p \\ |J|=q}} \alpha_{IJ}(z)\, u_{IJ}(z)\, dz_I \wedge d\bar{z}_J \otimes e(z)$

where $\alpha_{I,J}(z) = \sum_{i\in I} \lambda_i(z) + \sum_{j\in J} \lambda_j(z) - \sum_{1\le j\le n} \lambda_j(z)$

proof: exercice

hint: when computed with respect to $\left(\frac{\partial}{\partial z_j}\right)$ that is an orthonormal basis for $\omega(z)$ and orthogonal for $i\Theta_{E,h}(z)$

we can write $i\Theta_{E,h}(z) = i \sum_{1\le j\le n} \lambda_j(z)\, dz_j \wedge d\bar{z}_j$

$$\omega = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad i\Theta_{E,h} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

note: $\alpha_{IJ}(z) = \sum_{j\in J} \lambda_j(z) - \sum_{i\in {}^cI} \lambda_i(z)$

in the case u is an (n,q)-form $|I| = p = n$, then ${}^cI = \phi$

so $\alpha_{IJ}(z) = \sum_{j\in J} \lambda_j(z) \geq (\lambda_1 + \dots + \lambda_q)(z)$

so for (n,q)-forms $\int_X \langle [i\Theta_{E,h}, \Lambda_\omega] u, u\rangle\, dV_\omega \geq \int_X (\lambda_1 + \dots + \lambda_q)|u|^2\, dV_\omega$

Corollary: (Kodaira)

If (E,h) is a line bundle with $i\Theta_{E,h} > 0$ ($\Rightarrow$ Kähler) on X compact, then

$$H^{n,q}(X,E) = 0 \qquad \forall q \geq 1$$

Observation: If $\omega = i\Theta_{E,h} > 0$ is Kähler, then $\lambda_j(z) = 1$ identically

$\Rightarrow \alpha_{IJ}(z) = p+q-n$

Theorem (Akizuki – Kodaira – Nakano): (1953 – 1954)

If (E,h) has a positive curvature (>0), then

$H^{p,q}(X,E) = 0 \qquad \forall\, p,q$ with $p+q \geq n+1$

because $i\Theta_{E^*,h^*} = -i\Theta_{E,h}$ and by Serre duality $(E,h) > 0 \Rightarrow H^{p,q}(X,E^*) = 0$ for $p+q \leq n-1$

Observation: If X is non compact but $u \in \mathcal{D}_{p,q}(X,E) = C^\infty(X, \Lambda^{p,q} T_X^* \otimes E)$
(u is then compactly supported)

we have $\|\bar\partial_E u\|^2 + \|\bar\partial_E^* u\|^2 = \langle\langle \square_E u, u \rangle\rangle \geq \int_X \langle A^{p,q} u, u \rangle \, dV_\omega$

$A^{p,q} = A^{p,q}_{\substack{X,\omega \\ E,h}} = [i \Theta_{E,h}, \Lambda_\omega]$ (no problem here)

question: Can we apply this inequality by just assuming that $u \in \mathrm{Dom}_{L^2}(\bar\partial_E) \cap \mathrm{Dom}_{L^2}(\bar\partial_E^*)$
(so $u \in L^2$, $\bar\partial_E u \in L^2$ and $\bar\partial_E^* u \in L^2$ in sens of distributions)?

If we can, we could use Von Neumann theory.

↳ Taking $u \in \mathrm{Dom}_{L^2}(\bar\partial_E) \cap \mathrm{Dom}_{L^2}(\bar\partial_E^*)$

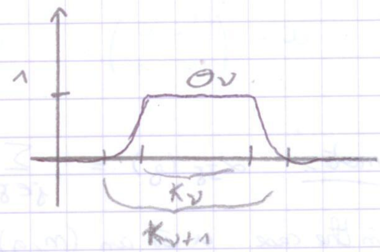
one would like to write $\begin{cases} u = \lim\limits_{\nu \to +\infty} u_\nu & \text{where } u_\nu \in \mathcal{D}_{p,q}(X,E) \\ \bar\partial_E u = \lim\limits_{\nu \to +\infty} \bar\partial_E u_\nu \\ \bar\partial_E^* u = \lim\limits_{\nu \to +\infty} \bar\partial_E^* u_\nu \end{cases}$

to do this, use a troncation:

$X = \bigcup K_\nu$ where $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ are compact sets

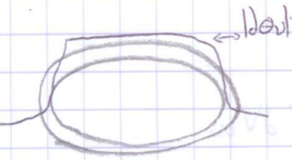
$\theta_\nu \in \mathcal{D}(X)$ $\theta_\nu = 1$ on K_ν and $\mathrm{Supp}\, \theta_\nu \subset K_{\nu+1}$

so $u = \lim\limits_{\nu \to +\infty} \theta_\nu u$ (OK in L^2)



but $\bar\partial_E(\theta_\nu u) = \underbrace{\bar\partial \theta_\nu \wedge u}_{\downarrow} + \underbrace{\theta_\nu \bar\partial_E u}_{\xrightarrow{L^2} \bar\partial_E u}$ OK

$\bar\partial \theta_\nu$ may become very large!
so this term don't do 0. problem!



Fact: For (M,g) a Riemannian manifold then the following are equivalent

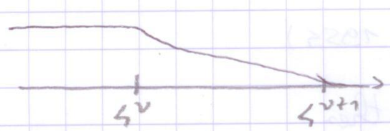
i: Geodesic distance d_g is complete (Hopf - Rinow lemma)
(so (M,g) is a complete metric space)

ii: $\forall a \in M$, $\forall R > 0$, $B_{d_g}(a,R) = \{x \in M \,/\, d_g(x,a) \leq R\}$ is compact

iii: Can write $M = \bigcup K_\nu$, $\exists\, \theta_\nu$ troncating functions $\theta_\nu = 1$ on K_ν and $\mathrm{Supp}(\theta_\nu) \subset K_{\nu+1}$ such that $\sup\limits_M |d\theta_\nu|_{d_g} \leq 2^{-\nu}$

so for $K_\nu = \{x \in \,/\, d_g(x,a) \leq 4^\nu\}$ (for example)

we have



the θ_ν "vanishes slowly" → so we can have $d\theta_\nu \xrightarrow[\nu \to +\infty]{} 0$

⇒ So by assuming that (X,ω) is a complete Kähler (ie: the godesic distance d_ω is complete), then $\forall u \in \mathrm{Dom}_{L^2}(\bar\partial_E) \cap \mathrm{Dom}_{L^2}(\bar\partial_E^*)$ we have
$\|\bar\partial_E u\|^2 + \|\bar\partial_E^* u\|^2 \geq \int_X \langle A^{p,q} u, u \rangle \, dV_\omega$ provided that $A^{p,q} \geq 0$

apply to $u_\nu = \rho_{1,\nu} * (\theta_\nu u)$

This applies to
- $X = \mathbb{C}^n$ $\quad \omega = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$
- $X = \mathbb{D}$ Poincaré disc model with metric $\frac{|dz|^2}{(1-|z|^2)^2}$ $\quad \int_t^1 \frac{1}{1-r^2} dr$ 
- $X = \mathbb{B}^n$ $\quad \omega = i\partial\bar\partial \log \frac{1}{1-|z|^2}$ is complete

<u>Definition :</u> $(X, \mathcal{O}_X)$ will be called weakly pseudoconvex if $\exists \Psi : X \longrightarrow [0;+\infty[$ exhaustion psh function

(Ψ exhaustion $\Leftrightarrow \forall c > 0 \ \{z \in X \,/\, \Psi(z) \leq c\} \subset\subset X$

$\forall A,\ \exists K_A$ compact in X such that $\Psi(z) > A$ on $X \setminus K_A$

<u>Proposition :</u> Assume (X, ω) is Kähler

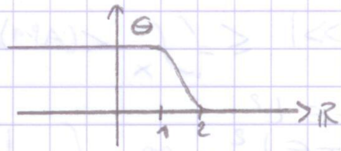
Define $\hat{\omega}_\varepsilon = \omega + \varepsilon i \partial\bar\partial(\Psi^2)$

$= \omega + \varepsilon 2i(\Psi\partial\bar\partial\Psi + \partial\Psi \wedge \bar\partial\Psi)$

take $\theta_\nu(z) = \theta(2^{-\nu}\Psi(z))$

$\theta_\nu = \begin{cases} 1 & \text{on } K_\nu = \{z \,/\, \Psi(z) \leq 2^\nu\} \\ 0 & \text{on } X \setminus K_{\nu+1} \end{cases}$

$\operatorname{Supp} \theta_\nu \subset K_{\nu+1}$



$d\theta_\nu = \partial\theta_\nu + \bar\partial\theta_\nu$

$\partial\theta_\nu = 2^{-\nu} \theta'(2^{-\nu}\Psi(z))\, \partial\Psi$ $\qquad \operatorname{Supp}(\theta') \subset [1,2]$

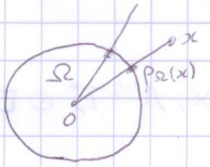
so $\hat{\omega}_\varepsilon \geq 2i\,\varepsilon\, \partial\Psi \wedge \bar\partial\Psi$

$\Rightarrow |\partial\Psi|^2_{\hat{\omega}_\varepsilon} \leq \frac{1}{2\varepsilon}$

$\sup_X |\partial\theta_\nu|_{\hat\omega_\varepsilon} \leq \frac{C}{\varepsilon} 2^{-\nu}$

$\Longrightarrow \hat{\omega}_\varepsilon = \omega + \varepsilon i \partial\bar\partial(\Psi^2)$ is a complete Kähler metric ✓

As a consequence, this tricks works for any convex domain $\Omega \subset \mathbb{C}^n$ with a convex exhaustion function Ψ.



$p_\Omega(x) = \inf\{\lambda > 0 \,/\, \frac{x}{\lambda} \in \Omega\}$

then $\Psi(x) = \frac{1}{1 - p_\Omega(x)}$ is an exhaustion function on $\Omega \subset \mathbb{C}^n$ convex.

Apply Von Neumann theory of L^2 adjoints

Assuming
- (X, ω) complete Kähler
- $A^{p,q} = A^{p,q}_{X,\omega,E,h} > 0$ on (p,q)-forms

Question: Solve $\bar\partial$ equation : $\bar\partial_E u = v$
where $v \in L^2(X, \Lambda^{p,q} T^*_X \otimes E)$ and $\bar\partial_E v = 0$

We look at the linear form $\bar\partial_E^* f \longrightarrow \langle\langle f, v\rangle\rangle$
for $f \in \mathrm{Dom}_{L^2}(\bar\partial_E^*) \subset L^2(X, \Lambda^{p,q} T^*_X \otimes E)$, $v \in \ker \bar\partial_E$ closed subset of $L^2(X, \Lambda^{p,q} T^*_X \otimes E)$

so
$$L^2(X, \Lambda^{p,q} T^*_X \otimes E) = (\ker \bar\partial_E) \oplus (\ker \bar\partial_E)^\perp$$
$$f = g \oplus h$$

Von Neumann theory $\Rightarrow (\ker \bar\partial_E)^\perp = \overline{\mathrm{Im}\, \bar\partial_E^*} \subset \ker \bar\partial_E^*$ because $(\bar\partial_E^*)^2 = 0$

then $\bar\partial_E^* h = 0$

We have
$$\langle\langle f, v\rangle\rangle = \langle\langle g, v\rangle\rangle$$
$$= \int_X \langle g(z), v(z)\rangle \, dV_\omega$$
$$= \int_X \langle (A^{p,q})^{1/2} g, (A^{p,q})^{-1/2} v\rangle \, dV_\omega \qquad A^{p,q} > 0$$

$$\Rightarrow |\langle\langle f, v\rangle\rangle| \leq \left(\int_X |(A^{p,q})^{1/2} g|^2 \, dV_\omega\right)^{1/2} \left(\int_X |(A^{p,q})^{-1/2} v|^2 dV_\omega\right)^{1/2} \text{ Cauchy-Schwarz}$$

We need $v \in L^2$, i.e. $\int_X |v|^2 dV_\omega < +\infty$, and
$$C = \int_X |(A^{p,q})^{-1/2} v|^2 dV_\omega = \int_X \langle (A^{p,q})^{-1} v, v\rangle \, dV_\omega < +\infty.$$

Then
$$|\langle\langle f, v\rangle\rangle| \leq C^{1/2} \left(\int_X \langle A^{p,q} g, g\rangle \, dV_\omega\right)^{1/2}$$
because $\langle A^{p,q} u, v\rangle = \langle A^{p,q\,1/2} u, A^{p,q\,1/2} v\rangle$

$$\underset{\text{BKN}}{\leq} C^{1/2} \left(\|\cancel{\bar\partial_E g}\|^2 + \|\bar\partial_E^* g\|^2\right)^{1/2}$$
$g \in \ker \bar\partial_E$

$$|\langle\langle f, v\rangle\rangle| \leq C^{1/2} \|\bar\partial_E^* g\|$$

remember $h \in (\ker \bar\partial_E)^\perp$ so $\bar\partial_E^* h = 0 \Rightarrow \bar\partial_E^* f = \bar\partial_E^* g$

$$\Rightarrow |\langle\langle f, v\rangle\rangle| \leq C^{1/2} \|\bar\partial_E^* f\|$$

$\Rightarrow \bar\partial_E^* f \longrightarrow \langle\langle f, v\rangle\rangle$ is a continuous linear form!

$\Rightarrow \bar\partial_E^* f \longrightarrow \langle\langle f, v\rangle\rangle$ is a well defined linear form on $\overline{\mathrm{Im}\, \bar\partial_E^*} \subset L^2(X, \Lambda^{p,q} T^*_X \otimes E)$

Theory of Hilbert spaces $\Rightarrow$ $\exists!\ u \in \overline{\mathrm{Im}\,\bar\partial_E^*}$ such that $\forall f \in \mathrm{Dom}(\bar\partial_E^*)$

$$\langle\langle f, v \rangle\rangle = \langle\langle \bar\partial_E^* f, u \rangle\rangle$$

we can take here $f \in \mathcal{D}(X, \Lambda^{p,q} T_X^* \otimes E)$

This means $\langle\langle f, v \rangle\rangle = \langle\langle f, \bar\partial_E u \rangle\rangle$ $\forall f$ in sens of distributions

$$\Rightarrow \boxed{\bar\partial_E u = v}$$

In fact, the solution $u \in \overline{\mathrm{Im}\,\bar\partial_E^*} \subset \ker \bar\partial_E^*$, $\overline{\mathrm{Im}\,\bar\partial_E^*} = (\ker \bar\partial_E)^\perp$

$$\Rightarrow \begin{cases} \bar\partial_E u = v \\ \bar\partial_E^* u = 0 \end{cases}$$

actually u is unique if taken in $(\ker \bar\partial_E)^\perp$

$$\Box_E u = \bar\partial_E \cancel{\bar\partial_E^* u} + \bar\partial_E^* \bar\partial_E u$$
$$= \bar\partial_E^* v$$

so if $v \in C^\infty$, by $\Box_E u = \bar\partial_E^* v$ and ellipticity of $\Box_E$, the solution u is C^∞

Since the norm of the linear form is $\|u\|$, we also get $\|u\| \le C^{1/2}$, that is,

$$\int_X |u|^2\, dV_\omega \le C = \int_X \langle (A^{p,q})^{-1} v, v\rangle\, dV_\omega.$$

We have therefore proved the following result.

Theorem (S. Bochner, K. Kodaira, S. Nakano, J. Kohn, A. Andreotti - E. Vesentini, L. Hörmander and continuators)
Let (X,ω) be a complete Kähler manifold and (E,h) a hermitian holomorphic vector bundle over X. Assume that the self-adjoint operator

$$A^{p,q} = A^{p,q}_{X,\omega\,;\,E,h} := [\Theta_{E,h}, \Lambda_\omega]$$

is positive definite on $\Lambda^{p,q}T^*_X \otimes E$. Then for every (p,q) form $v \in L^2(X, \Lambda^{p,q}T^*_X \otimes E)$ such that $\overline{\partial}_E v = 0$, the del-bar equation

$$\overline{\partial}_E u = v \tag{a}$$

admits a solution $u \in L^2(X, \Lambda^{p,q-1}T^*_X \otimes E)$ in the sense of distributions, such that

$$\int_X |u|^2\, dV_\omega \le \int_X \langle (A^{p,q})^{-1} v, v\rangle\, dV_\omega, \tag{b}$$

provided that the right hand side of (b) *is convergent.*

(c) *The solution of minimal L^2 norm is the one such that $u \in (\mathrm{Ker}\overline{\partial}_E)^\perp = \overline{\mathrm{Im}\overline{\partial}{}^*_E}$. This solution is unique and satisfies the additional property*

$$\overline{\partial}{}^*_E u = 0.$$

(d) *The minimal L^2 solution satisfies $\overline{\square}_E u = \overline{\partial}{}^*_E v$, therefore by ellipticity, one gets automatically $u \in C^\infty(X, \Lambda^{p,q-1}T^*_X \otimes E)$ if $v \in C^\infty(X, \Lambda^{p,q}T^*_X \otimes E)$.*

Corollary 1. *Let (X,ω) be a Kähler manifold (where ω is not necessarily complete), and let (E,h) be a hermitian holomomorphic line bundle such that $i\Theta_{E,h} > 0$ as a real $(1,1)$-form. Assume additionally that X is weakly pseudoconvex, i.e. that X possesses a smooth psh exhaustion function ψ. Then for every (n,q)-form v in $L^2_{\mathrm{loc}}(X, \Lambda^{p,q}T^*_X \otimes E)$ $(q \ge 1)$, such that $\overline{\partial}_E v = 0$ there exists v in $L^2_{\mathrm{loc}}(X, \Lambda^{p,q-1}T^*_X \otimes E)$ such that $\overline{\partial}_E u = v$ and*

$$\int_X |u|^2\, dV_\omega \le \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |v|^2\, dV_\omega$$

where $0 < \lambda_1(z) \le \cdots \le \lambda_n(z)$ are the eigenvalues of $i\Theta_{E,h}(z)$ with respect to $\omega(z)$.

Proof. When ω is complete and additionally $v \in L^2$, this is just a special case of the theorem. Otherwise, we can apply the theorem after replacing ω by $\widehat{\omega}_\varepsilon = \omega + \varepsilon i\partial\overline{\partial}(\psi^2)$ which is complete for any $\varepsilon > 0$. The integral involving v and $\widehat{\omega}_\varepsilon$ is then uniformly bounded by the same integral calculated for ω (exercise, see Lemma 6.3 in Chapter VIII of my online book). One then gets a L^2 solution u_ε with respect to $\widehat{\omega}_\varepsilon$. By weak compactness of closed balls in Hilbert spaces, it is easily shown that there is a weakly convergent sequence u_{ε_k}

converging to a solution u that is L^2 with respect to ω. In order to get rid of the global L^2 condition for v, one can likewise observe that $X_c = \{z \in X\,;\ \psi(z) < c\}$ is relatively compact in X and weakly pseudoconvex with psh exhaustion $\psi_c(z) = 1/(c - \psi(z))$. One then gets a solution u_c on X_c, and finally a global solution $u = \lim u_{c_k}$ as a weak limit for some subsequence $c_k \to +\infty$.

Corollary 2. *Let X be a Kähler weakly pseudoconvex manifold and (E, h) be a hermitian holomomorphic line bundle such that $i\Theta_{E,h} > 0$. Then $H^{p,q}(X, E) = 0$ for $p + q \geq n + 1$.*

Proof. Let ψ be a psh exhaustion. By replacing h with $h_\chi = h\, e^{-\chi\circ\psi}$ where $\chi : \mathbb{R} \to \mathbb{R}$ is a fast increasing convex function, and taking

$$\omega = \omega_\chi = i\theta_{E,h_\chi} = i\theta_{E,h} + i\partial\overline{\partial}\chi \circ \psi,$$

we can at the same time obtain that ω_χ is complete, and achieve the convergence of the integral

$$\int_X |v|^2_{h_\chi,\omega_\chi} dV_{\omega_\chi} \leq \int_X |v|^2_{h_\chi,\omega} dV_\omega = \int_X |v|^2_{h,\omega} e^{-\chi\circ\psi} dV_\omega$$

for any given $v \in C^\infty(X, \Lambda^{p,q}T^*_X \otimes E)$ with $\overline{\partial}_E v = 0$ (here the eigenvalues are equal to 1 and $A^{p,q} = (p + q - n)\,\mathrm{Id}\,)$.